PHSX 491: HW03

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February 22, 2022

Question 1

a) We know that we will need to take all nine combinations of coordinates and derivatives, so let's just calculate them all out right away:

$$\begin{aligned}
\frac{\partial x}{\partial r} &= \sin(\theta)\cos(\phi) & \frac{\partial y}{\partial r} &= \sin(\theta)\sin(\phi) & \frac{\partial z}{\partial r} &= \cos(\theta)\\
\frac{\partial x}{\partial \theta} &= r\cos(\theta)\cos(\phi) & \frac{\partial y}{\partial \theta} &= r\cos(\theta)\sin(\phi) & \frac{\partial z}{\partial \theta} &= -r\sin(\theta)\\
\frac{\partial x}{\partial \phi} &= -r\sin(\theta)\sin(\phi) & \frac{\partial y}{\partial \phi} &= r\sin(\theta)\cos(\phi) & \frac{\partial z}{\partial \phi} &= 0
\end{aligned}$$

Now, we can use the definition of a tensor to calculate the elements one by one. Since it is just a lot of computation, I will just rapid-fire them.

$$g_{\alpha'\beta'} = \frac{\partial \alpha}{\partial \alpha'}\frac{\partial \beta}{\partial \beta'} g_{\alpha\beta}$$

$$\begin{aligned}
g_{rr} = &\frac{\partial x}{\partial r}\frac{\partial x}{\partial r}g_{xx} + \frac{\partial x}{\partial r}\frac{\partial y}{\partial r}g_{xy} + \frac{\partial x}{\partial r}\frac{\partial z}{\partial r}g_{xz} + \frac{\partial y}{\partial r}\frac{\partial x}{\partial r}g_{yx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r}g_{yy} + \frac{\partial y}{\partial r}\frac{\partial z}{\partial r}g_{yz} + \frac{\partial z}{\partial r}\frac{\partial x}{\partial r}g_{zx}\\
&+ \frac{\partial z}{\partial r}\frac{\partial y}{\partial r}g_{zy} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r}g_{zz}\\
&= \left[\frac{\partial x}{\partial r}\right]^2 g_{xx} + \left[\frac{\partial y}{\partial r}\right]^2 g_{yy} + \left[\frac{\partial z}{\partial r}\right]^2 g_{zz}\\
&= \sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\theta)\\
&= \sin^2(\theta) + \cos^2(\theta) = 1
\end{aligned}$$

$$\begin{aligned}
g_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta}g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta}g_{yy} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}g_{zz} = g_{\theta r}\\
&= [\sin(\theta)\cos(\phi)][r\cos(\theta)\cos(\phi)] + [\sin(\theta)\sin(\phi)][r\cos(\theta)\sin(\phi)] + [\cos(\theta)][-r\sin(\theta)]\\
&= r\sin(\theta)\cos(\theta)\cos^2(\phi) + r\sin(\theta)\cos(\theta)\sin^2(\phi) - r\cos(\theta)\sin(\theta)\\
&= r\sin(\theta)\cos(\theta) - r\cos(\theta)\sin(\theta) = 0
\end{aligned}$$

$$g_{r\phi} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi}g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi}g_{yy} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \phi}g_{zz} = g_{\phi r}$$
$$= [\sin(\theta)\cos(\phi)][-r\sin(\theta)\sin(\phi)] + [\sin(\theta)\sin(\phi)][r\sin(\theta)\cos(\phi)] + 0 = 0$$

$$g_{\theta\phi} = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \phi}g_{xx} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \phi}g_{yy} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \phi}g_{zz} = g_{\phi\theta}$$
$$= [r\cos(\theta)\cos(\phi)][-r\sin(\theta)\sin(\phi)] + [r\cos(\theta)\sin(\phi)][r\sin(\theta)\cos(\phi)] + 0 = 0$$

