Modern Homework 25

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Nice work.

1. $\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (U(x) - E)\psi$ $\qquad$ $\frac{d^2}{dx^2}\psi = \frac{2m}{\hbar}[U(x) - E]\psi$

+4

$[P_x] = -i\hbar\frac{d}{dx}$

$\langle p^2\rangle = \int_{-\infty}^{\infty} \Psi^*[p]^2\Psi\, dx$

$= \int_{-\infty}^{\infty} \psi^*\left(i\hbar\frac{d}{dx}\right)^2\psi\, dx$

$= \int_{-\infty}^{\infty} \psi^*\left(-\hbar^2\frac{d^2}{dx^2}\right.\psi\, dx$

$= -\hbar^2\int_{-\infty}^{\infty}\psi^*\left[\frac{2m}{\hbar^2}\right][U(x) - E]\psi\, dx$

$= \cancel{-\hbar^2}\, -2m\int_{-\infty}^{\infty}\psi^*[U(x) - E]\psi\, dx$

$= 2m\int_{-\infty}^{\infty}\psi^*[E - U(x)]\psi\, dx$

$= \langle 2m[E - U(x)]\rangle$

2. $\Delta Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2}$

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$\langle Q\rangle = \int_{-\infty}^{\infty}\psi^*[Q]\psi\, dx$

$\langle p\rangle = \int_{-L/2}^{L/2}\sqrt{\frac{L}{2}}\sin\left(\frac{2\pi}{L}x\right)\left(-i\hbar\frac{d}{dx}\right)\left(\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi}{L}x\right)dx\right.$

$= \frac{2\hbar}{L}\int_{-L/2}^{L/2} -i\sin\left(\frac{2\pi}{L}x\right)\left(\frac{2\pi}{L}\right)\cos\left(\frac{2\pi}{L}x\right)dx$

$= \frac{4\hbar\pi}{L^2}\int_{-L/2}^{L/2} -i\,\frac{1}{2}\sin\left(\frac{4\pi}{L}x\right)dx \xrightarrow{\text{odd}} 0$

$\langle p^2\rangle = \int_{-L/2}^{L/2}\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi}{L}x\right)\left(-\hbar^2\frac{d^2}{dx^2}\right)\left(\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi}{L}x\right)\right)dx$

$= -\frac{2\hbar^2}{L}\int_{-L/2}^{L/2}\sin\left(\frac{2\pi}{L}x\right)\left(-\frac{4\pi^2}{L^2}\right)\sin\left(\frac{2\pi}{L}x\right)dx$

$$\langle p^2\rangle = \frac{8\hbar^2\pi^2}{L^3}\int_{-L/2}^{L/2} \sin^2\left(\frac{2\pi}{L}x\right)dx$$

You could have used the relation you found in problem 1.

$$= \frac{8\hbar^2\pi^2}{L^3}\int_{-L/2}^{L/2}\left[\frac{1}{2}-\frac{1}{2}\cos\left(\frac{4\pi}{L}x\right)\right]dx$$

$$= \frac{4\pi^2\hbar^2}{L^3}\left[x-\frac{L}{4\pi}\sin\left(\frac{4\pi}{L}x\right)\right]\Bigg|_{-L/2}^{L/2}$$

$$= \frac{4\hbar^2\pi^2}{L^2}$$

$$\Delta P = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$

$$= \sqrt{\frac{4\hbar^2\pi^2}{L^2}} = 2\frac{\hbar\pi}{L}$$

$$\langle x\rangle = \int_{-L/2}^{L/2}\sin\left(\frac{2\pi}{L}x\right)x\sin\left(\frac{2\pi}{L}x\right)dx$$

$$= \int_{-L/2}^{L/2}x\sin^2\left(\frac{2\pi}{L}x\right)dx \xrightarrow{\text{odd}} 0$$

$$\langle x^2\rangle = \frac{2}{L}\int_{-L/2}^{L/2}\sin\left(\frac{2\pi}{L}x\right)x^2\sin\left(\frac{2\pi}{L}x\right)dx$$

$$= \frac{2}{L}\int_{-L/2}^{L/2}x^2\sin^2\left(\frac{2\pi}{L}x\right)dx$$

$$= \frac{2}{L}\int_{-L/2}^{L/2}x^2\left(\frac{1}{2}-\frac{1}{2}\cos\left(\frac{4\pi}{L}x\right)\right)dx \xrightarrow{\text{even}} \frac{L}{2}\int_{-L/2}^{L/2}\left(x^2-x^2\cos\left(\frac{4\pi}{L}x\right)\right)dx$$

Integration by parts time!

