Tensor Algebra Using xAct

INTRODUCTION

Calculations in GR involves long, tedious calculations involving manipulations of tensor equations.
These calculations are prone to errors when done manually and also time consuming.
But, these kind of calculations are ideal for a computer algorithm. Using these computer algorithms to do these Tensor manipulations makes these calculations easier to reproduce.

Many Tensor algebra tools (Abstract and explicit component calculations) and have been developed over the years. Similar packages are available in almost all programming languages.
here we will be briefly discussing the xAct package. xAct is a suite of free packages for tensor computer algebra in the Wolfram Language. xAct implements state-of-the-art algorithms for fast manipulations of indices
and has been modelled on the current geometric approach to General Relativity. It is highly programmable and configurable. Since its first public release in March 2004, xAct has been intensively tested and has solved
a number of hard problems in GR.

INSTALLATION

As xAct as a package written for Mathematica , you first need to be able to run Mathematica on your device to use xAct.
The detailed instructions for installation of xAct once you have installed Mathematica is given on the official website.

xAct webpage:

```
Hyperlink["http://www.xact.es/index.html"]
```

```
Out[]= http://www.xact.es/index.html
```

Installation Notes :

```
In[]:= Hyperlink["http://www.xact.es/download/install"]
```

```
Out[]= http://www.xact.es/download/install
```

Step 1 : Download the compressed file from the webpage and unzip the files.
Step 2 : Place all the files in the directory xAct/ must be placed at one of the places Mathematica prepares for external
applications (look at the Installation Notes for your Operating system)

Why xAct ?

1. xAct is a free open source package for Mathematica
2. Extensive Documentation
3. Strong community of users (Can join the Google group from the webpage)
4. Friendly and helpful developer community
5. Newer packages are being developed and added to the Contributed packages
6. Going to stay open sourced

7. Extremely generic and powerful

    Only a few domain specific assumptions are made.

    Arbitrary dimensions , with and without a metric

8. Extensively tested and validated
9. Modular system

    Core-modules : xCore, xPerm, xTensor,xCoba,

    Problem specific modules : xPert , Harmonics ,xPand nd many more

Basics of xAct

We will now look at some simple examples to understand how to use this package

1. Loading the package

```
In[ ]:= << xAct`xTensor`
```

```
------------------------------------------------------------
Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}
CopyRight (C) 2003-2020, Jose M. Martin-Garcia, under the General Public License.
Connecting to external MinGW executable...
Connection established.
------------------------------------------------------------
Package xAct`xTensor`  version 1.1.5, {2021, 2, 28}
CopyRight (C) 2002-2021, Jose M. Martin-Garcia, under the General Public License.
------------------------------------------------------------
These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
```

2. Define a Manifold

Defining a 4 dimensional manifold named M4, with indices a, b ,c .... except {g and h}

In[ ]:=
```
DefManifold[M4, 4 , Complement[IndexRange[a, z], {g, h}]]
```

```
** DefManifold: Defining manifold M4.
** DefVBundle: Defining vbundle TangentM4.
```

3. Defining a Metric

Here CD is the covariant derivative associated with the metric g
Defining the metric automatically defines Riemann,Ricci,Weyl,Einstein tensor, Levi -Civita and Christoffel symbols.

In[ ]:=
```
DefMetric[-1, g[-a, -b], CD]
```

```
** DefTensor: Defining symmetric metric tensor g[-a, -b].
** DefTensor: Defining antisymmetric tensor epsilong[-a, -b, -c, -d].
** DefTensor: Defining tetrametric Tetrag[-a, -b, -c, -d].
** DefTensor: Defining tetrametric Tetrag† [-a, -b, -c, -d].
** DefCovD: Defining covariant derivative CD[-a].
** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -b, -c].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -b, -c].
** DefTensor: Defining Riemann tensor RiemannCD[-a, -b, -c, -d].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-a, -b].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-a, -b].
** DefTensor: Defining Weyl tensor WeylCD[-a, -b, -c, -d].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-a, -b].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detg[]. Determinant.
```

In[ ]:=
```
{g[-a, -b] , g[a, b], g[-a, -b] * g[a, c], g[-a, -b] * g[a, b]}
```

Out[ ]= $\{ g_{ab} ,\ g^{ab} ,\ \delta_b{}^c ,\ 4\}$

In[]:=
```
{RiemannCD[a, -b, -c, -d], RicciCD[-a, -b],
 EinsteinCD[-a, -b], epsilong[a, b, c, d], ChristoffelCD[a, -b, -c]}
```

Out[]= $\{R[\nabla]^{a}{}_{bcd}, R[\nabla]_{ab}, G[\nabla]_{ab}, \epsilon g^{abcd}, \Gamma[\nabla]^{a}{}_{bc}\}$

All the tensors defined along with the metric posses their symmetries. we don't have to explicitly define the symmetry properties of these tensors.

In[]:=
```
RiemannCD[-a, -b, -c, -d] + RiemannCD[-c, -d, -a, -b]
```

Out[]= $R[\nabla]_{abcd} + R[\nabla]_{cdab}$

4.Canonicalization

Canonicalization is not done automatically.One of the reasons why it is not made automatic is because canonicalization is a computationally expensive operation , so It would take longer to complete computations
if canonicalization is done at every step.
ToCanonical looks for symmetries in the expression and gives us the simplest form of the expression that we can obtain by using the symmetries of the tensors in the expression

In[]:=
```
ToCanonical[%]
```

Out[]= $2\ R[\nabla]_{abcd}$

In[]:=
```
RiemannCD[-a, -b, -c, d] // RiemannToChristoffel
```

Out[]= $\Gamma[\nabla]^{d}{}_{bz\$10409}\ \Gamma[\nabla]^{z\$10409}{}_{ac} - \Gamma[\nabla]^{d}{}_{az\$10409}\ \Gamma[\nabla]^{z\$10409}{}_{bc} - \partial_a\Gamma[\nabla]^{d}{}_{bc} + \partial_b\Gamma[\nabla]^{d}{}_{ac}$

So, xAct automatically takes these weird dummy indices if , not mentioned to not to do so.
To prevent this from happening

In[]:=
```
$PrePrint = ScreenDollarIndices;
```

In[]:=
```
RiemannCD[-a, -b, -c, -d] // RiemannToChristoffel
```

Out[]= $g_{de}\ \left(\Gamma[\nabla]^{e}{}_{bf}\ \Gamma[\nabla]^{f}{}_{ac} - \Gamma[\nabla]^{e}{}_{af}\ \Gamma[\nabla]^{f}{}_{bc} - \partial_a\Gamma[\nabla]^{e}{}_{bc} + \partial_b\Gamma[\nabla]^{e}{}_{ac}\right)$

In[]:=
```
RiemannCD[-a, -b, -c, -d] // RiemannToChristoffel // ChristoffelToGradMetric
```

Out[]=
$$g_{de}\ \Bigg(\frac{1}{4}\ g^{ei}\ g^{fj}\ (\partial_b g_{fi} + \partial_f g_{bi} - \partial_i g_{bf})\ (\partial_a g_{cj} + \partial_c g_{aj} - \partial_j g_{ac}) -$$
$$\frac{1}{4}\ g^{ek}\ g^{fl}\ (\partial_a g_{fk} + \partial_f g_{ak} - \partial_k g_{af})\ (\partial_b g_{cl} + \partial_c g_{bl} - \partial_l g_{bc}) +$$
$$\frac{1}{2}\left(-g^{em}\ (\partial_a\partial_b g_{cm} + \partial_a\partial_c g_{bm} - \partial_a\partial_m g_{bc}) + g^{en}\ g^{mo}\ \partial_a g_{no}\ (\partial_b g_{cm} + \partial_c g_{bm} - \partial_m g_{bc})\right) +$$
$$\frac{1}{2}\left(g^{ep}\ (\partial_b\partial_a g_{cp} + \partial_b\partial_c g_{ap} - \partial_b\partial_p g_{ac}) - g^{eq}\ g^{pr}\ \partial_b g_{qr}\ (\partial_a g_{cp} + \partial_c g_{ap} - \partial_p g_{ac})\right)\Bigg)$$

So now it uses all the indices available in the definition of the manifold for the required additional dummy indices.

