Limits of Cayley Graphs

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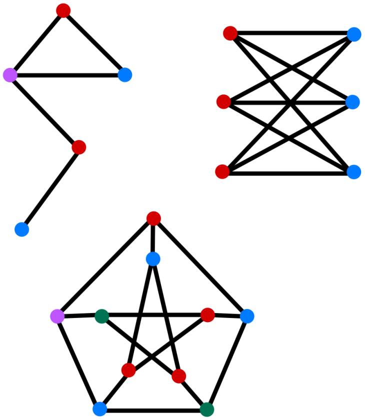
August 11, 2022

Graph Limit Theory

The study of graphs using analytic techniques

Graphs



- Vertex set and edge set

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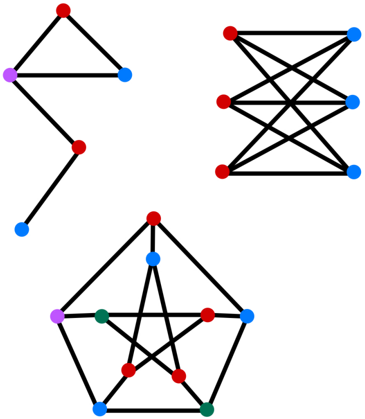

- Vertex set and edge set
- edges represent relationships between vertices

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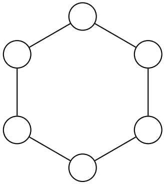
The graph C_6 (left) and its adjancency matrix (right)



$$Adj(C_6) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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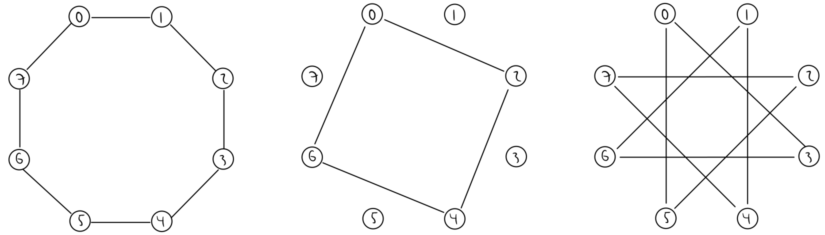

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- From there, find the limit object

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- We first attempt to express our limit as a *graphop* using the *Hausdorff* and *Levy-Prokhorov* metrics
- the set of all graphops contains all graphons and graphings which is why we believe that a graphop might be our desired limit

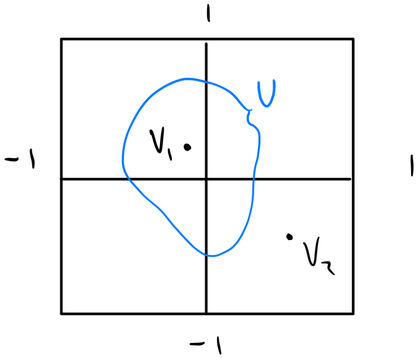
Necessary Definitions

The *joint empirical entry distribution* [1] of a k-tuple of vectors $(v_1, v_2, \ldots, v_k)$ is a probability measure on $\mathbb{R}^k$. It is defined as follows,

$$\mathcal{D}(v_1, v_2, \ldots, v_k) = \frac{1}{n} \sum_{j=1}^{n} \delta_{(v_{1,j}, v_{2,j}, \ldots, v_{k,j})}$$

$\delta_{(v_{1,j}, v_{2,j}, \ldots, v_{k,j})}$ is the Dirac measure of $(v_{1,j}, v_{2,j}, \ldots, v_{k,j})$ at some $U \in P(\mathbb{R}^k)$.

Let $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with v_1 and v_2 in $[0, 1]^2$.



In this case, $\mathcal{D}(V)$ of U is

$$\frac{1}{2}(\delta_{v_1} + \delta_{v_2}) = \frac{1}{2}(1 + 0) = \frac{1}{2}$$

K-Profile

The *K-profile* of a p-operator A is the set of all probability measures of the form

$$\mathcal{D}_A(v_1, \dots, v_k) := \mathcal{D}(v_1, \dots, v_k, Av_1, \dots, Av_k)$$

and it is denoted $\mathcal{S}_k(A)$.

