PHSX 425: HW11

William Jardee

December 3, 2021

In class, we solved the problem of relfection and refraction at oblique incidence to an interface between linear media, for the case of polarization in the plane of incidence.

Question 1

Derive E_{0R} and E_{0T} for the case of polarization perpendicular to the plane of incidence. Write your answer in terms of E_{0I}, α, and β.

$$\varepsilon_1(\mathbf{E}_{0I}+\mathbf{E}_{0R})_z = \varepsilon_2(\mathbf{E}_{0T})_z \tag{9.101 i}$$

$$(\mathbf{B}_{0I}+\mathbf{B}_{0R})_z = (\mathbf{B}_{0T})_z \tag{9.101 ii}$$

$$(\mathbf{E}_{0I}+\mathbf{E}_{0R})_{x,y} = (\mathbf{E}_{0T})_{x,y} \tag{9.101 iii}$$

$$\frac{1}{\mu_1}(\mathbf{B}_{0I}+\mathbf{B}_{0R})_{x,y} = \frac{1}{\mu_2}(\mathbf{B}_{0T})_{x,y} \tag{9.101 iv}$$

If we take the convention that is used in Griffiths, then the plane of incidence is in the xz-plane, and so the polarization of our EM wave will be perpendicular to this plane. So, the $\mathbf{E}$ doesn't have any components in the z-direction. So, equation 9.101 i becomes: $0 = 0$, and equation 9.101 iii becomes

$$\mathbf{E}_{0I}+\mathbf{E}_{0R} = \mathbf{E}_{0T} \tag{1}$$

To tackle the other two equation we have to recognize that $\frac{1}{v}(\mathbf{k}\times\mathbf{E}) = \mathbf{B}$. Equation 9.101 ii becomes:

$$(\mathbf{B}_{0I}+\mathbf{B}_{0R})_z = (\mathbf{B}_{0T})_z$$

$$\left(\frac{1}{v_1}\mathbf{E}_{0I}+\frac{1}{v_1}\mathbf{0R}\right)_z = \left(\frac{1}{v_2}\mathbf{E}_{0T}\right)_z$$

$$E_{0I}\sin(\theta_I) + E_{0R}\sin(\theta_R) = \frac{v_1}{v_2}E_{0T}\sin(\theta_T)$$

$$E_{0I} + E_{0R} = \frac{v_1}{v_2}\frac{\sin(\theta_T)}{\sin(\theta_R)}E_{0T} \tag{2}$$

Using a similar route, equation 9.101 iv becomes

$$\frac{1}{\mu_1}(\mathbf{B}_{0I} + \mathbf{B}_{0R})_{x,y} = \frac{1}{\mu_2}(\mathbf{B}_{0T})_{x,y}$$

$$\frac{1}{\mu_1}(B_{0I}(-\cos(\theta_I) + B_{0R}\cos(\theta_R)) = \frac{1}{\mu_2}(B_{0T}(-\cos(\theta_T))$$

$$\frac{1}{\mu_1}\frac{1}{v_1}\cos(\theta_I)(-E_{0I} + E_{0R}) = -\frac{1}{\mu_2}\frac{1}{v_2}E_{0T}\cos(\theta_T)$$

$$-E_{0I} + E_{0R} = -\frac{\mu_1 v_1}{\mu_2 v_2}\frac{\cos(\theta_R)}{\theta_I}E_{0T} \tag{3}$$

Using α and β to simplify the expressions, such that they are equation to:

$$\alpha = \frac{\cos(\theta_T)}{\cos(\theta_R)} \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

adding together (2) and (3)

$$2E_{0I} = (1 + \alpha\beta)E_{0T}$$

$$\boxed{E_{0T} = \frac{2}{1 + \alpha\beta}E_{0I}}$$

plugging this into (1)

$$E_{0I} + E_{0R} = \frac{2}{1 + \alpha\beta}E_0 I$$

$$\boxed{E_{0R} = \frac{2 - 1 - \alpha\beta}{1 + \alpha\beta}E_{0I} = \frac{1 - \alpha\beta}{1 + \alpha\beta}E_{0I}}$$

