PHSX 461: HW07

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October 26, 2021

A.2

Consider the collection of all polynomials (with complex coefficients) of degree $< N$ in x

a) *Does this set constitute a vector space (with the polynomials as "vectors")? If so, suggest a convenient basis, and give the dimension of the space. If not, which of the defining properties does it lack?*

This very much does constitute a vector space, as any linear combination of polynomials will be a polynomial. We also don't allow for the multiplication of vectors, so the maximum degree that is achievable is the largest degree initially given. A convenient basis would be a vector for every degree: $|1, 0, 0, \cdots\rangle, |0, x, 0, \cdots\rangle, |0, 0, x^2, \cdots\rangle$. There would be a total of N linearly independent vectors, so the dimensionality would be N.

b) *What if we require that the polynomials be even functions?*

We couldn't get an odd function out of a sum of even functions, so this would also be a vector space.

c) *what if we require that the leading coefficient (i.e. the number multiplying x^{N-1}) be 1?*

If we added together two vectors that were not linearly independent, we would no longer be in this space, so this one is not a vector space.

$$\text{i.e. } |0, 0, x^2, \cdots\rangle + |0, 0, x^2, \cdots\rangle = |0, 0, 2 \cdot x^2, \cdots\rangle$$

d) *What if we require that the polynomials have the value 0 at $x = 1$?*

This would be a vector space, as at the point $x = 1$ we would be summing the value of each vector, and $c \cdot 0 = 0$. So, this is closed over scalar multiplication and vector addition.

e) *What if we require that the polynomials have the value 1 at $x = 0$?*

This would not be, since we would be adding 1's at $x = 0$.

$$\text{i.e. } |1, 0, x^2, \cdots\rangle + |1, x, 0, \cdots\rangle = |2, x, x^2, \cdots\rangle \text{ which gives}$$
$$|2, x, x^2, \cdots\rangle\Big|_0 \neq |1, 0, 0, \cdots\rangle$$

3.2

a) *For what range $\boldsymbol{v}$ is the function $f(x) = x^{\boldsymbol{v}}$ in Hilbert space, on the interval (0,1)? Assume $\boldsymbol{v}$ is real, but not necessarily positive.*

The property of the Hilbert space that we need to satisfy is that the function is square integrate. That is

$$\int_0^1 [f(x)]^* f(x) dx < \infty$$

$$\int_0^1 (x^{\mathbf{v}})^* x^{\mathbf{v}} dx$$
$$\int_0^1 x^{2\mathbf{v}} dx$$
$$\frac{1}{2\mathbf{v}+1} x^{2\mathbf{v}+1}\Big|_{x=0}^{x=1}$$
$$\frac{1}{2\mathbf{v}+1}(1 + \lim_{R\to 0} R^{2\mathbf{v}+1})$$

So, we can say this diverges when the second term has the R in the denominator:

$$2\mathbf{v} + 1 > 0$$

$$\mathbf{v} > -\frac{1}{2}$$

b) *For the specific case $v = 1/2$, if $f(x)$ in this Hilbert space? What about $xf(x)$? How about $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$?*

- $f(x)$: since $\frac{1}{2} > -\frac{1}{2}$, $f(x)$ is in the Hilbert space.
- $xf(x)$: since $\mathbf{v} = \frac{1}{2}$ is in the Hilbert space, then $\mathbf{v} = \frac{3}{2}$ is also in the Hilbert space.
- $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$: $\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{1}{2}x^{-1/2}$, since $-\frac{1}{2} \not> \frac{1}{2}$ it is not in the Hilbert space.

3.3

Show that if $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$ for all h (in Hilbert space), then for all f and g $\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$(i.e. the two definition of "hermitian" are equivalent). Hint: first let $h = f + g$, and then let $h = f + ig$.

Since f and g can be **any** function in Hilbert space, we can construct another function $h = f + g$ that is in Hilbert space, and a $h' = f + ig$ that is also in Hilbert space.

$$\begin{aligned}
\langle h|\hat{Q}h\rangle &= \langle \hat{Q}h|h\rangle \\
\langle f+g|\hat{Q}(f+g)\rangle &= \langle \hat{Q}(f+g)|(f+g)\rangle \\
\int (f+g)^*[\hat{Q}(f+g)] &= \int [\hat{Q}(f+g)]^*(f+g) \\
\int [f^*+g^*][\hat{Q}(f+g)] &= \int [(\hat{Q}f)^* + (\hat{Q}g)^*](f+g) \\
\int f^*\hat{Q}f + f^*\hat{Q}g + g^*\hat{Q}f + g^*\hat{Q}g &= \int (\hat{Q}f)^*f + (\hat{Q}f)^*g + (\hat{Q}g)^*f + (\hat{Q}g)^*g \\
\int f^*\hat{Q}g + g^*\hat{Q}f &= \int (\hat{Q}f)^*g + (\hat{Q}g)^*f \qquad (1)
\end{aligned}$$

