Lecture 7. Notes on financial time series: empirical stylized facts

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March 15, 2018

1 Preface

The performance of stocks and markets over a certain time history is traditionally measured by the distribution of the $r\Delta t(t)$ logarithmic return, which gives us the generated return over a certain time period Δt. For individual stocks and market indices it is defined as the logarithmic price change over a fixed time interval, Δt:

$$r_{\Delta t}(t) = s(t + \Delta t) - s(t) = \ln \frac{S(t + \Delta t)}{S(t)},$$

where $s(t) = \ln(S(t))$ denotes the logarithmic index ($S(t)$ denotes the value of the index or the price of a stock). The standard deviation of $r_1(t)$ daily log-returns is called (daily) *volatility*. For the DJIA index, the historical daily volatility of the log-returns is about $\sigma = 0.011$, i.e. $\sigma \approx 1\%$.

Reading on general issues:

- Sándor, Bulcsú, et al. "Time-scale effects on the gain-loss asymmetry in stock indices." Physical Review E 94.2 (2016): 022311. https://arxiv.org/abs/1608.04506
- Cont, Rama. "Empirical properties of asset returns: stylized facts and statistical issues." (2001): 223-236. http://personal.fmipa.itb.ac.id/khreshna/files/2011/02/cont2001.pdf

2 Heavy tails

Definition. Consider any distribution $P(X)$ with cumulative distribution function $F(x) = 1 - \overline{F}(x)$ defined by $Pr(X > x) = \overline{F}(x)$, such that for some $\xi > 0$

$$\overline{F}(x) = x^{-1/\xi} L(x)$$

where $L(x)$ is some slowly varying function for large x. The tail index of the fat-tailed distribution $P(X)$ is by definition ξ. Although the freakonomics article calls this a *heavy-tailed* distribution, in Wikipedia and elsewhere this is called a *fat-tailed* distribution. The definition of a "slowly varying function" is that for all $a > 0$

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1.$$

So basically asymptotically $\overline{F}(x)$ is equivalent to $x^{-1/\xi}$.

Examples. A common example of one-sided heavy-tailed distribution is the Pareto distribution with the survival function

$$\overline{F}(x) = Pr(X > x) = \begin{cases} (x_m/x)^{\alpha}, & x \geqslant x_m, \\ 1, & x < x_m. \end{cases}$$

where x_m is the (necessarily positive) minimum possible value of X, and α is a positive parameter (*tail index*).

A common example of the two-sided heavy-tailed distributions is the Student's t. If $X_i, i = 1, \ldots, n$ is an i.i.d sample from a normal distribution and

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{and} \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

are the sample mean and the sample variance, then

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has Student's t-distribution with $n-1$ degrees of freedom. It has a PDF

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

where ν is the number of degrees of freedom and Γ is the Gamma function. For common financial time series, when the returns are nicely modelled by t-distribution with $\nu = 3$, we have the PDF $f(t) = \frac{6\sqrt{3}}{\pi(3+t^2)^2}$. For the t-distribution, the degrees of freedom equals the tail index.

Why we care? Empirical results show, that the distribution of logarithmic returns can be approximated by a Gaussian distribution (typically for larger Δt), although there are important differences, such as the presence of fat tails (most pronounced for shorter Δt). The fat tails correspond to a much larger probability for large price changes than what is to be expected from Gaussian statistics, an assumption made in the mainstream theoretical finance.

Reading on heavy tails:

- http://users.cms.caltech.edu/~adamw/papers/2013-SIGMETRICS-heavytails.pdf
- https://en.wikipedia.org/wiki/Heavy-tailed_distribution

3 Aggregational Gaussianity

5 minutes returns are leptokurtic and fat-tailed; then as you increase timeframe, returns become more and more normal. Yearly data is almost normal, if you have enough points.

Reading on aggregational Gaussianity:

- Xu, Dan, and Christian Beck. "Transition from lognormal to χ^2-superstatistics for financial time series." Physica A: Statistical Mechanics and its Applications 453 (2016): 173-183. https://arxiv.org/abs/1506.01660
- Alonso, Julio César, and Sebastián Montenegro. "Reconsidering Aggregational Gaussianity as an Stylized Fact of Assets' Returns." http://ciencias.bogota.unal.edu.co/fileadmin/content/eventos/simposioestadistica/documentos/memorias/MEMORIAS_2015/Comunicaciones/Econometria/Montenegro___Alonso_Gaussianity_Stylizad_facts.pdf

4 Autocorrelation

Autocorrelation is by definition the correlation of a stochastic process with a lagged version of itself. Let X_t be a stochastic process with mean function μ_t and variance function σ_t^2. Then its autocorrelation function between times s and t is defined as

$$R(s,t) = \frac{\mathrm{E}[(X_t - \mu_t)(X_s - \mu_s)]}{\sigma_t \sigma_s}.$$

Autocorrelation is estimated via

$$\widehat{R}(k) = \frac{1}{(n-k)\sigma^2} \sum_{t=1}^{n-k} (X_t - \mu)(X_{t+k} - \mu).$$