Expand Einstein tensor in terms of Ricci tensor

In[]:= **EinsteinCD[-a, -b] // EinsteinToRicci**

Out[]= $R[\nabla]_{ab} - \frac{1}{2} g_{ba} R[\nabla]$

5.Defining other tensors of our choice

In[]:= **DefTensor[V[a], M4]**

```
** DefTensor: Defining tensor V[a].
```

In[]:= **V[a]**

Out[]= V^a

In[]:= **delta[-a, b]**

Out[]= $\delta_a{}^b$

In[]:= **delta[-a, b] × V[a]**

Out[]= V^b

We can also define more complex tensors like defining a tensors and its symmetries.
we will define an antisymmetric covariant vector field:

In[]:= **DefTensor[F[-a, -b], M4, Antisymmetric[{-a, -b}]]**

```
** DefTensor: Defining tensor F[-a, -b].
```

In[]:= **F[-a, -b]**

Out[]= F_{ab}

In[]:= **F[-b, -a]**

Out[]= F_{ba}

In[]:= **% // ToCanonical**

Out[]= $-F_{ab}$

6. Undefining the defined quantities

In[]:= **UndefTensor[F]**

```
** UndefTensor: Undefined tensor F
```

In[]:= **UndefTensor[V]**

```
** UndefTensor: Undefined tensor V
```

In[]:= **UndefMetric[g]**

```
** UndefTensor: Undefined weight +2 density Detg
** UndefTensor: Undefined symmetric Christoffel tensor ChristoffelCD
** UndefTensor: Undefined symmetric Einstein tensor EinsteinCD
```

```
** UndefTensor: Undefined Kretschmann scalar KretschmannCD
** UndefTensor: Undefined symmetric Ricci tensor RicciCD
** UndefTensor: Undefined Ricci scalar RicciScalarCD
** UndefTensor: Undefined Riemann tensor RiemannCD
** UndefTensor: Undefined symmetric TFRicci tensor TFRicciCD
** UndefTensor: Undefined torsion tensor TorsionCD
** UndefTensor: Undefined Weyl tensor WeylCD
** UndefCovD: Undefined covariant derivative CD
** UndefTensor: Undefined tetrametric Tetrag†
** UndefTensor: Undefined tetrametric Tetrag
** UndefTensor: Undefined antisymmetric tensor epsilong
** UndefTensor: Undefined symmetric metric tensor g
```

7. Covariant Derivatives

In[∘]:= **DefCovD[CD[-a]]**

```
** DefCovD: Defining covariant derivative CD[-a].
** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -b, -c].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -b, -c].
** DefTensor: Defining Riemann tensor
 RiemannCD[-a, -b, -c, d]. Antisymmetric only in the first pair.
** DefTensor: Defining non-symmetric Ricci tensor RicciCD[-a, -b].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
```

Note : Several other objects are automatically defined along with Covariant derivates.

In[∘]:= **DefTensor[V[-a], M4]**

```
** DefTensor: Defining tensor V[-a].
```

In[∘]:= **CD[-a][V[b]]**

Out[∘]= $\nabla_a V^b$

In[∘]:= **DefTensor[F[-a, -b], M4]**

```
** DefTensor: Defining tensor F[-a, -b].
```

In[∘]:= **CD[-a][F[-b, -c] × V[c]]**

Out[∘]= $V^c \left(\nabla_a F_{bc}\right) + F_{bc} \left(\nabla_a V^c\right)$

In[∘]:= **% // ToCanonical**

Out[∘]= $V^c \left(\nabla_a F_{bc}\right) + F_{bc} \left(\nabla_a V^c\right)$

Note : The Leibnitz rule is applied automatically

Now Lets look at some of the tensors associated with the Covariant derivative.

In[ ]:=
```
{ChristoffelCD[a, -b, -c], RiemannCD[-a, -b, -c, d], RicciCD[-a, -b]}
```

Out[ ]= $\left\{\Gamma[\nabla]^{a}{}_{bc},\ R[\nabla]_{abc}{}^{d},\ R[\nabla]_{ab}\right\}$

Note : Now RiemannCD is not associated with the metric, Riemann Tensor associated with covariant derivative is only antisymmetric in one pair:

In[ ]:=
```
UndefTensor[{F, V}]
```

```
** UndefTensor: Undefined tensor F
** UndefTensor: Undefined tensor V
```

In[ ]:=
```
UndefCovD[CD]
```

```
** UndefTensor: Undefined symmetric Christoffel tensor ChristoffelCD
** UndefTensor: Undefined non-symmetric Ricci tensor RicciCD
** UndefTensor: Undefined Riemann tensor RiemannCD
** UndefTensor: Undefined torsion tensor TorsionCD
** UndefCovD: Undefined covariant derivative CD
```

In[ ]:=
```
UndefManifold[M4]
```

```
** UndefVBundle: Undefined vbundle TM4
** UndefManifold: Undefined manifold M4
```

In[ ]:=
```
exit
```

Out[ ]= exit

Verifying the Results obtained in GR notes using xACT

Now Lets look at some more solved examples from the GR notes and verify the results obtained using xAct.
We will again start from defining the Manifold and so on.

In[ ]:=
```
<< xAct`xTensor`
```

```
------------------------------------------------------------
Package xAct`xTensor`  version 1.1.5, {2021, 2, 28}
CopyRight (C) 2002-2021, Jose M. Martin-Garcia, under the General Public License.
------------------------------------------------------------
These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
```

Defining the Manifold and Metric

```
In[ ]:= DefManifold[M, 4, {α, β, μ, ρ, σ, ν, λ, ϕ, ψ}]
** DefManifold: Defining manifold M.
** DefVBundle: Defining vbundle TangentM.
In[ ]:= DefMetric[-1, g[-α, -β], CD]
** DefTensor: Defining symmetric metric tensor g[-α, -β].
** DefTensor: Defining antisymmetric tensor epsilong[-α, -β, -λ, -μ].
** DefTensor: Defining tetrametric Tetrag[-α, -β, -λ, -μ].
** DefTensor: Defining tetrametric Tetrag†[-α, -β, -λ, -μ].
** DefCovD: Defining covariant derivative CD[-α].
** DefTensor: Defining vanishing torsion tensor TorsionCD[α, -β, -λ].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[α, -β, -λ].
** DefTensor: Defining Riemann tensor RiemannCD[-α, -β, -λ, -μ].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-α, -β].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-α, -β].
** DefTensor: Defining Weyl tensor WeylCD[-α, -β, -λ, -μ].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-α, -β].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detg[]. Determinant.
```

Covariant Derivatives

Defining the covariant derivative

In[ ]:=
```
Options[DefCovD]
```

Out[ ]=
```
{SymbolOfCovD → {;, ∇}, Torsion → False, Curvature → True,
 FromMetric → Null, CurvatureRelations → True, ExtendedFrom → Null,
 OtherDependencies → {}, OrthogonalTo → {}, ProjectedWith → {},
 WeightedWithBasis → Null, ProtectNewSymbol :→ $ProtectNewSymbols,
 Master → Null, DefInfo → {covariant derivative, }}
```

In[ ]:=
```
DefCovD[cd[-α]]
```

```
** DefCovD: Defining covariant derivative cd[-α].
** DefTensor: Defining vanishing torsion tensor Torsioncd[α, -β, -λ].
** DefTensor: Defining symmetric Christoffel tensor Christoffelcd[α, -β, -λ].
** DefTensor: Defining Riemann tensor
 Riemanncd[-α, -β, -λ, μ]. Antisymmetric only in the first pair.
** DefTensor: Defining non-symmetric Ricci tensor Riccicd[-α, -β].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
```

Defining a contravariant vector field

In[ ]:=
```
DefTensor[V[μ], M]
V[μ]
```

ValidateSymbol: Symbol V is already used as a tensor.

Out[ ]= V^{μ}

Defining a contravariant Tensor of Rank 2

In[ ]:=
```
DefTensor[T[-μ, -ν], M]
T[-μ, -ν]
```

```
** DefTensor: Defining tensor T[-μ, -ν].
```

Out[ ]= $T_{\mu\nu}$

Defining a mixed tensor of Rank 3

In[ ]:=
```
DefTensor[J[-μ, -ν, ρ], M]
J[-μ, -ν, ρ]
```

```
** DefTensor: Defining tensor J[-μ, -ν, ρ].
```

Out[ ]= $J_{\mu\nu}{}^{\rho}$

Let us consider the covariant derivatives of a contravariant vector field V

In[ ]:=
```
$PrePrint = ScreenDollarIndices;
CD[-β][V[μ]]
```

Out[ ]= $\nabla_{\beta} V^{\mu}$

In[ ]:=
```
% // CovDToChristoffel
```

Out[ ]= $\Gamma[\nabla]^{\mu}{}_{\beta\alpha}\ V^{\alpha} + \partial_{\beta}V^{\mu}$

So we have checked the the expression of covariant derivative of a contravariant vector field V and got

$$\nabla_{\beta} V^{\mu} = \Gamma[\nabla]^{\mu}{}_{\beta\alpha}\ V^{\alpha} + \partial_{\beta}V^{\mu}$$

Let us try to write the expression covariant derivatives in terms of the metric

1. $\nabla_{\beta} V^{\mu}$

In[ ]:=
```
CD[-β][V[μ]] // CovDToChristoffel // ChristoffelToGradMetric
```

Out[ ]= $\partial_{\beta}V^{\mu} + \frac{1}{2}\, g^{\mu\lambda}\ V^{\alpha}\ \left(\partial_{\alpha}g_{\beta\lambda} + \partial_{\beta}g_{\alpha\lambda} - \partial_{\lambda}g_{\beta\alpha}\right)$

2. $\nabla_{\nu} V^{\mu}$

In[ ]:=
```
CD[-ν][V[μ]] // CovDToChristoffel // ChristoffelToGradMetric
```

Out[ ]= $\frac{1}{2}\, g^{\mu\beta}\ V^{\alpha}\ \left(\partial_{\alpha}g_{\nu\beta} - \partial_{\beta}g_{\nu\alpha} + \partial_{\nu}g_{\alpha\beta}\right) + \partial_{\nu}V^{\mu}$

3. $\nabla_{\alpha} \nabla_{\beta} V^{\mu}$

In[ ]:=
```
CD[-α][CD[-β][V[μ]]] // CovDToChristoffel // ChristoffelToGradMetric
```

