# Tensor Algebra Using xAct
# Metric Perturbations

## Importing the xPert Package

```
<< xAct`xPert`
```

------------------------------------------------------------

## Defining the Manifold and Metric

```
DefManifold[M, 4, {a, b, c, d, e, f, i, j, k, l}]
In[3]:= DefMetric[-1, g[-a, -b], cd]
```

## Perturbations of the metric

```
In[126]:= DefMetricPerturbation[g, h, ϵ]
In[138]:= Perturbed[g[-a, -b], 2]
```

Out[138]= $g_{ab} + \epsilon\, h^{1}{}_{ab} + \frac{1}{2}\epsilon^2\, h^{2}{}_{ab}$

```
In[139]:= Perturbation[g[-a, -b], 2]
```

Out[139]= $h^{2}{}_{ab}$

```
In[140]:= Unprotect[IndexForm];
IndexForm[LI[x_]] := ColorString[ToString[x], Red]
In[142]:= Perturbation[g[-a, -b], 3]
```

Out[142]= $h^{3}{}_{ab}$

```
In[143]:= Perturbed[g[a, b], 2] // ExpandPerturbation
```

Out[143]= $g^{ab} - \epsilon\, h^{1ab} + \frac{1}{2}\epsilon^2 \left(2\, h^{1ac}\, h^{1}{}_{c}{}^{b} - h^{2ab}\right)$

```
In[144]:= Perturbation[g[a, b], 2] // ExpandPerturbation
```

Out[144]= $2\, h^{1ac}\, h^{1}{}_{c}{}^{b} - h^{2ab}$

**The expression for metric up to 2nd order of perturbation is given by**

$$g^{ab} - \epsilon\, h^{1ab} + \frac{1}{2}\epsilon^2 \left(2\, h^{1ac}\, h^{1}{}_{c}{}^{b} - h^{2ab}\right)$$

## Perturbation of Riemann Curvature Tensor

### I. First Order

```
In[146]:= Perturbation[Riemanncd[a, -b, -c, -d], 1]
```

Out[146]= $\triangle\left[R[\nabla]^{a}{}_{bcd}\right]$

In[147]:= `Perturbation[Riemanncd[a, -b, -c, -d], 1] // ExpandPerturbation`

Out[147]= $$-h^{1ae}\,R[\nabla]_{ebcd}+g^{ae}\left(h^{1}{}_{fd}\,R[\nabla]_{ebc}{}^{f}+g_{fd}\left(\frac{1}{2}\left(h^{1f}{}_{e;c;b}+h^{1f}{}_{c;e;b}-h^{1}{}_{ce}{}^{;f}{}_{;b}\right)+\frac{1}{2}\left(-h^{1f}{}_{c;b;e}-h^{1f}{}_{b;c;e}+h^{1}{}_{cb}{}^{;f}{}_{;e}\right)\right)\right)$$

In[148]:= `SortCovDs[%] // ToCanonical`

Out[148]= $$\frac{1}{2}g^{ae}\,h^{1}{}_{e}{}^{f}\,R[\nabla]_{bcdf}+\frac{1}{2}g^{ae}\,h^{1}{}_{d}{}^{f}\,R[\nabla]_{bcef}-\frac{1}{2}g^{ae}\,h^{1}{}_{e}{}^{f}\,R[\nabla]_{bdcf}-\frac{1}{2}g^{ae}\,h^{1}{}_{c}{}^{f}\,R[\nabla]_{bdef}+h^{1ae}\,R[\nabla]_{becd}-\frac{1}{2}g^{ae}\,h^{1}{}_{d}{}^{f}\,R[\nabla]_{becf}+\frac{1}{2}g^{ae}\,h^{1}{}_{c}{}^{f}\,R[\nabla]_{bedf}+\frac{1}{2}g^{ae}\,h^{1}{}_{c}{}^{f}\,R[\nabla]_{bfed}+\frac{1}{2}g^{ae}\,h^{1}{}_{b}{}^{f}\,R[\nabla]_{cfed}+\frac{1}{2}g^{ae}\,h^{1}{}_{ed;b;c}-\frac{1}{2}g^{ae}\,h^{1}{}_{ce;b;d}+\frac{1}{2}g^{ae}\,h^{1}{}_{bc;e;d}-\frac{1}{2}g^{ae}\,h^{1}{}_{bd;c;e}$$

In[149]:= `% /. _Riemanncd → 0 // ContractMetric`

Out[149]= $$-\frac{1}{2}h^{1}{}_{bd;c}{}^{;a}+\frac{1}{2}h^{1a}{}_{d;b;c}+\frac{1}{2}h^{1}{}_{bc}{}^{;a}{}_{;d}-\frac{1}{2}h^{1}{}_{c}{}^{a}{}_{;b;d}$$

In[150]:= $$\mathbf{Simplify}\left[-\frac{1}{2}h^{1}{}_{bd;c}{}^{;a}+\frac{1}{2}h^{1a}{}_{d;b;c}+\frac{1}{2}h^{1}{}_{bc}{}^{;a}{}_{;d}-\frac{1}{2}h^{1}{}_{c}{}^{a}{}_{;b;d}\right]$$

Out[150]= $$\frac{1}{2}\left(-h^{1}{}_{bd;c}{}^{;a}+h^{1a}{}_{d;b;c}+h^{1}{}_{bc}{}^{;a}{}_{;d}-h^{1}{}_{c}{}^{a}{}_{;b;d}\right)$$

**The following is the expression for Riemann Curvature Tensor to First order**

$$\Delta\left[R[\nabla]^{a}{}_{bcd}\right]=\frac{1}{2}\left(-h^{1}{}_{bd;c}{}^{;a}+h^{1a}{}_{d;b;c}+h^{1}{}_{bc}{}^{;a}{}_{;d}-h^{1}{}_{c}{}^{a}{}_{;b;d}\right)$$

## II . Second Order

In[151]:= `Perturbation[Riemanncd[a, -b, -c, -d], 2]`

Out[151]= $$\Delta^{2}\left[R[\nabla]^{a}{}_{bcd}\right]$$

In[159]:= 
```
Perturbation[Riemanncd[a, -b, -c, -d], 2] // ExpandPerturbation
```

Out[159]=

$$
\begin{aligned}
&\left(2\, h^{1\,af}\, h^{1}{}_{f}{}^{e} - h^{2\,ae}\right) R[\nabla]_{ebcd} - \\
&2\, h^{1\,ae} \Bigg( h^{1}{}_{fd}\, R[\nabla]_{ebc}{}^{f} + g_{fd} \Bigg( \frac{1}{2} \left( h^{1\,f}{}_{e;c;b} + h^{1\,f}{}_{c;e;b} - h^{1}{}_{ce}{}^{;f}{}_{;b} \right) + \\
&\quad \frac{1}{2} \left( - h^{1\,f}{}_{c;b;e} - h^{1\,f}{}_{b;c;e} + h^{1}{}_{cb}{}^{;f}{}_{;e} \right) \Bigg) \Bigg) + g^{ae} \Bigg( h^{2}{}_{fd}\, R[\nabla]_{ebc}{}^{f} + \\
&2\, h^{1}{}_{fd} \left( \frac{1}{2} \left( h^{1\,f}{}_{e;c;b} + h^{1\,f}{}_{c;e;b} - h^{1}{}_{ce}{}^{;f}{}_{;b} \right) + \frac{1}{2} \left( - h^{1\,f}{}_{c;b;e} - h^{1\,f}{}_{b;c;e} + h^{1}{}_{cb}{}^{;f}{}_{;e} \right) \right) + \\
&g_{fd} \left( \frac{1}{2} \left( h^{2\,f}{}_{e;c;b} + h^{2\,f}{}_{c;e;b} - h^{2}{}_{ce}{}^{;f}{}_{;b} \right) + \frac{1}{2} \left( - h^{2\,f}{}_{c;b;e} - h^{2\,f}{}_{b;c;e} + h^{2}{}_{cb}{}^{;f}{}_{;e} \right) \right) - \\
&2 \times \Bigg( \frac{1}{2}\, h^{1\,fi} \left( h^{1}{}_{ie;c;b} + h^{1}{}_{ic;e;b} - h^{1}{}_{ce;i;b} \right) + \frac{1}{4} \left( h^{1}{}_{je;c} + h^{1}{}_{jc;e} - h^{1}{}_{ec;j} \right) \\
&\quad \left( h^{1\,jf}{}_{;b} + h^{1\,j}{}_{b}{}^{;f} - h^{1\,f}{}_{b}{}^{;j} \right) \Bigg) + 2 \times \Bigg( \frac{1}{2}\, h^{1\,fi} \left( h^{1}{}_{ic;b;e} + h^{1}{}_{ib;c;e} - h^{1}{}_{cb;i;e} \right) + \\
&\quad \frac{1}{4} \left( h^{1}{}_{jc;b} + h^{1}{}_{jb;c} - h^{1}{}_{bc;j} \right) \left( h^{1\,jf}{}_{;e} + h^{1\,j}{}_{e}{}^{;f} - h^{1\,f}{}_{e}{}^{;j} \right) \Bigg) \Bigg)
\end{aligned}
$$

