Linear Algebra Homework 5

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April 2020

1. Prove Theorem 5.8(d)

"If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is Q_1Q_2"

Proof:

If Q_1 and Q_2 are orthogonal, to show that Q_1Q_2 is orthogonal, we need to show that $(Q_1Q_2)^{-1} = (Q_1Q_2)^T$. Then the following:

$$(Q_1Q_2)^{-1} = Q_2^{-1}Q_1^{-1} = Q_2^TQ_1^T = (Q_1Q_2)^T$$

"Quack" ■

2. "Prove that null$(A^T) \perp$ col(A)"

Proof:

We must show that if $\mathbf{x} \in \mathbb{R}$, then $\mathbf{x}\cdot$(every vector in col(A)) $= 0$ and $\mathbf{x} \in$ Null(A^T) are equivalent. Let us begin $\mathbf{x}$ being in Null(A^T):

$$A^T\mathbf{x} = \mathbf{0}$$
$$(\mathbf{x}^TA)^T = \mathbf{0}$$
$$\mathbf{x}^TA = \mathbf{0}^T = \mathbf{0}$$

This last line means that $\mathbf{x} \cdot$ (each column of A) $= 0$, or x is perpendicular to each column of A.

"Quack" ■

3. Prove Theorem 5.9(c)

"Let W be a subspace of $\mathbb{R}^n$. then $W \cap W^\perp = \{\mathbf{0}\}$"

Proof:

It is enough to show that if $\mathbf{x} \in W$, then $\text{Proj}_{W^\perp}\mathbf{x} = \mathbf{0}$. For simplicity we will assume that W and $W^\perp$ are basis already. If not we can make both spaces into basis, as the basis will cover the same space and the proof holds.
If $\mathbf{x} \in W$, then $\mathbf{x}$ can be written as

$$\mathbf{x} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + ... + c_n\mathbf{w}_n$$

Then: $\text{Proj}_{W^\perp}\mathbf{x}$

$$\begin{gathered}
= (\mathbf{x}\cdot\mathbf{w}_1^\perp)\mathbf{w}_1^\perp + (\mathbf{x}\cdot\mathbf{w}_2^\perp)\mathbf{w}_2^\perp + ... + (\mathbf{x}\cdot\mathbf{w}_k^\perp)\mathbf{w}_k^\perp \\
= ((c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + ... + c_n\mathbf{w}_n)\cdot\mathbf{w}_1^\perp)\mathbf{w}_1^\perp + ... \\
= (c_1\mathbf{w}_1\cdot\mathbf{w}_1^\perp + c_2\mathbf{w}_2\cdot\mathbf{w}_1^\perp + ... + c_n\mathbf{w}_n\cdot\mathbf{w}_1^\perp)\mathbf{w}_1^\perp + ... \\
= (c_1\cdot 0 + c_2\cdot 0 + ... + c_n\cdot 0)\mathbf{w}_1^\perp + ... \\
= 0\cdot\mathbf{w}_1^\perp + 0\cdot\mathbf{w}_2^\perp + ... + 0\cdot\mathbf{w}_k^\perp = 0
\end{gathered}$$

So the only vector $\mathbf{x}$ that can be could be in both W and $W^\perp$ is $\mathbf{0}$. So the union of the two is $\{\mathbf{0}\}$.

"Quack" ■

4. Use Gram Schmidt to find an orthonormal basis for the column space of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \tfrac{\mathbf{v}_1\cdot\mathbf{x}_2}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \tfrac{\mathbf{v}_1\cdot\mathbf{x}_3}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 - \tfrac{\mathbf{v}_2\cdot\mathbf{x}_3}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{4}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \tfrac{1}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{6} \\ \frac{3}{6} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Orthonormal basis for $A = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\right)$.

5. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(a) Prove that A is orthogonally diagonalizable.
All symmetric matrices are orthogonally diagonalizable, so A is.

