Load Required Packages

In[1]:=
```
<< xAct`xTensor`
<< xAct`xPert`
<< xAct`xCoba`
```

```
------------------------------------------------------------

Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}

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Connecting to external MinGW executable...

Connection established.

------------------------------------------------------------

Package xAct`xTensor`  version 1.1.5, {2021, 2, 28}

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Package xAct`xPert`  version 1.0.6, {2018, 2, 28}

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** Variable $PrePrint assigned value ScreenDollarIndices

** Variable $CovDFormat changed from Prefix to Postfix

** Option AllowUpperDerivatives of ContractMetric changed from False to True

** Option MetricOn of MakeRule changed from None to All

** Option ContractMetrics of MakeRule changed from False to True

------------------------------------------------------------

Package xAct`xCoba`  version 0.8.6, {2021, 2, 28}

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------------------------------------------------------------
```

In this notebook we try to compute the expressions of components of Christoffel symbols, Riemann curvature tensor etc for a perturbed metric.

the metric is composed of a background metric and its perturbations

In[4]:=
```
$PrePrint = ScreenDollarIndices
```

Out[4]=
```
ScreenDollarIndices
```

Abstract part

First we need a 4D manifold:

In[5]:=
```
DefManifold[M, 4, {α, β, γ, μ, ν, λ, ρ, σ, η}]
```

```
** DefManifold: Defining manifold M.
** DefVBundle: Defining vbundle TangentM.
```

Defining an abstract metric

In[6]:=
```
DefMetric[-1, background[-μ, -ν], cd, PrintAs → "g"]
```

```
** DefTensor: Defining symmetric metric tensor background[-μ, -ν].
** DefTensor: Defining antisymmetric tensor epsilonbackground[-α, -β, -γ, -η].
** DefTensor: Defining tetrametric Tetrabackground[-α, -β, -γ, -η].
** DefTensor: Defining tetrametric Tetrabackground†[-α, -β, -γ, -η].
** DefCovD: Defining covariant derivative cd[-μ].
** DefTensor: Defining vanishing torsion tensor Torsioncd[α, -β, -γ].
** DefTensor: Defining symmetric Christoffel tensor Christoffelcd[α, -β, -γ].
** DefTensor: Defining Riemann tensor Riemanncd[-α, -β, -γ, -η].
** DefTensor: Defining symmetric Ricci tensor Riccicd[-α, -β].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarcd[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor Einsteincd[-α, -β].
** DefTensor: Defining Weyl tensor Weylcd[-α, -β, -γ, -η].
** DefTensor: Defining symmetric TFRicci tensor TFRiccicd[-α, -β].
** DefTensor: Defining Kretschmann scalar Kretschmanncd[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detbackground[]. Determinant.
```

Construct a perturbative structure:

In[7]:=
```
DefParameter[ϵ]
```

```
** DefParameter: Defining parameter ϵ.
```

In[8]:=
```
DefMetricPerturbation[background, pert, ϵ, PrintAs → "h"]
```

```
** DefTensor: Defining tensor pert[LI[order], -α, -β].
```

In[9]:=
```
Unprotect[IndexForm];
IndexForm[LI[x_]] := ColorString[ToString[x], Red]
```

We will be interested in Christoffel symbols,Riemann Curvature tensor, Ricci tensor, Ricci scalar for this metric. here 10stands for order 0. similar methods can be used for obtaining the expressions for higher orders in the same manner

----

We will be interested in Christoffel symbols,Riemann Curvature tensor, Ricci tensor, Ricci scalar for this metric. here 10stands for order 0. similar methods can be used for obtaining the expressions for higher orders in the same manner
As we are not interested in perturbations in this case we will be considering all the zeroth order terms (only the terms corresponding to the background)
The advantage of using this procedure instead of straightforward component computation using just xCoba is that here we have the freedom to introduce perturbations to the background metric and obtain the results to any order of perturbations we need by just changing the order in the following lines.
whereas if we just use xCoba we wont have the freedom to introduce perturbations.