Out[ ]=
$$\begin{aligned}
&\partial_{\alpha}\partial_{\beta}V^{\mu} + \\
&\frac{1}{2}\, V^{\lambda}\ \left(g^{\mu\nu}\ \left(\partial_{\alpha}\partial_{\beta}g_{\lambda\nu} + \partial_{\alpha}\partial_{\lambda}g_{\beta\nu} - \partial_{\alpha}\partial_{\nu}g_{\beta\lambda}\right) - g^{\mu\rho}\ g^{\nu\sigma}\ \partial_{\alpha}g_{\rho\sigma}\ \left(\partial_{\beta}g_{\lambda\nu} + \partial_{\lambda}g_{\beta\nu} - \partial_{\nu}g_{\beta\lambda}\right)\right) + \\
&\frac{1}{2}\, g^{\mu\nu}\ \partial_{\alpha}V^{\lambda}\ \left(\partial_{\beta}g_{\lambda\nu} + \partial_{\lambda}g_{\beta\nu} - \partial_{\nu}g_{\beta\lambda}\right) + \\
&\frac{1}{2}\, g^{\mu\rho}\ \left(\partial_{\alpha}g_{\nu\rho} + \partial_{\nu}g_{\alpha\rho} - \partial_{\rho}g_{\alpha\nu}\right)\left(\partial_{\beta}V^{\nu} + \frac{1}{2}\, g^{\nu\sigma}\ V^{\lambda}\ \left(\partial_{\beta}g_{\lambda\sigma} + \partial_{\lambda}g_{\beta\sigma} - \partial_{\sigma}g_{\beta\lambda}\right)\right) - \\
&\frac{1}{2}\, g^{\nu\rho}\ \left(\partial_{\alpha}g_{\beta\rho} + \partial_{\beta}g_{\alpha\rho} - \partial_{\rho}g_{\alpha\beta}\right)\left(\partial_{\nu}V^{\mu} + \frac{1}{2}\, g^{\mu\sigma}\ V^{\lambda}\ \left(\partial_{\lambda}g_{\nu\sigma} + \partial_{\nu}g_{\lambda\sigma} - \partial_{\sigma}g_{\nu\lambda}\right)\right)
\end{aligned}$$

4. $\nabla_{\alpha} T_{\mu}{}^{\alpha}$

In[ ]:=
```
CD[-α][T[-μ, α]] // CovDToChristoffel // ChristoffelToGradMetric
```

Out[ ]= $\partial_{\alpha}T_{\mu}{}^{\alpha} + \frac{1}{2}\, g^{\alpha\lambda}\ T_{\mu}{}^{\beta}\ \left(\partial_{\alpha}g_{\beta\lambda} + \partial_{\beta}g_{\alpha\lambda} - \partial_{\lambda}g_{\alpha\beta}\right) - \frac{1}{2}\, g^{\beta\lambda}\ T_{\beta}{}^{\alpha}\ \left(\partial_{\alpha}g_{\mu\lambda} - \partial_{\lambda}g_{\alpha\mu} + \partial_{\mu}g_{\alpha\lambda}\right)$

$\nabla_{\alpha} J_{\mu}{}^{\beta}{}_{\nu}$

5. $\nabla_\alpha J_\mu{}^\beta{}_\nu$

In[ ]:=
```
CD[-α][J[-μ, β, -ν]] // CovDToChristoffel // ChristoffelToGradMetric
```

Out[ ]= $\partial_\alpha J_\mu{}^\beta{}_\nu + \frac{1}{2} g^{\beta\rho} J_\mu{}^\lambda{}_\nu (\partial_\alpha g_{\lambda\rho} + \partial_\lambda g_{\alpha\rho} - \partial_\rho g_{\alpha\lambda}) - \frac{1}{2} g^{\lambda\rho} J_\lambda{}^\beta{}_\nu (\partial_\alpha g_{\mu\rho} + \partial_\mu g_{\alpha\rho} - \partial_\rho g_{\alpha\mu}) - \frac{1}{2} g^{\lambda\rho} J_\mu{}^\beta{}_\lambda (\partial_\alpha g_{\nu\rho} + \partial_\nu g_{\alpha\rho} - \partial_\rho g_{\alpha\nu})$

Christoffel Symbols in terms of the metric

1. $\Gamma[\nabla]^\lambda{}_{\mu\nu}$

```
Christoffel[CD, PD][λ, -μ, -ν]
```

Out[ ]= $\Gamma[\nabla]^\lambda{}_{\mu\nu}$

In[ ]:=
```
% // ChristoffelToGradMetric
```

Out[ ]= $\frac{1}{2} g^{\lambda\alpha} (-\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha})$

$$\Gamma[\nabla]^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} (-\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha})$$

2. $\Gamma[\nabla]^\mu{}_{\mu\lambda}$

In[ ]:=
```
Christoffel[CD, PD][μ, -μ, -λ]
```

Out[ ]= $\Gamma[\nabla]^\mu{}_{\mu\lambda}$

In[ ]:=
```
% // ChristoffelToGradMetric
```

Out[ ]= $\frac{1}{2} g^{\mu\alpha} (-\partial_\alpha g_{\mu\lambda} + \partial_\lambda g_{\mu\alpha} + \partial_\mu g_{\lambda\alpha})$

$$\Gamma[\nabla]^\mu{}_{\mu\lambda} = \frac{1}{2} g^{\mu\alpha} (-\partial_\alpha g_{\mu\lambda} + \partial_\lambda g_{\mu\alpha} + \partial_\mu g_{\lambda\alpha})$$

Covariant Derivative of the metric

In[ ]:=
```
CD[-μ][g[ν, λ]]
```

Out[ ]= 0

i.e. Covariant Derivative of Metric = 0

Now lets consider taking the covariant derivative twice (Writing the expressions in terms of the Christoffel Symbols

In[ ]:=
```
CD[-α][CD[-β][V[μ]]]
```

Out[∘]= $\nabla_\alpha \nabla_\beta V^\mu$

In[∘]:=
```
% // CovDToChristoffel // ToCanonical
```

ToCanonical: Detected metric–incompatible derivatives {PD}.

Out[∘]= $\Gamma[\nabla]^{\mu}{}_{\alpha}{}^{\nu}\ \Gamma[\nabla]_{\nu\beta\lambda}\ V^\lambda - \Gamma[\nabla]^{\mu}{}_{\lambda\nu}\ \Gamma[\nabla]^{\nu}{}_{\alpha\beta}\ V^\lambda + V^\lambda\ \partial_\alpha \Gamma[\nabla]^{\mu}{}_{\beta\lambda} + \Gamma[\nabla]^{\mu}{}_{\beta\lambda}\ \partial_\alpha V^\lambda + \Gamma[\nabla]^{\mu}{}_{\alpha\lambda}\ \partial_\beta V^\lambda + \partial_\beta\partial_\alpha V^\mu - \Gamma[\nabla]_{\lambda\alpha\beta}\ \partial^\lambda V^\mu$

$$\nabla_\alpha \nabla_\beta V^\mu = \Gamma[\nabla]^{\mu}{}_{\alpha}{}^{\nu}\ \Gamma[\nabla]_{\nu\beta\lambda}\ V^\lambda - \Gamma[\nabla]^{\mu}{}_{\lambda\nu}\ \Gamma[\nabla]^{\nu}{}_{\alpha\beta}\ V^\lambda + V^\lambda\ \partial_\alpha \Gamma[\nabla]^{\mu}{}_{\beta\lambda} + \Gamma[\nabla]^{\mu}{}_{\beta\lambda}\ \partial_\alpha V^\lambda + \Gamma[\nabla]^{\mu}{}_{\alpha\lambda}\ \partial_\beta V^\lambda + \partial_\beta\partial_\alpha V^\mu - \Gamma[\nabla]_{\lambda\alpha\beta}\ \partial^\lambda V^\mu$$

$$\nabla_\alpha \nabla_\beta V^\mu$$

$$= \nabla_\alpha \left(\nabla_\beta V^\mu\right)$$

$$= \frac{\partial}{\partial x^\alpha}\left(\nabla_\beta V^\mu\right) - \Gamma^\sigma_{\beta\alpha}\nabla_\sigma V^\mu + \Gamma^\mu_{\sigma\alpha}\nabla_\beta V^\sigma$$

Here we have used covariant derivative of mixed tensor $\nabla_\beta V^\mu$

$$= \frac{\partial}{\partial x^\alpha}\left\{\frac{\partial V^\mu}{\partial x^\beta} + \Gamma^\mu_{\beta\rho} V^\rho\right\}$$

$$- \Gamma^\sigma_{\beta\alpha}\left\{\frac{\partial V^\mu}{\partial x^\sigma} + \Gamma^\mu_{\sigma\rho} V^\rho\right\}$$

$$+ \Gamma^\mu_{\sigma\alpha}\left\{\frac{\partial V^\sigma}{\partial x^\beta} + \Gamma^\sigma_{\beta\rho} V^\rho\right\}.$$

Here we have used covariant derivative of contravariant component of the vector V

$$= \frac{\partial^2 V^\mu}{\partial x^\alpha \partial x^\beta} + \left(\frac{\partial}{\partial x^\alpha}\Gamma^\mu_{\beta\rho}\right)V^\rho + \Gamma^\mu_{\beta\rho}\frac{\partial V^\rho}{\partial x^\alpha}$$

$$- \Gamma^\sigma_{\beta\alpha}\frac{\partial V^\mu}{\partial x^\sigma} - \Gamma^\sigma_{\beta\alpha}\Gamma^\mu_{\sigma\rho} V^\rho$$

$$+ \Gamma^\mu_{\sigma\alpha}\frac{\partial V^\sigma}{\partial x^\beta} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\beta\rho} V^\rho$$

Now Consider the following commutator of covariant derivative ($\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu$)

In[∘]:=
```
CD[-α]@CD[-β][V[μ]] - CD[-β]@CD[-α][V[μ]]
```

Out[•]= $\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu$

In[•]:=
```
% // SortCovDs
```

Out[•]= $R[\nabla]_{\beta\alpha\lambda}{}^{\mu}\; V^\lambda$

In[•]:=
```
CD[-α]@CD[-β][V[μ]] - CD[-β]@CD[-α][V[μ]]
```

Out[•]= $\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu$

In[•]:=
```
% // CovDToChristoffel // ToCanonical // Simplify
```

ToCanonical: Detected metric–incompatible derivatives {PD}.