In[160]:= 
```
SortCovDs[%] // ToCanonical
```

Out[160]=

$$
\begin{aligned}
&- h^{1\,ae}\, h^{1}{}_{e}{}^{f}\, R[\nabla]_{bcdf} + \frac{1}{2} g^{ae}\, h^{2}{}_{e}{}^{f}\, R[\nabla]_{bcdf} - h^{1\,ae}\, h^{1}{}_{d}{}^{f}\, R[\nabla]_{bcef} + \frac{1}{2} g^{ae}\, h^{2}{}_{d}{}^{f}\, R[\nabla]_{bcef} + \\
&g^{ae}\, h^{1}{}_{d}{}^{i}\, h^{1}{}_{e}{}^{f}\, R[\nabla]_{bcfi} + g^{ae}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{i}\, R[\nabla]_{bcfi} + h^{1\,ae}\, h^{1}{}_{e}{}^{f}\, R[\nabla]_{bdcf} - \\
&\frac{1}{2} g^{ae}\, h^{2}{}_{e}{}^{f}\, R[\nabla]_{bdcf} + h^{1\,ae}\, h^{1}{}_{c}{}^{f}\, R[\nabla]_{bdef} - \frac{1}{2} g^{ae}\, h^{2}{}_{c}{}^{f}\, R[\nabla]_{bdef} + h^{2\,ae}\, R[\nabla]_{becd} + \\
&h^{1\,ae}\, h^{1}{}_{d}{}^{f}\, R[\nabla]_{becf} - \frac{1}{2} g^{ae}\, h^{2}{}_{d}{}^{f}\, R[\nabla]_{becf} - h^{1\,ae}\, h^{1}{}_{c}{}^{f}\, R[\nabla]_{bedf} + \frac{1}{2} g^{ae}\, h^{2}{}_{c}{}^{f}\, R[\nabla]_{bedf} - \\
&2\, h^{1\,ae}\, h^{1}{}_{e}{}^{f}\, R[\nabla]_{bfcd} - g^{ae}\, h^{1}{}_{d}{}^{f}\, h^{1}{}_{e}{}^{i}\, R[\nabla]_{bfci} + h^{1\,ae}\, h^{1}{}_{c}{}^{f}\, R[\nabla]_{bfde} + \\
&\frac{1}{2} g^{ae}\, h^{2}{}_{c}{}^{f}\, R[\nabla]_{bfed} + g^{ae}\, h^{1}{}_{d}{}^{i}\, h^{1}{}_{e}{}^{f}\, R[\nabla]_{bicf} + h^{1\,ae}\, h^{1}{}_{b}{}^{f}\, R[\nabla]_{cfde} + \\
&\frac{1}{2} g^{ae}\, h^{2}{}_{b}{}^{f}\, R[\nabla]_{cfed} - \frac{1}{2} g^{ae}\, h^{1}{}_{d}{}^{f}{}_{;b}\, h^{1}{}_{ef;c} - h^{1\,ae}\, h^{1}{}_{de;b;c} + \frac{1}{2} g^{ae}\, h^{2}{}_{ed;b;c} - \\
&\frac{1}{2} g^{ae}\, h^{1}{}_{ef;c}\, h^{1}{}_{b}{}^{f}{}_{;d} + \frac{1}{2} g^{ae}\, h^{1}{}_{c}{}^{f}{}_{;b}\, h^{1}{}_{ef;d} + \frac{1}{2} g^{ae}\, h^{1}{}_{b}{}^{f}{}_{;c}\, h^{1}{}_{ef;d} + h^{1\,ae}\, h^{1}{}_{ce;b;d} - \\
&\frac{1}{2} g^{ae}\, h^{2}{}_{ce;b;d} - h^{1\,ae}\, h^{1}{}_{bc;e;d} + \frac{1}{2} g^{ae}\, h^{2}{}_{bc;e;d} - \frac{1}{2} g^{ae}\, h^{1}{}_{b}{}^{f}{}_{;d}\, h^{1}{}_{cf;e} - \\
&\frac{1}{2} g^{ae}\, h^{1}{}_{df;b}\, h^{1}{}_{c}{}^{f}{}_{;e} + \frac{1}{2} g^{ae}\, h^{1}{}_{c}{}^{f}{}_{;b}\, h^{1}{}_{df;e} + \frac{1}{2} g^{ae}\, h^{1}{}_{b}{}^{f}{}_{;c}\, h^{1}{}_{df;e} + h^{1\,ae}\, h^{1}{}_{bd;c;e} - \\
&\frac{1}{2} g^{ae}\, h^{2}{}_{bd;c;e} + \frac{1}{2} g^{ae}\, h^{1}{}_{b}{}^{f}{}_{;d}\, h^{1}{}_{ce;f} - \frac{1}{2} g^{ae}\, h^{1}{}_{c}{}^{f}{}_{;b}\, h^{1}{}_{ed;f} - \frac{1}{2} g^{ae}\, h^{1}{}_{b}{}^{f}{}_{;c}\, h^{1}{}_{ed;f} - \\
&\frac{1}{2} g^{ae}\, h^{1}{}_{ef;d}\, h^{1}{}_{bc}{}^{;f} - \frac{1}{2} g^{ae}\, h^{1}{}_{df;e}\, h^{1}{}_{bc}{}^{;f} + \frac{1}{2} g^{ae}\, h^{1}{}_{ed;f}\, h^{1}{}_{bc}{}^{;f} + \\
&\frac{1}{2} g^{ae}\, h^{1}{}_{ef;c}\, h^{1}{}_{bd}{}^{;f} + \frac{1}{2} g^{ae}\, h^{1}{}_{cf;e}\, h^{1}{}_{bd}{}^{;f} - \frac{1}{2} g^{ae}\, h^{1}{}_{ce;f}\, h^{1}{}_{bd}{}^{;f} + \frac{1}{2} g^{ae}\, h^{1}{}_{df;b}\, h^{1}{}_{ce}{}^{;f}
\end{aligned}
$$

In[161]:= 
```
% /. _Riemanncd → 0 // ContractMetric
```

Out[161]=

$$-\frac{1}{2} h^{2}{}_{bd;c}{}^{;a}+\frac{1}{2} h^{1}{}_{de}{}^{;a} h^{1}{}_{c}{}^{e}{}_{;b}-\frac{1}{2} h^{1}{}_{c}{}^{e;a} h^{1}{}_{de;b}-\frac{1}{2} h^{1}{}_{d}{}^{e}{}_{;b} h^{1a}{}_{e;c}+$$
$$\frac{1}{2} h^{1}{}_{de}{}^{;a} h^{1}{}_{b}{}^{e}{}_{;c}-h^{1ae} h^{1}{}_{de;b;c}+\frac{1}{2} h^{2a}{}_{d;b;c}+\frac{1}{2} h^{1}{}_{c}{}^{e}{}_{;b} h^{1a}{}_{e;d}+\frac{1}{2} h^{1}{}_{b}{}^{e}{}_{;c} h^{1a}{}_{e;d}-$$
$$\frac{1}{2} h^{1}{}_{ce}{}^{;a} h^{1}{}_{b}{}^{e}{}_{;d}-\frac{1}{2} h^{1a}{}_{e;c} h^{1}{}_{b}{}^{e}{}_{;d}+\frac{1}{2} h^{2}{}_{bc}{}^{;a}{}_{;d}+h^{1ae} h^{1}{}_{ce;b;d}-\frac{1}{2} h^{2}{}_{c}{}^{a}{}_{;b;d}-$$
$$h^{1ae} h^{1}{}_{bc;e;d}-\frac{1}{2} h^{1}{}_{c}{}^{e}{}_{;b} h^{1a}{}_{d;e}-\frac{1}{2} h^{1}{}_{b}{}^{e}{}_{;c} h^{1a}{}_{d;e}+\frac{1}{2} h^{1}{}_{b}{}^{e}{}_{;d} h^{1}{}_{c}{}^{a}{}_{;e}+$$
$$h^{1ae} h^{1}{}_{bd;c;e}-\frac{1}{2} h^{1}{}_{de}{}^{;a} h^{1}{}_{bc}{}^{;e}-\frac{1}{2} h^{1a}{}_{e;d} h^{1}{}_{bc}{}^{;e}+\frac{1}{2} h^{1a}{}_{d;e} h^{1}{}_{bc}{}^{;e}+$$
$$\frac{1}{2} h^{1}{}_{ce}{}^{;a} h^{1}{}_{bd}{}^{;e}+\frac{1}{2} h^{1a}{}_{e;c} h^{1}{}_{bd}{}^{;e}-\frac{1}{2} h^{1}{}_{c}{}^{a}{}_{;e} h^{1}{}_{bd}{}^{;e}+\frac{1}{2} h^{1}{}_{de;b} h^{1}{}_{c}{}^{a;e}$$