NOTE: Change the order to 1 if we need to get results to to the first order of perturbation
if we need to any higher order of perturbation, we need to define the corresponding matrix for perturbation in xCoba and add it in the rules section.

Christoffel symbol (order 0):

In[11]:=
```
C1[α, -β, -γ] =
 ToCanonical[Perturbed[Christoffelcd[α, -β, -γ], 0] // ExpandPerturbation //
   SeparateMetric[background], UseMetricOnVBundle → None]
```

Out[11]= $\Gamma[\nabla]^{\alpha}{}_{\beta\gamma}$

Similarly we can define the abstract expressions for the other tensors we are interested in:

Riemann Curvature tensor(order 0):

In[12]:=
```
Riem[-μ, -ν, -ρ, -σ]  =
 ToCanonical[Perturbed[Riemanncd[-μ, -ν, -ρ, -σ], 0] // ExpandPerturbation //
   SeparateMetric[background], UseMetricOnVBundle → None]
```

Out[12]= $R[\nabla]_{\mu\nu\rho\sigma}$

Ricci Tensor (order 0) :

In[13]:=
```
Ricci1[-μ, -ν]  = ToCanonical[Perturbed[Riccicd[-μ, -ν], 0] // ExpandPerturbation //
   SeparateMetric[background], UseMetricOnVBundle → None]
```

Out[13]= $R[\nabla]_{\mu\nu}$

Ricci Scalar (order 0):

In[14]:=
```
RS = ToCanonical[Perturbed[RicciScalarcd[], 0] // ExpandPerturbation //
   SeparateMetric[background], UseMetricOnVBundle → None]
```

Out[14]= $R[\nabla]$

Component Part

Since we will be working with coordinates, we need to define our chart, it will be named as ch, the coordinates are numerated as {0,1,2,3},they will be t,x,y,and z (notice that we need to add [] after each coordinate as they will be defined as scaler fields).

Since we will be working with coordinates, we need to define our chart, it will be named as ch, the coordinates are numerated as {0,1,2,3},they will be t,x,y,and z (notice that we need to add [] after each coordinate as they will be defined as scaler fields).

In[15]:=
```
DefChart[ch, M, {0, 1, 2, 3}, {t[], r[], θ[], ϕ[]}, ChartColor → Purple]
```

```
** DefChart: Defining chart ch.
** DefTensor: Defining coordinate scalar t[].
** DefTensor: Defining coordinate scalar r[].
** DefTensor: Defining coordinate scalar θ[].
** DefTensor: Defining coordinate scalar ϕ[].
** DefMapping: Defining mapping ch.
** DefMapping: Defining inverse mapping ich.
** DefTensor: Defining mapping differential tensor dich[-a, ichα].
** DefTensor: Defining mapping differential tensor dch[-α, cha].
** DefBasis: Defining basis ch. Coordinated basis.
** DefCovD: Defining parallel derivative PDch[-α].
** DefTensor: Defining vanishing torsion tensor TorsionPDch[α, -β, -γ].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelPDch[α, -β, -γ].
** DefTensor: Defining vanishing Riemann tensor RiemannPDch[-α, -β, -γ, η].
** DefTensor: Defining vanishing Ricci tensor RicciPDch[-α, -β].
** DefTensor: Defining antisymmetric +1 density etaUpch[α, β, γ, η].
** DefTensor: Defining antisymmetric -1 density etaDownch[-α, -β, -γ, -η].
```

For the 4D metric, we'll need to define the required scalar functions, constants etc

In[16]:=
```
DefScalarFunction[Ω]
```

```
** DefScalarFunction: Defining scalar function Ω.
```

In[17]:=
```
DefScalarFunction[f]
DefScalarFunction[g]
DefScalarFunction[h]
```

```
** DefScalarFunction: Defining scalar function f.
** DefScalarFunction: Defining scalar function g.
** DefScalarFunction: Defining scalar function h.
```

In[20]:=
```
DefScalarFunction[A]
DefScalarFunction[B]
```

```
** DefScalarFunction: Defining scalar function A.
** DefScalarFunction: Defining scalar function B.
```