ToCanonical: Detected metric–incompatible derivatives {PD}.

Out[•]= $V^\lambda \left(-\Gamma[\nabla]^{\mu}{}_{\beta}{}^{\nu}\, \Gamma[\nabla]_{\nu\alpha\lambda} + \Gamma[\nabla]^{\mu}{}_{\alpha}{}^{\nu}\, \Gamma[\nabla]_{\nu\beta\lambda} + \partial_\alpha \Gamma[\nabla]^{\mu}{}_{\beta\lambda} - \partial_\beta \Gamma[\nabla]^{\mu}{}_{\alpha\lambda}\right)$

So, we got $\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = R[\nabla]_{\beta\alpha\rho}{}^{\mu}\; V^\rho$

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu$$

$$= \left\{ \frac{\partial^2 V^\mu}{\partial x^\alpha \partial x^\beta} + \left(\frac{\partial}{\partial x^\alpha}\Gamma^\mu_{\beta\rho}\right)V^\rho + \Gamma^\mu_{\beta\rho}\frac{\partial V^\rho}{\partial x^\alpha} - \Gamma^\sigma_{\beta\alpha}\frac{\partial V^\mu}{\partial x^\sigma} - \Gamma^\sigma_{\beta\alpha}\Gamma^\mu_{\sigma\rho}V^\rho + \Gamma^\mu_{\sigma\alpha}\frac{\partial V^\sigma}{\partial x^\beta} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\beta\rho}V^\rho \right\}$$

$$- \left\{ \frac{\partial^2 V^\mu}{\partial x^\beta \partial x^\alpha} + \left(\frac{\partial}{\partial x^\beta}\Gamma^\mu_{\alpha\rho}\right)V^\rho + \Gamma^\mu_{\alpha\rho}\frac{\partial V^\rho}{\partial x^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial V^\mu}{\partial x^\sigma} - \Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\rho}V^\rho + \Gamma^\mu_{\sigma\rho}\frac{\partial V^\sigma}{\partial x^\alpha} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\rho}V^\rho \right\}$$

$$= \left[\frac{\partial}{\partial x^\alpha}\Gamma^\mu_{\beta\rho} - \frac{\partial}{\partial x^\beta}\Gamma^\mu_{\alpha\rho} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\beta\rho} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\rho}\right]V^\rho$$

$$R^\mu{}_{\rho\alpha\beta}V^\rho = R^\mu{}_{\nu\alpha\beta}V^\nu \quad \text{......(a)}$$ [Replacing ρ by ν]

$$= \left[\frac{\partial}{\partial x^\alpha}\Gamma^\mu_{\beta\rho} - \frac{\partial}{\partial x^\beta}\Gamma^\mu_{\alpha\rho} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\beta\rho} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\rho}\right] V^\rho$$

$$= R^\mu{}_{\rho\alpha\beta} V^\rho = R^\mu{}_{\nu\alpha\beta} V^\nu \quad \text{......(a)}$$ [Replacing ρ by ν]

Similarly we can extend these results by following the same procedure

1. Covariant Vector field

In[∘]:=
```
CD[-α]@CD[-β][V[-μ]] - CD[-β]@CD[-α][V[-μ]]
```

Out[∘]= $\nabla_\alpha \nabla_\beta V_\mu - \nabla_\beta \nabla_\alpha V_\mu$

In[∘]:=
```
% // SortCovDs
```

Out[∘]= $-R[\nabla]_{\beta\alpha\mu}{}^{\lambda} V_\lambda$

i.e. $\nabla_\alpha \nabla_\beta V_\mu - \nabla_\beta \nabla_\alpha V_\mu = - R[\nabla]_{\beta\alpha\mu}{}^{\rho} V_\rho$

2. Covariant Tensor

In[∘]:=
```
CD[-α]@CD[-β][T[-μ, -ν]] - CD[-β]@CD[-α][T[-μ, -ν]]
% // SortCovDs
```

Out[∘]= $\nabla_\alpha \nabla_\beta T_{\mu\nu} - \nabla_\beta \nabla_\alpha T_{\mu\nu}$

Out[∘]= $-R[\nabla]_{\beta\alpha\mu}{}^{\lambda} T_{\lambda\nu} - R[\nabla]_{\beta\alpha\nu}{}^{\lambda} T_{\mu\lambda}$

i.e. $\nabla_\alpha \nabla_\beta T_{\mu\nu} - \nabla_\beta \nabla_\alpha T_{\mu\nu} = - R[\nabla]_{\beta\alpha\nu}{}^{\rho} T_{\mu\rho} - R[\nabla]_{\beta\alpha\mu}{}^{\rho} T_{\rho\nu}$

Consider a Covariant Tensor defined as $F_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$

In[∘]:=
```
DefTensor[F[-μ, -ν], M]
```

** DefTensor: Defining tensor F[-μ, -ν].

In[∘]:=
```
F[-μ, -ν] = CD[-μ][V[-ν]] - CD[-ν][V[-μ]]
F[-ν, -λ] = CD[-ν][V[-λ]] - CD[-λ][V[-ν]]
F[-λ, -μ] = CD[-λ][V[-μ]] - CD[-μ][V[-λ]]
```

Out[∘]= $\nabla_\mu V_\nu - \nabla_\nu V_\mu$

Out[∘]= $-(\nabla_\lambda V_\nu) + \nabla_\nu V_\lambda$

Out[∘]= $\nabla_\lambda V_\mu - \nabla_\mu V_\lambda$

In[∘]:=
```
CD[-μ][F[-ν, -λ]] + CD[-λ][F[-μ, -ν]] + CD[-ν][F[-λ, -μ]]
```

Out[ ]= $\nabla_\lambda \nabla_\mu V_\nu - \nabla_\lambda \nabla_\nu V_\mu - \nabla_\mu \nabla_\lambda V_\nu + \nabla_\mu \nabla_\nu V_\lambda + \nabla_\nu \nabla_\lambda V_\mu - \nabla_\nu \nabla_\mu V_\lambda$

