PHSX 491: HW06

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Question 1

a) My understanding of this idea is still being developed, but the route that I've been using to justify to myself is that in a flat-spacetime: $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. So, if we maximize the negative portion of the differential then we minimize the total differential. Thus, we want $\mathrm{d}t$ to be as large as possible. This explanation doesn't quite sit right with me, however.

b) Let's just start with the definition of proper time and justify everything at the end.

$$\begin{aligned}
\Delta\tau &= \int_0^1 \mathrm{d}\sigma \sqrt{-g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\sigma}\frac{\mathrm{d}t}{\mathrm{d}\sigma} - g_{xx}\frac{\mathrm{d}x}{\mathrm{d}\sigma}\frac{\mathrm{d}x}{\mathrm{d}\sigma} - g_{yy}\frac{\mathrm{d}y}{\mathrm{d}\sigma}\frac{\mathrm{d}y}{\mathrm{d}\sigma} - g_{zz}\frac{\mathrm{d}z}{\mathrm{d}\sigma}\frac{\mathrm{d}z}{\mathrm{d}\sigma}} \\
&= \int_0^1 \mathrm{d}\sigma \left(\frac{\mathrm{d}t}{\mathrm{d}\sigma}\right) \sqrt{\left(\frac{\mathrm{d}t}{\mathrm{d}\sigma}\right)^2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}x}{\mathrm{d}\sigma}\right)^2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}\sigma}\right)^2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}\sigma}\right)^2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2} \\
&= \int_{t_A}^{t_B} \mathrm{d}t \sqrt{\left(\frac{\mathrm{d}t}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \\
&\boxed{= \int_{t_A}^{t_B} \mathrm{d}t \sqrt{1 - v_x^2}}
\end{aligned}$$

Notice when we changed the integral from $\mathrm{d}\sigma$ to $\mathrm{d}t$, we also had to change the bounds to the starting and stopping t to match. It was also given that $v_y = v_z = 0$, so we could drop the y and z terms. Finally, we are justified by multiplying the right side by $\frac{\mathrm{d}t}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$ because that is just 1.

c) When we take a small perturbation we are changing the path, but still need to keep the same end points. Thus: $x^\mu_{\text{pert}}(0) = x^\mu(0)$ and $x^\mu_{\text{pert}}(1) = x^\mu(1)$. This means that

$$\boxed{f^\mu(0) = f^\mu(1) = 0}$$

d) Taking $x^\mu_{\text{pert}}(\sigma) = x^\mu(\sigma) + \varepsilon f^\mu(\sigma)$ and $\dot{x}^\mu_{\text{pert}}(\sigma) = \dot{x}^\mu(\sigma) + \varepsilon \dot{f}^\mu(\sigma)$, we can use the first order Taylor expansion

$$f(x,y) \approx f(x_0,\, y_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0,\, y_0}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{x_0, y_0}(y - y_0)$$

to say

$$\boxed{\mathcal{L}\left(x^\mu_{\text{pert}}, \dot{x}^\mu_{\text{pert}}\right) \approx \mathcal{L}\left(x^\mu, \dot{x}^\mu\right) + \left.\frac{\partial \mathcal{L}}{\partial x^\mu}\right|_{x^\mu,\, \dot{x}^\mu}\left(\varepsilon f^\mu(\sigma)\right) + \left.\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right|_{x^\mu,\, \dot{x}^\mu}\left(\varepsilon \dot{f}^\mu(\sigma)\right)}$$

e) Starting with the original definition of proper time, then "upgrading" it to the perturbed path:

$$\begin{aligned}
\Delta\tau &= \int_0^1 \mathcal{L}\left(x^\mu(\sigma), \dot{x}^\mu(\sigma)\right) \mathrm{d}\sigma \\
&\to \int_0^1 \mathcal{L}\left(x^\mu_{\text{pert}}, \dot{x}^\mu_{\text{pert}}\right) \mathrm{d}\sigma \\
&= \int_0^1 \left[\mathcal{L}\left(x^\mu, \dot{x}^\mu\right) + \left.\frac{\partial\mathcal{L}}{\partial x^\mu}\right|_{x^\mu,\, \dot{x}^\mu} \left(\varepsilon f^\mu(\sigma)\right) + \left.\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\right|_{x^\mu,\, \dot{x}^\mu} \left(\varepsilon \dot{f}^\mu(\sigma)\right)\right] \mathrm{d}\sigma \\
&= \boxed{\left[\int_0^1 \mathcal{L}\left(x^\mu, \dot{x}^\mu\right) \mathrm{d}\sigma\right] + \varepsilon\left[\int_0^1 \left.\frac{\partial\mathcal{L}}{\partial x^\mu}\right|_{x^\mu,\, \dot{x}^\mu} \left(f^\mu(\sigma)\right) + \left.\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\right|_{x^\mu,\, \dot{x}^\mu} \left(\dot{f}^\mu(\sigma)\right) \mathrm{d}\sigma\right]}
\end{aligned}$$

From this point on I will be dropping the evaluation bar and the clarification of where $\mathcal{L}$ is evaluated at. It should be assumed that, when not otherwise stated, they are evaluated at x^μ and $\dot{x}^\mu$.

f) We need to show that $\int_0^1 \left[\frac{\partial\mathcal{L}}{\partial x^\alpha} f^\mu + \frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha} \dot{f}^\mu\right] \mathrm{d}\sigma = \int_0^1 f^\mu \left[\frac{\partial\mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha}\right)\right] \mathrm{d}\sigma$. It might be helpful to remember that $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}v$.