(b) Orthongonally diagonalize A.
To first look for the eigenvalues:

$$
\begin{aligned}
0 &= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1-\lambda & 1 \end{vmatrix} \\
&= (1-\lambda)[(1-\lambda)(1-\lambda)-1] - ((1-\lambda)-1) + (1-(1-\lambda)) \\
&= (1-\lambda)^3 - (1-\lambda) - (1-\lambda) + 1 + 1 - (1-\lambda) \\
&= (1-\lambda)(1-2\lambda+\lambda^2) - 3 + 3\lambda + 2 \\
&= (1-2\lambda+\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 1 + 3\lambda \\
&= -\lambda^3 + 3\lambda^2
\end{aligned}
$$

$\lambda = 0$, with mult. 2 and $\lambda = 3$

Solving for $\lambda = 0$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$x + y + z = 0 \qquad \rightarrow \qquad \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Solving for $\lambda = 3$:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the related eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

6. $A = \begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

(a) Prove that A is not orthogonally diagonalizable.
The matrix is not symmetric, so it is not orthogonally diagonalizable.

(b) Prove that A is diagonalizable.
To prove that the matrix is diagonalizable, it must be shown that the algebraic multiplicity of the eigenvalues is equal to their geometric multiplicity.
$0 = (4-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -1 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & -2 \\ -1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ 1-\lambda & -2 \end{vmatrix}$
$= (4-\lambda)((1-\lambda)^2 - 2) - 2(-(1-\lambda) - 2) + (2 + 2(1-\lambda))$
$= (4-\lambda)(1 - 2\lambda + \lambda^2 - 2) - 2(-3+\lambda) + (4 - 2\lambda)$
$= (-\lambda+4)(\lambda^2 - 2\lambda - 1) - 2\lambda + 6 + 4 - 2\lambda$
$= -\lambda^3 + 2\lambda^2 + \lambda + 4\lambda^2 - 8\lambda - 4 - 4\lambda + 10$
$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$
$= (\lambda-1)(\lambda-2)(\lambda-3)$

$$\lambda = 1, 2, 3$$

Each eigenvalue has a algebraic multiplicity of 1, and each one has a geometric multiplicity of at least 1, and the total number of vectors is 3. Since each algebraic multiplicity is equal to their geometric multiplicity, the matrix is diagonalizable.

(c) For $\lambda = 1$:

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$2x - 2z = 0$, and $x - y = 0$, so the eigenvector, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$:

$$\begin{bmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 2 & -1 & -2 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 3$:

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -2 & -2 \\ 1 & -1 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ 2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the eigenvector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$$

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) = -1 + 1 + 1 = 1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \frac{1}{1} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

7. $A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 3 \end{bmatrix}$

(a) Prove that A is not orthogonally diagonalizable
A is not symmetric, so A is not orthogonally diagonalizable

(b) Prove that A is not diagonalizable.
To show A is not diagonalizable, the geometric multiplicity is not equal to the algebraic multiplicity.
$0 = \det(A) = (3-\lambda)(3-\lambda)(3-\lambda)$
So $\lambda = 3$, with multiplicity 3.

Checking for the geometric multiplicity:

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}, \text{ then } \mathbf{x} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So the algebraic multiplicity of $\lambda = 3$ is 2, but the geometric multiplicity is 2; so the matrix A is not diagonalizable.

(c) Prove that A is orthogonally triangularizable.
For the matrix to be orthogonally triangularizable, the matrix has to have real eigenvalues. All the eigenvalues are 3, which is real, so the matrix A is orthogonally triangularizable.

(d) To get the orthogonal matrix, the eigenvectors make up the first two columns. To complete the orthonormal basis the last orthogonal vector can be used, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. So

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

To verify Q is orthogonal, $Q^T Q = I$. This is obviously true.
To get the Triangular matrix, $T = Q^T A Q$.

$$
\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}3&0&0\\-1&3&-1\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}
$$

$$
= \begin{bmatrix}3&-1&0\\0&-1&3\\0&3&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}
$$

$$
= \begin{bmatrix}3&0&-1\\0&3&-1\\0&0&3\end{bmatrix} = T
$$

So $A = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}3&0&-1\\0&3&-1\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

8. "Let A be an $n \times n$ real matrix, all of whose eigenvalues are real. Prove that there exist an orthogonal matrix Q and an upper triangular matrix T such that $Q^T AQ = T$."