In our case,

$$\mathcal{D}_{A_n}(v_1, \dots, v_k) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{(v_{i,1}, \dots, v_{i,k}, v_{\overline{i-1},1} + v_{\overline{i+1},1}, \dots, v_{\overline{i-1},k} + v_{\overline{i+1},k})}$$

where $\overline{i+1} \equiv i+1 \mod (n)$

Hausdorff Metric

The *Hausdorff metric* measures the distance between two sets X and Y, where $X, Y \subseteq \mathcal{P}(\mathbb{R}^k)$ [1]. It is defined as

$$d_{\mathrm{H}}(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} \mathrm{d}(x, y), \sup_{y \in Y} \inf_{x \in X} \mathrm{d}(a, b) \right\}$$

In Words

We want to show that for any probability measure μ in $\mathcal{S}_k(A_n)$ that there is a probability measure $\tilde{\mu}$ in $\mathcal{S}_k(A_m)$ such that

$$\tilde{\mu}(U^{\varepsilon}) + \varepsilon \geq \mu(U) \tag{1}$$

$$\mu(U^{\varepsilon}) + \varepsilon \geq \tilde{\mu}(U) \tag{2}$$

This has proven to be difficult to show. But there were some techniques that I found useful while attempting this.

First approach

For the forward direction

- Start with $\mu \in \mathcal{S}_k(A_n)$
- μ is a measure of n points in $\mathcal{Q}_{2k}$
- The last k values of each point are uniquely determined by the first k
- We want to approximate μ on n points with $\tilde{\mu}$ on m points
- So we just choose from the original n points with repetition

For the backward direction

- Likely the same, just starting with m points

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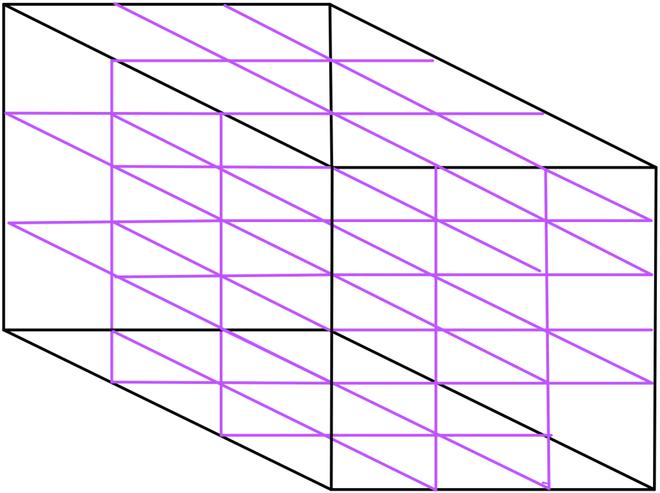

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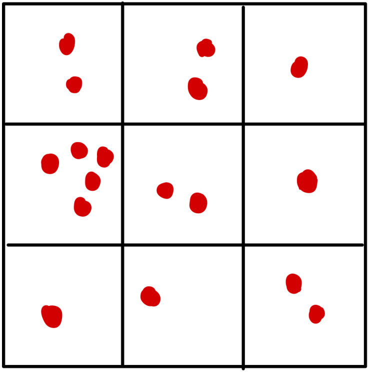
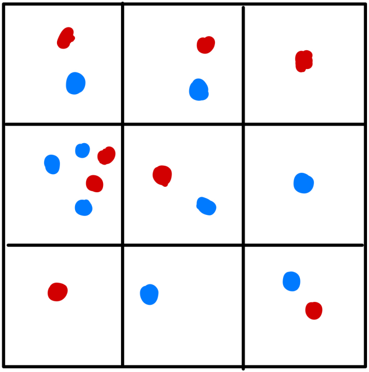
Example, when k is 2, m is 18, and n is 9





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For any probability measure μ on $\mathcal{Q}_k$ of n points $\{p_i \in \mathbb{R}^k\}_{i=1}^n$ there exists some probability measure $\tilde{\mu}$ on Q_k of m points $\{\tilde{p}_i \in \mathbb{R}^k\}_{i=1}^m$ such that

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for any $U \in \mathcal{P}(\mathbb{R}^k)$ for every $m > n$.