$$\begin{aligned}
\int_0^1 \frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha} \dot{f}^\mu &= \left.\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha} f^\mu\right|_0^1 - \int_0^1 f^\mu \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha}\right) \mathrm{d}\sigma \\
&= \left.\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha}\right|_1 (0) - \left.\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha}\right|_0 (0) - \int_0^1 f^\mu \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha}\right) \mathrm{d}\sigma \\
&= -\int_0^1 f^\mu \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha}\right) \mathrm{d}\sigma
\end{aligned}$$

Substituting this into the left side of the equation above:

$$\begin{aligned}
\int_0^1 \left[\frac{\partial\mathcal{L}}{\partial x^\alpha} f^\mu + \frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha} \dot{f}^\mu\right] \mathrm{d}\sigma &= \int_0^1 \left[\frac{\partial\mathcal{L}}{\partial x^\alpha} f^\mu - f^\mu \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha}\right)\right] \mathrm{d}\sigma \\
&\boxed{= \int_0^1 f^\mu \left[\frac{\partial\mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha}\right)\right] \mathrm{d}\sigma}
\end{aligned}$$

g) To find the maxima/minima/saddles of a function, we have to take the derivative with respect to the independent variable in mind and set it equal to zero. We are trying to find the ε that minimizes τ, so it makes sense to look for $\frac{\mathrm{d}\tau}{\mathrm{d}\varepsilon} = 0$. I'm not sure what the $\varepsilon \to 0$ is eluding to, other than the fact that we have assumed a small perturbation from the start (so that the Taylor expansion was valid).

h) We are going to do this by staring with our equation above, then taking the derivative with respect to ε:

$$
\begin{aligned}
\Delta\tau &= \int_0^1 \mathcal{L}\,\mathrm{d}\sigma + \varepsilon \int_0^1 f^\mu \left[\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right)\right] \mathrm{d}\sigma \\
\frac{\mathrm{d}\Delta\tau}{\mathrm{d}\varepsilon} = 0 &= \cancelto{0}{\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_0^1 \mathcal{L}\,\mathrm{d}\sigma} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varepsilon \int_0^1 f^\mu \left[\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right)\right] \mathrm{d}\sigma \\
\frac{\mathrm{d}}{\mathrm{d}\sigma}(0) &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\int_0^1 f^\mu \left[\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right)\right] \mathrm{d}\sigma \\
\frac{1}{f^\mu}\,0 &= f^\mu \left[\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right)\right] \frac{1}{f^\mu} \\
0 &= \frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right)
\end{aligned}
$$

$$
\boxed{\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) = 0}
$$

i) We will simplify each component individually, then combine the two. But, before doing that I will show that $1 = \frac{\mathrm{d}\tau}{\mathrm{d}\tau} = \frac{\mathrm{d}\tau}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} \to 1/\mathcal{L} = 1/\frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = \frac{\mathrm{d}\sigma}{\mathrm{d}\tau}$, since $\tau = \int \mathcal{L}\,\mathrm{d}\sigma$.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} &= \frac{\partial}{\partial \dot{x}^\alpha}\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} \\
&= \frac{1}{2\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\left[-g_{\mu\nu}\frac{\partial}{\partial \dot{x}^\alpha}\left(\dot{x}^\mu\dot{x}^\nu\right)\right] \\
&= \frac{-g_{\mu\nu}}{2\mathcal{L}}\left[\delta^\mu_\alpha\dot{x}^\nu + \delta^\nu_\beta\dot{x}^\mu\right] \\
&= \frac{-1}{2\mathcal{L}}\left[\left(\delta^\mu_\alpha g_{\mu\nu}\right)\dot{x}^\nu + \left(\delta^\nu_\beta g_{\mu\nu}\right)\dot{x}^\mu\right] \\
&= \frac{-1}{2\mathcal{L}}\left[g_{\alpha\nu}\dot{x}^\nu + g_{\mu\beta}\dot{x}^\mu\right] && \text{(using our delta rules)} \\
&= \frac{-1}{2\mathcal{L}}\left[g_{\nu\alpha}\dot{x}^\nu + g_{\mu\beta}\dot{x}^\mu\right] && \text{(because the metric is symmetric)} \\
&= \frac{-1}{2\mathcal{L}}\left[g_{\mu\alpha}\dot{x}^\mu + g_{\mu\alpha}\dot{x}^\mu\right] && \text{(using our renaming rules)} \\
&= \frac{-1}{\mathcal{L}}\left[g_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\right] \\
&= -\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\left[g_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\right] \\
&= \boxed{-g_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}}
\end{aligned}
$$

For this next one, we need to remember that when we defined the Lagrangian we declared x to be independent of $\dot{x}$.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\alpha} &= \frac{\partial}{\partial x^\alpha}\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} \\
&= \frac{1}{2\sqrt{-g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\frac{\partial}{\partial x^\alpha}\left[-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right] \\
&= \frac{-\dot{x}^\mu\dot{x}^\nu}{2\mathcal{L}}\frac{\partial}{\partial x^\alpha}g_{\mu\nu} \\
&= \left[\partial_\alpha g_{\mu\nu}\right]\frac{1}{2\mathcal{L}}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \\
&= \frac{1}{2}\left[\partial_\alpha g_{\mu\nu}\right]\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \\
&= \boxed{\frac{1}{2}\left[\partial_\alpha g_{\mu\nu}\right]\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}
\end{aligned}$$

Substituting both of these back into the equation:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) \\
&= -\frac{1}{2}\left[\partial_\alpha g_{\mu\nu}\right]\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} + \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(g_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) \\
\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\cdot(0) &= -\frac{1}{2}\left[\partial_\alpha g_{\mu\nu}\right]\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} + \frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(g_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right] - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right] - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0} \quad \checkmark$$