In[162]:= 

```
% // Simplify
```

Out[162]=

$$\frac{1}{2}\Big(-h^{2}{}_{bd;c}{}^{;a}+h^{1}{}_{df}{}^{;a} h^{1}{}_{c}{}^{f}{}_{;b}-h^{1}{}_{c}{}^{f;a} h^{1}{}_{df;b}-h^{1}{}_{d}{}^{f}{}_{;b} h^{1a}{}_{f;c}+$$
$$h^{1}{}_{df}{}^{;a} h^{1}{}_{b}{}^{f}{}_{;c}-2 h^{1ae} h^{1}{}_{de;b;c}+h^{2a}{}_{d;b;c}+h^{1}{}_{c}{}^{f}{}_{;b} h^{1a}{}_{f;d}+h^{1}{}_{b}{}^{f}{}_{;c} h^{1a}{}_{f;d}-$$
$$h^{1}{}_{cf}{}^{;a} h^{1}{}_{b}{}^{f}{}_{;d}-h^{1a}{}_{f;c} h^{1}{}_{b}{}^{f}{}_{;d}+h^{2}{}_{bc}{}^{;a}{}_{;d}+2 h^{1ae} h^{1}{}_{ce;b;d}-h^{2}{}_{c}{}^{a}{}_{;b;d}-$$
$$2 h^{1ae} h^{1}{}_{bc;e;d}-2 h^{1}{}_{d}{}^{e}\left(h^{1}{}_{be;c}{}^{;a}-h^{1a}{}_{e;b;c}-h^{1}{}_{bc}{}^{;a}{}_{;e}+h^{1}{}_{c}{}^{a}{}_{;b;e}\right)+$$
$$2 h^{1ae} h^{1}{}_{bd;c;e}-h^{1}{}_{c}{}^{f}{}_{;b} h^{1a}{}_{d;f}-h^{1}{}_{b}{}^{f}{}_{;c} h^{1a}{}_{d;f}+h^{1}{}_{b}{}^{f}{}_{;d} h^{1}{}_{c}{}^{a}{}_{;f}+$$
$$2 h^{1}{}_{d}{}^{f}\left(h^{1}{}_{bf;c}{}^{;a}-h^{1a}{}_{f;b;c}-h^{1}{}_{bc}{}^{;a}{}_{;f}+h^{1}{}_{c}{}^{a}{}_{;b;f}\right)-h^{1}{}_{df}{}^{;a} h^{1}{}_{bc}{}^{;f}-h^{1a}{}_{f;d} h^{1}{}_{bc}{}^{;f}+$$
$$h^{1a}{}_{d;f} h^{1}{}_{bc}{}^{;f}+h^{1}{}_{cf}{}^{;a} h^{1}{}_{bd}{}^{;f}+h^{1a}{}_{f;c} h^{1}{}_{bd}{}^{;f}-h^{1}{}_{c}{}^{a}{}_{;f} h^{1}{}_{bd}{}^{;f}+h^{1}{}_{df;b} h^{1}{}_{c}{}^{a;f}\Big)$$

The Following is the expression for Riemann Curvature Tensor to second order

$$\triangle^{2}\left[R[\nabla]^{a}{}_{bcd}\right]$$


$$=\frac{1}{2}\Big(-h^{2}{}_{bd;c}{}^{;a}+h^{1}{}_{df}{}^{;a} h^{1}{}_{c}{}^{f}{}_{;b}-h^{1}{}_{c}{}^{f;a} h^{1}{}_{df;b}-h^{1}{}_{d}{}^{f}{}_{;b} h^{1a}{}_{f;c}+$$
$$h^{1}{}_{df}{}^{;a} h^{1}{}_{b}{}^{f}{}_{;c}-2 h^{1ae} h^{1}{}_{de;b;c}+h^{2a}{}_{d;b;c}+h^{1}{}_{c}{}^{f}{}_{;b} h^{1a}{}_{f;d}+h^{1}{}_{b}{}^{f}{}_{;c} h^{1a}{}_{f;d}-$$
$$h^{1}{}_{cf}{}^{;a} h^{1}{}_{b}{}^{f}{}_{;d}-h^{1a}{}_{f;c} h^{1}{}_{b}{}^{f}{}_{;d}+h^{2}{}_{bc}{}^{;a}{}_{;d}+2 h^{1ae} h^{1}{}_{ce;b;d}-h^{2}{}_{c}{}^{a}{}_{;b;d}-$$
$$2 h^{1ae} h^{1}{}_{bc;e;d}-2 h^{1}{}_{d}{}^{e}\left(h^{1}{}_{be;c}{}^{;a}-h^{1a}{}_{e;b;c}-h^{1}{}_{bc}{}^{;a}{}_{;e}+h^{1}{}_{c}{}^{a}{}_{;b;e}\right)+$$
$$2 h^{1ae} h^{1}{}_{bd;c;e}-h^{1}{}_{c}{}^{f}{}_{;b} h^{1a}{}_{d;f}-h^{1}{}_{b}{}^{f}{}_{;c} h^{1a}{}_{d;f}+h^{1}{}_{b}{}^{f}{}_{;d} h^{1}{}_{c}{}^{a}{}_{;f}+$$
$$2 h^{1}{}_{d}{}^{f}\left(h^{1}{}_{bf;c}{}^{;a}-h^{1a}{}_{f;b;c}-h^{1}{}_{bc}{}^{;a}{}_{;f}+h^{1}{}_{c}{}^{a}{}_{;b;f}\right)-h^{1}{}_{df}{}^{;a} h^{1}{}_{bc}{}^{;f}-h^{1a}{}_{f;d} h^{1}{}_{bc}{}^{;f}+$$
$$h^{1a}{}_{d;f} h^{1}{}_{bc}{}^{;f}+h^{1}{}_{cf}{}^{;a} h^{1}{}_{bd}{}^{;f}+h^{1a}{}_{f;c} h^{1}{}_{bd}{}^{;f}-h^{1}{}_{c}{}^{a}{}_{;f} h^{1}{}_{bd}{}^{;f}+h^{1}{}_{df;b} h^{1}{}_{c}{}^{a;f}\Big)$$


## Perturbation of Ricci Tensor

### I. First Order

In[193]:=
```
Perturbation[Riccicd[-a, -b], 1]
```

Out[193]= $\triangle\left[R[\nabla]_{ab}\right]$

In[194]:=
```
Perturbation[Riccicd[-a, -b], 1] // ExpandPerturbation
```

Out[194]= $$\frac{1}{2}\left(-h^{1c}{}_{c;b;a}-h^{1c}{}_{b;c;a}+h^{1}{}_{bc}{}^{;c}{}_{;a}\right)+\frac{1}{2}\left(h^{1c}{}_{b;a;c}+h^{1c}{}_{a;b;c}-h^{1}{}_{ba}{}^{;c}{}_{;c}\right)$$

In[195]:=
```
SortCovDs[%] // ToCanonical
```

Out[195]= $$-\frac{1}{2}h^{1c}{}_{c;a;b}+\frac{1}{2}h^{1}{}_{b}{}^{c}{}_{;a;c}+\frac{1}{2}h^{1}{}_{a}{}^{c}{}_{;b;c}-\frac{1}{2}h^{1}{}_{ab}{}^{;c}{}_{;c}$$

In[196]:=
```
% /. _Riccicd → 0 // ContractMetric
```

Out[196]= $$-\frac{1}{2}h^{1c}{}_{c;a;b}+\frac{1}{2}h^{1}{}_{b}{}^{c}{}_{;a;c}+\frac{1}{2}h^{1}{}_{a}{}^{c}{}_{;b;c}-\frac{1}{2}h^{1}{}_{ab}{}^{;c}{}_{;c}$$

In[197]:=
```
% // Simplify
```

Out[197]= $$\frac{1}{2}\left(-h^{1c}{}_{c;a;b}+h^{1}{}_{b}{}^{c}{}_{;a;c}+h^{1}{}_{a}{}^{c}{}_{;b;c}-h^{1}{}_{ab}{}^{;c}{}_{;c}\right)$$

**The Following is the expansion of Ricci Tensor up to First order perturbation**

$$\triangle\left[R[\nabla]_{ab}\right]=\frac{1}{2}\left(-h^{1c}{}_{c;a;b}+h^{1}{}_{b}{}^{c}{}_{;a;c}+h^{1}{}_{a}{}^{c}{}_{;b;c}-h^{1}{}_{ab}{}^{;c}{}_{;c}\right)$$