In[ ]:= **% // CovDToChristoffel**

Out[ ]=
$$\begin{aligned}
&- \Gamma[\nabla]^{\beta}{}_{\mu\nu} \left(- \Gamma[\nabla]^{\alpha}{}_{\beta\lambda} V_\alpha + \partial_\beta V_\lambda\right) + \Gamma[\nabla]^{\beta}{}_{\nu\mu} \left(- \Gamma[\nabla]^{\alpha}{}_{\beta\lambda} V_\alpha + \partial_\beta V_\lambda\right) + \\
&\Gamma[\nabla]^{\beta}{}_{\lambda\nu} \left(- \Gamma[\nabla]^{\alpha}{}_{\beta\mu} V_\alpha + \partial_\beta V_\mu\right) - \Gamma[\nabla]^{\beta}{}_{\nu\lambda} \left(- \Gamma[\nabla]^{\alpha}{}_{\beta\mu} V_\alpha + \partial_\beta V_\mu\right) - \\
&\Gamma[\nabla]^{\beta}{}_{\lambda\mu} \left(- \Gamma[\nabla]^{\alpha}{}_{\beta\nu} V_\alpha + \partial_\beta V_\nu\right) + \Gamma[\nabla]^{\beta}{}_{\mu\lambda} \left(- \Gamma[\nabla]^{\alpha}{}_{\beta\nu} V_\alpha + \partial_\beta V_\nu\right) - \\
&V_\alpha\, \partial_\lambda \Gamma[\nabla]^{\alpha}{}_{\mu\nu} + V_\alpha\, \partial_\lambda \Gamma[\nabla]^{\alpha}{}_{\nu\mu} - \Gamma[\nabla]^{\alpha}{}_{\mu\nu}\, \partial_\lambda V_\alpha + \Gamma[\nabla]^{\alpha}{}_{\nu\mu}\, \partial_\lambda V_\alpha + \\
&\Gamma[\nabla]^{\beta}{}_{\mu\nu} \left(- \Gamma[\nabla]^{\alpha}{}_{\lambda\beta} V_\alpha + \partial_\lambda V_\beta\right) - \Gamma[\nabla]^{\beta}{}_{\nu\mu} \left(- \Gamma[\nabla]^{\alpha}{}_{\lambda\beta} V_\alpha + \partial_\lambda V_\beta\right) + \partial_\lambda \partial_\mu V_\nu - \\
&\partial_\lambda \partial_\nu V_\mu + V_\alpha\, \partial_\mu \Gamma[\nabla]^{\alpha}{}_{\lambda\nu} - V_\alpha\, \partial_\mu \Gamma[\nabla]^{\alpha}{}_{\nu\lambda} + \Gamma[\nabla]^{\alpha}{}_{\lambda\nu}\, \partial_\mu V_\alpha - \Gamma[\nabla]^{\alpha}{}_{\nu\lambda}\, \partial_\mu V_\alpha - \\
&\Gamma[\nabla]^{\beta}{}_{\lambda\nu} \left(- \Gamma[\nabla]^{\alpha}{}_{\mu\beta} V_\alpha + \partial_\mu V_\beta\right) + \Gamma[\nabla]^{\beta}{}_{\nu\lambda} \left(- \Gamma[\nabla]^{\alpha}{}_{\mu\beta} V_\alpha + \partial_\mu V_\beta\right) - \partial_\mu \partial_\lambda V_\nu + \\
&\partial_\mu \partial_\nu V_\lambda - V_\alpha\, \partial_\nu \Gamma[\nabla]^{\alpha}{}_{\lambda\mu} + V_\alpha\, \partial_\nu \Gamma[\nabla]^{\alpha}{}_{\mu\lambda} - \Gamma[\nabla]^{\alpha}{}_{\lambda\mu}\, \partial_\nu V_\alpha + \Gamma[\nabla]^{\alpha}{}_{\mu\lambda}\, \partial_\nu V_\alpha + \\
&\Gamma[\nabla]^{\beta}{}_{\lambda\mu} \left(- \Gamma[\nabla]^{\alpha}{}_{\nu\beta} V_\alpha + \partial_\nu V_\beta\right) - \Gamma[\nabla]^{\beta}{}_{\mu\lambda} \left(- \Gamma[\nabla]^{\alpha}{}_{\nu\beta} V_\alpha + \partial_\nu V_\beta\right) + \partial_\nu \partial_\lambda V_\mu - \partial_\nu \partial_\mu V_\lambda
\end{aligned}$$

In[ ]:= **% // ToCanonical**

ToCanonical: Detected metric–incompatible derivatives {PD}.

ToCanonical: Detected metric–incompatible derivatives {PD}.

ToCanonical: Detected metric–incompatible derivatives {PD}.

General: Further output of ToCanonical::cmods will be suppressed during this calculation.

Out[ ]= 0

i.e. $\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} =$
0

Similarly, we can show

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V_\mu = R_\mu{}^{\nu}{}_{\alpha\beta} V_\nu \quad \text{---(b)}$$

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) T_{\mu\nu} = R_\mu{}^{\lambda}{}_{\alpha\beta} T_{\lambda\nu} + R_\nu{}^{\lambda}{}_{\alpha\beta} T_{\mu\lambda} \quad \text{---(c)}$$

$$\nabla_\mu F_{\nu\lambda} + \nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} = 0 \quad \text{---(d)}$$

[Here $F_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$]

LIST OF IDENTITIES

The following are the list of identities that we have verified:
These identities can be brought to the same form in the Notes using the Symmetries.

1. $\nabla_\alpha \nabla_\beta V^\mu = V^\lambda\, \partial_\alpha \Gamma[\nabla]^{\mu}{}_{\beta\lambda} + \Gamma[\nabla]^{\mu}{}_{\beta\lambda}\, \partial_\alpha V^\lambda + \partial_\alpha \partial_\beta V^\mu + \Gamma[\nabla]^{\mu}{}_{\alpha\nu} \left(\Gamma[\nabla]^{\nu}{}_{\beta\lambda} V^\lambda + \partial_\beta V^\nu\right) - \Gamma[\nabla]^{\nu}{}_{\alpha\beta} \left(\Gamma[\nabla]^{\mu}{}_{\nu\lambda} V^\lambda + \partial_\nu V^\mu\right)$

2. $\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = R[\nabla]_{\beta\alpha\rho}{}^{\mu} \; V^\rho$

3. $\nabla_\alpha \nabla_\beta V_\mu - \nabla_\beta \nabla_\alpha V_\mu = - R[\nabla]_{\beta\alpha\mu}{}^{\rho} \; V_\rho$

4. $\nabla_\alpha \nabla_\beta T_{\mu\nu} - \nabla_\beta \nabla_\alpha T_{\mu\nu} = - R[\nabla]_{\beta\alpha\nu}{}^{\rho} \; T_{\mu\rho} - R[\nabla]_{\beta\alpha\mu}{}^{\rho} \; T_{\rho\nu}$

5. $\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0$ where $F_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$

Properties of Riemann Curvature Tensor

1. $R[\nabla]_{\mu\nu\lambda\rho} + R[\nabla]_{\nu\mu\lambda\rho} = 0$

In[ ]:=
```
RiemannCD[-μ, -ν, -λ, -ρ] + RiemannCD[-ν, -μ, -λ, -ρ]
```

Out[ ]= $R[\nabla]_{\mu\nu\lambda\rho} + R[\nabla]_{\nu\mu\lambda\rho}$

In[ ]:=
```
% // RiemannToChristoffel
```

Out[ ]= $g_{\rho\alpha} \left(-\Gamma[\nabla]^{\alpha}{}_{\nu\beta}\;\Gamma[\nabla]^{\beta}{}_{\mu\lambda} + \Gamma[\nabla]^{\alpha}{}_{\mu\beta}\;\Gamma[\nabla]^{\beta}{}_{\nu\lambda} + \partial_\mu\Gamma[\nabla]^{\alpha}{}_{\nu\lambda} - \partial_\nu\Gamma[\nabla]^{\alpha}{}_{\mu\lambda}\right) + g_{\rho\alpha} \left(\Gamma[\nabla]^{\alpha}{}_{\nu\beta}\;\Gamma[\nabla]^{\beta}{}_{\mu\lambda} - \Gamma[\nabla]^{\alpha}{}_{\mu\beta}\;\Gamma[\nabla]^{\beta}{}_{\nu\lambda} - \partial_\mu\Gamma[\nabla]^{\alpha}{}_{\nu\lambda} + \partial_\nu\Gamma[\nabla]^{\alpha}{}_{\mu\lambda}\right)$

In[ ]:=
```
% // ToCanonical
```

Out[ ]= 0

i.e. $R[\nabla]_{\mu\nu\lambda\rho} + R[\nabla]_{\nu\mu\lambda\rho} = 0$

2. $R[\nabla]_{\mu\nu\lambda\rho} + R[\nabla]_{\mu\nu\rho\lambda} = 0$

In[ ]:= `RiemannCD[-μ, -ν, -λ, -ρ] + RiemannCD[-μ, -ν, -ρ, -λ]`

Out[ ]= $R[\nabla]_{\mu\nu\lambda\rho} + R[\nabla]_{\mu\nu\rho\lambda}$

In[ ]:= `% // RiemannToChristoffel // ChristoffelToGradMetric`

Out[ ]=
$$\begin{aligned}
&g_{\rho\alpha}\Big(\frac{1}{4}\, g^{\alpha\sigma}\, g^{\beta\phi}\, (\partial_\beta g_{\nu\sigma} + \partial_\nu g_{\beta\sigma} - \partial_\sigma g_{\nu\beta})\,(\partial_\lambda g_{\mu\phi} + \partial_\mu g_{\lambda\phi} - \partial_\phi g_{\mu\lambda}) - \\
&\quad \frac{1}{4}\, g^{\alpha\psi}\, g^{\beta\psi1}\, (\partial_\beta g_{\mu\psi} + \partial_\mu g_{\beta\psi} - \partial_\psi g_{\mu\beta})\,(\partial_\lambda g_{\nu\psi1} + \partial_\nu g_{\lambda\psi1} - \partial_{\psi1} g_{\nu\lambda}) + \\
&\quad \frac{1}{2}\,\big(-g^{\alpha\psi2}\,(\partial_\mu\partial_\lambda g_{\nu\psi2} + \partial_\mu\partial_\nu g_{\lambda\psi2} - \partial_\mu\partial_{\psi2} g_{\nu\lambda}) + g^{\alpha\psi3}\, g^{\psi2\psi4}\, \partial_\mu g_{\psi3\psi4} \\
&\qquad (\partial_\lambda g_{\nu\psi2} + \partial_\nu g_{\lambda\psi2} - \partial_{\psi2} g_{\nu\lambda})\big) + \frac{1}{2}\,\big(g^{\alpha\psi5}\,(\partial_\nu\partial_\lambda g_{\mu\psi5} + \partial_\nu\partial_\mu g_{\lambda\psi5} - \partial_\nu\partial_{\psi5} g_{\mu\lambda}) - \\
&\qquad g^{\alpha\psi6}\, g^{\psi5\psi7}\, \partial_\nu g_{\psi6\psi7}\,(\partial_\lambda g_{\mu\psi5} + \partial_\mu g_{\lambda\psi5} - \partial_{\psi5} g_{\mu\lambda})\big)\Big) + \\
&g_{\lambda\alpha}\Big(\frac{1}{4}\, g^{\alpha\sigma}\, g^{\beta\phi}\, (\partial_\beta g_{\nu\sigma} + \partial_\nu g_{\beta\sigma} - \partial_\sigma g_{\nu\beta})\,(\partial_\mu g_{\rho\phi} + \partial_\rho g_{\mu\phi} - \partial_\phi g_{\mu\rho}) - \\
&\quad \frac{1}{4}\, g^{\alpha\psi}\, g^{\beta\psi1}\, (\partial_\beta g_{\mu\psi} + \partial_\mu g_{\beta\psi} - \partial_\psi g_{\mu\beta})\,(\partial_\nu g_{\rho\psi1} + \partial_\rho g_{\nu\psi1} - \partial_{\psi1} g_{\nu\rho}) + \\
&\quad \frac{1}{2}\,\big(-g^{\alpha\psi2}\,(\partial_\mu\partial_\nu g_{\rho\psi2} + \partial_\mu\partial_\rho g_{\nu\psi2} - \partial_\mu\partial_{\psi2} g_{\nu\rho}) + g^{\alpha\psi3}\, g^{\psi2\psi4}\, \partial_\mu g_{\psi3\psi4} \\
&\qquad (\partial_\nu g_{\rho\psi2} + \partial_\rho g_{\nu\psi2} - \partial_{\psi2} g_{\nu\rho})\big) + \frac{1}{2}\,\big(g^{\alpha\psi5}\,(\partial_\nu\partial_\mu g_{\rho\psi5} + \partial_\nu\partial_\rho g_{\mu\psi5} - \partial_\nu\partial_{\psi5} g_{\mu\rho}) - \\
&\qquad g^{\alpha\psi6}\, g^{\psi5\psi7}\, \partial_\nu g_{\psi6\psi7}\,(\partial_\mu g_{\rho\psi5} + \partial_\rho g_{\mu\psi5} - \partial_{\psi5} g_{\mu\rho})\big)\Big)
\end{aligned}$$