In[22]:=

NOTE : Defining the functions f(r), g(r) and h(r) . Here I have taken functions f(r), and g(r) as -A(r) and B(r) as mentioned in the
Make the required changes in the following block to define the required background metric :

In[23]:=
```
f[r[]] = -A[r[]]
g[r[]] = B[r[]]
h[r[]] = r[]^2
```
Out[23]= -A[r]

Out[24]= B[r]

Out[25]= r^2

Defining the function Ω[r, θ, ϕ] :

NOTE : Make changes in the following block to define the required function in the perturbations

In[26]:=
```
Ω[r[], θ[], ϕ[] ] = θ[] × ϕ[]
```
Out[26]= θ ϕ

In[27]:=
```
DefConstantSymbol[speedc, PrintAs → "C"]
```
```
** DefConstantSymbol: Defining constant symbol speedc.
```

In[28]:=
```
DefConstantSymbol[mass, PrintAs → "M"]
```
```
** DefConstantSymbol: Defining constant symbol mass.
```

Defining the elements of background metric :
you can remove the assumption of metric being a diagonal metric and choose any metric of your choice:

In[29]:=
```
backgroundmetric =
 CTensor[DiagonalMatrix[{f[r[]], g[r[]], h[r[]], h[r[]] Sin[θ[]]^2}], {-ch, -ch}]
```
Out[29]= CTensor[
 {{-A[r], 0, 0, 0}, {0, B[r], 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[θ]^2}}, {-ch, -ch}, 0]

In[30]:=
```
backgroundmetric[-μ, -ν]
```
Out[30]=

$$\begin{pmatrix} -A[r] & 0 & 0 & 0 \\ 0 & B[r] & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \, \mathrm{Sin}[\theta]^2 \end{pmatrix}_{\mu\nu}$$

In[31]:=
```
MetricCompute[backgroundmetric, ch, All]
```

In[32]:= `bgcd = CovDOfMetric[backgroundmetric]`

Out[32]= $\text{CCovD}\Big[\text{PDch}, \text{CTensor}\Big[\Big\{\Big\{\Big\{0, \frac{A'[r]}{2\,A[r]}, 0, 0\Big\}, \Big\{\frac{A'[r]}{2\,A[r]}, 0, 0, 0\Big\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\Big\},$
$\Big\{\Big\{\frac{A'[r]}{2\,B[r]}, 0, 0, 0\Big\}, \Big\{0, \frac{B'[r]}{2\,B[r]}, 0, 0\Big\}, \Big\{0, 0, -\frac{r}{B[r]}, 0\Big\}, \Big\{0, 0, 0, -\frac{r\,\text{Sin}[\theta]^2}{B[r]}\Big\}\Big\},$
$\Big\{\{0, 0, 0, 0\}, \Big\{0, 0, \frac{1}{r}, 0\Big\}, \Big\{0, \frac{1}{r}, 0, 0\Big\}, \{0, 0, 0, -\text{Cos}[\theta]\,\text{Sin}[\theta]\}\Big\},$
$\Big\{\{0, 0, 0, 0\}, \Big\{0, 0, 0, \frac{1}{r}\Big\}, \{0, 0, 0, \text{Cot}[\theta]\}, \Big\{0, \frac{1}{r}, \text{Cot}[\theta], 0\Big\}\Big\}\Big\},$
$\{\text{ch}, -\text{ch}, -\text{ch}\}, 0\Big], \text{CTensor}\Big[\{\{-A[r], 0, 0, 0\}, \{0, B[r], 0, 0\},$
$\{0, 0, r^2, 0\}, \{0, 0, 0, r^2\,\text{Sin}[\theta]^2\}\}, \{-\text{ch}, -\text{ch}\}, 0\Big]\Big]$

Defining the perturbations

NOTE : Increase the LargeComponentSize if you observe that the tensors aren't being computed properly while computing the elements of Riemann Curvature tensor etc.. (This can be one of the issues)

