PHSX 491: HW03

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Question 1

a)

$$\frac{\partial x}{\partial r} = \sin(\theta)\cos(\phi) \qquad \frac{\partial y}{\partial r} = \sin(\theta)\sin(\phi) \qquad \frac{\partial z}{\partial r} = \cos(\theta)$$
$$\frac{\partial x}{\partial \theta} = r\cos(\theta)\cos(\phi) \qquad \frac{\partial y}{\partial \theta} = r\cos(\theta)\sin(\phi) \qquad \frac{\partial z}{\partial \theta} = -r\sin(\theta)$$
$$\frac{\partial x}{\partial \phi} = -r\sin(\theta)\sin(\phi) \qquad \frac{\partial y}{\partial \phi} = r\sin(\theta)\cos(\phi) \qquad \frac{\partial z}{\partial \phi} = 0$$

$$\begin{aligned}
g_{rr} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial r}g_{xx} + \frac{\partial x}{\partial r}\frac{\partial y}{\partial r}g_{xy} + \frac{\partial x}{\partial r}\frac{\partial z}{\partial r}g_{xz} + \frac{\partial y}{\partial r}\frac{\partial x}{\partial r}g_{yx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r}g_{yy} + \frac{\partial y}{\partial r}\frac{\partial z}{\partial r}g_{yz} + \frac{\partial z}{\partial r}\frac{\partial x}{\partial r}g_{zx} \\
&\quad + \frac{\partial z}{\partial r}\frac{\partial y}{\partial r}g_{zy} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r}g_{zz} \\
&= \left[\frac{\partial x}{\partial r}\right]^2 g_{xx} + \left[\frac{\partial y}{\partial r}\right]^2 g_{yy} + \left[\frac{\partial z}{\partial r}\right]^2 g_{zz} \\
&= \sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\theta) \\
&= \sin^2(\theta) + \cos^2(\theta) = 1
\end{aligned}$$

$$\begin{aligned}
g_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta}g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta}g_{yy} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}g_{zz} = g_{\theta r} \\
&= [\sin(\theta)\cos(\phi)][r\cos(\theta)\cos(\phi)] + [\sin(\theta)\sin(\phi)][r\cos(\theta)\sin(\phi)] + [\cos(\theta)][-r\sin(\theta)] \\
&= r\sin(\theta)\cos(\theta)\cos^2(\phi) + r\sin(\theta)\cos(\theta)\sin^2(\phi) - r\cos(\theta)\sin(\theta) \\
&= r\sin(\theta)\cos(\theta) - r\cos(\theta)\sin(\theta) = 0
\end{aligned}$$

$$\begin{aligned}
g_{r\phi} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi}g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi}g_{yy} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi}g_{zz} = g_{\phi r} \\
&= [\sin(\theta)\cos(\phi)][-r\sin(\theta)\sin(\phi)] + [\sin(\theta)\sin(\phi)][r\sin(\theta)\cos(\phi)] + 0 = 0
\end{aligned}$$

$$g_{\theta\phi} = \frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\phi}g_{xx} + \frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\phi}g_{yy} + \frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\phi}g_{zz} = g_{\phi\theta}$$
$$= [r\cos(\theta)\cos(\phi)][-r\sin(\theta)\sin(\phi)] + [r\cos(\theta)\sin(\phi)][r\sin(\theta)\cos(\phi)] + 0 = 0$$

$$g_{\theta\theta} = \left[\frac{\partial x}{\partial\theta}\right]^2 g_{xx} + \left[\frac{\partial y}{\partial\theta}\right]^2 g_{yy} + \left[\frac{\partial z}{\partial\theta}\right]^2 g_{zz}$$
$$= [r\cos(\theta)\cos(\phi)]^2 + [r\cos(\theta)\sin(\phi)]^2 + [-r\sin(\theta)]^2$$
$$= r^2\cos^2(\theta)\cos^2(\phi) + r^2\cos^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)$$
$$= r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$$

$$g_{\phi\phi} = \left[\frac{\partial x}{\partial\phi}\right]^2 g_{xx} + \left[\frac{\partial y}{\partial\phi}\right]^2 g_{yy} + \left[\frac{\partial z}{\partial\phi}\right]^2 g_{zz}$$
$$= [-r\sin(\theta)\sin(\phi)]^2 + [r\sin(\theta)\cos(\phi)]^2 + [0]^2$$
$$= r^2\sin^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)\cos^2(\phi) = r^2\sin^2(\theta)$$

$$\boxed{g_{\alpha'\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}}$$

b)

$$\boxed{g^{\alpha'\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2(\theta)} \end{bmatrix}}$$

c) $A_x = 1 \cdot g_{xx} + 1 \cdot g_{xy} + 1 \cdot g_{xz} = 1$
$A_y = 1 \cdot g_{yx} + 1 \cdot g_{yy} + 1 \cdot g_{yz} = 1$
$A_z = 1 \cdot g_{zx} + 1 \cdot g_{zy} + 1 \cdot g_{zz} = 1$

$$\boxed{\tilde{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}$$

$A^2 = A^\alpha g_{\alpha\beta} A^\beta = A^\alpha A_\alpha = 3$

$$A^2 = 3$$

d)

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \phi = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\vec{A} \longrightarrow \begin{bmatrix} \sqrt{3} \\ \frac{\pi}{4} \\ \tan^{-1}(\sqrt{2}) \end{bmatrix}$$

$$A_\alpha = g_{\alpha\beta} A^\beta$$

$$\begin{aligned}
A_r &= \sqrt{3} \cdot g_{rr} + \frac{\pi}{4} \cdot g_{r\theta} + \tan^{-1}\left(\sqrt{2}\right) g_{r\phi} = \sqrt{3} \\
A_\theta &= \sqrt{3} \cdot g_{\theta r} + \frac{\pi}{4} g_{\theta\theta} + \tan^{-1}\left(\sqrt{2}\right) g_{\theta\phi} = r^2 \frac{\pi}{4} \\
A_\phi &= \sqrt{3} \cdot g_{\phi r} + \frac{\pi}{4} g_{\phi\theta} + \tan^{-1}\left(\sqrt{2}\right) g_{\phi\phi} = r^2 \sin^2(\theta) \tan^{-1}\left(\sqrt{2}\right)
\end{aligned}$$

$$\boxed{\tilde{A} = \begin{bmatrix} \sqrt{3} & r^2 \frac{\pi}{4} & r^2 \sin^2(\theta) \tan^{-1}(\sqrt{2}) \end{bmatrix}}$$

$$\begin{aligned}
A_\alpha A^\alpha &= A_r A^r + A_\theta A^\theta + A_\phi A^\phi \\
&= \sqrt{3} \cdot \sqrt{3} + \frac{\pi}{4} \cdot r^2 \cdot \frac{\pi}{4} + \tan^{-1}\left(\sqrt{2}\right) \cdot r^2 \sin^2(\theta) \cdot \tan^{-1}\left(\sqrt{2}\right)
\end{aligned}$$

$$\boxed{A^2 = 3}$$

e) Invariant

Question 2