PHSX 491: HW05

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Question 1

a)

$$
\begin{aligned}
\Delta\tau &= \int \sqrt{-\,\mathrm{d}s^2} \\
&= \int \sqrt{-\left[-\left(1-\frac{r_s}{R}\right)\mathrm{d}t^2 + \left(1-\frac{r_s}{R}\right)^{-1}\mathrm{d}r^2 + R^2\,\mathrm{d}\theta^2 + R^2\,\mathrm{d}\phi^2\right]} \\
&= \sqrt{1-\frac{r_s}{R}}\int \mathrm{d}t \\
&= \left(1-\frac{r_s}{R}\right)^{1/2}\Delta t
\end{aligned}
$$

b) At $R \to \infty$, the time goes to $1^{1/2}\,\Delta t = \Delta t$. As the radius get's larger and larger the time step get's closer to 1. A.k.a. the closer to the event horizon, the slower the time, since the coefficient is always less than 1.

c) We are doing actual calculations, so we need to get some numbers to plug in.

$$
G = 6.6704\times 10^{-11}\,\mathrm{m}^2/\mathrm{kg\,s}^2 \qquad\qquad c = 3.00\times 10^{8}\,\mathrm{m/s}
$$

using this:

$$
\Delta\tau = \sqrt{1-\frac{2GM}{c^2R}} \longrightarrow \qquad \Delta\tau(\alpha) \approx \sqrt{1-\frac{1.48\times 10^{-27}}{\alpha}}\Delta t
$$

$$
\boxed{
\begin{aligned}
\Delta\tau\left(\tfrac{5}{2}\right) &= \sqrt{1-1.93\times 10^{-28}}\Delta t \\
\Delta\tau(3) &= \sqrt{1-4.94\times 10^{-28}}\Delta t \\
\Delta\tau(6) &= \sqrt{1-2.47\times 10^{-28}}\Delta t \\
\Delta\tau(100) &= \sqrt{1-1.48\times 10^{-29}}\Delta t \\
\Delta\tau(1000) &= \sqrt{1-1.48\times 10^{-30}}\Delta t
\end{aligned}
}
$$

But, this is quite unwieldy. So, let's do this analysis by letting $G = c = 1$:

$$
\boxed{
\begin{aligned}
\Delta\tau\left(\tfrac{5}{2}\right) &= 0.775\Delta t &&= 0.775 \text{ hours} \\
\Delta\tau(3) &= 0.816\Delta t &&= 0.816 \text{ hours} \\
\Delta\tau(6) &= 0.913\Delta t &&= 0.913 \text{ hours} \\
\Delta\tau(100) &= 0.995\Delta t &&= 0.995 \text{ hours} \\
\Delta\tau(1000) &= 0.999\Delta t &&= 0.999 \text{ hours}
\end{aligned}
}
$$

d) So, time flows the slowest closer to r_s.

e) Inside the Schwarzchild radius the radical turns negative and thus we have an imaginary time. What does this mean? Isn't that a great question...

f) See Fig 1
My answers to the previous part doesn't change at all. In fact, graphing it was how I justified my answers to myself.

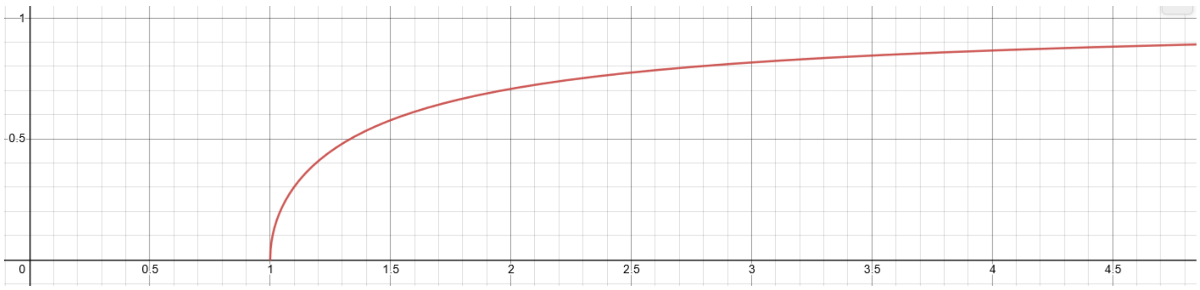

Figure 1: A qualitative graph of g_{tt}. Here $r_s = 1$, so the plot is effectively $\sqrt{1 - \dfrac{1}{x}}$.