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follows as a corollary.

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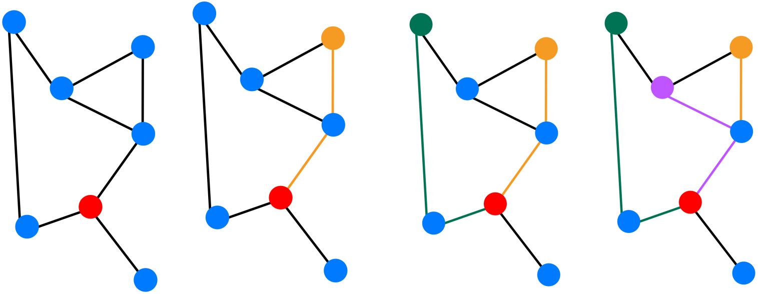

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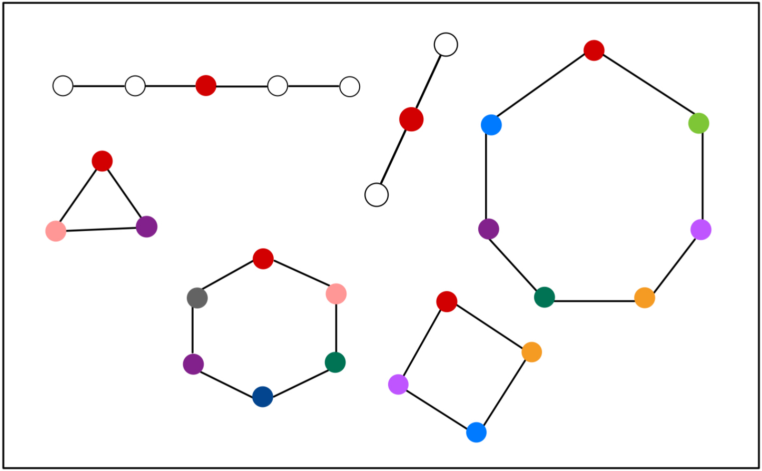

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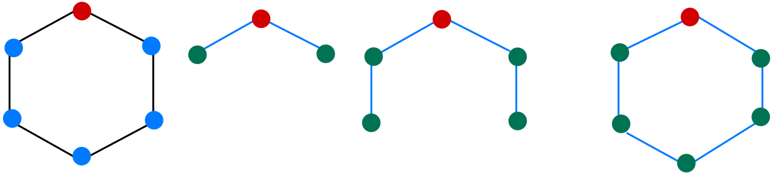

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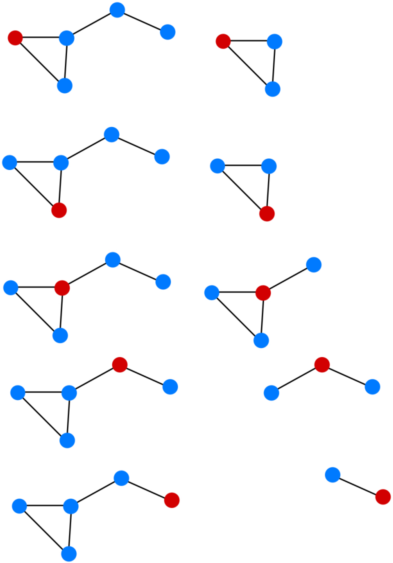
- For a given graph G and radius r, we consider $P_{G,r}$ as the distribution of $N_{G,r}$ on U^r.

Suppose $r = 1$. $N_{G,1}(v)$ for all possible roots of the graph below.