$$\langle x^2\rangle = \frac{2}{L}\left(\frac{1}{3}x^3\right)\Big|_0^{L/2} - \left[\frac{L}{4\pi}x^2\sin\left(\frac{4\pi}{L}x\right)\Big|_0^{L/2} - \int_0^{L/2}\frac{1}{2}x\left(\frac{L}{4\pi}\right)\sin\left(\frac{4\pi}{L}x\right)dx\right]$$

$$= \frac{2}{L}\left(\frac{1}{3}x^3\right)\Big|_0^{L/2} - \left[0+\frac{1}{2}x\left(\frac{L^2}{16\pi^2}\right)\cos\left(\frac{4\pi}{L}x\right)\Big|_0^{L/2} - \int_0^{L/2}\frac{1}{2}\left(\frac{L^2}{16\pi^2}\right)\cos\left(\frac{4\pi}{L}x\right)dx\right]$$

$$= \frac{2}{L}\left(\frac{1}{3}\left(\frac{L^3}{8}\right)\right) - \left[\frac{L^3}{64\pi^2} - \frac{1}{2}\left(\frac{L^3}{64\pi^3}\right)\sin\left(\frac{4\pi}{L}x\right)\Big|_0^{L/2}\right]$$

$$= \frac{2}{L}\left(\frac{1}{3}\left(\frac{L^3}{8}\right) - \frac{L^3}{64\pi^2}\right) = \frac{2}{L}\left(\frac{L^3}{24}-\frac{L^3}{64\pi^2}\right) = \frac{2}{L}\left(L^3\left(\frac{1}{24}-\frac{1}{64\pi^3}\right)\right)$$

$$= L^2\left(\frac{1}{12}-\frac{1}{32}\pi^2\right)$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{L}{2}\sqrt{\frac{1}{3} - \frac{1}{8\pi^2}}$$

$$\Delta x \Delta p = \frac{L}{2}\sqrt{\frac{1}{3} - \frac{1}{8\pi^2}}\,\frac{\hbar\pi}{L}\,2$$

$$= \underbrace{\sqrt{\frac{\pi^2}{3} - \frac{1}{8}}}_{>1}\,\hbar$$

$$\text{so} \quad \Delta x \Delta p > \hbar \geq \frac{\hbar}{2}$$

Idk, did you actually calculate it?

3. a) $\omega \equiv [\frac{1}{s}]$ $\quad m \equiv [kg]$ $\quad \hbar \equiv [\frac{kg\,m^2}{s}]$ $\quad P \equiv [\frac{kg\,m}{s}]$

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$$[m] = \left([\tfrac{kg\,m^2}{s}][\tfrac{1}{kg}][s]\right)^{1/2} \Rightarrow \left(\frac{\hbar}{m\omega}\right)^{1/2}$$

$$\left[\frac{kg\,m}{s}\right] = \left([\tfrac{kg\,m^2}{s}][kg][\tfrac{1}{s}]\right)^{1/2} \Rightarrow (\hbar m\omega)^{1/2}$$

b) $$\langle p\rangle = \left(\frac{1}{2\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\right)\int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar}(2x)\left(-i\hbar\frac{d}{dx}\right)\left(e^{-m\omega x^2/2\hbar}\right)(2x)\,dx$$

$$= \left(\frac{1}{2\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\right)\int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar}(-i\hbar 2x)\left(2e^{-m\omega^2/2\hbar} - \frac{2x^2}{2\hbar}m\omega\, e^{-m\omega x^2/2\hbar}\right)dx$$

$$= \left(\frac{1}{2\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\right)\int_{-\infty}^{\infty} -4i\hbar\left(x - \frac{x^3 m\omega}{\hbar}\right)e^{-m\omega x^2/\hbar}\,dx$$

$$\rightarrow \text{odd} \qquad = 0$$

$$\langle p^2\rangle = \langle 2m(E-U(x))\rangle = \langle 2m(\tfrac{3}{2}\hbar m - \tfrac{1}{2}m\omega^2x^2)\rangle$$

Same as with last problem, there's an easier way.

$$= 2m\left(\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\right)\int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar}\, x\left(\tfrac{3}{2}\hbar\omega - \tfrac{1}{2}m\omega^2x^2\right)e^{-m\omega x^2/2\hbar}\,dx$$

$$= \frac{2m}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\int_{-\infty}^{\infty}(3\hbar\omega - m\omega^2x^2)\,x^2 e^{-m\omega x^2/\hbar}\,dx$$

$$= \frac{4m}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\left[\int_0^{\infty} 3\hbar\omega x^2 e^{-m\omega x^2/\hbar}\,dx - m\omega^2\int_0^{\infty} x^4 e^{-m\omega x^2/\hbar}\,dx\right]$$

$$= \frac{4m}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\left[3\hbar\omega\left(\tfrac{1}{4}\sqrt{\pi}\,\lambda^{-3/2}\right) - m\omega^2\left[\tfrac{3}{8}\sqrt{\pi}\,\lambda^{-5/2}\right]\right]$$

$$= \frac{3}{2}m\left(\frac{m\omega}{\hbar}\right)^{3/2}\left[2\hbar\omega\left(\frac{\hbar}{m\omega}\right)^{3/2} - m\omega^2\left(\frac{\hbar}{m\omega}\right)^{5/2}\right]$$