## II. Second Order

In[209]:=
```
Perturbation[Riccicd[-a, -b], 2]
```

Out[209]= $\triangle^{2}\left[R[\nabla]_{ab}\right]$

In[210]:=
```
Perturbation[Riccicd[-a, -b], 2] // ExpandPerturbation
```

Out[210]= $$\begin{aligned}&\frac{1}{2}\left(-h^{2c}{}_{c;b;a}-h^{2c}{}_{b;c;a}+h^{2}{}_{bc}{}^{;c}{}_{;a}\right)+\\&\frac{1}{2}\left(h^{2c}{}_{b;a;c}+h^{2c}{}_{a;b;c}-h^{2}{}_{ba}{}^{;c}{}_{;c}\right)+2\times\left(\frac{1}{2}h^{1cd}\left(h^{1}{}_{dc;b;a}+h^{1}{}_{db;c;a}-h^{1}{}_{bc;d;a}\right)+\right.\\&\quad\left.\frac{1}{4}\left(h^{1}{}_{ec;b}+h^{1}{}_{eb;c}-h^{1}{}_{cb;e}\right)\left(h^{1ec}{}_{;a}+h^{1e}{}_{a}{}^{;c}-h^{1c}{}_{a}{}^{;e}\right)\right)-\\&2\times\left(\frac{1}{2}h^{1cd}\left(h^{1}{}_{db;a;c}+h^{1}{}_{da;b;c}-h^{1}{}_{ba;d;c}\right)+\right.\\&\quad\left.\frac{1}{4}\left(h^{1}{}_{eb;a}+h^{1}{}_{ea;b}-h^{1}{}_{ab;e}\right)\left(h^{1ec}{}_{;c}+h^{1e}{}_{c}{}^{;c}-h^{1c}{}_{c}{}^{;e}\right)\right)\end{aligned}$$

In[211]:=
```
SortCovDs[%] // ToCanonical
```

Out[211]= $$\begin{aligned}&\frac{1}{2}h^{1cd}{}_{;a}\,h^{1}{}_{cd;b}+h^{1cd}\,h^{1}{}_{cd;a;b}-\frac{1}{2}h^{2c}{}_{c;a;b}+\frac{1}{2}h^{1}{}_{b}{}^{c}{}_{;a}\,h^{1d}{}_{d;c}+\\&\frac{1}{2}h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1d}{}_{d;c}+\frac{1}{2}h^{2}{}_{b}{}^{c}{}_{;a;c}+\frac{1}{2}h^{2}{}_{a}{}^{c}{}_{;b;c}-\frac{1}{2}h^{2}{}_{ab}{}^{;c}{}_{;c}-\frac{1}{2}h^{1d}{}_{d;c}\,h^{1}{}_{ab}{}^{;c}-\\&h^{1}{}_{b}{}^{c}{}_{;a}\,h^{1}{}_{c}{}^{d}{}_{;d}-h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1}{}_{c}{}^{d}{}_{;d}+h^{1}{}_{ab}{}^{;c}\,h^{1}{}_{c}{}^{d}{}_{;d}-h^{1cd}\,h^{1}{}_{bc;a;d}-\\&h^{1cd}\,h^{1}{}_{ac;b;d}+h^{1cd}\,h^{1}{}_{ab;c;d}-h^{1}{}_{bd;c}\,h^{1}{}_{a}{}^{c;d}+h^{1}{}_{bc;d}\,h^{1}{}_{a}{}^{c;d}\end{aligned}$$

In[212]:= `% /. _Riccicd → 0 // ContractMetric`

Out[212]= $$\frac{1}{2} h^{1cd}{}_{;a}\, h^{1}{}_{cd;b} + h^{1cd}\, h^{1}{}_{cd;a;b} - \frac{1}{2} h^{2c}{}_{c;a;b} + \frac{1}{2} h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1d}{}_{d;c} + \frac{1}{2} h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1d}{}_{d;c} + \frac{1}{2} h^{2}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2} h^{2}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2} h^{2}{}_{ab}{}^{;c}{}_{;c} - \frac{1}{2} h^{1d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} + h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1cd}\, h^{1}{}_{bc;a;d} - h^{1cd}\, h^{1}{}_{ac;b;d} + h^{1cd}\, h^{1}{}_{ab;c;d} - h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d}$$

In[213]:= `% // Simplify`

Out[213]= $$\frac{1}{2} \Big( h^{1cd}{}_{;a}\, h^{1}{}_{cd;b} - h^{2c}{}_{c;a;b} + h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1d}{}_{d;c} + h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1d}{}_{d;c} + h^{2}{}_{b}{}^{c}{}_{;a;c} + h^{2}{}_{a}{}^{c}{}_{;b;c} - h^{2}{}_{ab}{}^{;c}{}_{;c} - h^{1d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - 2\, h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1}{}_{c}{}^{d}{}_{;d} - 2\, h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} + 2\, h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} + 2\, h^{1cd} \left( h^{1}{}_{cd;a;b} - h^{1}{}_{bc;a;d} - h^{1}{}_{ac;b;d} + h^{1}{}_{ab;c;d} \right) - 2\, h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + 2\, h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d} \Big)$$

**The Following is the expression for Ricci Tensor to second order**

$$\Delta^{2}\left[ R[\nabla]_{ab} \right] = \frac{1}{2} \Big( h^{1cd}{}_{;a}\, h^{1}{}_{cd;b} - h^{2c}{}_{c;a;b} + h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1d}{}_{d;c} + h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1d}{}_{d;c} + h^{2}{}_{b}{}^{c}{}_{;a;c} + h^{2}{}_{a}{}^{c}{}_{;b;c} - h^{2}{}_{ab}{}^{;c}{}_{;c} - h^{1d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - 2\, h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1}{}_{c}{}^{d}{}_{;d} - 2\, h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} + 2\, h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} + 2\, h^{1cd} \left( h^{1}{}_{cd;a;b} - h^{1}{}_{bc;a;d} - h^{1}{}_{ac;b;d} + h^{1}{}_{ab;c;d} \right) - 2\, h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + 2\, h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d} \Big)$$

# Perturbation of Ricci Scalar

## I. First Order

In[203]:= `Perturbation[Riccicd[a, -a], 1]`

Out[203]= $\Delta[R[\nabla]]$

In[204]:= `Perturbation[Riccicd[a, -a], 1] // ExpandPerturbation`

Out[204]= $$-h^{1ab}\, R[\nabla]_{ab} + g^{ab} \left( \frac{1}{2} \left( -h^{1c}{}_{c;b;a} - h^{1c}{}_{b;c;a} + h^{1}{}_{bc}{}^{;c}{}_{;a} \right) + \frac{1}{2} \left( h^{1c}{}_{b;a;c} + h^{1c}{}_{a;b;c} - h^{1}{}_{ba}{}^{;c}{}_{;c} \right) \right)$$

In[205]:= `SortCovDs[%] // ToCanonical`

Out[205]= $$-h^{1ab}\, R[\nabla]_{ab} - \frac{1}{2} g^{ab}\, h^{1c}{}_{c;a;b} + g^{ab}\, h^{1}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2} g^{ab}\, h^{1}{}_{ab}{}^{;c}{}_{;c}$$

In[206]:= `% /. _Riccicd → 0 // ContractMetric`

Out[206]= $$-\frac{1}{2} h^{1b}{}_{b}{}^{;a}{}_{;a} + h^{1ab}{}_{;a;b} - \frac{1}{2} h^{1}{}_{a}{}^{a;b}{}_{;b}$$

In[207]:= `% // Simplify`

$$-\frac{1}{2} h^{1b}{}_{b}{}^{;a}{}_{;a} + h^{1ab}{}_{;a;b} - \frac{1}{2} h^{1}{}_{a}{}^{a;b}{}_{;b}$$

