PHSX 491: HW02

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Thanks for the feedback on the last assignment about the margins. I realize that for homework the margin might have been a little bigger than ideal. I appreciate the feedback, so please keep them coming!

The objective of this homework is show that

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

by using the Levi-Civita symbol and index notation.

a) Show that $[\vec{A} \times \vec{B}]_i = \varepsilon_{ijk} A^j B^k$:

$$\begin{aligned}
[\vec{A} \times \vec{B}] &= [A_y B_z - A_z B_y]\hat{x} - [A_x B_z - A_z B_x]\hat{y} + [A_x B_y - A_y B_x]\hat{z} \\
[\vec{A} \times \vec{B}]_x &= A_y B_z - A_z B_y \\
&= \varepsilon_{xyz} A_y B_z + \varepsilon_{xzy} A_z B_y + \cancelto{0}{\varepsilon_{xxy} A_x B_y} + \cancelto{0}{\ldots} \\
&= \varepsilon_{xjk} A^j B^k
\end{aligned}$$

using a very similar logic:

$$\begin{aligned}
[\vec{A} \times \vec{B}]_y &= A_z B_x - A_x B_z \\
&= \varepsilon_{yzx} A_z B_x + \varepsilon_{yxz} A_x B_z + \cancelto{0}{\ldots} \\
&= \varepsilon_{yjk} A^j B^k \\
[\vec{A} \times \vec{B}]_z &= \varepsilon_{zjk} A^j B^k
\end{aligned}$$

So, putting all three of these together:

$$\boxed{[\vec{A} \times \vec{B}]_i = \varepsilon_{ijk} A^j B^k}$$

b) Show that $\varepsilon^{123} = 1$:

We know that there is the rule $g^{ij} A_i = A^j$, where g is the metric of our space. For regular, Cartesian space:

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Jumping to the conclusion:

$$\varepsilon^{123} = g^{33}\varepsilon^{12}{}_{3} = g^{33}g^{22}\varepsilon^{1}{}_{23} = g^{33}g^{22}g^{11}\varepsilon_{123}$$
$$= 1 \cdot 1 \cdot 1 \cdot \varepsilon_{123} = 1 \quad \checkmark$$

c) Write down the ith component of $\vec{A} \times (\vec{B} \times \vec{C})$ using index notation.:

$$\vec{A} \times [\vec{B} \times \vec{C}]_i = \varepsilon_{ijk} A^j [\vec{B} \times \vec{C}]^k$$
$$g_{kk}g_{jj}g^{kk}g^{ii}g^{jj}\vec{A} \times [\vec{B} \times \vec{C}]_i = g_{kk}g_{jj}g^{kk}g^{ii}g^{jj}\varepsilon^{ijk} A^j [\vec{B} \times \vec{C}]^k$$
$$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \vec{A} \times [\vec{B} \times \vec{C}]^i = \varepsilon^{ijk} A_j [\vec{B} \times \vec{C}]_k$$
$$\vec{A} \times [\vec{B} \times \vec{C}]^i = \varepsilon^{ijk} A_j \varepsilon_{kmn} B^m C^n$$
$$= \varepsilon^{kij}\varepsilon_{kmn} A_j B^m C^n$$

$$\boxed{[\vec{A} \times \vec{B} \times \vec{C}]^i = \varepsilon^{kij}\varepsilon_{kmn} A_j B^m C^n}$$

d) Write down the ith component of $(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ using index notation.:

$$\boxed{[(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}]^i = A^\alpha C_\alpha B_i - A^\alpha B_\alpha C_i}$$

e) Finish out showing that the two sides are equal:

Let's pick back up the left side:

$$[\vec{A} \times \vec{B} \times \vec{C}]^i = \varepsilon^{kij}\varepsilon_{kmn} A_j B^m C^n$$
$$= [\delta^i_m \delta^j_n - \delta^j_m \delta^i_n]\, A_j B^m C^n$$
$$= \delta^i_m \delta^j_n A_j B^m C^n - \delta^j_m \delta^i_n A_j B^m C^n$$
$$= A_n C^n B^i - A_m B^m C^i$$
$$g_{ii}[\vec{A} \times \vec{B} \times \vec{C}]^i = g_{ii}[A_n C^n B^i - A_m B^m C^i]$$
$$= A_\alpha B^\alpha B_i - A_\alpha B^\alpha C_i$$
$$= [(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}]^i$$

Thus:

$$\boxed{\vec{A} \times [\vec{B} \times \vec{C}] = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}}$$