A common question arising when working with real world time series is whether they are independently distributed (null hypothesis H_0) vs. non independently distributed (alternative hypothesis H_a). To test this, we can compute the statistic

$$Q = n\,(n+2) \sum_{k=1}^{h} \frac{\hat{\rho}_k^2}{n-k}$$

where n is the sample size, $\widehat{\rho}_k$ is the sample autocorrelation at lag k, and h is the number of lags being tested. Under H_0 the statistic Q follows a $\chi^2_{(h)}$. Therefore, we can reject the null hypothesis when $Q > \chi^2_{1-\alpha,h}$.

Reading on autocorrelations:

- https://en.wikipedia.org/wiki/Ljung-Box_test

5 Autocorrelation in realized volatility

Absolute daily log-returns are (very) noisy proxies for daily volatilities σ_t and squared daily log-returns are (very) noisy proxies for daily variances σ_t^2. The Oxford-Man Institute of Quantitative Finance makes historical realized variance estimates for 21 different stock indices freely available at http://realized.oxford-man.ox.ac.uk. These estimates are updated daily. We may then investigate the time series properties of σ_t^2 empirically.

The empirical ACF of volatility does not decay as a power-law. In fact, SPX realized variance has the following amazingly simple scaling property:

$$m(\Delta, q) = \mathrm{E}[|\log \sigma_{t+\Delta} - \log \sigma_t|^q] = A\Delta^{qH}$$

This simple scaling property holds for all 21 indices in the Oxford-Man dataset. It also holds for crude oil, gold and Bund futures. For SPX over 14 years, $H \approx 0.14$ and $A \approx 0.38$. As a consequence it may be shown that the autocorrelation function should take the form ($q = 2$):

$$\rho(\Delta) \sim \exp\left\{ -\frac{1}{2}\nu^2\Delta^{2H} \right\}$$

Reading on autocorrelations in volatility:

- Gatheral, Jim, and Roel CA Oomen. "Zero-intelligence realized variance estimation." Finance and Stochastics 14.2 (2010): 249-283. https://link.springer.com/content/pdf/10.1007/s00780-009-0120-1.pdf
- Bergomi,Lorenzo."SmiledynamicsII."(2005).

6 Gain-loss asymmetry

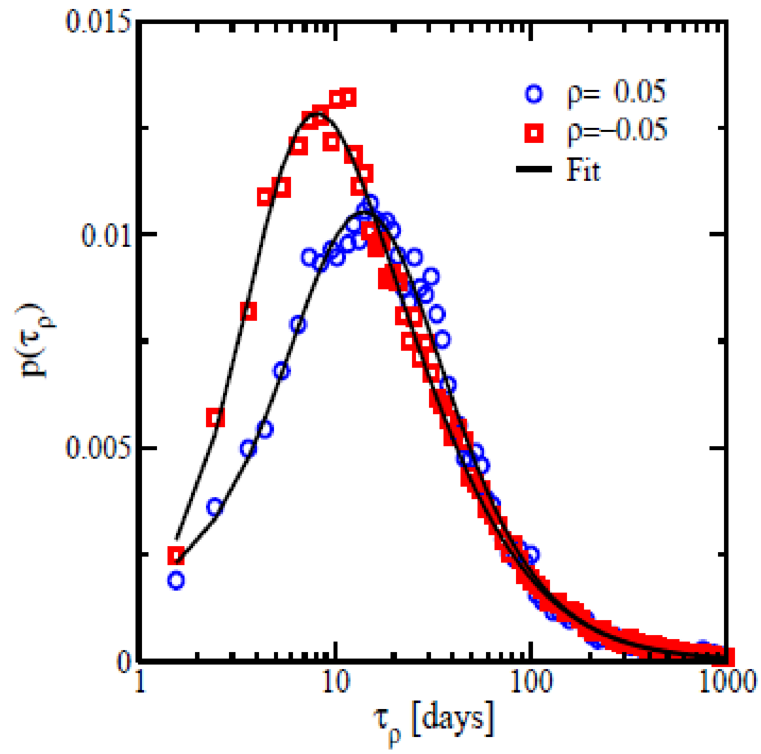
The gain-loss asymmetry, observed in the inverse statistics of stock indices is present for logarithmic return levels that are over 2%, and it is the result of the non-Pearson type auto-correlations in the index.