Now, doing the same analysis on h':

$$\langle h'|\hat{Q}h'\rangle = \langle \hat{Q}h'|h'\rangle$$

$$\langle f+ig|\hat{Q}(f+ig)\rangle = \langle \hat{Q}(f+ig)|(f+ig)\rangle$$

$$\int (f+ig)^*[\hat{Q}(f+ig)] = \int [\hat{Q}(f+ig)]^*(f+ig)$$
$$\int [f^* + (ig)^*][\hat{Q}(f+ig)] = \int [(\hat{Q}f)^* + (\hat{Q}ig)^*](f+ig)$$
$$\int f^*\hat{Q}f + f^*\hat{Q}ig - ig^*\hat{Q}f - ig^*\hat{Q}ig = \int (\hat{Q}f)^*f + (\hat{Q}f)^*ig - i(\hat{Q}g)^*f - i(\hat{Q}g)^*ig$$
$$\int f^*\hat{Q}g - g^*\hat{Q}f = \int (\hat{Q}f)^*g - (\hat{Q}g)^*f \tag{2}$$

Summing together 1 and 2, it turns into

$$2\int f^*\hat{Q}g = 2\int (\hat{Q}f)^*g$$
$$f^*\hat{Q}g = \int (\hat{Q}f)^*g$$
$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

3.5

a) *Find the hermitian conjugates of* x, i, *and* $\mathrm{d}/\mathrm{d}x$

- x: $\langle f|xg\rangle = \int f^*xg = \int (xf)^*g = \langle xf|g\rangle$, $x^\dagger = x$
- i: $\langle f|ig\rangle = \int f^*ig = \int (-if)^*g = \langle -if|g\rangle$, $i^\dagger = -i$
- i: $\left\langle f\middle|\frac{\mathrm{d}}{\mathrm{d}x}g\right\rangle = \int f^*\, \mathrm{d}g/\mathrm{d}x$ this one we will have to be a tad more clever and use integration by parts to get: $f^*g\Big|_{-\infty}^{\infty} - \int \frac{\mathrm{d}}{\mathrm{d}x}(f)^*g = 0 + \int (-\frac{\mathrm{d}f}{\mathrm{d}x})^*g = \left\langle -\frac{\mathrm{d}}{\mathrm{d}x}f\middle|g\right\rangle$, $(\frac{\mathrm{d}}{\mathrm{d}x})^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x}$

b) *Show that* $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$ *(notice the reversed order),* $(\hat{Q}+\hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger$, *and* $(c\hat{Q})^\dagger = c^*\hat{Q}^\dagger$ *for a complex number c.*

- To show $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$:

$$\left\langle f\middle|(\hat{Q}\hat{R})^\dagger g\right\rangle$$
$$\int f^*(\hat{Q}\hat{R}^\dagger g$$

$$\int (\hat{Q}\hat{R}f)^* g$$

$$\int (\hat{R}f)^* \hat{Q}^\dagger g$$

$$\int f^* \hat{R}^\dagger \hat{Q}^\dagger g$$

$$\left\langle f \middle| \hat{R}^\dagger \hat{Q}^\dagger g \right\rangle$$

Thus $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\,\hat{Q}^\dagger$

- To show $(\hat{Q} + \hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger$:

$$\left\langle f \middle| (\hat{Q} + \hat{R})^\dagger g \right\rangle$$

$$\int f^* (\hat{Q} + \hat{R})^\dagger g$$

$$\int ((\hat{Q} + \hat{R}) f)^* g$$

$$\int (\hat{Q}f)^* g + \int (\hat{R}f)^* g$$

$$\int f^* \hat{Q}^\dagger g + \int f^* \hat{R}^\dagger g$$

$$\int f^* (\hat{Q}^\dagger + \hat{R}^\dagger) g$$

$$\left\langle f \middle| (\hat{Q}^\dagger + \hat{R}^\dagger) g \right\rangle$$

Thus $(\hat{Q} + \hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger$

- To show $(c\,\hat{Q})^\dagger = c^*\,\hat{Q}^\dagger$:

$$\left\langle f \middle| (c\hat{Q})^\dagger g \right\rangle$$

$$\int f^* (c\hat{Q})^\dagger g$$

$$\int (c\hat{Q}f)^* g$$

$$\int c^*(\hat{Q}f)^* g$$

$$\int f^*(c\hat{Q}^\dagger) g$$

$$\left\langle f \middle| (c^*\hat{Q}^\dagger) g \right\rangle$$

Thus $(c\,\hat{Q})^\dagger = c^*\,\hat{Q}^\dagger$

c) *Construct the hermitian conjugate of* a_+

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x)$$

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)$$

$$(a_+)^\dagger = \left(\frac{1}{\sqrt{2\hbar m\omega}}\right)^* [(-i\hat{p})^\dagger + (m\omega x)^\dagger]$$

$$(a_+)^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}[i(\hat{p})^\dagger + (m\omega x)^\dagger]$$

$$(a_+)^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega x) = a_-$$

Which I am glad we got to $(a_+)^\dagger = a_-$ because this was a point made back in Chapter 2.