# PHSX 425, HW 09

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## Question 1

*consider the 3D scalar wave equation*

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

*Find the general solution $f(x, y, x, t)$, in terms of complex exponentials, by separation of variables.*

Let's start by saying that $f$ is seperable, that is

$$f(x, y, x, t) = f_x(x)f_y(y)f_z(z)f_t(t)$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] f_x(x)f_y(y)f_z(z)f_t(t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} f_x(x)f_y(y)f_z(z)f_t(t)$$

$$f_t\left[f_y f_z \frac{\partial^2 f_x}{\partial x^2} + f_x f_z \frac{\partial^2 f_y}{\partial y^2} + f_x f_y \frac{\partial^2 f_z}{\partial z^2}\right] = f_x f_y f_z \frac{\partial^2 f_t}{\partial t^2}$$

$$\frac{1}{f_x}\frac{\partial^2 f_x}{\partial x^2} + \frac{1}{f_y}\frac{\partial^2 f_y}{\partial y^2} + \frac{1}{f_z}\frac{\partial^2 f_z}{\partial z^2} = \frac{1}{v^2}\frac{1}{f_t}\frac{\partial^2 f_t}{\partial t^2}$$

Since the left and right side sides have to be equal for all spacial and temporal setups, let us say that they both have to be $k^2$, where $k \in \mathbb{C}$. Let's tackle the right hand side first.

$$\frac{1}{v^2}\frac{1}{f_t}\frac{\partial^2 f_t}{\partial t^2} = k^2$$

$$\frac{\partial^2 f_t}{\partial t^2} = k^2 v^2 f_t$$

$$\Rightarrow f_t(t) = A_t e^{kvt} + B_t e^{-kvt}$$

Now to catch the right side.

$$\frac{1}{f_x}\frac{\partial^2 f_x}{\partial x^2} + \frac{1}{f_y}\frac{\partial^2 f_y}{\partial y^2} + \frac{1}{f_z}\frac{\partial^2 f_z}{\partial z^2} = k^2$$

Since this has to be true for all $x$, $y$, and $z$, then let's just say that

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

The derivation then becomes identical for all three:

$$\frac{1}{f_x}\frac{\partial^2 f_x}{\partial x^2} = k_x^2$$

$$\frac{\partial^2 f_x}{\partial x^2} = k_x^2 f_x$$

$$\Rightarrow f_x(x) = A_x e^{k_x x} + B_x e^{-k_x x}$$

Putting this all together to get one real big equation:

$$f(x,y,z,t) = \left(A_x e^{k_x x} + B_x e^{-k_x x}\right)\left(A_y e^{k_y y} + B_y e^{-k_y y}\right)\times \left(A_z e^{k_z z} + B_z e^{-k_z z}\right)\left(A_t e^{kvt} + B_t e^{-kvt}\right)$$

# Question 2

*A sound wave represented by* $\varpi_I = Ae^{ik(x-ut)}$ *is incident on a boundary at* $x = 0$*, where the speed speed changes abruptly:*

$$c_s = \left\{ \begin{array}{ll} u, & x < 0 \\ 2u, & x > 0 \end{array} \right\} \tag{1}$$

*The boundary conditions at* $x = 0$ *are continuity in both* $\varpi$ *and* $\partial\varpi/\partial x$*. Solve for the reflected and transmitted waves,* $\varpi_R$ *and* $\varpi_T$*.*

To not beat around the bush; we want to satisfy the boundary equations:

$$\varpi_I(0) + \varpi_R(0) = \varpi_T(0)$$

$$\left.\frac{\partial\varpi_I}{\partial x}\right|_0 + \left.\frac{\partial\varpi_R}{\partial x}\right|_0 = \left.\frac{\partial\varpi_T}{\partial x}\right|_0$$

