PHSX 462: HW07

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Question 1

We provided

$$\mathrm{d}N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}\,\mathrm{d}E$$

in class, so let's start with that.

$$\begin{aligned}
E_{\text{total}} &= \int_0^N E\,\mathrm{d}N \rightarrow \int_0^{E_F} E\left[\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}\,\mathrm{d}E\right] \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{E_F} E^{3/2}\,\mathrm{d}E \\
&= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{5}\left(\frac{\hbar^2}{2m}\left(\frac{3N\pi^2}{V}\right)^{2/3}\right)^{5/2} \\
&= \frac{\hbar^2}{10\pi^2 m}\frac{(3\pi^2 N)^{5/3}}{V^{-2/3}} \qquad \checkmark
\end{aligned}$$

Griffiths 5.21

a) We know that $E_F = \frac{\hbar^2}{2m}\left(3\rho\pi^2\right)^{2/3}$. The density, ρ, here is the density of electron; so, it will be

$$\rho = 8.96\frac{\text{g}}{\text{cm}^3}\cdot\frac{1}{63.5}\frac{\text{mole}}{\text{g}} \approx 0.1411\frac{\text{mole}}{\text{cm}^3} \approx \boxed{8.4974\times10^{28}\,\frac{\text{electrons}}{\text{m}^3}}.$$

Using the value of $m = 0.511\text{Mev}/\text{c}^2$ and $\hbar = 6.582\times10^{-16}\text{eV}\,\text{s}$ the calculation yields

$$\boxed{E_F \approx 7.842\times10^{-17}\,\text{eV}}.$$

b) We can assume that there is no potential energy, and these are non-relativistic particles, so

$$E_F = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{2E}{m}} \approx 1.752\times10^{-11}\,c \approx \boxed{5.256\times10^{-3}\frac{\text{m}}{\text{s}}}.$$

c) This is as simple as evaluating

$$T = \frac{E_F}{k_B} \approx \boxed{9.0998\times10^{-13}\,\text{K}}.$$

d) Again, just “plug and chug”:

$$P = \frac{\left(3\pi^2\right)^{2/3}\hbar^2}{5m}\rho^{5/3} \approx \boxed{2.3988\times10^{29}\frac{\text{eV}}{\text{m}^3}}.$$

Griffiths 5.30

We need to start by describing the wave function of our 2D well as

$$\psi_{n_x\,n_y} = \sqrt{\frac{4}{A}}\sin(k_x\,x)\sin(k_y\,y).$$

With a corresponding energy $E_k = \dfrac{\hbar^2|\mathbf{k}|^2}{2m}$, where $\mathbf{k} = k_x\hat{x} + k_y\hat{y}$ is the wavenumber. Considering this as a problem in a 2D wavenumber space, opposed to a quantum number space, the area spanned by each wave number is

$$\frac{\pi}{l_x}\cdot\frac{\pi}{l_y} = \frac{\pi^2}{A},$$

where l_x and l_y are the length of the well in x and y directions, respectively.

The total area covered by the ground state, in the wavenumber space, is $A_F = \dfrac{1}{4}\left(\pi k_F\right)^2$, where k_F is the largest wavenumber reached, relating to the Fermi energy, etc. The total number of electron "squares" needed will be

$$N_{\text{electrons}} = 2\cdot N_{\text{squares}} = 2\,\frac{A_F}{(\text{area per cube})} = 2\,\frac{\frac{1}{4}\left(\pi k_F\right)^2}{\frac{\pi^2}{A}} = \frac{Ak_F^2}{2\pi}.$$

Solving this for the wavenumber of the Fermi energy gives

$$k_F = \sqrt{\frac{2\pi N}{A}} = \sqrt{2\pi\sigma},$$

where σ is the density of states. The Fermi energy is then

$$E_F = \frac{\hbar^2}{2m}k_F^2 = \boxed{\frac{\hbar^2\pi}{m}\sigma}$$

Griffiths 5.35

a) Starting with the total energy of the system

$$E_{\text{total}} = \frac{\hbar^2 \left(3\pi^2 N\right)^{5/3}}{10\pi^2 m} V^{-2/3}$$

and substituting what we know, overloading some of the terms,

$$E_{\text{total}} = \frac{\hbar^2 \left(3\pi^2 N d\right)^{5/3}}{10\pi^2 m} \left(\frac{4}{3}\pi r^3\right)^{-2/3}.$$