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- For a given graph G and radius r, we consider $P_{G,r}$ as the distribution of $N_{G,r}$ on U^r.
- For a sequence $\{G_n\}$, if $P_{G_n,r}$ converges to a limit distribution as $n \to \infty$ for any fixed $r \geq 0$ then $\{G_n\}$ is locally convergent [2]

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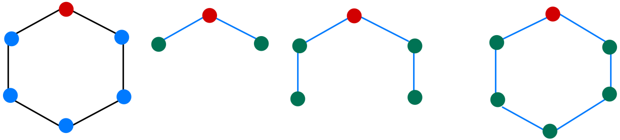

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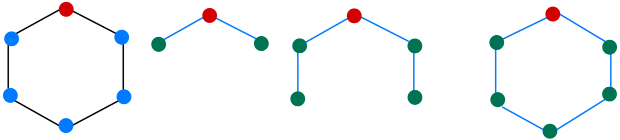

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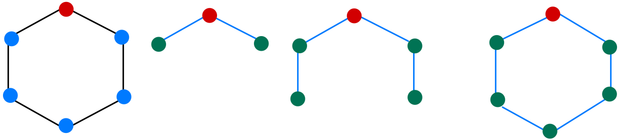

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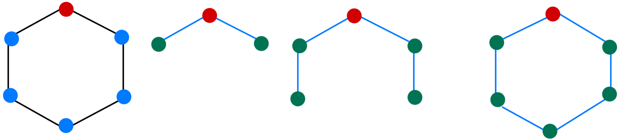

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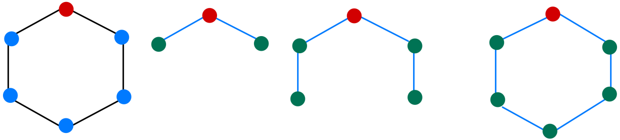

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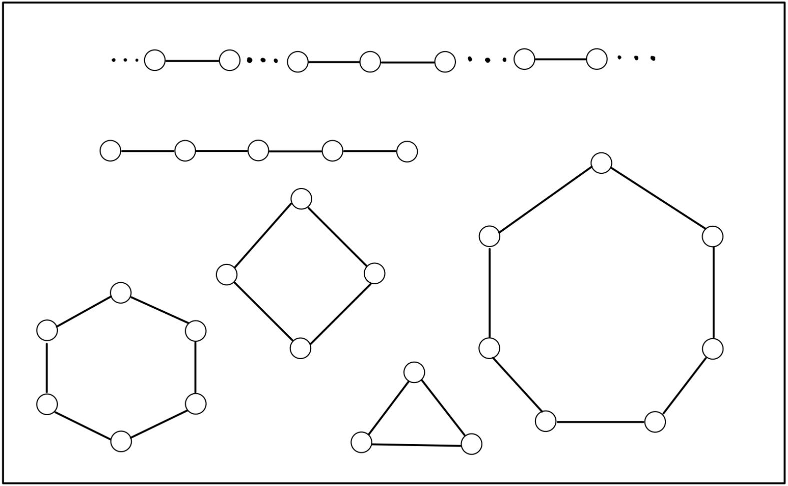
- So our sequence is locally convergent

Finding the Limit

- We view the limit object as a probability measure on the space of isomorphism classes of rooted graphs with maximum degree 2, denoted $\mathcal{G}$



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- $\mathcal{G}(B, o)$ contains all (G, o) satisfying $N_{G,r}(o) \cong (B, o)$

