PHSX 491: HW03

William Jardee

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Question 1

$$\Phi(x, y) = bxy$$

a) Find the components of the gradient for Φ in Cartesian coordinates $\partial_\mu \Phi$.

$\partial_\mu \Phi \rightarrow \left[\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right] \Phi = by\hat{x} + bx\hat{y}$

$$\boxed{\begin{aligned} A_x &= by \\ A_y &= bx \end{aligned}}$$

b) Convert Φ to the polar coordinates (r, θ).

We can use the relationships: $x = r\cos(\theta)$ and $y = r\sin(\theta)$

$$\boxed{\Phi(r, \theta) = br^2 \cos(\theta)\sin(\theta)}$$

c) Calculate $\partial_r \Phi$ and $\partial_\theta \Phi$.

To this one right, I figured out that we have to use the gradient in polar coordinates:

$$\partial_\mu \rightarrow \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}$$

Using this, and grouping terms, we get:

$$\boxed{\begin{aligned} A_r &= 2br\cos(\theta)\sin(\theta) \\ A_\theta &= br^2[\cos^2(\theta) - \sin^2(\theta)] \end{aligned}}$$

d) If we transform the gradient from Cartesian to polar coordinates, do we get the components found above? That is, do the above components transform like a covector?

So, I see two ways to do this question. I will just go about both of them and hopefully satisfy that the gradient does transform like a covector:

$by\hat{x} + bx\hat{y} = br\sin(\theta)\hat{x} + br\cos(\theta)\hat{y}$

using the fact that $\hat{x} = \cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}$ and $\hat{y} = \sin(\theta)\hat{r} + \cos(\theta)\hat{\theta}$:

$$
\begin{aligned}
by\hat{x} + bx\hat{y} &= br\sin(\theta)[\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}] + br\cos(\theta)[\sin(\theta)\hat{r} + \cos(\theta)\hat{\theta}] \\
&= 2br\cos(\theta)\sin(\theta)\hat{r} + br[-\sin^2(\theta) + \cos^2(\theta)]\hat{\theta}
\end{aligned}
$$

So, by this method, it works. Now, let's do it with a more sophisticated approach:

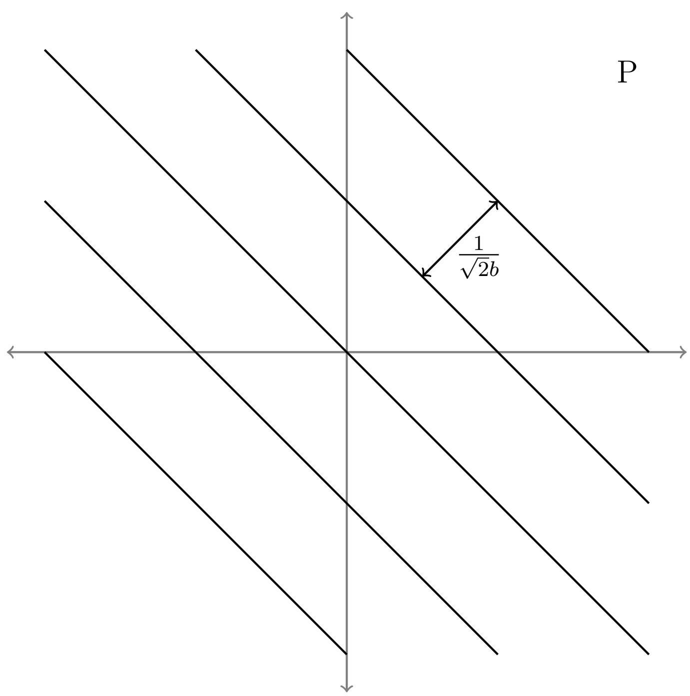
$$A_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} A_{\mu}$$