Question 2

Also derive the reflection and transmission coefficients. Show that they sum to unity.
We know that $I = \frac{1}{2}v\varepsilon E^2$, so for the transmission:[1]

$$T = \frac{I_T}{I_I} = \frac{v_2\varepsilon_2}{v_1\varepsilon_1}\frac{\cos(\theta_T)}{\cos(\theta_R)}\Big(\frac{E_T^2}{E_I^2}\Big)$$

$$\boxed{T = \beta\alpha\Big(\frac{2}{1+\alpha\beta}\Big)^2}$$

And for reflection:

$$R = \frac{I_R}{I_I} = \frac{v_1\varepsilon_1}{v_1\varepsilon_1}\frac{\cos(\theta_R)}{\cos(\theta_T)}\Big(\frac{E_R^2}{E_T^2}\Big)$$

$$\boxed{R = \Big(\frac{1-\alpha\beta}{1+\alpha\beta}\Big)^2}$$

Checking that these sum to 1:

$$\begin{aligned} T + R &= \beta\alpha\Big(\frac{2}{1+\alpha\beta}\Big)^2 + \Big(\frac{1-\alpha\beta}{1+\alpha\beta}\Big)^2 \\ &= \frac{4\alpha\beta + 1 - 2\alpha\beta + (\alpha\beta)^2}{(1+\alpha\beta)^2} \\ &= \frac{1 + 2\alpha\beta + (\alpha\beta)^2}{(1+\alpha\beta)^2} \\ &= \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1 \checkmark \end{aligned}$$

Question 3

Using the values $n_1 = 1$, $n_2 = 1.5$, with $\mu_1 = \mu_2 = \mu_0$, plot E_{0R}/E_{0T} and E_{0T}/E_{0I} as a function of the incidence angle, θ_I.

Question 4

Also plot the reflection and transmission coefficients as a function of θ_I.

[1] I take for granted the term $\cos(\theta_T)/\cos(\theta_R)$ because it is explained in Griffiths. The basis of which is about the fact that the conservation of energy into the interface is dependent of the average rate the wave front hits the interface - which is at an angle.