In[ ]:= `% // ToCanonical`

Out[ ]= 0

i.e $R[\nabla]_{\mu\nu\lambda\rho} + R[\nabla]_{\mu\nu\rho\lambda} = 0$

3. $R[\nabla]_{\mu\lambda\rho\nu} + R[\nabla]_{\mu\nu\lambda\rho} + R[\nabla]_{\mu\rho\nu\lambda} = 0$

In[ ]:= `RiemannCD[-μ, -ν, -λ, -ρ] + RiemannCD[-μ, -ρ, -ν, -λ] + RiemannCD[-μ, -λ, -ρ, -ν]`

Out[ ]= $R[\nabla]_{\mu\lambda\rho\nu} + R[\nabla]_{\mu\nu\lambda\rho} + R[\nabla]_{\mu\rho\nu\lambda}$

In[ ]:= `% // RiemannToChristoffel // ChristoffelToGradMetric`

Out[∘]= $g_{\nu\alpha}\left(-\frac{1}{4}\,g^{\alpha\sigma}\,g^{\beta\phi}\,(\partial_\beta g_{\mu\sigma}+\partial_\mu g_{\beta\sigma}-\partial_\sigma g_{\mu\beta})\,(\partial_\lambda g_{\rho\phi}+\partial_\rho g_{\lambda\phi}-\partial_\phi g_{\lambda\rho})+\frac{1}{4}\,g^{\alpha\psi}\,g^{\beta\psi1}\,(\partial_\beta g_{\lambda\psi}+\partial_\lambda g_{\beta\psi}-\partial_\psi g_{\lambda\beta})\,(\partial_\mu g_{\rho\psi1}+\partial_\rho g_{\mu\psi1}-\partial_{\psi1} g_{\mu\rho})+\frac{1}{2}\left(g^{\alpha\psi2}\,(\partial_\lambda\partial_\mu g_{\rho\psi2}+\partial_\lambda\partial_\rho g_{\mu\psi2}-\partial_\lambda\partial_{\psi2} g_{\mu\rho})-g^{\alpha\psi3}\,g^{\psi2\psi4}\,\partial_\lambda g_{\psi3\psi4}\,(\partial_\mu g_{\rho\psi2}+\partial_\rho g_{\mu\psi2}-\partial_{\psi2} g_{\mu\rho})\right)+\frac{1}{2}\left(-g^{\alpha\psi5}\,(\partial_\mu\partial_\lambda g_{\rho\psi5}+\partial_\mu\partial_\rho g_{\lambda\psi5}-\partial_\mu\partial_{\psi5} g_{\lambda\rho})+g^{\alpha\psi6}\,g^{\psi5\psi7}\,\partial_\mu g_{\psi6\psi7}\,(\partial_\lambda g_{\rho\psi5}+\partial_\rho g_{\lambda\psi5}-\partial_{\psi5} g_{\lambda\rho})\right)\right)+$
$g_{\rho\alpha}\left(\frac{1}{4}\,g^{\alpha\sigma}\,g^{\beta\phi}\,(\partial_\beta g_{\nu\sigma}+\partial_\nu g_{\beta\sigma}-\partial_\sigma g_{\nu\beta})\,(\partial_\lambda g_{\mu\phi}+\partial_\mu g_{\lambda\phi}-\partial_\phi g_{\mu\lambda})-\frac{1}{4}\,g^{\alpha\psi}\,g^{\beta\psi1}\,(\partial_\beta g_{\mu\psi}+\partial_\mu g_{\beta\psi}-\partial_\psi g_{\mu\beta})\,(\partial_\lambda g_{\nu\psi1}+\partial_\nu g_{\lambda\psi1}-\partial_{\psi1} g_{\nu\lambda})+\frac{1}{2}\left(-g^{\alpha\psi2}\,(\partial_\mu\partial_\lambda g_{\nu\psi2}+\partial_\mu\partial_\nu g_{\lambda\psi2}-\partial_\mu\partial_{\psi2} g_{\nu\lambda})+g^{\alpha\psi3}\,g^{\psi2\psi4}\,\partial_\mu g_{\psi3\psi4}\,(\partial_\lambda g_{\nu\psi2}+\partial_\nu g_{\lambda\psi2}-\partial_{\psi2} g_{\nu\lambda})\right)+\frac{1}{2}\left(g^{\alpha\psi5}\,(\partial_\nu\partial_\lambda g_{\mu\psi5}+\partial_\nu\partial_\mu g_{\lambda\psi5}-\partial_\nu\partial_{\psi5} g_{\mu\lambda})-g^{\alpha\psi6}\,g^{\psi5\psi7}\,\partial_\nu g_{\psi6\psi7}\,(\partial_\lambda g_{\mu\psi5}+\partial_\mu g_{\lambda\psi5}-\partial_{\psi5} g_{\mu\lambda})\right)\right)+$
$g_{\lambda\alpha}\left(\frac{1}{4}\,g^{\alpha\sigma}\,g^{\beta\phi}\,(\partial_\beta g_{\rho\sigma}+\partial_\rho g_{\beta\sigma}-\partial_\sigma g_{\rho\beta})\,(\partial_\mu g_{\nu\phi}+\partial_\nu g_{\mu\phi}-\partial_\phi g_{\mu\nu})-\frac{1}{4}\,g^{\alpha\psi}\,g^{\beta\psi1}\,(\partial_\beta g_{\mu\psi}+\partial_\mu g_{\beta\psi}-\partial_\psi g_{\mu\beta})\,(\partial_\nu g_{\rho\psi1}+\partial_\rho g_{\nu\psi1}-\partial_{\psi1} g_{\rho\nu})+\frac{1}{2}\left(-g^{\alpha\psi2}\,(\partial_\mu\partial_\nu g_{\rho\psi2}+\partial_\mu\partial_\rho g_{\nu\psi2}-\partial_\mu\partial_{\psi2} g_{\rho\nu})+g^{\alpha\psi3}\,g^{\psi2\psi4}\,\partial_\mu g_{\psi3\psi4}\,(\partial_\nu g_{\rho\psi2}+\partial_\rho g_{\nu\psi2}-\partial_{\psi2} g_{\rho\nu})\right)+\frac{1}{2}\left(g^{\alpha\psi5}\,(\partial_\rho\partial_\mu g_{\nu\psi5}+\partial_\rho\partial_\nu g_{\mu\psi5}-\partial_\rho\partial_{\psi5} g_{\mu\nu})-g^{\alpha\psi6}\,g^{\psi5\psi7}\,\partial_\rho g_{\psi6\psi7}\,(\partial_\mu g_{\nu\psi5}+\partial_\nu g_{\mu\psi5}-\partial_{\psi5} g_{\mu\nu})\right)\right)$

In[∘]:= **% // ToCanonical**

Out[∘]= 0

i.e. $R[\nabla]_{\mu\lambda\rho\nu}+R[\nabla]_{\mu\nu\lambda\rho}+R[\nabla]_{\mu\rho\nu\lambda}=0$

4. $-R[\nabla]_{\lambda\rho\mu\nu}+R[\nabla]_{\mu\nu\lambda\rho}=0$

In[∘]:= **RiemannCD[-μ, -ν, -λ, -ρ] - RiemannCD[-λ, -ρ, -μ, -ν]**

Out[∘]= $-R[\nabla]_{\lambda\rho\mu\nu}+R[\nabla]_{\mu\nu\lambda\rho}$

In[∘]:= **% // RiemannToChristoffel // ChristoffelToGradMetric // ToCanonical**

Out[∘]= 0

i.e $- R[\nabla]_{\lambda\rho\mu\nu} + R[\nabla]_{\mu\nu\lambda\rho} = 0$

5. $\nabla_\lambda R[\nabla]_{\mu\nu\rho\sigma} + \nabla_\rho R[\nabla]_{\mu\nu\sigma\lambda} + \nabla_\sigma R[\nabla]_{\mu\nu\lambda\rho} = 0$

Finding The expressions used in the derivation of the above equation (FROM GR NOTES)
The variable corresponding to each expression are same same as the equation numbers in the GR Notes.