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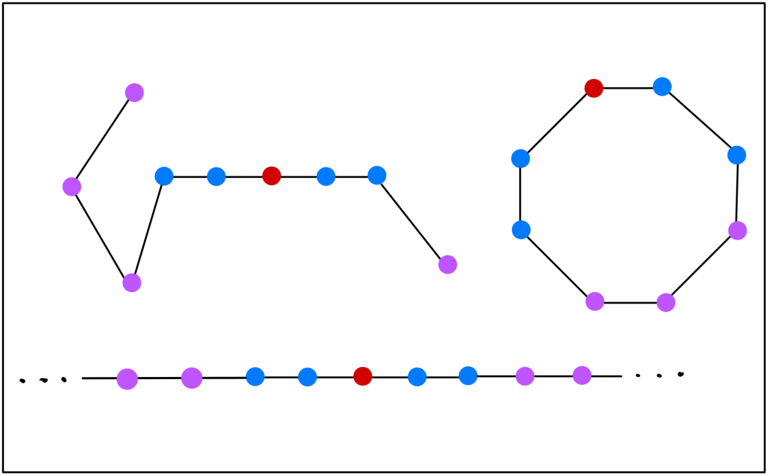
Then $\mathcal{G}(B, o)$ looks something like



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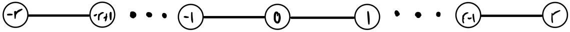

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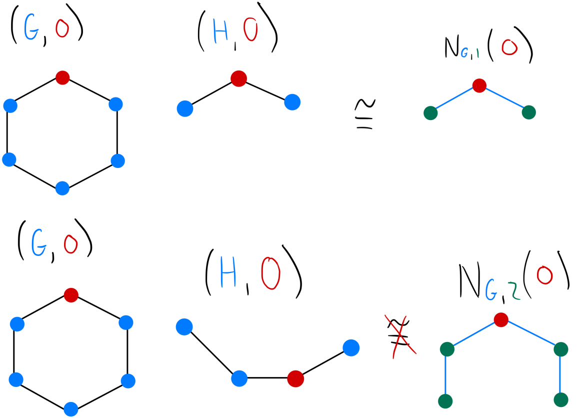
- To visualize



Finding the Limit

The local limit of our sequence is the probability measure δ on $\mathcal{G}$ defined by

$$\delta(A) = \begin{cases} 1 & \text{if } P_\infty \in A \\ 0 & \text{otherwise} \end{cases}$$

where P_∞ is the infinite graph $\text{Cay}(\mathbb{Z}, \{-1, 1\})$ rooted at 0 and $A \subseteq \mathcal{G}$.

Confirming our suspicions

From Hatami, Lovás, and Szegedy in [2] we know that the *Benjamini-Schramm* limit of a locally-convergent graph sequence is a probability measure ν on the borel sets of $\mathcal{G}$ satisfying

$$\lim_{n\to\infty} P_{G_n,r}(B,o) = \nu(\mathcal{G}(B,o))$$

for all $r > 0$ and every rooted graph (G,o) with radius r.

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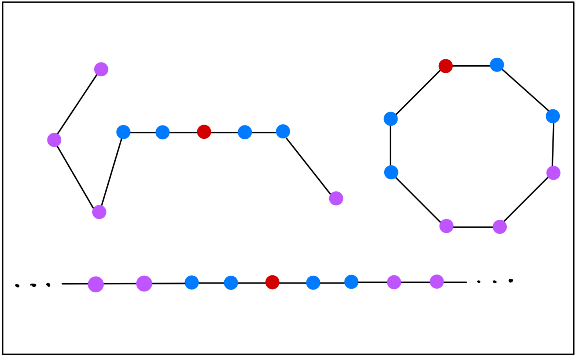

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$P_{G_n,r}(B,o) = 1 \implies \delta(\mathcal{G}(B,o)) = 1$

- Suppose for graph (B,o) that $P_{G_n,r}(B,o) = 1$
- Then, $(B,o) \cong \mathrm{Path}(-r,0,r)$
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- Then, $(B,o) \cong \mathrm{Path}(-r,0,r)$
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- $\mathcal{G}(B,o) \subseteq \mathcal{G}$
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- Suppose for graph (B,o) that $\delta(\mathcal{G}(B,o)) = 1$
- Then P_∞ must be locally similar to (B,o)
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- Meaning that $N_{G_n,r}(B,o) = 1$

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- Thanks for listening!

References

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