$$= \frac{3}{2}m\left(2\hbar\omega - m\omega^2\left(\frac{\hbar}{m\omega}\right)\right)$$

$$\langle p^2 \rangle = \frac{3}{2}(2\hbar m\omega - m\omega\hbar) = \frac{3}{2}(\hbar m\omega)$$

$$\Delta p = \sqrt{\left(\frac{3}{2}\hbar m\omega\right) - 0} = \sqrt{\left(\frac{3}{2}\right)(\hbar m\omega)}$$ units work

$$\langle x \rangle = \left(\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\right)\int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar}\, x^3\, dx \xrightarrow{\text{odd}} 0$$

$$\langle x^2 \rangle = \left(\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\right)\int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar}\, x^4\, dx$$

$$\overset{\text{even}}{=} \left(\frac{4}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\right)\int_{0}^{\infty} e^{-m\omega x^2/\hbar}\, x^4\, dx$$

$$= \left(\frac{4}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}\right)\left(\frac{3}{8}\sqrt{\pi}\left(\frac{m\omega}{\hbar}\right)^{-5/2}\right)$$

$$= \frac{3}{2}\left(\frac{\hbar}{m\omega}\right)$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\left(\frac{3}{2}\right)\left(\frac{\hbar}{m\omega}\right)}$$

$$\boxed{\Delta p \Delta x = \sqrt{\left(\frac{3}{2}\right)(\hbar\omega m)}\sqrt{\left(\frac{3}{2}\right)\left(\frac{\hbar}{m\omega}\right)} = \frac{3}{2}\hbar \geq \frac{\hbar}{2}}$$

c) $\langle U(x) \rangle = \frac{1}{2}m\omega^2 \langle x^2 \rangle = \frac{1}{2}m\omega^2\left(\frac{3}{2}\frac{\hbar}{m\omega}\right) = \frac{3}{4}\omega\hbar$

$\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2m}\langle p^2 \rangle = \frac{1}{2m}\left(\frac{3}{2}\hbar m\omega\right) = \frac{3}{4}\omega\hbar$ $\quad \langle U(x) \rangle = \left\langle \frac{p^2}{2m} \right\rangle$

Classically?

4. a) $\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left(\frac{\hbar}{i}\right)\frac{d}{dx}\psi\, dx$

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As we discussed in class, Ae^{iqx} is not normalizable. For this reason, ψ cannot be written as a probability density $|\psi|^2$ in deriving $\langle p \rangle$ eq. the $|\psi|^2$ ideal was fundamental. No $|\psi|^2$, No $\langle p \rangle$ eq.

b) $\psi(x) = Ae^{i\alpha x}$

$$P\psi(x) = -i\hbar\frac{d}{dx}\psi(x)$$

$$= -i\hbar(i\alpha)\psi(x)$$

$P = \alpha\hbar$ this is the only possible solution.

c) $\psi(x) = Be^{-i\beta x}$

$$P\psi(x) = -i\hbar \frac{d}{dx}\psi(x)$$

$$= -i\hbar(-i\beta)\psi(x)$$

$P = -\beta\hbar$ this is the only possible solution

D) $$\langle p \rangle = \frac{1}{N}\sum_i n_i t_i$$

$$= \sum_i \frac{n_i}{N} t_i$$

$$= \sum_i P_i t_i$$

all straight forward

* as shown with the tails-head example $\langle T \rangle = P_{heads} - P_{tails}$

$$= P_A(\alpha\hbar) - P_B(\beta\hbar)$$

$\cancel{= \frac{1}{2}(\alpha - \beta)\hbar}$

~~Both are 100%, and A ≠ B, so they have equal chance to happen, so $P_A = P_B = \frac{1}{2}$~~

Argh, the scratch work is killing me.

$\cancel{P(\psi) = -i\hbar \frac{d}{dx}\psi(x)}$

$\cancel{P = -i\hbar(\alpha i - i\beta)}$

$\cancel{P = \hbar(\alpha - \beta)}$

$\cancel{P(Ae^{i\alpha x} + Be^{-i\beta x}) = -i\hbar(i\alpha Ae^{i\alpha x} - i\beta Be^{-i\beta x})}$

$\cancel{P(A+B}$ $\cancel{P(Ae^{\alpha} + Be^{-\beta}) = (A\hbar\alpha e^{\alpha} - \hbar\beta Be^{-\beta})}$

$\cancel{P = \hbar(A\alpha e^{\alpha}}$

Close. $|A|^2$ and $|B|^2$ are the probabilities of measuring α and β, respectively. The expectation then is the sum of the probabilities times the values.

Complete (logical?) guess:

$$P_A = \frac{|A|^2}{|A|^2 + |B|^2} \qquad P_B = \frac{|B|^2}{|A|^2 + |B|^2}$$

So $$\langle p \rangle = \frac{|A|^2}{|A|^2 + |B|^2}(\alpha\hbar) - \frac{|B|^2}{|A|^2 + |B|^2}(\beta\hbar)$$