In[ ]:=
```
CD[-σ][RiemannCD[-μ, -ν, -λ, -ρ]] +
 CD[-ρ][RiemannCD[-μ, -ν, -σ, -λ]] + CD[-λ][RiemannCD[-μ, -ν, -ρ, -σ]]
```

Out[ ]= $\nabla_\lambda R[\nabla]_{\mu\nu\rho\sigma} + \nabla_\rho R[\nabla]_{\mu\nu\sigma\lambda} + \nabla_\sigma R[\nabla]_{\mu\nu\lambda\rho}$

In[ ]:=
```
% // CovDToChristoffel
```

Out[ ]= $$- \Gamma[\nabla]^{\alpha}{}_{\sigma\mu} R[\nabla]_{\alpha\nu\lambda\rho} - \Gamma[\nabla]^{\alpha}{}_{\lambda\mu} R[\nabla]_{\alpha\nu\rho\sigma} - \Gamma[\nabla]^{\alpha}{}_{\rho\mu} R[\nabla]_{\alpha\nu\sigma\lambda} - \Gamma[\nabla]^{\alpha}{}_{\sigma\nu} R[\nabla]_{\mu\alpha\lambda\rho} - \Gamma[\nabla]^{\alpha}{}_{\lambda\nu} R[\nabla]_{\mu\alpha\rho\sigma} - \Gamma[\nabla]^{\alpha}{}_{\rho\nu} R[\nabla]_{\mu\alpha\sigma\lambda} - \Gamma[\nabla]^{\alpha}{}_{\rho\sigma} R[\nabla]_{\mu\nu\alpha\lambda} - \Gamma[\nabla]^{\alpha}{}_{\sigma\lambda} R[\nabla]_{\mu\nu\alpha\rho} - \Gamma[\nabla]^{\alpha}{}_{\lambda\rho} R[\nabla]_{\mu\nu\alpha\sigma} - \Gamma[\nabla]^{\alpha}{}_{\sigma\rho} R[\nabla]_{\mu\nu\lambda\alpha} - \Gamma[\nabla]^{\alpha}{}_{\lambda\sigma} R[\nabla]_{\mu\nu\rho\alpha} - \Gamma[\nabla]^{\alpha}{}_{\rho\lambda} R[\nabla]_{\mu\nu\sigma\alpha} + \partial_\lambda R[\nabla]_{\mu\nu\rho\sigma} + \partial_\rho R[\nabla]_{\mu\nu\sigma\lambda} + \partial_\sigma R[\nabla]_{\mu\nu\lambda\rho}$$

1. Consider a tensor $\nabla_\mu V_\nu$

evaluate

In[ ]:=
```
a = CD[-α][CD[-β][CD[-μ][V[-ν] ]]] - CD[-β][CD[-α][CD[-μ][V[-ν] ]]]
```

Out[ ]= $\nabla_\alpha \nabla_\beta \nabla_\mu V_\nu - \nabla_\beta \nabla_\alpha \nabla_\mu V_\nu$

In[ ]:=
```
i = % // SortCovDs
```

Out[ ]= $- R[\nabla]_{\beta\alpha\mu}{}^{\lambda} (\nabla_\lambda V_\nu) - R[\nabla]_{\beta\alpha\nu}{}^{\lambda} (\nabla_\mu V_\lambda)$

In[ ]:=
```
b = CD[-β][CD[-μ]@CD[-α][V[-ν] ]] - CD[-μ][CD[-β]@CD[-α][V[-ν] ]]
```

Out[ ]= $\nabla_\beta \nabla_\mu \nabla_\alpha V_\nu - \nabla_\mu \nabla_\beta \nabla_\alpha V_\nu$

In[ ]:=
```
j = % // SortCovDs
```

Out[ ]= $- R[\nabla]_{\mu\beta\nu}{}^{\lambda} (\nabla_\alpha V_\lambda) - R[\nabla]_{\mu\beta\alpha}{}^{\lambda} (\nabla_\lambda V_\nu)$

In[ ]:=
```
c = CD[-μ][CD[-α]@CD[-β][V[-ν] ]] - CD[-α][CD[-μ]@CD[-β][V[-ν] ]]
```

Out[ ]= $-\left(\nabla_\alpha \nabla_\mu \nabla_\beta V_\nu\right) + \nabla_\mu \nabla_\alpha \nabla_\beta V_\nu$

In[ ]:= **k = % // SortCovDs**

Out[ ]= $R[\nabla]_{\mu\alpha\nu}{}^{\lambda} \left(\nabla_\beta V_\lambda\right) + R[\nabla]_{\mu\alpha\beta}{}^{\lambda} \left(\nabla_\lambda V_\nu\right)$

on taking the sum of expressions i , j, k , we get :

In[ ]:= **l = i + j + k // ToCanonical**

Out[ ]= $R[\nabla]_{\beta\mu\nu\lambda} \left(\nabla_\alpha V^{\lambda}\right) - R[\nabla]_{\alpha\mu\nu\lambda} \left(\nabla_\beta V^{\lambda}\right) + R[\nabla]_{\alpha\beta\mu\lambda} \left(\nabla^{\lambda} V_\nu\right) + R[\nabla]_{\alpha\lambda\beta\mu} \left(\nabla^{\lambda} V_\nu\right) - R[\nabla]_{\alpha\mu\beta\lambda} \left(\nabla^{\lambda} V_\nu\right) + R[\nabla]_{\alpha\beta\nu\lambda} \left(\nabla_\mu V^{\lambda}\right)$

In[ ]:= **a + b + c**

Out[ ]= $\nabla_\alpha \nabla_\beta \nabla_\mu V_\nu - \nabla_\alpha \nabla_\mu \nabla_\beta V_\nu - \nabla_\beta \nabla_\alpha \nabla_\mu V_\nu + \nabla_\beta \nabla_\mu \nabla_\alpha V_\nu + \nabla_\mu \nabla_\alpha \nabla_\beta V_\nu - \nabla_\mu \nabla_\beta \nabla_\alpha V_\nu$

so we got (ex
1)

$$\nabla_\alpha \nabla_\beta \nabla_\mu V_\nu - \nabla_\alpha \nabla_\mu \nabla_\beta V_\nu - \nabla_\beta \nabla_\alpha \nabla_\mu V_\nu + \nabla_\beta \nabla_\mu \nabla_\alpha V_\nu + \nabla_\mu \nabla_\alpha \nabla_\beta V_\nu - \nabla_\mu \nabla_\beta \nabla_\alpha V_\nu =$$

$$- R[\nabla]_{\mu\beta\nu}{}^{\lambda} \left(\nabla_\alpha V_\lambda\right) + R[\nabla]_{\mu\alpha\nu}{}^{\lambda} \left(\nabla_\beta V_\lambda\right) - R[\nabla]_{\beta\alpha\mu}{}^{\lambda} \left(\nabla_\lambda V_\nu\right) + R[\nabla]_{\mu\alpha\beta}{}^{\lambda} \left(\nabla_\lambda V_\nu\right) - R[\nabla]_{\mu\beta\alpha}{}^{\lambda} \left(\nabla_\lambda V_\nu\right) - R[\nabla]_{\beta\alpha\nu}{}^{\lambda} \left(\nabla_\mu V_\lambda\right)$$

Similarly consider the identity 3 from the list of identities

In[ ]:= **d = -CD[-α][CD[-μ]@CD[-β][V[-ν]]] + CD[-α][CD[-β]@CD[-μ][V[-ν]]]**

Out[ ]= $\nabla_\alpha \nabla_\beta \nabla_\mu V_\nu - \nabla_\alpha \nabla_\mu \nabla_\beta V_\nu$

In[ ]:= **p = % // SortCovDs**

Out[ ]= $- R[\nabla]_{\mu\beta\nu}{}^{\lambda} \left(\nabla_\alpha V_\lambda\right) - V_\lambda \left(\nabla_\beta R[\nabla]_{\mu\alpha\nu}{}^{\lambda}\right) - R[\nabla]_{\beta\alpha\mu}{}^{\lambda} \left(\nabla_\lambda V_\nu\right) + R[\nabla]_{\mu\alpha\beta}{}^{\lambda} \left(\nabla_\lambda V_\nu\right) - R[\nabla]_{\mu\beta\alpha}{}^{\lambda} \left(\nabla_\lambda V_\nu\right) + V_\lambda \left(\nabla_\mu R[\nabla]_{\beta\alpha\nu}{}^{\lambda}\right)$