Simplifying this dramatically yields

$$\boxed{E_{\text{total}} = \frac{\hbar^2}{10m}\left(\frac{3\pi}{4}\right)^{2/3} \frac{(3Nd)^{5/3}}{r^2}}.$$

b) Grabbing this equation from the all-knowing Google

$$W = -\frac{3}{5}\frac{G\,(M')^2}{R}.$$

This is the work it takes to create a dense sphere of radius R that has mass M. For our problem this looks like

$$\boxed{U = -\frac{3}{5}\frac{G(NM)^2}{R}}.$$

c) The equation for the total energy is

$$E(r) = \frac{\hbar^2}{10m}\left(\frac{3\pi}{4}\right)^{2/3} \frac{(3Nd)^{5/3}}{r^2} - \frac{3}{5}\frac{G(NM)^2}{r}.$$

Maximizing this, with respect to r gives

$$\frac{\mathrm{d}E}{\mathrm{d}r} = -\frac{\hbar}{5m}\left(\frac{3\pi}{4}\right)^{2/3} \frac{(3Nd)^{5/3}}{r^3} + \frac{3}{5}\frac{G(NM)^2}{r^2} = 0,$$

$$r = \frac{\frac{\hbar^2}{m}\left(\frac{3\pi}{4}\right)^{2/3}(3Nd)^{5/3}}{3G(NM)^2} = \boxed{\left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2 d^{5/3}}{GmM^2N^{1/3}}}.$$

Plugging in $m = 0.511\text{MeV}/c^2$, $M = 939\text{MeV}/c^2$, $d = 1/2$, $G = 6.674\times 10^{-11}\text{N m}^2/\text{kg}^2$ give $\boxed{R \approx 7.6\times 10^{25}\, N^{-1/3}\,\text{m}}$.

d) Using the mass of the sun, $M_{\text{Sun}} = 1.989\times 10^{30}\text{kg}$, and the fact that $N = M_{\text{Sun}}/M$,

$$\boxed{R \approx 7.16\times 10^6\text{m}}.$$

e) The Fermi energy is

$$E_F = \frac{\hbar^2}{2m}\left(3\rho\pi^2\right)^{2/3} = \frac{\hbar^2}{2m}\left(3\frac{N}{\frac{4}{3}\pi R^3}\pi^2\right)^{2/3}$$

where R and N are the same as the last part. Plugging in values we get

$$\boxed{E_F \approx 1.94\times 10^5\text{eV}}.$$

Griffiths 5.25

We want the lowest energy, so that will be the solution to

$$\cos(qa) = cos(ka) + \frac{m\alpha}{\hbar k}\sin(ka)$$

when the n embedded in q is zero. Since $q = \dfrac{2\pi n}{Na} \to 0$, and we are given that β (the jazz in front of the sin) is 10, the problem is really

$$1 = \cos(z) + 10\frac{\sin(z)}{z}$$

where $z = ka$. Using digital software, $z \approx 2.628$.

Given that $k = \dfrac{\sqrt{2mE}}{\hbar}$, $\beta = \dfrac{m\alpha a}{\hbar^2}$, $\beta = 10$, and $\alpha/a = 1$ eV, E can be solved for. Rearanging the β equation gives that

$$a^2 = a\frac{\beta\hbar^2}{m\alpha} = \frac{\beta\hbar^2}{m}\left(\frac{\alpha}{a}\right)^{-1} = \frac{10\hbar^2}{m}\ \text{eV}\,.$$

Solving for E from the k equation, and substituting what a^2 is gives

$$\begin{aligned} E &= \frac{z^2\hbar^2}{a^2}\frac{1}{2m} = \frac{z^2 h\hbar^2 m}{10\hbar^2 2m} = \frac{(2.628)^2}{20}\text{eV} \\ &\approx \boxed{0.345\ \text{eV}} \end{aligned}$$

Griffiths 5.26

We get back to the same differential equation as in the book;

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -\frac{2mE}{\hbar^2}\psi\,.$$

