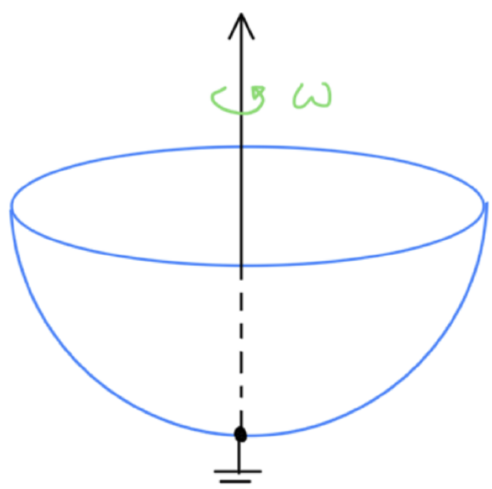
PHSX 425, Exam 02

William Jardee

October 28, 2021

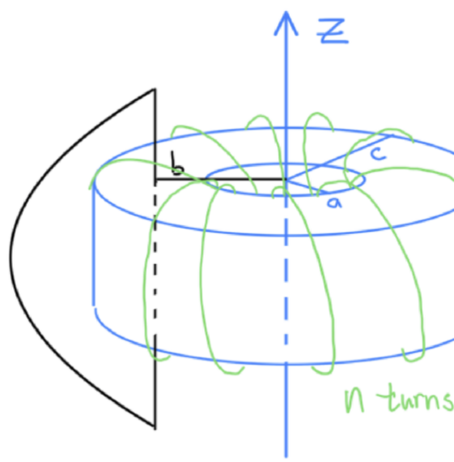
Question 1:

An empty, hemispherical, stainless steel salad bowl of radius R is electrically grounded at its point of contact with the table. The bowl spins with angular velocity ω about this point of contact, in a uniform vertical magnetic field of strength B. Find the elctrical potential at the rim of the bowl.



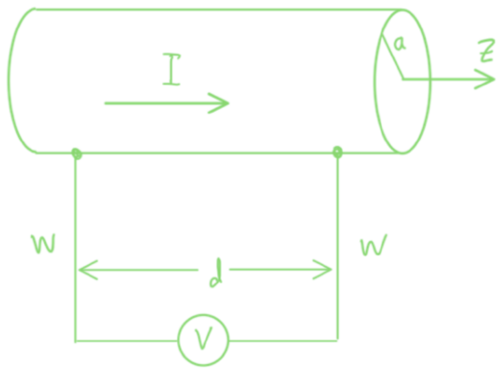
Question 2:

A toroidal coil with rectangular cross section has n turns, with inner radius a and outer radius c as shown. It is wound on a form of non-magnetic material, $\mu = \mu_0$. At $s = b\,(a < b < c)$, an insulated wire is threaded between the windings of the coil, parallel to the z-axis. The rest of this wire is formed into a continuous loop, which closes outside the toroid. Find teh mutual inductance between the wire loop and the coil.



Question 3:

A high-impedance voltmeter is connected to a wire of radius a as shown. Assume that the current I in the wire is uniformly distributed.



a) *If I is constant, and the wire's resistivity is k, what voltage is measured?*

The first step I took was to calculate the mutual inductance, that way I could check if there is any change in the voltage due to an EMF.

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$B \cdot 2\pi s = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi s}$$

$$\begin{aligned}\Phi &= \int_a^{a+w} \frac{\mu_0 I}{2\pi s} d\,\mathrm{d}s \\ &= \frac{\mu_0 I d}{2\pi} \ln\left(\frac{a+w}{a}\right) \\ &= \frac{\mu_0 I d}{2\pi} \ln\left(1 + \frac{w}{a}\right)\end{aligned}$$

$$M = \frac{\mu_0 d}{2\pi} \ln\left(1 + \frac{w}{a}\right) \tag{1}$$

Since $\mathcal{E} = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = 0$, so the only voltage measured is from the Ohms Law.

$$\Delta V = \frac{I}{k\pi a^2}$$

b) *Find the measure voltage $V(t)$ if the current is $I = I_0 e^{i\omega t}$. Assume that $w\omega \ll c$, where c is the speed of light; why is this important?*

We have two sources, both pointed out in the part a.

$$\Delta V = \frac{I}{k\pi a^2} - \mathcal{E}$$

$$\frac{I_0 e^{i\omega t}}{k\pi a^2} - \frac{\mu_0 d}{2\pi} \ln\left(1 + \frac{w}{a}\right) I_0 i\omega e^{i\omega t}$$

$$\Delta V = \left[\frac{1}{k\pi a^2} - i\omega\frac{\mu_0 d}{2\pi} \ln\left(1 + \frac{w}{a}\right)\right] I_0 e^{i\omega t}$$

It is important that $w\omega \ll c$ because otherwise the speed that the electrons would have to near/overcome would be the speed of light. And, as we all know from our time with special relativity, lengths and time scales get weird when we near $\frac{1}{10}c$. Especially since we haven't dealt this this concept formally in class, we wouldn't know exactly how to deal with it.[1]

[1] It's an interesting thought, however. My first guess would be that there would be a battle between time dilation and length contraction going on. I'm not confident to say whether this idea is actually important, but, thinking about electrons skipping over atoms entirely in a pseudo runaway event. At this point we would be dealing with a plasma situation and I don't think simple wire setup like this is built to deal with that.