In[33]:= `$LargeComponentSize = 10000;`

In[34]:=
```
perturbations = CTensor[
  DiagonalMatrix[{-2 * Ω[r[], θ[], ϕ[]] / speedc^2, -2 * Ω[r[], θ[], ϕ[]] / speedc^2,
    -2 * Ω[r[], θ[], ϕ[]] / speedc^2, -2 * Ω[r[], θ[], ϕ[]] / speedc^2}], {-ch, -ch}]
```

Out[34]= $\text{CTensor}\Big[\Big\{\Big\{-\frac{2\,\theta\,\phi}{c^2}, 0, 0, 0\Big\}, \Big\{0, -\frac{2\,\theta\,\phi}{c^2}, 0, 0\Big\}, \Big\{0, 0, -\frac{2\,\theta\,\phi}{c^2}, 0\Big\}, \Big\{0, 0, 0, -\frac{2\,\theta\,\phi}{c^2}\Big\}\Big\},$
$\{-\text{ch}, -\text{ch}\}, 0\Big]$

In[35]:= `perturbations[-μ, -ν]`

Out[35]=
$$\begin{pmatrix} -\frac{2\,\theta\,\phi}{c^2} & 0 & 0 & 0 \\ 0 & -\frac{2\,\theta\,\phi}{c^2} & 0 & 0 \\ 0 & 0 & -\frac{2\,\theta\,\phi}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{2\,\theta\,\phi}{c^2} \end{pmatrix}_{\mu\nu}$$

So , now we have the expressions for background metric and perturbations. Any general background metric and perturbations can be used, here I have used a very simple case.

So, the next step is to use the defined background metric and perturbations in the expression of Christoffel Symbols, Riemann Curvature tensor etc obtained using xPert in the beginning of this notebook.

NOTE : We need to replace abstract indices with component values. right now this metric g and perturbations h in the above expression and the CTensors background and perturbations we have defined using xCoba are different. We need to specify these as the same objects and compute the expressions.

In[36]:=
```
rules = {
    background[-μ_Symbol, -ν_Symbol] → backgroundmetric[-μ, -ν],
    background[μ_Symbol, ν_Symbol] → Inv[backgroundmetric][μ, ν],
    cd → bgcd,
    Riccicd[inds__] → Ricci[bgcd][inds],
    RicciScalarcd[] → RicciScalar[bgcd][],
    Einsteincd[inds__] → Einstein[bgcd][inds],
    Christoffelcd[inds__] → Christoffel[bgcd][inds],
    Riemanncd[inds__] → Riemann[bgcd][inds],
    pert[LI[1], inds__] → perturbations[inds]
   };
```

In[37]:=
```
$CVSimplify = Expand;
```

Christoffel Symbol:

In[38]:=
```
PChristoffel1 = Collect[C1[α, -β, -γ] /. rules, ε, Simplify]
```

Out[38]=

$$\Gamma[\mathcal{D}]^{\alpha}{}_{\beta\gamma} + \left(\begin{array}{cccc}
\begin{pmatrix} 0 \\ \frac{A'[r]}{2\,A[r]} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{A'[r]}{2\,A[r]} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{A'[r]}{2\,B[r]} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{B'[r]}{2\,B[r]} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -\frac{r}{B[r]} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{r\,\mathrm{Sin}[\theta]^2}{B[r]} \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{Cot}[\theta] \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ \mathrm{Cot}[\theta] \\ 0 \end{pmatrix}
\end{array}\right)^{\alpha}{}_{\beta\gamma}$$

Riemann Curvature Tensor:

In[39]:= **Priemann1 = Collect[Riem[-μ, -ν, -ρ, -σ] /. rules, ϵ, Simplify]**

Out[39]=

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{g_{01}\left(A[r]\,A'[r]\,B'[r]+B[r]\left(A'[r]^2-2A[r]\,A''[r]\right)\right)}{4A[r]\,B[r]^2} & \frac{g_{11}\left(A[r]\,A'[r]\,B'[r]+B[r]\left(A'[r]^2-2A[r]\,A''[r]\right)\right)}{4A[r]\,B[r]^2} & \frac{g_{21}\left(A[r]\,A'[r]\,B'[r]+B[r]\left(A'[r]^2-2A[r]\,A''[r]\right)\right)}{4A[r]\,B[r]^2} & g_{31}\,(A[r] \\ \frac{g_{00}\left(A[r]\,A'[r]\,B'[r]+B[r]\left(A'[r]^2-2A[r]\,A''[r]\right)\right)}{4A[r]^2\,B[r]} & \frac{g_{10}\left(A[r]\,A'[r]\,B'[r]+B[r]\left(A'[r]^2-2A[r]\,A''[r]\right)\right)}{4A[r]^2\,B[r]} & \frac{g_{20}\left(A[r]\,A'[r]\,B'[r]+B[r]\left(A'[r]^2-2A[r]\,A''[r]\right)\right)}{4A[r]^2\,B[r]} & g_{30}\,(A[r] \\ 0&0&0& \\ 0&0&0& \end{pmatrix}$$

$$\begin{pmatrix} -\frac{g_{02}\,A'[r]}{2B[r]\,r} & -\frac{g_{12}\,A'[r]}{2B[r]\,r} & -\frac{g_{22}\,A'[r]}{2B[r]\,r} & -\frac{g_{32}\,A'[r]}{2B[r]\,r} \\ 0&0&0&0 \\ -\frac{g_{00}\,r\,A'[r]}{2A[r]\,B[r]} & -\frac{g_{10}\,r\,A'[r]}{2A[r]\,B[r]} & -\frac{g_{20}\,r\,A'[r]}{2A[r]\,B[r]} & -\frac{g_{30}\,r\,A'[r]}{2A[r]\,B[r]} \\ 0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{g_{03}\,A'[r]}{2B[r]\,r} & -\frac{g_{13}\,A'[r]}{2B[r]\,r} & -\frac{g_{23}\,A'[r]}{2B[r]\,r} & -\frac{g_{33}\,A'[r]}{2B[r]\,r} \\ 0&0&0&0 \\ 0&0&0&0 \\ -\frac{g_{00}\,r\,\mathrm{Sin}[\theta]^2\,A'[r]}{2A[r]\,B[r]} & -\frac{g_{10}\,r\,\mathrm{Sin}[\theta]^2\,A'[r]}{2A[r]\,B[r]} & -\frac{g_{20}\,r\,\mathrm{Sin}[\theta]^2\,A'[r]}{2A[r]\,B[r]} & -\frac{g_{30}\,r\,\mathrm{Sin}[\theta]^2\,A'[r]}{2A[r]\,B[r]} \end{pmatrix}$$

Ricci Tensor:

In[40]:= **Pricci1 = Collect[Ricci1[-μ, -ν] /. rules, ε] // Expand**

Out[40]=

$$\begin{pmatrix} \frac{A'[r]}{B[r]\,r}-\frac{A'[r]^2}{4A[r]\,B[r]}-\frac{A'[r]\,B'[r]}{4B[r]^2}+\frac{A''[r]}{2B[r]} & 0 & 0 & 0 \\ 0 & \frac{A'[r]^2}{4A[r]^2}+\frac{B'[r]}{B[r]\,r}+\frac{A'[r]\,B'[r]}{4A[r]\,B[r]}-\frac{A''[r]}{2A[r]} & 0 & 0 \\ 0 & 0 & 1-\frac{1}{B[r]}-\frac{r\,A'[r]}{2A[r]\,B[r]}+\frac{r\,B'[r]}{2B[r]^2} & 0 \\ 0&0&0& \mathrm{Sin}[\theta]^2-\frac{\mathrm{Sin}[\theta]^2}{B[r]}-\frac{r\,\mathrm{Sin}[\theta]^2\,A'[r]}{2A[r]\,B[r]}+\frac{r\,\mathrm{Sin}[\theta]^2\,B'[r]}{2B[r]^2} \end{pmatrix}_{\mu}$$

Ricci Scalar:

In[41]:= **PRS1 = Collect[RS /. rules, ε, Simplify]**

Out[41]=

$$\frac{1}{2A[r]^2\,B[r]^2\,r^2}\Big(B[r]\,r^2\,A'[r]^2+4A[r]^2\left(-B[r]+B[r]^2+r\,B'[r]\right)+A[r]\,r\,\left(r\,A'[r]\,B'[r]-2B[r]\,\left(2A'[r]+r\,A''[r]\right)\right)\Big)$$