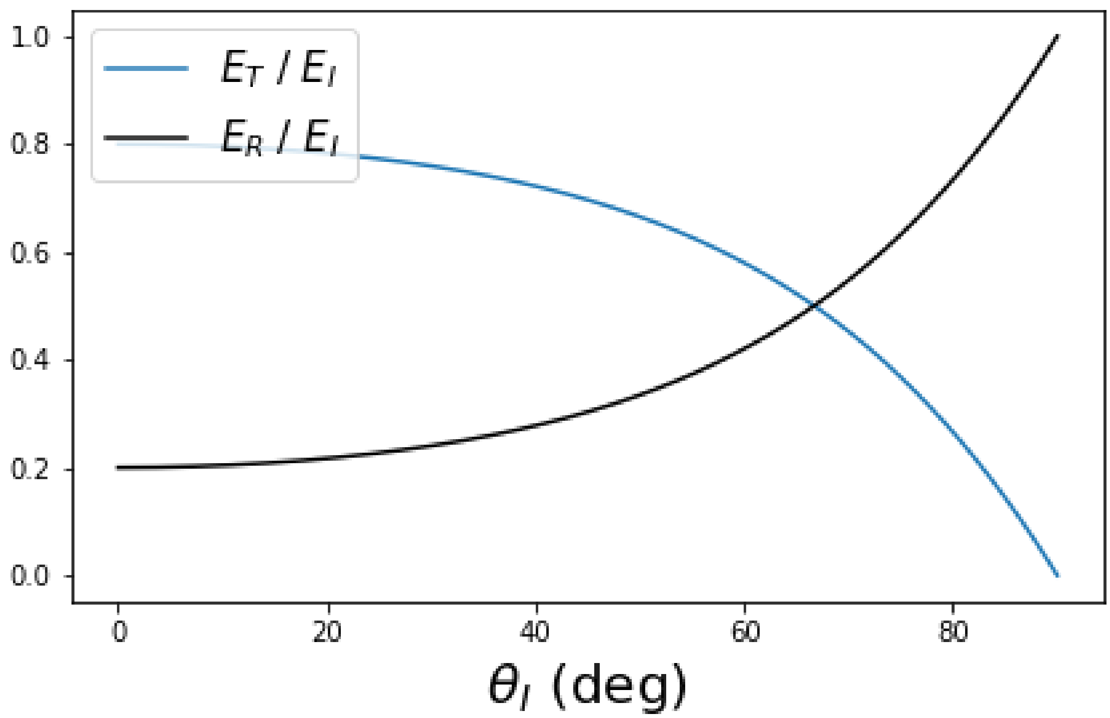

Figure 1: graph for **Question 3**

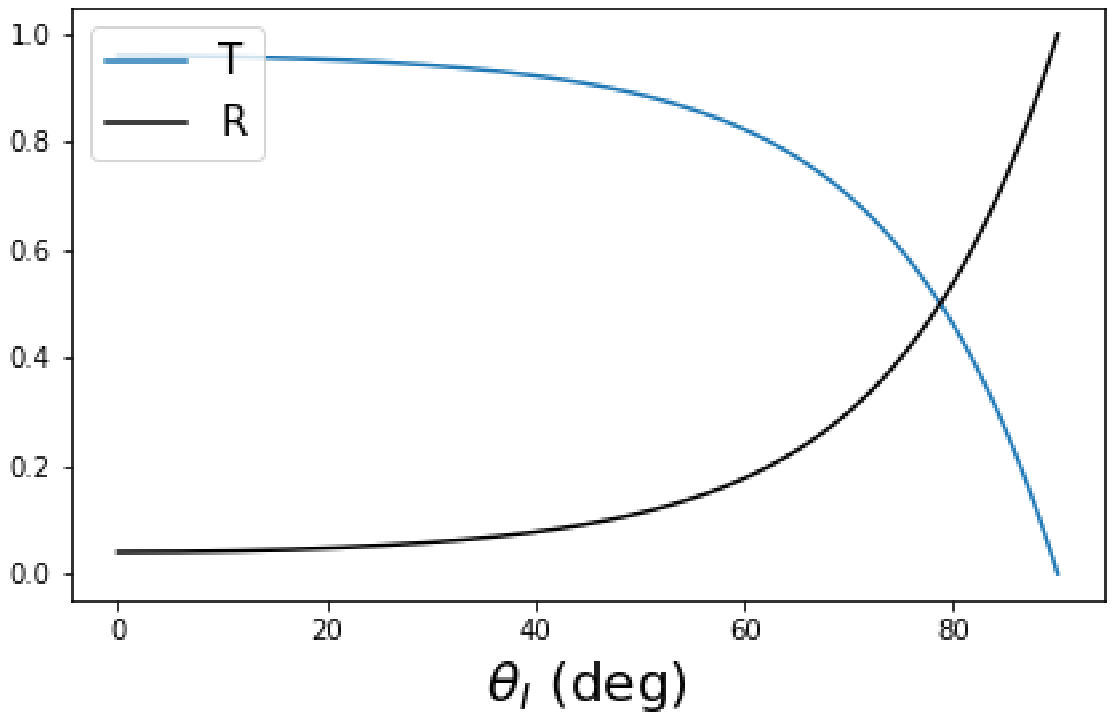

Figure 2: graph for **Question 4**

The code used to plot both graphs:

```
import astropy.units as u
import numpy as np
from astropy.constants import mu0
import matplotlib.pyplot as plt

n1 = 1
n2 = 1.5
mu1 = mu0
mu2 = mu0
beta = (mu1*n2)/(mu2*n1)
t_T = lambda t_I: (np.arcsin(n1 * np.sin(t_I)/n2)).to(u.deg)
alpha = lambda t_I: np.cos(t_T(t_I))/np.cos(t_I)
E_T = lambda t_I : abs(2./(1+alpha(t_I)*beta))
E_R = lambda t_I: abs((1-alpha(t_I)*beta)/(1+alpha(t_I)*beta))

T = lambda t_I: beta*alpha(t_I)*(E_T(t_I))**2
R = lambda t_I: (E_R(t_I))**2

x = np.linspace(0, 90, 1000)*u.deg
plt.plot(x, E_T(x), label=r'$E_T$ / $E_I$')
plt.plot(x, E_R(x), color='black', label=r'$E_R$ / $E_I$')
plt.xlabel(r"$\theta_I$ (deg)", fontsize=20)
plt.legend(fontsize=15, loc='upper left')
plt.tight_layout()
plt.savefig('phsx425_hw11_3')
plt.show()

x = np.linspace(0, 90, 1000)*u.deg
plt.plot(x, T(x), label=r'T')
plt.plot(x, R(x), color='black', label=r'R')
plt.xlabel(r"$\theta_I$ (deg)", fontsize=20)
plt.legend(fontsize=15, loc='upper left')
plt.tight_layout()
plt.savefig('phsx425_hw11_4')
plt.show()
```