$$g_{\theta\theta} = \left[\frac{\partial x}{\partial \theta}\right]^2 g_{xx} + \left[\frac{\partial y}{\partial \theta}\right]^2 g_{yy} + \left[\frac{\partial z}{\partial \theta}\right]^2 g_{zz}$$
$$= [r\cos(\theta)\cos(\phi)]^2 + [r\cos(\theta)\sin(\phi)]^2 + [-r\sin(\theta)]^2$$
$$= r^2\cos^2(\theta)\cos^2(\phi) + r^2\cos^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)$$
$$= r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$$

$$g_{\phi\phi} = \left[\frac{\partial x}{\partial \phi}\right]^2 g_{xx} + \left[\frac{\partial y}{\partial \phi}\right]^2 g_{yy} + \left[\frac{\partial z}{\partial \phi}\right]^2 g_{zz}$$
$$= [-r\sin(\theta)\sin(\phi)]^2 + [r\sin(\theta)\cos(\phi)]^2 + [0]^2$$
$$= r^2\sin^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)\cos^2(\phi) = r^2\sin^2(\theta)$$

$$\boxed{g_{\alpha'\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}}$$

Where the coordinates α', β' are in (r, θ, ϕ).

b) We know that the inverse of a diagonal matrix is that matrix with each diagonal element replaced with its reciprocal:

$$\boxed{g^{\alpha'\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2(\theta)} \end{bmatrix}}$$

c) Let's be nit-picky and calculate the covector of the vector. This should be the transpose of the vector, and indeed it is. $A_x = 1 \cdot g_{xx} + 1 \cdot g_{xy} + 1 \cdot g_{xz} = 1$
$A_y = 1 \cdot g_{yx} + 1 \cdot g_{yy} + 1 \cdot g_{yz} = 1$
$A_z = 1 \cdot g_{zx} + 1 \cdot g_{zy} + 1 \cdot g_{zz} = 1$

$$\boxed{\tilde{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}$$

Invoking the dot-product:
$A^2 = A^\alpha g_{\alpha\beta} A^\beta = A^\alpha A_\alpha = 3$

$$\boxed{A^2 = 3}$$

d) In order to find the magnitude in spherical coordinates, we must transform the vector into those coordinates. Here is the general equation for transforming Cartesian to spherical coordinates:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \phi = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

Doing this calculation:

$$\vec{A} \longrightarrow \begin{bmatrix} \sqrt{3} \\ \frac{\pi}{4} \\ \tan^{-1}(\sqrt{2}) \end{bmatrix}$$

Calculating the covector in spherical:

$$A_\alpha = g_{\alpha\beta} A^\beta$$

$$\begin{aligned}
A_r &= \sqrt{3} \cdot g_{rr} + \frac{\pi}{4} \cdot g_{r\theta} + \tan^{-1}\left(\sqrt{2}\right) g_{r\phi} = \sqrt{3} \\
A_\theta &= \sqrt{3} \cdot g_{\theta r} + \frac{\pi}{4} g_{\theta\theta} + \tan^{-1}\left(\sqrt{2}\right) g_{\theta\phi} = r^2 \frac{\pi}{4} \\
A_\phi &= \sqrt{3} \cdot g_{\phi r} + \frac{\pi}{4} g_{\phi\theta} + \tan^{-1}\left(\sqrt{2}\right) g_{\phi\phi} = r^2 \sin^2(\theta) \tan^{-1}\left(\sqrt{2}\right)
\end{aligned}$$

$$\boxed{\tilde{A} = \begin{bmatrix} \sqrt{3} & r^2 \frac{\pi}{4} & r^2 \sin^2(\theta) \tan^{-1}(\sqrt{2}) \end{bmatrix}}$$

Calculating the magnitude in the spherical:

$$\begin{aligned}
A_\alpha A^\alpha &= A_r A^r + A_\theta A^\theta + A_\phi A^\phi \\
&= \sqrt{3} \cdot \sqrt{3} + \frac{\pi}{4} \cdot r^2 \cdot \frac{\pi}{4} + \tan^{-1}\left(\sqrt{2}\right) \cdot r^2 \sin^2(\theta) \cdot \tan^{-1}\left(\sqrt{2}\right)
\end{aligned}$$

To measure the magnitude we will need to orientate the vector at the origin. At the origin $r = 0$, and thus the latter two terms drop out. So, the magnitude is:

$$\boxed{A^2 = 3}$$

e) From this one example this is pretty strong extrapolation, but logically it makes sense that when we transform coordinates the length is <u>invariant</u>.