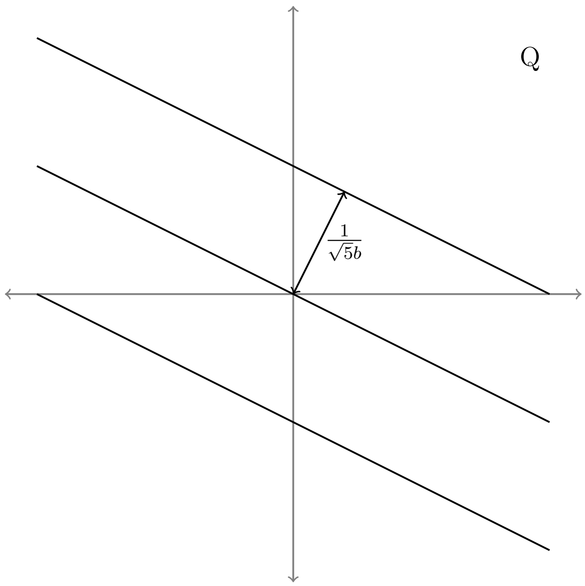
$$
\begin{aligned}
A_r &= \tfrac{\partial x}{\partial r} A_x + \tfrac{\partial y}{\partial r} A_y = \cos(\theta) br\sin(\theta) + \sin(\theta) br\cos(\theta) = 2br\cos(\theta)\sin(\theta) \\
A_\theta &= \tfrac{\partial x}{\partial \theta} A_x + \tfrac{\partial y}{\partial \theta} A_y = -\sin(\theta) br\sin(\theta) + \cos(\theta) br\cos(\theta) = br[\cos^2(\theta) - \sin^2(\theta)]
\end{aligned}
$$

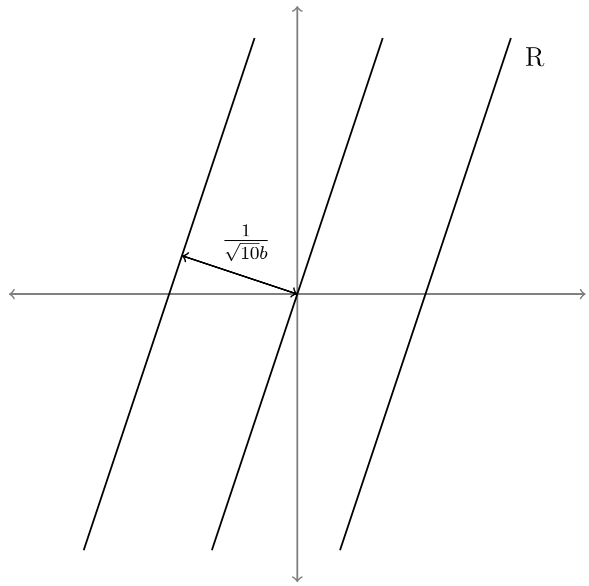
$$\boxed{\begin{aligned} A_r &= 2br\cos(\theta)\sin(\theta) \\ A_\theta &= br^2[\cos^2(\theta) - \sin^2(\theta)] \end{aligned}}$$

So, it looks that the gradient does transform like a covector! (*Phew!*)

e) Consider the points $P = (1, 1)$, $Q = (-2, -1)$, and $R = (1, -3)$. How do the collection of surfaces described by the gradient behave at these points?







I tried to do the graphing in $\LaTeX$, so it is a bit scuffed. However, I think it kinda worked.

For point P: The gradient turns into the set $A_x = b$, $A_y = b$. So, the covector will be perpendicular to this, with a slope -1.

For point Q: The gradient turns into the set $A_x = -b$, $A_y = -2b$. So, the covector will have the slope $-1/2$.

For point R: The gradient turns into the set $A_x = -3b$, $A_y = b$. So, the covector will have the slope 3.

	Point	∂_μ	Slope	Density
increasing $\downarrow$	P	$[b, b]$	-1	$\sqrt{2}b$
	Q	$[-b, -2b]$	$-\frac{1}{2}$	$\sqrt{5}b$
	R	$[-3b, b]$	3	$\sqrt{10}b$