PART 2

(Using specific functions for A(r) and B(r))

Now we will try to define the functions A(r) and B(r) and try to get these results :

In[42]:=
```
A[r[]] = r[]^3 + r[]^2
B[r[]] = r[]^3
```

Out[42]= $r^2 + r^3$

Out[43]= r^3

In[44]:=

In[45]:=
```
backgroundmetric2 =
 CTensor[DiagonalMatrix[{f[r[]], g[r[]], h[r[]], h[r[]] Sin[θ[]]^2}], {-ch, -ch}]
```

Out[45]=
```
CTensor[
 {{-r^2 - r^3, 0, 0, 0}, {0, r^3, 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[θ]^2}}, {-ch, -ch}, 0]
```

In[46]:=
```
backgroundmetric2[-μ, -ν]
```

Out[46]=

$$\begin{pmatrix} -r^2-r^3 & 0 & 0 & 0 \\ 0 & r^3 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}_{\mu\nu}$$

In[47]:=
```
MetricCompute[backgroundmetric2, ch, All]
bgcd2 = CovDOfMetric[backgroundmetric2]
```

Out[48]=

$$\text{CCovD}\Big[\text{PDch}, \text{CTensor}\Big[\Big\{\Big\{\Big\{\Big\{0, -\frac{r^8 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} - \frac{3 r^9 \text{Sin}[\theta]^2}{2\left(-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2\right)}, 0, 0\Big\},$$

$$\Big\{-\frac{r^8 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} - \frac{3 r^9 \text{Sin}[\theta]^2}{2\left(-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2\right)}, 0, 0, 0\Big\},$$

$$\{0, 0, 0, 0\}, \{0, 0, 0, 0\}\Big\}, \Big\{\Big\{-\frac{r^7 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} -$$

$$\frac{5 r^8 \text{Sin}[\theta]^2}{2\left(-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2\right)} - \frac{3 r^9 \text{Sin}[\theta]^2}{2\left(-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2\right)}, 0, 0, 0\Big\},$$

$$\Big\{0, -\frac{3 r^8 \text{Sin}[\theta]^2}{2\left(-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2\right)} - \frac{3 r^9 \text{Sin}[\theta]^2}{2\left(-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2\right)}, 0, 0\Big\},$$

$$\Big\{0, 0, \frac{r^7 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} + \frac{r^8 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2}, 0\Big\},$$

$$\Big\{0, 0, 0, \frac{r^7 \text{Sin}[\theta]^4}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} + \frac{r^8 \text{Sin}[\theta]^4}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2}\Big\}\Big\},$$

$$\Big\{\{0, 0, 0, 0\}, \Big\{0, 0, -\frac{r^8 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} - \frac{r^9 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2}, 0\Big\},$$

$$\Big\{0, -\frac{r^8 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} - \frac{r^9 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2}, 0, 0\Big\},$$

$$\Big\{0, 0, 0, \frac{\text{Cos}[\theta]\, r^9 \text{Sin}[\theta]^3}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} + \frac{\text{Cos}[\theta]\, r^{10} \text{Sin}[\theta]^3}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2}\Big\}\Big\},$$

$$\Big\{\{0, 0, 0, 0\}, \Big\{0, 0, 0, -\frac{r^8 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} - \frac{r^9 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2}\Big\},$$

$$\Big\{0, 0, 0, -\frac{\text{Cos}[\theta]\, r^9 \text{Sin}[\theta]}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} - \frac{\text{Cos}[\theta]\, r^{10} \text{Sin}[\theta]}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2}\Big\},$$

$$\Big\{0, -\frac{r^8 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} - \frac{r^9 \text{Sin}[\theta]^2}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2},$$

$$-\frac{\text{Cos}[\theta]\, r^9 \text{Sin}[\theta]}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2} - \frac{\text{Cos}[\theta]\, r^{10} \text{Sin}[\theta]}{-r^9 \text{Sin}[\theta]^2 - r^{10} \text{Sin}[\theta]^2}, 0\Big\}\Big\}\Big\}, \{\text{ch}, -\text{ch}, -\text{ch}\}, 0\Big],$$