For the case when $E > 0$, the solution follows exactly as in the book. If we assume that $E < 0$, then the general solution is instead

$$\psi(x) = A\sinh(kx) + B\cosh(kx)\,.$$

Using Bloch's theorem, we know that if we move left a cell the solution can also be

$$\psi(x) = e^{-iqa}\left[A\sinh(kx) + B\cosh(kx)\right]\,.$$

At the point $x = 0$, these solution must be equivalent; that is

$$B = e^{-iqa}\left[A\sinh(ka) + B\cosh(ka)\right]\,.$$

Solving for A gives

$$A = \left[Be^{iqa} - B\cosh(ka)\right](\sinh(ka))^{-1}\,.$$

Following equation 2.128 in the book $\left(\Delta\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{2m\alpha}{\hbar^2}\psi(0)\right)$, we get

$$kA - e^{-iqa}k\left[A\cosh(ka) + B\sinh(ka)\right] = \frac{2m\alpha}{\hbar^2}B\,.$$

Substituting in the A that we got earlier, and doing some simplification gives

$$k\left[Be^{iqa} - B\cosh(ka)\right] - e^{-iqa}\left[\left(Be^{iqa} - B\cosh(ka)\right)\cosh(ka) + B\sinh^2(ka)\right] = \frac{2m\alpha}{\hbar^2}B\sinh(ka)\,,$$

$$k\left[e^{iqa} - 2\cosh(ka) + e^{-iqa}\right] = \frac{2m\alpha}{\hbar^2}\sinh(ka)\,,$$

$$\boxed{\cos(qa) = \frac{m\alpha}{\hbar^2 k}\sinh(ka) + \cosh(ka)}\,.$$

The negative solution is monotonically decreasing, so there is only one band that is added here. Just like with the positive bands, there will be a total of N intersections, and thus N states.

See next page for graph.

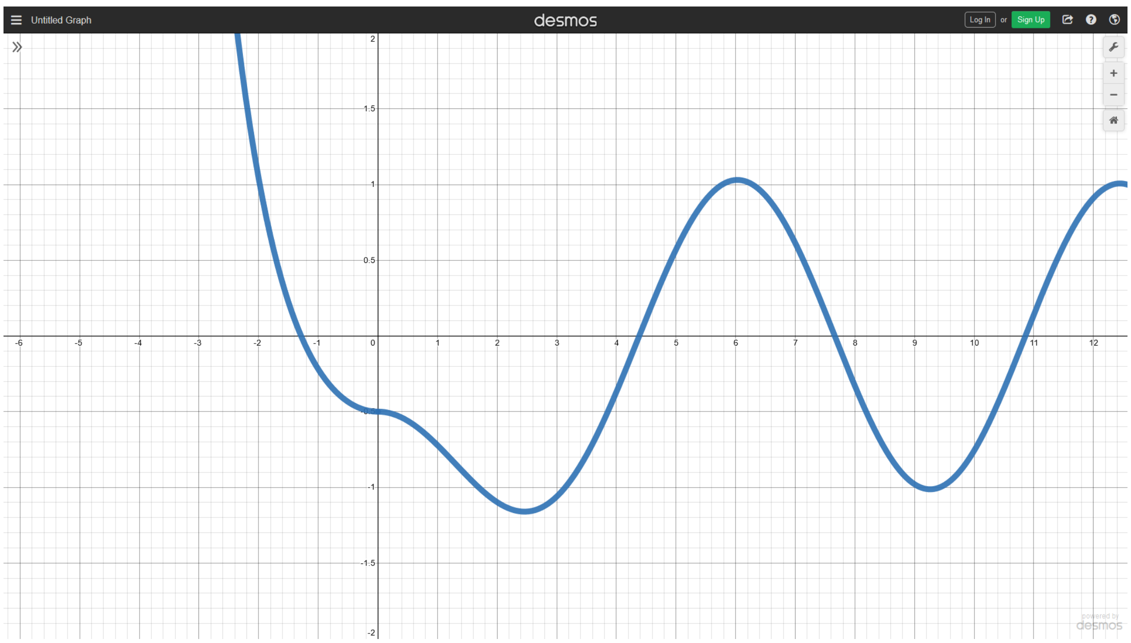

Figure 1: The plot for the dirac delta wells. The x-axis, which represents z, is plotted from -2π to 4π with negative z being the negative energy solution and positive the solution given in the book. The y-axis runs from -2 to 2.