

PHSX 491: HW03

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Question 1

- a) We know that we will need to take all nine combinations of coordinates and derivatives, so let's just calculate them all out right away:

$$\begin{array}{lll} \frac{\partial x}{\partial r} = \sin(\theta) \cos(\phi) & \frac{\partial y}{\partial r} = \sin(\theta) \sin(\phi) & \frac{\partial z}{\partial r} = \cos(\theta) \\ \frac{\partial x}{\partial \theta} = r \cos(\theta) \cos(\phi) & \frac{\partial y}{\partial \theta} = r \cos(\theta) \sin(\phi) & \frac{\partial z}{\partial \theta} = -r \sin(\theta) \\ \frac{\partial x}{\partial \phi} = -r \sin(\theta) \sin(\phi) & \frac{\partial y}{\partial \phi} = r \sin(\theta) \cos(\phi) & \frac{\partial z}{\partial \phi} = 0 \end{array}$$

Now, we can use the definition of a tensor to calculate the elements one by one. Since it is just a lot of computation, I will just rapid-fire them.

$$g_{\alpha'\beta'} = \frac{\partial \alpha}{\partial \alpha'} \frac{\partial \beta}{\partial \beta'} g_{\alpha\beta}$$

$$\begin{aligned} g_{rr} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial r} g_{xx} + \frac{\partial x}{\partial r} \frac{\partial y}{\partial r} g_{xy} + \frac{\partial x}{\partial r} \frac{\partial z}{\partial r} g_{xz} + \frac{\partial y}{\partial r} \frac{\partial x}{\partial r} g_{yx} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial r} g_{yy} + \frac{\partial y}{\partial r} \frac{\partial z}{\partial r} g_{yz} + \frac{\partial z}{\partial r} \frac{\partial x}{\partial r} g_{zx} \\ &\quad + \frac{\partial z}{\partial r} \frac{\partial y}{\partial r} g_{zy} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial r} g_{zz} \\ &= \left[\frac{\partial x}{\partial r} \right]^2 g_{xx} + \left[\frac{\partial y}{\partial r} \right]^2 g_{yy} + \left[\frac{\partial z}{\partial r} \right]^2 g_{zz} \\ &= \sin^2(\theta) \cos^2(\phi) + \sin^2(\theta) \sin^2(\phi) + \cos^2(\theta) \\ &= \sin^2(\theta) + \cos^2(\theta) = 1 \end{aligned}$$

$$\begin{aligned} g_{r\theta} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} g_{xx} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} g_{yy} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \theta} g_{zz} = g_{\theta r} \\ &= [\sin(\theta) \cos(\phi)][r \cos(\theta) \cos(\phi)] + [\sin(\theta) \sin(\phi)][r \cos(\theta) \sin(\phi)] + [\cos(\theta)][-r \sin(\theta)] \\ &= r \sin(\theta) \cos(\theta) \cos^2(\phi) + r \sin(\theta) \cos(\theta) \sin^2(\phi) - r \cos(\theta) \sin(\theta) \\ &= r \sin(\theta) \cos(\theta) - r \cos(\theta) \sin(\theta) = 0 \end{aligned}$$

$$\begin{aligned}
g_{r\phi} &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial \phi} g_{xx} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \phi} g_{yy} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \phi} g_{zz} = g_{\phi r} \\
&= [\sin(\theta) \cos(\phi)][-r \sin(\theta) \sin(\phi)] + [\sin(\theta) \sin(\phi)][r \sin(\theta) \cos(\phi)] + 0 = 0
\end{aligned}$$

$$\begin{aligned}
g_{\theta\phi} &= \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \phi} g_{xx} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \phi} g_{yy} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \phi} g_{zz} = g_{\phi\theta} \\
&= [r \cos(\theta) \cos(\phi)][-r \sin(\theta) \sin(\phi)] + [r \cos(\theta) \sin(\phi)][r \sin(\theta) \cos(\phi)] + 0 = 0
\end{aligned}$$

$$\begin{aligned}
g_{\theta\theta} &= \left[\frac{\partial x}{\partial \theta} \right]^2 g_{xx} + \left[\frac{\partial y}{\partial \theta} \right]^2 g_{yy} + \left[\frac{\partial z}{\partial \theta} \right]^2 g_{zz} \\
&= [r \cos(\theta) \cos(\phi)]^2 + [r \cos(\theta) \sin(\phi)]^2 + [-r \sin(\theta)]^2 \\
&= r^2 \cos^2(\theta) \cos^2(\phi) + r^2 \cos^2(\theta) \sin^2(\phi) + r^2 \sin^2(\theta) \\
&= r^2 \cos^2(\theta) + r^2 \sin^2(\theta) = r^2
\end{aligned}$$