$$\text{CTensor}\Big[\Big\{\Big\{-r^2 - r^3, 0, 0, 0\Big\}, \Big\{0, r^3, 0, 0\Big\}, \Big\{0, 0, r^2, 0\Big\}, \Big\{0, 0, 0, r^2 \text{Sin}[\theta]^2\Big\}\Big\},$$

$$\{-\text{ch}, -\text{ch}\}, 0\Big]\Big]$$

In[49]:=
```
rules = {
    background[-μ_Symbol, -ν_Symbol] → backgroundmetric2[-μ, -ν],
    background[μ_Symbol, ν_Symbol] → Inv[backgroundmetric2][μ, ν],
    cd → bgcd2,
    Riccicd[inds__] → Ricci[bgcd2][inds],
    RicciScalarcd[] → RicciScalar[bgcd2][],
    Einsteincd[inds__] → Einstein[bgcd2][inds],
    Christoffelcd[inds__] → Christoffel[bgcd2][inds],
    Riemanncd[inds__] → Riemann[bgcd2][inds],
    pert[LI[1], inds__] → perturbations[inds]
   };
```

In[50]:=
```
$CVSimplify = Expand;
```

Christoffel Symbol:

In[51]:=
```
PChristoffel2 = Collect[C1[α, -β, -γ] /. rules, ε, Simplify]
```

Out[51]=

$$\Gamma[\mathcal{D}]^{\alpha}{}_{\beta\gamma} + \left(\begin{array}{cccc}
\begin{pmatrix} 0 \\ \frac{2+3 r}{2 r+2 r^2} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{2+3 r}{2 r+2 r^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{2+3 r}{2 r^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{3}{2 r} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{r^2} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\mathrm{Sin}[\theta]^2}{r^2} \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{Cos}[\theta]\ \mathrm{Sin}[\theta] \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{Cot}[\theta] \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ \mathrm{Cot}[\theta] \\ 0 \end{pmatrix}
\end{array}\right)^{\alpha}{}_{\beta\gamma}$$

Riemann Curvature Tensor:

In[52]:= `Priemann2 = Collect[Riem[-μ, -ν, -ρ, -σ] /. rules, ϵ, Simplify]`

Out[52]=

$$\left(\begin{array}{cc}
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} -\frac{g_{01}(6+11r+6r^2)}{4r^3(1+r)} & -\frac{g_{11}(6+11r+6r^2)}{4r^3(1+r)} & -\frac{g_{21}(6+11r+6r^2)}{4r^3(1+r)} \\ -\frac{g_{00}(6+11r+6r^2)}{4r^2(1+r)^2} & -\frac{g_{10}(6+11r+6r^2)}{4r^2(1+r)^2} & -\frac{g_{20}(6+11r+6r^2)}{4r^2(1+r)^2} \\ 0&0&0\\0&0&0 \end{pmatrix} \\
\begin{pmatrix} \frac{g_{01}(6+11r+6r^2)}{4r^3(1+r)} & \frac{g_{11}(6+11r+6r^2)}{4r^3(1+r)} & \frac{g_{21}(6+11r+6r^2)}{4r^3(1+r)} & \frac{g_{31}(6+11r+6r^2)}{4r^3(1+r)} \\ \frac{g_{00}(6+11r+6r^2)}{4r^2(1+r)^2} & \frac{g_{10}(6+11r+6r^2)}{4r^2(1+r)^2} & \frac{g_{20}(6+11r+6r^2)}{4r^2(1+r)^2} & \frac{g_{30}(6+11r+6r^2)}{4r^2(1+r)^2} \\ 0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\
\begin{pmatrix} -\frac{g_{02}(2+3r)}{2r^3} & -\frac{g_{12}(2+3r)}{2r^3} & -\frac{g_{22}(2+3r)}{2r^3} & -\frac{g_{32}(2+3r)}{2r^3} \\ 0&0&0&0 \\ -\frac{g_{00}(2+3r)}{2r^3(1+r)} & -\frac{g_{10}(2+3r)}{2r^3(1+r)} & -\frac{g_{20}(2+3r)}{2r^3(1+r)} & -\frac{g_{30}(2+3r)}{2r^3(1+r)} \\ 0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0 \\ -\frac{3g_{02}}{2r^2} & -\frac{3g_{12}}{2r^2} & -\frac{3g_{22}}{2r^2} & -\frac{3g_{32}}{2r^2} \\ \frac{3g_{01}}{2r^3} & \frac{3g_{11}}{2r^3} & \frac{3g_{21}}{2r^3} & \frac{3g_{31}}{2r^3} \\ 0&0&0&0 \end{pmatrix} \\
\begin{pmatrix} -\frac{g_{03}(2+3r)}{2r^3} & -\frac{g_{13}(2+3r)}{2r^3} & -\frac{g_{23}(2+3r)}{2r^3} & -\frac{g_{33}(2+3r)}{2r^3} \\ 0&0&0&0 \\ 0&0&0&0 \\ -\frac{g_{00}(2+3r)\,\mathrm{Sin}[\theta]^2}{2r^3(1+r)} & -\frac{g_{10}(2+3r)\,\mathrm{Sin}[\theta]^2}{2r^3(1+r)} & -\frac{g_{20}(2+3r)\,\mathrm{Sin}[\theta]^2}{2r^3(1+r)} & -\frac{g_{30}(2+3r)\,\mathrm{Sin}[\theta]^2}{2r^3(1+r)} \end{pmatrix} &
\begin{pmatrix} 0&0&0 \\ -\frac{3g_{03}}{2r^2} & -\frac{3g_{13}}{2r^2} & -\frac{3g_{23}}{2r^2} \\ 0&0&0 \\ \frac{3g_{01}\,\mathrm{Sin}[\theta]^2}{2r^3} & \frac{3g_{11}\,\mathrm{Sin}[\theta]^2}{2r^3} & \frac{3g_{21}\,\mathrm{Sin}[\theta]^2}{2r^3} \end{pmatrix}
\end{array}\right)$$

Ricci Tensor:

In[53]:= `$LargeComponentSize = 10000;`

Note : Increase the LargeComponent size if the result of any of these computations returns a matrix with weird zeros.

In[54]:= `Pricci2 = Collect[Ricci1[-μ, -ν] /. rules, ε]`

Out[54]=

$$\begin{pmatrix} -\frac{r^{15}\,\mathrm{Sin}[\theta]^4}{2\left(-r^9\,\mathrm{Sin}[\theta]^2-r^{10}\,\mathrm{Sin}[\theta]^2\right)^2}-\frac{9\,r^{16}\,\mathrm{Sin}[\theta]^4}{4\left(-r^9\,\mathrm{Sin}[\theta]^2-r^{10}\,\mathrm{Sin}[\theta]^2\right)^2}-\frac{13\,r^{17}\,\mathrm{Sin}[\theta]^4}{4\left(-r^9\,\mathrm{Sin}[\theta]^2-r^{10}\,\mathrm{Sin}[\theta]^2\right)^2}-\frac{3\,r^{18}\,\mathrm{Sin}[\theta]^4}{2\left(-r^9\,\mathrm{Sin}[\theta]^2-r^{10}\,\mathrm{Sin}[\theta]^2\right)^2}-\frac{r^6\,\mathrm{Sin}[\theta]^2}{-r^9\,\mathrm{Sin}[\theta]^2-r^{10}\,\mathrm{Sin}[\theta]^2}-\frac{4\,r^7\,\mathrm{Sin}[\theta]^2}{-r^9\,\mathrm{Sin}[\theta]^2-r^{10}\,\mathrm{Sin}[\theta]^2}-\ldots \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Ricci Scalar :

In[55]:= `PRS2 = Collect[RS /. rules, ε, Simplify]`

Out[55]=

$$\frac{6+7\,r+2\,r^2+4\,r^3+8\,r^4+4\,r^5}{2\,r^5\,(1+r)^2}$$