In[208]:= **% // ToCanonical**

Out[208]= $h^{1ab}{}_{;a;b} - h^{1a}{}_{a}{}^{;b}{}_{;b}$

**The Following is the expression for Ricci Scalar to first order**

$$\triangle[R[\nabla]] = h^{1ab}{}_{;a;b} - h^{1a}{}_{a}{}^{;b}{}_{;b}$$

## II. Second Order

In[215]:= **Perturbation[Riccicd[a, -a], 2]**

Out[215]= $\triangle^2[R[\nabla]]$

In[216]:= **Perturbation[Riccicd[a, -a], 2] // ExpandPerturbation**

Out[216]=
$$\left(2\, h^{1ac}\, h^{1}{}_{c}{}^{b} - h^{2ab}\right) R[\nabla]_{ab} - 2\, h^{1ab} \left(\frac{1}{2}\left(-h^{1c}{}_{c;b;a} - h^{1c}{}_{b;c;a} + h^{1}{}_{bc}{}^{;c}{}_{;a}\right) + \frac{1}{2}\left(h^{1c}{}_{b;a;c} + h^{1c}{}_{a;b;c} - h^{1}{}_{ba}{}^{;c}{}_{;c}\right)\right) +$$
$$g^{ab}\left(\frac{1}{2}\left(-h^{2c}{}_{c;b;a} - h^{2c}{}_{b;c;a} + h^{2}{}_{bc}{}^{;c}{}_{;a}\right) + \frac{1}{2}\left(h^{2c}{}_{b;a;c} + h^{2c}{}_{a;b;c} - h^{2}{}_{ba}{}^{;c}{}_{;c}\right) +\right.$$
$$2 \times \left(\frac{1}{2} h^{1cd}\left(h^{1}{}_{dc;b;a} + h^{1}{}_{db;c;a} - h^{1}{}_{bc;d;a}\right) + \frac{1}{4}\left(h^{1}{}_{ec;b} + h^{1}{}_{eb;c} - h^{1}{}_{cb;e}\right)\right.$$
$$\left.\left(h^{1ec}{}_{;a} + h^{1e}{}_{a}{}^{;c} - h^{1c}{}_{a}{}^{;e}\right)\right) - 2 \times \left(\frac{1}{2} h^{1cd}\left(h^{1}{}_{db;a;c} + h^{1}{}_{da;b;c} - h^{1}{}_{ba;d;c}\right) +\right.$$
$$\left.\left.\frac{1}{4}\left(h^{1}{}_{eb;a} + h^{1}{}_{ea;b} - h^{1}{}_{ab;e}\right)\left(h^{1ec}{}_{;c} + h^{1e}{}_{c}{}^{;c} - h^{1c}{}_{c}{}^{;e}\right)\right)\right)$$

In[217]:= **SortCovDs[%] // ToCanonical**

Out[217]=
$$-h^{2ab} R[\nabla]_{ab} + 2\, h^{1}{}_{a}{}^{c}\, h^{1ab} R[\nabla]_{bc} + \frac{1}{2} g^{ab}\, h^{1cd}{}_{;a}\, h^{1}{}_{cd;b} +$$
$$g^{ab}\, h^{1cd}\, h^{1}{}_{cd;a;b} + h^{1ab}\, h^{1c}{}_{c;a;b} - \frac{1}{2} g^{ab}\, h^{2c}{}_{c;a;b} + g^{ab}\, h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1d}{}_{d;c} -$$
$$2\, h^{1ab}\, h^{1}{}_{a}{}^{c}{}_{;b;c} + g^{ab}\, h^{2}{}_{a}{}^{c}{}_{;b;c} + h^{1ab}\, h^{1}{}_{ab}{}^{;c}{}_{;c} - \frac{1}{2} g^{ab}\, h^{2}{}_{ab}{}^{;c}{}_{;c} -$$
$$\frac{1}{2} g^{ab}\, h^{1d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - 2\, g^{ab}\, h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} + g^{ab}\, h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} -$$
$$2\, g^{ab}\, h^{1cd}\, h^{1}{}_{ac;b;d} + g^{ab}\, h^{1cd}\, h^{1}{}_{ab;c;d} - g^{ab}\, h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + g^{ab}\, h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d}$$

In[218]:= **% /. _Riccicd → 0 // ContractMetric**

Out[218]=
$$h^{1bc}\, h^{1}{}_{bc}{}^{;a}{}_{;a} - \frac{1}{2} h^{2b}{}_{b}{}^{;a}{}_{;a} + \frac{1}{2} h^{1bc}{}_{;a}\, h^{1}{}_{bc}{}^{;a} + h^{1ab}{}_{;a}\, h^{1c}{}_{c;b} + h^{1ab}\, h^{1c}{}_{c;a;b} + h^{2ab}{}_{;a;b} -$$
$$\frac{1}{2} h^{2}{}_{a}{}^{a;b}{}_{;b} - \frac{1}{2} h^{1c}{}_{c;b}\, h^{1}{}_{a}{}^{a;b} - 2\, h^{1ab}{}_{;a}\, h^{1}{}_{b}{}^{c}{}_{;c} + h^{1}{}_{a}{}^{a;b}\, h^{1}{}_{b}{}^{c}{}_{;c} - 2\, h^{1bc}\, h^{1a}{}_{b;a;c} +$$
$$h^{1bc}\, h^{1}{}_{a}{}^{a}{}_{;b;c} - 2\, h^{1ab}\, h^{1}{}_{a}{}^{c}{}_{;b;c} + h^{1ab}\, h^{1}{}_{ab}{}^{;c}{}_{;c} - h^{1a}{}_{c;b}\, h^{1}{}_{a}{}^{b;c} + h^{1a}{}_{b;c}\, h^{1}{}_{a}{}^{b;c}$$

In[219]:=
```
% // Simplify
```

Out[219]=

$$h^{1bc}\, h^{1}{}_{bc}{}^{;a}{}_{;a} - \frac{1}{2} h^{2b}{}_{b}{}^{;a}{}_{;a} + \frac{1}{2} h^{1bc}{}_{;a}\, h^{1}{}_{bc}{}^{;a} + h^{1ab}{}_{;a}\, h^{1c}{}_{c;b} + h^{2ab}{}_{;a;b} - \frac{1}{2} h^{2}{}_{a}{}^{a;b}{}_{;b} - \frac{1}{2} h^{1c}{}_{c;b}\, h^{1}{}_{a}{}^{a;b} - 2 h^{1ab}{}_{;a}\, h^{1}{}_{b}{}^{c}{}_{;c} + h^{1}{}_{a}{}^{a;b}\, h^{1}{}_{b}{}^{c}{}_{;c} - 2 h^{1bc}\, h^{1a}{}_{b;a;c} + h^{1bc}\, h^{1}{}_{a}{}^{a}{}_{;b;c} + h^{1ab}\left(h^{1c}{}_{c;a;b} - 2 h^{1}{}_{a}{}^{c}{}_{;b;c} + h^{1}{}_{ab}{}^{;c}{}_{;c}\right) - h^{1a}{}_{c;b}\, h^{1}{}_{a}{}^{b;c} + h^{1a}{}_{b;c}\, h^{1}{}_{a}{}^{b;c}$$

$$-\frac{1}{2} h^{1b}{}_{b}{}^{;a}{}_{;a} + h^{1ab}{}_{;a;b} - \frac{1}{2} h^{1}{}_{a}{}^{a;b}{}_{;b}$$

In[220]:=
```
% // ToCanonical
```

Out[220]=

$$2 h^{1ab}\, h^{1c}{}_{c;a;b} + h^{2ab}{}_{;a;b} - h^{2a}{}_{a}{}^{;b}{}_{;b} - 2 h^{1ab}\, h^{1}{}_{a}{}^{c}{}_{;c;b} - \frac{1}{2} h^{1c}{}_{c;b}\, h^{1a}{}_{a}{}^{;b} - 2 h^{1ab}{}_{;a}\, h^{1}{}_{b}{}^{c}{}_{;c} + 2 h^{1a}{}_{a}{}^{;b}\, h^{1}{}_{b}{}^{c}{}_{;c} - 2 h^{1ab}\, h^{1}{}_{a}{}^{c}{}_{;b;c} + 2 h^{1ab}\, h^{1}{}_{ab}{}^{;c}{}_{;c} - h^{1}{}_{ac;b}\, h^{1ab;c} + \frac{3}{2} h^{1}{}_{ab;c}\, h^{1ab;c}$$

**The Following is the expression for Ricci Scalar to second order**

$$\triangle^{2}[R[\nabla]] =$$

$$2 h^{1ab}\, h^{1c}{}_{c;a;b} + h^{2ab}{}_{;a;b} - h^{2a}{}_{a}{}^{;b}{}_{;b} - 2 h^{1ab}\, h^{1}{}_{a}{}^{c}{}_{;c;b} - \frac{1}{2} h^{1c}{}_{c;b}\, h^{1a}{}_{a}{}^{;b} - 2 h^{1ab}{}_{;a}\, h^{1}{}_{b}{}^{c}{}_{;c} + 2 h^{1a}{}_{a}{}^{;b}\, h^{1}{}_{b}{}^{c}{}_{;c} - 2 h^{1ab}\, h^{1}{}_{a}{}^{c}{}_{;b;c} + 2 h^{1ab}\, h^{1}{}_{ab}{}^{;c}{}_{;c} - h^{1}{}_{ac;b}\, h^{1ab;c} + \frac{3}{2} h^{1}{}_{ab;c}\, h^{1ab;c}$$

# Perturbation of Einstein Tensor

## I. First Order

In[222]:=
```
Perturbation[Einsteincd[-a, -b], 1]
```

Out[222]=

$$\triangle\left[G[\nabla]_{ab}\right]$$

In[223]:=
```
Perturbation[Einsteincd[-a, -b], 1] // ExpandPerturbation
```

Out[223]=

$$\frac{1}{2}\left(-h^{1c}{}_{c;b;a} - h^{1c}{}_{b;c;a} + h^{1}{}_{bc}{}^{;c}{}_{;a}\right) + \frac{1}{2}\left(h^{1c}{}_{b;a;c} + h^{1c}{}_{a;b;c} - h^{1}{}_{ba}{}^{;c}{}_{;c}\right) + \frac{1}{2}\left(-h^{1}{}_{ab} R[\nabla] - g_{ab}\left(-h^{1cd} R[\nabla]_{cd} + g^{cd}\left(\frac{1}{2}\left(-h^{1e}{}_{e;d;c} - h^{1e}{}_{d;e;c} + h^{1}{}_{de}{}^{;e}{}_{;c}\right) + \frac{1}{2}\left(h^{1e}{}_{d;c;e} + h^{1e}{}_{c;d;e} - h^{1}{}_{dc}{}^{;e}{}_{;e}\right)\right)\right)\right)$$