Proof by induction:

Take the base case as the 1×1 works, as any 1×1 matrix is already a orthogonally triangularizable with $Q = [1]$.

For the inductive step, assume the $k \times k$ matrix A is able to be written as $Q^T AQ = T$. Then for the $(k+1) \times (k+1)$, take the real eigenvalue be λ_1, then $\mathbf{v}_1$ is a real eigenvector related to λ_1, such that $\mathbf{v}_1$ is a unit vector.

Use the Gram-Schmit method to develop an orthogonal basis such that

$$
Q = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{k+1}]
$$

$$
Q^T AQ = \begin{bmatrix}\mathbf{v}_1^T\\ \mathbf{v}_2^T\\ \cdots\\ \mathbf{v}_{k+1}^T\end{bmatrix} A \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{k+1}\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1^T\\ \mathbf{v}_2^T\\ \cdots\\ \mathbf{v}_{k+1}^T\end{bmatrix} \begin{bmatrix}\lambda_1\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_{k+1}\end{bmatrix}
$$

$$
= \begin{bmatrix}\lambda_1 & * \\ 0 & B_{k\times k}\end{bmatrix}
$$

Such that $B_{k\times k} = \begin{bmatrix}\mathbf{v}_2^T\\ \cdots\\ \mathbf{v}_{k+1}^T\end{bmatrix} A \begin{bmatrix}\mathbf{v}_2 & \cdots & \mathbf{v}_{k+1}\end{bmatrix} = V^T AV$, where V is an orthogonal matrix.

So now, $Q^T AQ = \begin{bmatrix}\lambda_1 & * \\ 0 & V^T AV\end{bmatrix} = \begin{bmatrix}\lambda_1 & * \\ 0 & T\end{bmatrix}$

So for any $(k+1) \times (k+1)$ matrix A, $Q^T AT = T'$, where T' is an upper triangular matrix. We can generalize this to any $n \times n$ matrix A can be written as $Q^T AQ = T$.

"Quack" ■

9. "Let $A = QTQ^T$ be the Schur Triangulation of a square matrix A, where Q is orthogonal and T is upper triangular. Prove that the eigenvalues of A are on the diagonal of T."

Proof:

$0 = \det(A - \Lambda) = \det(QTQ^T - \lambda QQ^T)$
$= \det(QTQ^T - Q\lambda IQ^T) = \det(Q(T - \lambda I)Q^T)$
$= \det(Q)\det(T - \lambda I)\det(Q^T)$
$= \det(Q)\det(Q^{-1})\det(T - \lambda I)$
$= \det(T - \lambda I)$

Since A and T have the same characteristic equation, they have the same eigenvalues. T is a triangular matrix, so the eigenvalues are on the diagonal. So it follows that the eigenvalues of A are on the diagonal of T.

"Quack" ■

10. "Let $A = QTQ^T$ be the Schur Triangulation of a square matrix A, where Q is orthogonal and T is upper triangular. Prove that if A is invertible, then $A^{-1} = QT^{-1}Q^T$."

Proof:
If $A = QTQ^T$ and A is invertible, then:

$$A^{-1} = (QTQ^T)^{-1} = (Q^T)^{-1}T^{-1}Q^{-1} = QT^{-1}Q^T$$

This assumes that T^{-1} exists. We can verify this assumption by seeing that A and T have the same eigenvalues, using the result from Question 9, and consequently the same determinants. So, because A is invertible, $\det(A)$ is not 0, so $\det(T)$ is not 0, and we can see T is invertible.

"Quack" ■