In[ ]:= **e = CD[-β][CD[-μ]@CD[-α][V[-ν]]] - CD[-β][CD[-α]@CD[-μ][V[-ν]]]**

Out[ ]= $-\left(\nabla_\beta \nabla_\alpha \nabla_\mu V_\nu\right) + \nabla_\beta \nabla_\mu \nabla_\alpha V_\nu$

In[ ]:= **q = % // SortCovDs**

Out[ ]= $V_\lambda \left(\nabla_\beta R[\nabla]_{\mu\alpha\nu}{}^{\lambda}\right) + R[\nabla]_{\mu\alpha\nu}{}^{\lambda} \left(\nabla_\beta V_\lambda\right)$

In[ ]:= **f = CD[-μ][CD[-α]@CD[-β][V[-ν]]] - CD[-μ][CD[-β]@CD[-α][V[-ν]]]**

Out[ ]= $\nabla_\mu \nabla_\alpha \nabla_\beta V_\nu - \nabla_\mu \nabla_\beta \nabla_\alpha V_\nu$

In[ ]:= **r = % // SortCovDs**

Out[ ]= $-V_{\lambda}\left(\nabla_{\mu} R[\nabla]_{\beta\alpha\nu}{}^{\lambda}\right)-R[\nabla]_{\beta\alpha\nu}{}^{\lambda}\left(\nabla_{\mu} V_{\lambda}\right)$

In[ ]:= **s = p + q + r // Simplify**

Out[ ]= $-R[\nabla]_{\mu\beta\nu}{}^{\lambda}\left(\nabla_{\alpha} V_{\lambda}\right)+R[\nabla]_{\mu\alpha\nu}{}^{\lambda}\left(\nabla_{\beta} V_{\lambda}\right)-R[\nabla]_{\beta\alpha\mu}{}^{\lambda}\left(\nabla_{\lambda} V_{\nu}\right)+R[\nabla]_{\mu\alpha\beta}{}^{\lambda}\left(\nabla_{\lambda} V_{\nu}\right)-R[\nabla]_{\mu\beta\alpha}{}^{\lambda}\left(\nabla_{\lambda} V_{\nu}\right)-R[\nabla]_{\beta\alpha\nu}{}^{\lambda}\left(\nabla_{\mu} V_{\lambda}\right)$

In[ ]:= **d + e + f**

Out[ ]= $\nabla_{\alpha}\nabla_{\beta}\nabla_{\mu}V_{\nu}-\nabla_{\alpha}\nabla_{\mu}\nabla_{\beta}V_{\nu}-\nabla_{\beta}\nabla_{\alpha}\nabla_{\mu}V_{\nu}+\nabla_{\beta}\nabla_{\mu}\nabla_{\alpha}V_{\nu}+\nabla_{\mu}\nabla_{\alpha}\nabla_{\beta}V_{\nu}-\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}V_{\nu}$

So we got (ex
2)

$$\nabla_{\alpha}\nabla_{\beta}\nabla_{\mu}V_{\nu}-\nabla_{\alpha}\nabla_{\mu}\nabla_{\beta}V_{\nu}-\nabla_{\beta}\nabla_{\alpha}\nabla_{\mu}V_{\nu}+\nabla_{\beta}\nabla_{\mu}\nabla_{\alpha}V_{\nu}+\nabla_{\mu}\nabla_{\alpha}\nabla_{\beta}V_{\nu}-\nabla_{\mu}\nabla_{\beta}\nabla_{\alpha}V_{\nu}=$$

$$-R[\nabla]_{\mu\beta\nu}{}^{\lambda}\left(\nabla_{\alpha} V_{\lambda}\right)+R[\nabla]_{\mu\alpha\nu}{}^{\lambda}\left(\nabla_{\beta} V_{\lambda}\right)-R[\nabla]_{\beta\alpha\mu}{}^{\lambda}\left(\nabla_{\lambda} V_{\nu}\right)+R[\nabla]_{\mu\alpha\beta}{}^{\lambda}\left(\nabla_{\lambda} V_{\nu}\right)-R[\nabla]_{\mu\beta\alpha}{}^{\lambda}\left(\nabla_{\lambda} V_{\nu}\right)-R[\nabla]_{\beta\alpha\nu}{}^{\lambda}\left(\nabla_{\mu} V_{\lambda}\right)$$

Now consider the LHS of ex 1 and ex 2 :

In[ ]:= **a + b + c - d - e - f**

Out[ ]= 0

i.e. The LHS of both the expressions are equal
This implies the RHS of both the expression must be equal

In[ ]:= **l == s**

Out[ ]= True

These were the expressions calculated in the GR notes to arrive at this result
$\nabla_{\lambda} R[\nabla]_{\mu\nu\rho\sigma}+\nabla_{\rho} R[\nabla]_{\mu\nu\sigma\lambda}+\nabla_{\sigma} R[\nabla]_{\mu\nu\lambda\rho}=0$

This Same result can be verified using xACT

In[ ]:= **Antisymmetrize[cd[-σ][Riemanncd[-λ, -ρ, -μ, ν]], {-σ, -λ, -ρ}]**

Out[•]= $\frac{1}{6}\left(\nabla_{\lambda} R[\nabla]_{\rho\sigma\mu}{}^{\nu} - \nabla_{\lambda} R[\nabla]_{\sigma\rho\mu}{}^{\nu} - \nabla_{\rho} R[\nabla]_{\lambda\sigma\mu}{}^{\nu} + \nabla_{\rho} R[\nabla]_{\sigma\lambda\mu}{}^{\nu} + \nabla_{\sigma} R[\nabla]_{\lambda\rho\mu}{}^{\nu} - \nabla_{\sigma} R[\nabla]_{\rho\lambda\mu}{}^{\nu}\right)$

In[•]:= **3 * % // ToCanonical**

Out[•]= $\nabla_{\lambda} R[\nabla]_{\rho\sigma\mu}{}^{\nu} - \nabla_{\rho} R[\nabla]_{\lambda\sigma\mu}{}^{\nu} + \nabla_{\sigma} R[\nabla]_{\lambda\rho\mu}{}^{\nu}$

In[•]:= **% // CovDToChristoffel**

Out[•]= $-\Gamma[\nabla]^{\alpha}{}_{\sigma\lambda}\, R[\nabla]_{\alpha\rho\mu}{}^{\nu} - \Gamma[\nabla]^{\alpha}{}_{\lambda\rho}\, R[\nabla]_{\alpha\sigma\mu}{}^{\nu} + \Gamma[\nabla]^{\alpha}{}_{\rho\lambda}\, R[\nabla]_{\alpha\sigma\mu}{}^{\nu} +$
$\Gamma[\nabla]^{\alpha}{}_{\rho\sigma}\, R[\nabla]_{\lambda\alpha\mu}{}^{\nu} - \Gamma[\nabla]^{\alpha}{}_{\sigma\rho}\, R[\nabla]_{\lambda\alpha\mu}{}^{\nu} - \Gamma[\nabla]^{\alpha}{}_{\sigma\mu}\, R[\nabla]_{\lambda\rho\alpha}{}^{\nu} +$
$\Gamma[\nabla]^{\nu}{}_{\sigma\alpha}\, R[\nabla]_{\lambda\rho\mu}{}^{\alpha} + \Gamma[\nabla]^{\alpha}{}_{\rho\mu}\, R[\nabla]_{\lambda\sigma\alpha}{}^{\nu} - \Gamma[\nabla]^{\nu}{}_{\rho\alpha}\, R[\nabla]_{\lambda\sigma\mu}{}^{\alpha} - \Gamma[\nabla]^{\alpha}{}_{\lambda\sigma}\, R[\nabla]_{\rho\alpha\mu}{}^{\nu} -$
$\Gamma[\nabla]^{\alpha}{}_{\lambda\mu}\, R[\nabla]_{\rho\sigma\alpha}{}^{\nu} + \Gamma[\nabla]^{\nu}{}_{\lambda\alpha}\, R[\nabla]_{\rho\sigma\mu}{}^{\alpha} + \partial_{\lambda} R[\nabla]_{\rho\sigma\mu}{}^{\nu} - \partial_{\rho} R[\nabla]_{\lambda\sigma\mu}{}^{\nu} + \partial_{\sigma} R[\nabla]_{\lambda\rho\mu}{}^{\nu}$

In[•]:= **% // RiemannToChristoffel // ToCanonical**

Out[•]= 0

i.e. $\nabla_{\lambda} R[\nabla]_{\rho\sigma\mu}{}^{\nu} - \nabla_{\rho} R[\nabla]_{\lambda\sigma\mu}{}^{\nu} + \nabla_{\sigma} R[\nabla]_{\lambda\rho\mu}{}^{\nu} = 0$

Properties of Riemann Curvature tensor

1. $R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho}$
2. $R_{\mu\nu\lambda\rho} = -R_{\mu\nu\rho\lambda}$
3. $R_{\mu\nu\lambda\rho} + R_{\mu\rho\nu\lambda} + R_{\mu\lambda\rho\nu} = 0$
4. $\nabla_{\sigma} R_{\mu\nu\lambda\rho} + \nabla_{\rho} R_{\mu\nu\sigma\lambda} + \nabla_{\lambda} R_{\mu\nu\rho\sigma} = 0$
5. $R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$