What are the inverse statistics of the time series? Let us consider the inverse question: what is the typical waiting time to generate a fluctuation of a given size in the price? To answer this question, we have to determine for an index or a stock the distribution of τ_ρ time intervals needed to obtain a predefined return level ρ. Practically, if given a fixed logarithmic return target ρ (proposed by the investor) for a stock or an index, as well as a fixed investment date (when the investor buys some assets), by the inverse statistics the time span is estimated for which the log-return of the stock or index reaches for the first time the desired level ρ. This is also called the *first passage time* through the level ρ. In a mathematical formulation this is equivalent to:

$$\tau_\rho(t) = \inf\{\Delta t | |r_{\Delta t}(t)| \geqslant \rho\}$$

where $\rho > 0$ (i.e. absolute log-return exceeds the level ρ by becoming either greater than ρ or smaller than $-\rho$).

The waiting time $\tau_\rho(t)$ is the momentary investment horizon for the proposed ρ log-return value, indicating the time interval an investor has to wait if the investment was made at time t, and he/she wants to achieve a ρ log-return value at time $t + \tau_\rho$. In the literature, its time-averaged value is the *investment horizon*. The normalized histogram of the first passage times for many t starting times, gives the $p(\tau_\rho)$ probability distribution of the momentary investment horizons. The method described above is called the method of inverse statistics. The distribution of the momentary investment horizons for the DJIA index in case of $|\rho| = 5\sigma$ (i.e. $\approx 5\%$ return) is depicted in the left panel of 6. The maximum of the distribution function determines the most probable waiting time for that log-return (τ_ρ^*), or in other words the *optimal investment horizon* for that stock or index.



(a) DJIA (1896.5–2001.7)

Figure 1: Investment horizon distribution of the original DJIA index.

A simple Brownian motion approximation for the log-prices would yield for the first passage time distribution:

$$p(\tau_\rho) = \frac{|\rho|}{\sqrt{4\pi D \tau_\rho^3}} \exp[-\frac{\rho^2}{4D\tau_\rho}]$$

with D a generalized diffusion constant. Since the first moment diverges, we determine the most probable first passage time:

$$\tau_\rho^* = \frac{1}{6D}\rho^2 \sim \rho^\gamma.$$

In constructing the inverse statistics of the DJIA index also for the negative return levels (i.e. $\rho = -5\%$), it was found that the distribution of investment horizons is similar in shape to the one for positive levels. However, there is one important difference: for negative return levels the maximum of the probability distribution is shifted to the left, generating about a $\Delta\tau_\rho \approx 13$ trading days difference in the optimal investment horizons. In the Fig. 6 this asymmetry of the inverse statistics is presented for $\rho = +5\sigma$ and $\rho = -5\sigma$ log-returns. It was found, that the asymmetry of inverse statistics is present for all the established stock indices, thus stock markets present a universal feature, called the *gain-loss asymmetry*. Contrary to indices, stock prices show a smaller degree of asymmetry. The asymmetry of the inverse statistics of stock markets is still a central problem of applied mathematics, econophysics and economics.

Reading on gain/loss asymmetry:

- Sándor, Bulcsú, et al. "Time-scale effects on the gain-loss asymmetry in stock indices." Physical Review E 94.2 (2016): 022311. https://arxiv.org/abs/1608.04506

- Simonsen, Ingve, Mogens H. Jensen, and Anders Johansen. "Optimal investment horizons." The European Physical Journal B-Condensed Matter and Complex Systems 27.4 (2002): 583-586. https://link.springer.com/content/pdf/10.1140/epjb/e2002-00193-x.pdf

7 Conditional heavy tails

The celebrated ARCH models of Engle (1982) were designed to capture the phenomenon of volatility clustering in the returns series. An ARCH(p) model can be described by the following equation:

$$h_t = \varepsilon_t \sqrt{a + \sum_{i=1}^{p} a_i h_{t-i}^2}$$

The original assumption was that the series $\{\varepsilon_t\}$ is i.i.d. $\mathcal{N}(0,1)$. Nevertheless, it was soon observed that the residuals, say $\{\widehat{\varepsilon}_t\}$, from a fitted ARCH($p$) model do not appear to be in accordance to the normality assumption as they are typically heavy-tailed. Residuals are given by $\widehat{\varepsilon}_t = h_t / \sqrt{\widehat{a} + \sum_{i=1}^{p} \widehat{a}_i h_{t-i}^2}$.

Consequently, practitioners have been resorting to ARCH models with heavy-tailed errors. A popular assumption for the distribution of the $\{Z_t\}$ is the t-distribution with degrees of freedom empirically chosen to match the apparent degree of heavy tails as measured by higher-order moments such as the kurtosis.

Reading on conditional heavy tails:

- Politis, Dimitris N. "A heavy-tailed distribution for ARCH residuals with application to volatility prediction." (2004). http://aeconf.com/articles/nov2004/aef050206.pdf

- Bollerslev, Tim, Ray Y. Chou, and Kenneth F. Kroner. "ARCH modeling in finance: A review of the theory and empirical evidence." Journal of econometrics 52.1-2 (1992): 5-59. http://www.academia.edu/download/6044309/bollerslev_chou_kroner_1992.pdf