Where

$$\varpi_R(x,t) = Be^{ik(-x-ut)}$$

$$\varpi_T(x,t) = Ce^{ik'(x-2ut)}$$

So, the first boundary condition becomes:

$$Ae^{-ikut} + Be^{-ikut} = Ce^{-ik'2ut}$$

Since this has to be true for all $t$, we can get

$$A + B = C$$

For the second boundary condition:

$$Aike^{-ikut} - Bike^{-ikut} = Cik'e^{-ik'2ut}$$

$$\Rightarrow Aik - Bik = Cik'$$

$k$ represents our wavenumber. So, if we double the wave speed then our wave number is going to get cut in half. So, $k'_k = 2$

$$A + B = \frac{k'}{k}C = \frac{1}{2}C$$

By combining equations:

$$2A = \frac{3}{2}C \longrightarrow C = \frac{4}{3}A$$

$$B = \frac{1}{3}A$$

$$\boxed{\varpi_R(x,t) = \frac{1}{3}Ae^{ik(-x-ut)}}$$

$$\boxed{\varpi_T(x,t) = \frac{4}{3}Ae^{ik(x/2-ut)}}$$

# Griffiths 9.9

*Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude $E_0$, frequency $\omega$, and phase angle zero that is (a) traveling in the negative $x$ direction and polarized in the $z$ direction; (b) traveling in the direction from the origin to the point $(1,1,1)$, with polarization parallel to the $xz$ plane. In each case, sketch the wave, and give the explicit Cartesian components of* $\mathbf{k}$ *and* $\hat{\mathbf{n}}$

We can use the general equations that

$$\mathbf{E} = E_0 \hat{n} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\mathbf{B} = \frac{1}{c} E_0 (\hat{k} \times \hat{n}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

a)

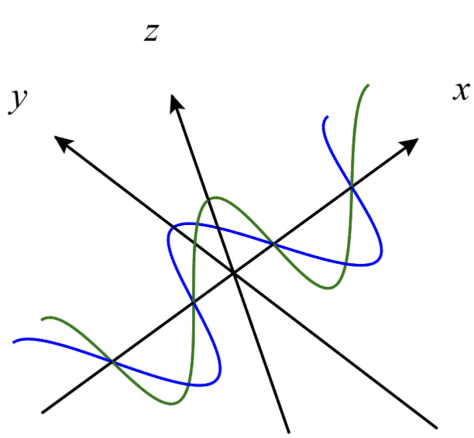
$$\hat{k} = -\hat{x} \qquad \hat{n} = \hat{z}$$

$$\boxed{\mathbf{E} = E_0 e^{i(-x-\omega t)}\ \hat{z}}$$

$$\mathbf{B} = \frac{1}{c} E_0 (-\hat{x} \times \hat{z}) e^{i(-x-\omega t)}$$

$$\boxed{\mathbf{B} = \frac{1}{c} E_0 e^{i(-x-\omega t)} \hat{y}}$$

Figure 1: EM wave for question 9.9 part a. Green: E-field. Blue: B-field



b)

$$\hat{k} = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z}) \qquad \hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{z})$$

$$\boxed{\mathbf{E} = \frac{1}{\sqrt{2}} E_0 e^{i((x+y+z)/\sqrt{3} - \omega t)} \, (\hat{x} - \hat{z})}$$

$$\mathbf{B} = \frac{1}{c} E_0 \Big( \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z}) \times \frac{1}{\sqrt{2}}(\hat{x} - \hat{z}) \Big) e^{i((x+y+z)/\sqrt{3} - \omega t)}$$

Quick side note to do that cross-product:

$$(\hat{x} + \hat{y} + \hat{z})) \times (\hat{x} - \hat{z})$$

$$= [(\hat{x} \times \hat{x}) + (\hat{y} \times \hat{x}) + (\hat{z} \times \hat{x})] - [(\hat{x} \times \hat{z}) + (\hat{y} \times \hat{z}) + (\hat{z} \times \hat{z})]$$

$$= (-\hat{z} + \hat{y}) - (\hat{y} + \hat{x})$$

$$= (-\hat{z} - \hat{x}) = -(\hat{z} + \hat{x})$$

Back to the derivation:

$$\boxed{\mathbf{B} = -\frac{1}{\sqrt{6}\, c} E_0 e^{i((x+y+z)/\sqrt{3} - \omega t)} \, (\hat{x} + \hat{z})}$$

Figure 2: EM wave for question 9.9 part b. This one is quick a bit more tricky to draw. the idea is that the Poynting vector is in the (1, 1, 1) direction and the E and B parts oare in the proper planes orthogonal to each other.