Question 2

a) See Fig. 1

b)

$$\begin{aligned}
q &= y - cx^2 = y - cp^2 \\
\rightarrow y &= q + cp^2
\end{aligned}$$

$$\boxed{\begin{aligned} x(p, q) &= p \\ y(p, q) &= q + cp^2 \end{aligned}}$$

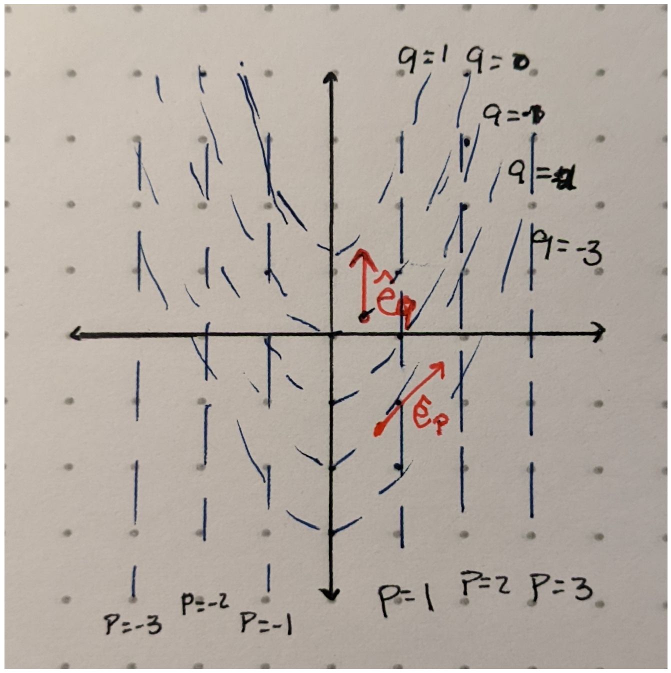

Figure 1: Our new $p-q$ coordinates. Constant q looks like a parabola, and constant p are values of x.

c) If we "zoom" into a point, the smoothness of the space allows us to use the Pythagorean theorem:

$$\begin{aligned} \mathrm{d}s^2 &\approx \mathrm{d}x^2 + \mathrm{d}y^2 \\ &= (\mathrm{d}p)^2 + (\mathrm{d}q + 2cp\,\mathrm{d}p)^2 \\ &= (\mathrm{d}p)^2 + (\mathrm{d}q)^2 + 4cp\,\mathrm{d}p\,\mathrm{d}q + 4c^2p^2\,(\mathrm{d}p)^2 \end{aligned}$$

Using this to find the tensor definition of the dot product and assuming that $g_{pq} = g_{qp}$:

$$\begin{aligned} g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta &\to g_{pp}\,(\mathrm{d}p)^2 + g_{pq}\,\mathrm{d}p\,\mathrm{d}q + g_{qp}\,\mathrm{d}q\,\mathrm{d}p + g_{qq}\,(\mathrm{d}q)^2 \\ \left(1+4c^2p^2\right)(\mathrm{d}p)^2 + 4cp\,\mathrm{d}p\,\mathrm{d}q + (\mathrm{d}q)^2 &= g_{pp}\,(\mathrm{d}p)^2 + 2g_{pq}\,\mathrm{d}p\,\mathrm{d}q + g_{qq}\,(\mathrm{d}q)^2 \\ g_{pp} &= 1 + 4c^2p^2 \\ g_{pq} &= 2cp \\ g_{qq} &= 1 \end{aligned}$$

Putting this into a matrix representation of the tensor:

$$\boxed{M_{qp} = \begin{bmatrix} 1 & 2cp \\ 2cp & 1+4c^2p^2 \end{bmatrix}}$$

d) The primary takeaways is that **a)** the vectors are not <u>orthogonal</u> (this is from the fact that off diagonal elements are not zero) and **b)** that $\hat{e}_p$ is not normalized (a.k.a. the basis is not ortho<u>normal</u>).