In[224]:=
```
SortCovDs[%] // ToCanonical
```

Out[224]=

$$\frac{1}{2} g_{ab}\, h^{1cd}\, R[\nabla]_{cd} - \frac{1}{2} h^{1}{}_{ab}\, R[\nabla] - \frac{1}{2} h^{1c}{}_{c;a;b} + \frac{1}{2} h^{1}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2} h^{1}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2} h^{1}{}_{ab}{}^{;c}{}_{;c} + \frac{1}{4} g_{ab}\, g^{cd}\, h^{1e}{}_{e;c;d} - \frac{1}{2} g_{ab}\, g^{cd}\, h^{1}{}_{c}{}^{e}{}_{;d;e} + \frac{1}{4} g_{ab}\, g^{cd}\, h^{1}{}_{cd}{}^{;e}{}_{;e}$$

In[225]:= `% /. _Riccicd → 0 // ContractMetric`

Out[225]= $-\frac{1}{2} h^{1}{}_{ab} R[\nabla] - \frac{1}{2} h^{1c}{}_{c;a;b} + \frac{1}{2} h^{1}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2} h^{1}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2} h^{1}{}_{ab}{}^{;c}{}_{;c} + \frac{1}{4} g_{ab} h^{1d}{}_{d}{}^{;c}{}_{;c} - \frac{1}{2} g_{ab} h^{1cd}{}_{;c;d} + \frac{1}{4} g_{ab} h^{1}{}_{c}{}^{c;d}{}_{;d}$

In[226]:= `% // Simplify`

Out[226]= $\frac{1}{4} \times \left(-2 h^{1}{}_{ab} R[\nabla] - 2 h^{1c}{}_{c;a;b} + 2 h^{1}{}_{b}{}^{c}{}_{;a;c} + 2 h^{1}{}_{a}{}^{c}{}_{;b;c} - 2 h^{1}{}_{ab}{}^{;c}{}_{;c} + g_{ab} h^{1e}{}_{e}{}^{;d}{}_{;d} - 2 g_{ab} h^{1de}{}_{;d;e} + g_{ab} h^{1}{}_{c}{}^{c;e}{}_{;e}\right)$

In[227]:= `% // ToCanonical`

Out[227]= $-\frac{1}{2} h^{1}{}_{ab} R[\nabla] - \frac{1}{2} h^{1c}{}_{c;a;b} + \frac{1}{2} h^{1}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2} h^{1}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2} h^{1}{}_{ab}{}^{;c}{}_{;c} - \frac{1}{2} g_{ab} h^{1cd}{}_{;c;d} + \frac{1}{2} g_{ab} h^{1c}{}_{c}{}^{;d}{}_{;d}$

**The Following is the expression for Einstein Tensor to first order**

$$\triangle\left[G[\nabla]_{ab}\right] = -\frac{1}{2} h^{1}{}_{ab} R[\nabla] - \frac{1}{2} h^{1c}{}_{c;a;b} + \frac{1}{2} h^{1}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2} h^{1}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2} h^{1}{}_{ab}{}^{;c}{}_{;c} - \frac{1}{2} g_{ab} h^{1cd}{}_{;c;d} + \frac{1}{2} g_{ab} h^{1c}{}_{c}{}^{;d}{}_{;d}$$

## II. Second Order

In[228]:= `Perturbation[Einsteincd[-a, -b], 2]`

Out[228]= $\triangle^{2}\left[G[\nabla]_{ab}\right]$

In[242]:=
```
Perturbation[Einsteincd[-a, -b], 2] // ExpandPerturbation
```

Out[242]=

$$\frac{1}{2}\left(-h^{2c}{}_{c;b;a}-h^{2c}{}_{b;c;a}+h^{2}{}_{bc}{}^{;c}{}_{;a}\right)+$$

$$\frac{1}{2}\left(h^{2c}{}_{b;a;c}+h^{2c}{}_{a;b;c}-h^{2}{}_{ba}{}^{;c}{}_{;c}\right)+2\times\Big(\frac{1}{2}\,h^{1cd}\left(h^{1}{}_{dc;b;a}+h^{1}{}_{db;c;a}-h^{1}{}_{bc;d;a}\right)+$$

$$\frac{1}{4}\left(h^{1}{}_{ec;b}+h^{1}{}_{eb;c}-h^{1}{}_{cb;e}\right)\left(h^{1ec}{}_{;a}+h^{1e}{}_{a}{}^{;c}-h^{1c}{}_{a}{}^{;e}\right)\Big)-$$

$$2\times\Big(\frac{1}{2}\,h^{1cd}\left(h^{1}{}_{db;a;c}+h^{1}{}_{da;b;c}-h^{1}{}_{ba;d;c}\right)+$$

$$\frac{1}{4}\left(h^{1}{}_{eb;a}+h^{1}{}_{ea;b}-h^{1}{}_{ab;e}\right)\left(h^{1ec}{}_{;c}+h^{1e}{}_{c}{}^{;c}-h^{1c}{}_{c}{}^{;e}\right)\Big)+$$

$$\frac{1}{2}\Big(-h^{2}{}_{ab}\,R[\nabla]-g_{ab}\,\Big(\left(2\,h^{1ce}\,h^{1}{}_{e}{}^{d}-h^{2cd}\right)R[\nabla]_{cd}-$$

$$2\,h^{1cd}\left(\frac{1}{2}\left(-h^{1f}{}_{f;d;c}-h^{1f}{}_{d;f;c}+h^{1}{}_{df}{}^{;f}{}_{;c}\right)+\frac{1}{2}\left(h^{1f}{}_{d;c;f}+h^{1f}{}_{c;d;f}-h^{1}{}_{dc}{}^{;f}{}_{;f}\right)\right)+$$

$$g^{cd}\,\Big(\frac{1}{2}\left(-h^{2i}{}_{i;d;c}-h^{2i}{}_{d;i;c}+h^{2}{}_{di}{}^{;i}{}_{;c}\right)+\frac{1}{2}\left(h^{2i}{}_{d;c;i}+h^{2i}{}_{c;d;i}-h^{2}{}_{dc}{}^{;i}{}_{;i}\right)+$$

$$2\times\Big(\frac{1}{2}\,h^{1ij}\left(h^{1}{}_{ji;d;c}+h^{1}{}_{jd;i;c}-h^{1}{}_{di;j;c}\right)+\frac{1}{4}\left(h^{1}{}_{ki;d}+h^{1}{}_{kd;i}-h^{1}{}_{id;k}\right)$$

$$\left(h^{1ki}{}_{;c}+h^{1k}{}_{c}{}^{;i}-h^{1i}{}_{c}{}^{;k}\right)\Big)-2\times\Big(\frac{1}{2}\,h^{1ij}\left(h^{1}{}_{jd;c;i}+h^{1}{}_{jc;d;i}-h^{1}{}_{dc;j;i}\right)+$$

$$\frac{1}{4}\left(h^{1}{}_{kd;c}+h^{1}{}_{kc;d}-h^{1}{}_{cd;k}\right)\left(h^{1ki}{}_{;i}+h^{1k}{}_{i}{}^{;i}-h^{1i}{}_{i}{}^{;k}\right)\Big)\Big)\Big)\Big)-$$

$$2\,h^{1}{}_{ab}\,\Big(-h^{1l\,l1}\,R[\nabla]_{l\,l1}+g^{l\,l1}\,\Big(\frac{1}{2}\left(-h^{1l2}{}_{l2;l1;l}-h^{1l2}{}_{l1;l2;l}+h^{1}{}_{l1\,l2}{}^{;l2}{}_{;l}\right)+$$

$$\frac{1}{2}\left(h^{1l2}{}_{l1;l;l2}+h^{1l2}{}_{l;l1;l2}-h^{1}{}_{l\,l1}{}^{;l2}{}_{;l2}\right)\Big)\Big)\Big)$$