$$\begin{aligned}
g_{\phi\phi} &= \left[\frac{\partial x}{\partial \phi} \right]^2 g_{xx} + \left[\frac{\partial y}{\partial \phi} \right]^2 g_{yy} + \left[\frac{\partial z}{\partial \phi} \right]^2 g_{zz} \\
&= [-r \sin(\theta) \sin(\phi)]^2 + [r \sin(\theta) \cos(\phi)]^2 + [0]^2 \\
&= r^2 \sin^2(\theta) \sin^2(\phi) + r^2 \sin^2(\theta) \cos^2(\phi) = r^2 \sin^2(\theta)
\end{aligned}$$

$$g_{\alpha'\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix}$$

Where the coordinates α', β' are in (r, θ, ϕ) .

- b) We know that the inverse of a diagonal matrix is that matrix with each diagonal element replaced with its reciprocal:

$$g^{\alpha'\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2(\theta)} \end{bmatrix}$$

- c) Let's be nit-picky and calculate the covector of the vector. This should be the transpose of the vector, and indeed it is. $A_x = 1 \cdot g_{xx} + 1 \cdot g_{xy} + 1 \cdot g_{xz} = 1$

$$A_y = 1 \cdot g_{yx} + 1 \cdot g_{yy} + 1 \cdot g_{yz} = 1$$

$$A_z = 1 \cdot g_{zx} + 1 \cdot g_{zy} + 1 \cdot g_{zz} = 1$$

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Invoking the dot-product:

$$A^2 = A^\alpha g_{\alpha\beta} A^\beta = A^\alpha A_\alpha = 3$$

$$A^2 = 3$$

- d) In order to find the magnitude in spherical coordinates, we must transform the vector into those coordinates. Here is the general equation for transforming Cartesian to spherical coordinates:

$$r = \sqrt{x^2 + y^2 + z^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \phi = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

Doing this calculation:

$$\vec{A} \rightarrow \begin{bmatrix} \sqrt{3} \\ \frac{\pi}{4} \\ \tan^{-1}(\sqrt{2}) \end{bmatrix}$$

Calculating the covector in spherical:

$$A_\alpha = g_{\alpha\beta} A^\beta$$

$$A_r = \sqrt{3} \cdot g_{rr} + \frac{\pi}{4} \cdot g_{r\theta} + \tan^{-1}(\sqrt{2}) g_{r\phi} = \sqrt{3}$$

$$A_\theta = \sqrt{3} \cdot g_{\theta r} + \frac{\pi}{4} g_{\theta\theta} + \tan^{-1}(\sqrt{2}) g_{\theta\phi} = r^2 \frac{\pi}{4}$$

$$A_\phi = \sqrt{3} \cdot g_{\phi r} + \frac{\pi}{4} g_{\phi\theta} + \tan^{-1}(\sqrt{2}) g_{\phi\phi} = r^2 \sin^2(\theta) \tan^{-1}(\sqrt{2})$$

$$\boxed{\tilde{A} = \begin{bmatrix} \sqrt{3} & r^2 \frac{\pi}{4} & r^2 \sin^2(\theta) \tan^{-1}(\sqrt{2}) \end{bmatrix}}$$

Calculating the magnitude in the spherical:

$$\begin{aligned} A_\alpha A^\alpha &= A_r A^r + A_\theta A^\theta + A_\phi A^\phi \\ &= \sqrt{3} \cdot \sqrt{3} + \frac{\pi}{4} \cdot r^2 \cdot \frac{\pi}{4} + \tan^{-1}(\sqrt{2}) \cdot r^2 \sin^2(\theta) \cdot \tan^{-1}(\sqrt{2}) \end{aligned}$$

To measure the magnitude we will need to orientate the vector at the origin. At the origin $r = 0$, and thus the latter two terms drop out. So, the magnitude is:

$$\boxed{A^2 = 3}$$

- e) From this one example this is pretty strong extrapolation, but logically it makes sense that when we transform coordinates the length is invariant.

Question 2