Question 4:

At $t = 0$, an infinite conducting medium has free charge distribution $\rho_0(\boldsymbol{r}, 0)$. The medium is also a linear dielectric, with dielectric constant ε_r, but $\mu = \mu_0$.

a) *Find the free and bound charge densities as a function of time.*

Let's first time the $\rho_b(\mathbf{r}, 0)$, then I will provide an argument for how these two initial densities evolve with time. We know in a linear dielectric that

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = -\boldsymbol{\nabla} \cdot \varepsilon_0 \frac{\chi_e}{\varepsilon} \mathbf{D} = -\Big(\frac{\chi_e}{1 + \chi_e}\Big)\rho_f = \frac{1 - \varepsilon_r}{\varepsilon_r}\rho_f$$

$$\rho_b(\mathbf{r}, 0) = \frac{1 - \varepsilon_r}{\varepsilon_r}\rho_f(\mathbf{r}, 0)$$

We already know that $\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0)e^{-t/k\varepsilon_0} = (\rho_f(\mathbf{r}, 0) + \rho_b(\mathbf{r}, 0))e^{-t/k\varepsilon_0}$. At any point in time $\rho_{\text{tot}} = \rho_f + \rho_b$, so the free and bound densities must evolve in time in the same way that the total charge density, so we can just say:

$$\rho_f(\mathbf{r}, t) = \rho_f(\mathbf{r}, 0)e^{-t/k\varepsilon_0}$$

$$\rho_b(\mathbf{r}, t) = \frac{1 - \varepsilon_r}{\varepsilon_r}\rho_f(\mathbf{r}, 0)e^{-t/k\varepsilon_0}$$

b) *How are the free and polarization currents ($\boldsymbol{J}_f, \boldsymbol{J}_p$) related to the charge densities?*

The first thing to note for this question is that $\mu = \mu_0(1 + \chi_m) = \mu_0 \rightarrow \chi_m = 0$. So, $\mathbf{M} = 0$ and

$$J_{\text{bound}} = \boldsymbol{\nabla} \times \mathbf{M} = 0$$

$$J = J_{\text{free}} + J_{\text{bound}} = J_{\text{free}}$$

So, since we know that $\mathrm{d}\rho/\mathrm{d}t = -\boldsymbol{\nabla} \cdot J$,

$$-\boldsymbol{\nabla} \cdot \mathbf{J} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big[\frac{1 - \varepsilon_r}{\varepsilon_r} + 1\Big]\rho_f(\mathbf{r}, 0)e^{-t/k\varepsilon_0}\Big)$$

$$= \Big[\frac{1}{\varepsilon_r}\Big]\rho_f(\mathbf{r}, 0)\Big(\frac{-\sigma}{\varepsilon_0}\Big)e^{-t/k\varepsilon_0}$$

$$= \frac{-\sigma}{\varepsilon_0\varepsilon_r}\rho_f(\mathbf{r}, 0)e^{-t/k\varepsilon_0}$$

So, altogether

$$\mathbf{J}_{\text{bound}} = \mathbf{0}$$

$$\boldsymbol{\nabla} \cdot \mathbf{J}_{\text{free}} = \frac{\sigma}{\varepsilon_0\varepsilon_r}\rho_f(\mathbf{r}, 0)e^{-t/k\varepsilon_0} \tag{2}$$

c) *Is there a resulting magnetic field?*

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla \cdot \mathbf{E} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho}{\varepsilon_0}$$

This gives the same result as 2. So, when calculating $\nabla \times \mathbf{B} = \mu_0\Big(\mathbf{J} + \varepsilon_0 \frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t}\Big)$, we get 0. Since $\nabla \cdot \mathbf{B} = 0$ as well. So, put together:

$$\mathbf{B} = 0$$

d) *Do your results agree with all four Maxwell's equations in matter, as given in Griffiths?* From Griffiths:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{3}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$
$$\nabla \times \mathbf{H} = \mu_0\Big(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\Big) \tag{6}$$

To save being redundant, we used 3 to justify calculating ρ_b, so it must be consistent. We built $\mathbf{B} = 0$ from combining 4 and 6[2], so those are consistent. Finally, we can recognize that there is no source for a curl to $\mathbf{E}$, so it is good that is equal to zero.

[2]when we realize that $\mathbf{M} = 0 \to \mu_0\mathbf{H} = \mathbf{B}$, then we get back to the equation that we used initially, which is one of the normal Maxwell's equations