In[243]:= `SortCovDs[%] // ToCanonical`

Out[243]=

$$h^{1}{}_{ab}\, h^{1cd}\, R[\nabla]_{cd} + \frac{1}{2} g_{ab}\, h^{2cd}\, R[\nabla]_{cd} - g_{ab}\, h^{1}{}_{c}{}^{e}\, h^{1cd}\, R[\nabla]_{de} - \frac{1}{2} h^{2}{}_{ab}\, R[\nabla] +$$
$$\frac{1}{2} h^{1cd}{}_{;a}\, h^{1}{}_{cd;b} + h^{1cd}\, h^{1}{}_{cd;a;b} - \frac{1}{2} h^{2c}{}_{c;a;b} + \frac{1}{2} h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1d}{}_{d;c} + \frac{1}{2} h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1d}{}_{d;c} +$$
$$\frac{1}{2} h^{2}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2} h^{2}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2} h^{2}{}_{ab}{}^{;c}{}_{;c} - \frac{1}{2} h^{1d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} +$$
$$h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} - \frac{1}{4} g_{ab}\, g^{cd}\, h^{1ef}{}_{;c}\, h^{1}{}_{ef;d} - h^{1cd}\, h^{1}{}_{bc;a;d} - h^{1cd}\, h^{1}{}_{ac;b;d} +$$
$$h^{1cd}\, h^{1}{}_{ab;c;d} - \frac{1}{2} g_{ab}\, g^{cd}\, h^{1ef}\, h^{1}{}_{ef;c;d} + \frac{1}{2} g^{cd}\, h^{1}{}_{ab}\, h^{1e}{}_{e;c;d} - \frac{1}{2} g_{ab}\, h^{1cd}\, h^{1e}{}_{e;c;d} +$$
$$\frac{1}{4} g_{ab}\, g^{cd}\, h^{2e}{}_{e;c;d} - h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d} - \frac{1}{2} g_{ab}\, g^{cd}\, h^{1}{}_{c}{}^{e}{}_{;d}\, h^{1f}{}_{f;e} -$$
$$g^{cd}\, h^{1}{}_{ab}\, h^{1}{}_{c}{}^{e}{}_{;d;e} + g_{ab}\, h^{1cd}\, h^{1}{}_{c}{}^{e}{}_{;d;e} - \frac{1}{2} g_{ab}\, g^{cd}\, h^{2}{}_{c}{}^{e}{}_{;d;e} + \frac{1}{2} g^{cd}\, h^{1}{}_{ab}\, h^{1}{}_{cd}{}^{;e}{}_{;e} -$$
$$\frac{1}{2} g_{ab}\, h^{1cd}\, h^{1}{}_{cd}{}^{;e}{}_{;e} + \frac{1}{4} g_{ab}\, g^{cd}\, h^{2}{}_{cd}{}^{;e}{}_{;e} + \frac{1}{4} g_{ab}\, g^{cd}\, h^{1f}{}_{f;e}\, h^{1}{}_{cd}{}^{;e} +$$
$$g_{ab}\, g^{cd}\, h^{1}{}_{c}{}^{e}{}_{;d}\, h^{1}{}_{e}{}^{f}{}_{;f} - \frac{1}{2} g_{ab}\, g^{cd}\, h^{1}{}_{cd}{}^{;e}\, h^{1}{}_{e}{}^{f}{}_{;f} + g_{ab}\, g^{cd}\, h^{1ef}\, h^{1}{}_{ce;d;f} -$$
$$\frac{1}{2} g_{ab}\, g^{cd}\, h^{1ef}\, h^{1}{}_{cd;e;f} + \frac{1}{2} g_{ab}\, g^{cd}\, h^{1}{}_{df;e}\, h^{1}{}_{c}{}^{e;f} - \frac{1}{2} g_{ab}\, g^{cd}\, h^{1}{}_{de;f}\, h^{1}{}_{c}{}^{e;f}$$

In[244]:= `% /. _Einsteincd → 0 // ContractMetric`

Out[244]=

$$h^{1}{}_{ab}\, h^{1cd}\, R[\nabla]_{cd} + \frac{1}{2} g_{ab}\, h^{2cd}\, R[\nabla]_{cd} - g_{ab}\, h^{1}{}_{c}{}^{e}\, h^{1cd}\, R[\nabla]_{de} - \frac{1}{2} h^{2}{}_{ab}\, R[\nabla] +$$
$$\frac{1}{2} h^{1cd}{}_{;a}\, h^{1}{}_{cd;b} + h^{1cd}\, h^{1}{}_{cd;a;b} - \frac{1}{2} h^{2c}{}_{c;a;b} + \frac{1}{2} h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1d}{}_{d;c} + \frac{1}{2} h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1d}{}_{d;c} +$$
$$\frac{1}{2} h^{2}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2} h^{2}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2} g_{ab}\, h^{1de}\, h^{1}{}_{de}{}^{;c}{}_{;c} + \frac{1}{2} h^{1}{}_{ab}\, h^{1d}{}_{d}{}^{;c}{}_{;c} - \frac{1}{2} h^{2}{}_{ab}{}^{;c}{}_{;c} +$$
$$\frac{1}{4} g_{ab}\, h^{2d}{}_{d}{}^{;c}{}_{;c} - \frac{1}{2} h^{1d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - \frac{1}{4} g_{ab}\, h^{1de}{}_{;c}\, h^{1}{}_{de}{}^{;c} - h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1}{}_{c}{}^{d}{}_{;d} -$$
$$h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} + h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} - \frac{1}{2} g_{ab}\, h^{1cd}{}_{;c}\, h^{1e}{}_{e;d} - h^{1cd}\, h^{1}{}_{bc;a;d} - h^{1cd}\, h^{1}{}_{ac;b;d} +$$
$$h^{1cd}\, h^{1}{}_{ab;c;d} - h^{1}{}_{ab}\, h^{1cd}{}_{;c;d} - \frac{1}{2} g_{ab}\, h^{1cd}\, h^{1e}{}_{e;c;d} - \frac{1}{2} g_{ab}\, h^{2cd}{}_{;c;d} + \frac{1}{2} h^{1}{}_{ab}\, h^{1}{}_{c}{}^{c;d}{}_{;d} +$$
$$\frac{1}{4} g_{ab}\, h^{2}{}_{c}{}^{c;d}{}_{;d} - h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d} + \frac{1}{4} g_{ab}\, h^{1e}{}_{e;d}\, h^{1}{}_{c}{}^{c;d} +$$
$$g_{ab}\, h^{1cd}{}_{;c}\, h^{1}{}_{d}{}^{e}{}_{;e} - \frac{1}{2} g_{ab}\, h^{1}{}_{c}{}^{c;d}\, h^{1}{}_{d}{}^{e}{}_{;e} + g_{ab}\, h^{1de}\, h^{1c}{}_{d;c;e} - \frac{1}{2} g_{ab}\, h^{1de}\, h^{1}{}_{c}{}^{c}{}_{;d;e} +$$
$$g_{ab}\, h^{1cd}\, h^{1}{}_{c}{}^{e}{}_{;d;e} - \frac{1}{2} g_{ab}\, h^{1cd}\, h^{1}{}_{cd}{}^{;e}{}_{;e} + \frac{1}{2} g_{ab}\, h^{1c}{}_{e;d}\, h^{1}{}_{c}{}^{d;e} - \frac{1}{2} g_{ab}\, h^{1c}{}_{d;e}\, h^{1}{}_{c}{}^{d;e}$$

In[245]:= **% // Simplify**

Out[245]=
$$\begin{aligned}\frac{1}{4}\times\Big(&-2\big(h^{2}{}_{ab}\,R[\nabla] - h^{1cd}{}_{;a}\,h^{1}{}_{cd;b} - 2\,h^{1cd}\,h^{1}{}_{cd;a;b} + h^{2c}{}_{c;a;b} -\\
&h^{1}{}_{b}{}^{c}{}_{;a}\,h^{1d}{}_{d;c} - h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1d}{}_{d;c} - h^{2}{}_{b}{}^{c}{}_{;a;c} - h^{2}{}_{a}{}^{c}{}_{;b;c} + h^{2}{}_{ab}{}^{;c}{}_{;c} + h^{1d}{}_{d;c}\,h^{1}{}_{ab}{}^{;c} +\\
&2\,h^{1}{}_{b}{}^{c}{}_{;a}\,h^{1}{}_{c}{}^{d}{}_{;d} + 2\,h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1}{}_{c}{}^{d}{}_{;d} - 2\,h^{1}{}_{ab}{}^{;c}\,h^{1}{}_{c}{}^{d}{}_{;d} + 2\,h^{1cd}\,h^{1}{}_{bc;a;d} +\\
&2\,h^{1cd}\,h^{1}{}_{ac;b;d} - 2\,h^{1cd}\,h^{1}{}_{ab;c;d} + 2\,h^{1}{}_{bd;c}\,h^{1}{}_{a}{}^{c;d} - 2\,h^{1}{}_{bc;d}\,h^{1}{}_{a}{}^{c;d}\big) +\\
&2\,h^{1}{}_{ab}\,\big(2\,h^{1cd}\,R[\nabla]_{cd} + h^{1e}{}_{e}{}^{;d}{}_{;d} - 2\,h^{1de}{}_{;d;e} + h^{1}{}_{c}{}^{c;e}{}_{;e}\big) +\\
&g_{ab}\,\big(2\,h^{2cd}\,R[\nabla]_{cd} - 4\,h^{1}{}_{c}{}^{e}\,h^{1cd}\,R[\nabla]_{de} - h^{1ef}{}_{;c}\,h^{1}{}_{ef}{}^{;c} - 2\,h^{1cd}\,h^{1e}{}_{e;c;d} -\\
&2\,h^{1ef}\,h^{1}{}_{ef}{}^{;d}{}_{;d} + h^{2e}{}_{e}{}^{;d}{}_{;d} - 2\,h^{1de}{}_{;d}\,h^{1f}{}_{f;e} + 4\,h^{1cd}\,h^{1}{}_{c}{}^{e}{}_{;d;e} - 2\,h^{2de}{}_{;d;e} -\\
&2\,h^{1cd}\,h^{1}{}_{cd}{}^{;e}{}_{;e} + h^{2}{}_{c}{}^{c;e}{}_{;e} + h^{1f}{}_{f;e}\,h^{1}{}_{c}{}^{c;e} + 4\,h^{1de}{}_{;d}\,h^{1}{}_{e}{}^{f}{}_{;f} - 2\,h^{1}{}_{c}{}^{c;e}\,h^{1}{}_{e}{}^{f}{}_{;f} +\\
&4\,h^{1ef}\,h^{1d}{}_{e;d;f} - 2\,h^{1ef}\,h^{1}{}_{c}{}^{c}{}_{;e;f} + 2\,h^{1c}{}_{f;e}\,h^{1}{}_{c}{}^{e;f} - 2\,h^{1c}{}_{e;f}\,h^{1}{}_{c}{}^{e;f}\big)\Big)\end{aligned}$$

In[246]:= **% // ToCanonical**

Out[246]=
$$\begin{aligned}&h^{1}{}_{ab}\,h^{1cd}\,R[\nabla]_{cd} + \frac{1}{2}\,g_{ab}\,h^{2cd}\,R[\nabla]_{cd} - g_{ab}\,h^{1}{}_{c}{}^{e}\,h^{1cd}\,R[\nabla]_{de} -\\
&\frac{1}{2}\,h^{2}{}_{ab}\,R[\nabla] + \frac{1}{2}\,h^{1cd}{}_{;a}\,h^{1}{}_{cd;b} + h^{1cd}\,h^{1}{}_{cd;a;b} - \frac{1}{2}\,h^{2c}{}_{c;a;b} + \frac{1}{2}\,h^{1}{}_{b}{}^{c}{}_{;a}\,h^{1d}{}_{d;c} +\\
&\frac{1}{2}\,h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1d}{}_{d;c} + \frac{1}{2}\,h^{2}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2}\,h^{2}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2}\,h^{2}{}_{ab}{}^{;c}{}_{;c} - \frac{1}{2}\,h^{1d}{}_{d;c}\,h^{1}{}_{ab}{}^{;c} -\\
&h^{1}{}_{b}{}^{c}{}_{;a}\,h^{1}{}_{c}{}^{d}{}_{;d} - h^{1}{}_{a}{}^{c}{}_{;b}\,h^{1}{}_{c}{}^{d}{}_{;d} + h^{1}{}_{ab}{}^{;c}\,h^{1}{}_{c}{}^{d}{}_{;d} - h^{1cd}\,h^{1}{}_{bc;a;d} - h^{1cd}\,h^{1}{}_{ac;b;d} +\\
&h^{1cd}\,h^{1}{}_{ab;c;d} - h^{1}{}_{ab}\,h^{1cd}{}_{;c;d} - g_{ab}\,h^{1cd}\,h^{1e}{}_{e;c;d} - \frac{1}{2}\,g_{ab}\,h^{2cd}{}_{;c;d} +\\
&h^{1}{}_{ab}\,h^{1c}{}_{c}{}^{;d}{}_{;d} + \frac{1}{2}\,g_{ab}\,h^{2c}{}_{c}{}^{;d}{}_{;d} + g_{ab}\,h^{1cd}\,h^{1}{}_{c}{}^{e}{}_{;e;d} - h^{1}{}_{bd;c}\,h^{1}{}_{a}{}^{c;d} +\\
&h^{1}{}_{bc;d}\,h^{1}{}_{a}{}^{c;d} + \frac{1}{4}\,g_{ab}\,h^{1e}{}_{e;d}\,h^{1c}{}_{c}{}^{;d} + g_{ab}\,h^{1cd}{}_{;c}\,h^{1}{}_{d}{}^{e}{}_{;e} - g_{ab}\,h^{1c}{}_{c}{}^{;d}\,h^{1}{}_{d}{}^{e}{}_{;e} +\\
&g_{ab}\,h^{1cd}\,h^{1}{}_{c}{}^{e}{}_{;d;e} - g_{ab}\,h^{1cd}\,h^{1}{}_{cd}{}^{;e}{}_{;e} + \frac{1}{2}\,g_{ab}\,h^{1}{}_{ce;d}\,h^{1cd;e} - \frac{3}{4}\,g_{ab}\,h^{1}{}_{cd;e}\,h^{1cd;e}\end{aligned}$$

**The Following is the expression for Einstein Tensor to second order**

$\triangle^{2}\left[G[\nabla]_{ab}\right] =$

$$
\begin{aligned}
& h^{1}{}_{ab}\; h^{1cd}\; R[\triangledown]_{cd} + \frac{1}{2}\, g_{ab}\; h^{2cd}\; R[\triangledown]_{cd} - g_{ab}\; h^{1}{}_{c}{}^{e}\; h^{1cd}\; R[\triangledown]_{de} - \\
& \frac{1}{2}\, h^{2}{}_{ab}\; R[\triangledown] + \frac{1}{2}\, h^{1cd}{}_{;a}\; h^{1}{}_{cd;b} + h^{1cd}\; h^{1}{}_{cd;a;b} - \frac{1}{2}\, h^{2c}{}_{c;a;b} + \frac{1}{2}\, h^{1}{}_{b}{}^{c}{}_{;a}\; h^{1d}{}_{d;c} + \\
& \frac{1}{2}\, h^{1}{}_{a}{}^{c}{}_{;b}\; h^{1d}{}_{d;c} + \frac{1}{2}\, h^{2}{}_{b}{}^{c}{}_{;a;c} + \frac{1}{2}\, h^{2}{}_{a}{}^{c}{}_{;b;c} - \frac{1}{2}\, h^{2}{}_{ab}{}^{;c}{}_{;c} - \frac{1}{2}\, h^{1d}{}_{d;c}\; h^{1}{}_{ab}{}^{;c} - \\
& h^{1}{}_{b}{}^{c}{}_{;a}\; h^{1}{}_{c}{}^{d}{}_{;d} - h^{1}{}_{a}{}^{c}{}_{;b}\; h^{1}{}_{c}{}^{d}{}_{;d} + h^{1}{}_{ab}{}^{;c}\; h^{1}{}_{c}{}^{d}{}_{;d} - h^{1cd}\; h^{1}{}_{bc;a;d} - h^{1cd}\; h^{1}{}_{ac;b;d} + \\
& h^{1cd}\; h^{1}{}_{ab;c;d} - h^{1}{}_{ab}\; h^{1cd}{}_{;c;d} - g_{ab}\; h^{1cd}\; h^{1e}{}_{e;c;d} - \frac{1}{2}\, g_{ab}\; h^{2cd}{}_{;c;d} + \\
& h^{1}{}_{ab}\; h^{1c}{}_{c}{}^{;d}{}_{;d} + \frac{1}{2}\, g_{ab}\; h^{2c}{}_{c}{}^{;d}{}_{;d} + g_{ab}\; h^{1cd}\; h^{1}{}_{c}{}^{e}{}_{;e;d} - h^{1}{}_{bd;c}\; h^{1}{}_{a}{}^{c;d} + \\
& h^{1}{}_{bc;d}\; h^{1}{}_{a}{}^{c;d} + \frac{1}{4}\, g_{ab}\; h^{1e}{}_{e;d}\; h^{1c}{}_{c}{}^{;d} + g_{ab}\; h^{1cd}{}_{;c}\; h^{1}{}_{d}{}^{e}{}_{;e} - g_{ab}\; h^{1c}{}_{c}{}^{;d}\; h^{1}{}_{d}{}^{e}{}_{;e} + \\
& g_{ab}\; h^{1cd}\; h^{1}{}_{c}{}^{e}{}_{;d;e} - g_{ab}\; h^{1cd}\; h^{1}{}_{cd}{}^{;e}{}_{;e} + \frac{1}{2}\, g_{ab}\; h^{1}{}_{ce;d}\; h^{1cd;e} - \frac{3}{4}\, g_{ab}\; h^{1}{}_{cd;e}\; h^{1cd;e}
\end{aligned}
$$