Modern Homework 28 Will Jordee

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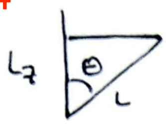
Excellent

1. +4



$$\cos\theta = \frac{L_z}{|L|} = \frac{m_\ell \hbar}{\sqrt{\ell(\ell+1)}\,\hbar} = \frac{m_\ell}{\sqrt{\ell(\ell+1)}}$$

θ_{min} is when cos is close as possible to 1, so $m_\ell = \ell$

$$\cos\theta = \frac{\ell}{\sqrt{\ell(\ell+1)}}$$

$$\boxed{\theta_{min} = \cos^{-1}\left(\frac{\ell}{\sqrt{\ell(\ell+1)}}\right)}$$

let us first use a triangle to write $\sin\theta_{min}$

$$\sin\theta_{min} = \frac{\sqrt{\ell(\ell+1)-\ell^2}}{\sqrt{\ell(\ell+1)}} = \frac{\sqrt{\ell}}{\sqrt{\ell(\ell+1)}} = \frac{1}{\sqrt{\ell+1}}$$

as $\ell \to \infty$, $\sin(\theta_{min}) \to \theta_{min}$ for small angle approx and as ℓ gets large, the 1 is negligable.

$$\theta_{min} \simeq \frac{1}{\sqrt{\ell+1}} \simeq \frac{1}{\sqrt{\ell}}$$

as ℓ gets very large we ~~get~~ approach a sharp value of θ and any value of θ is allowed, as classical physics would say!

2. +4 a) we know, from defining L, that $|\vec{L}|^2 = \ell(\ell+1)\hbar^2$

and that $L_z = m_\ell \hbar$

$$|\vec{L}|^2 = L_x^2 + L_y^2 + L_z^2 = \ell(\ell+1)\hbar^2$$

$$L_x^2 + L_y^2 = \ell(\ell+1)\hbar^2 - m_\ell^2\hbar^2$$

max when $m_\ell = 0$ min when $m_\ell = \pm 2$

$$\boxed{L_x^2 + L_y^2 = 6\hbar^2} \qquad \boxed{L_x^2 + L_y^2 = 2\hbar^2}$$

b) $\boxed{L_x^2 + L_y^2 = 5\hbar^2}$

No they cannot. If we know one, we know the other. this means all three components of the momentum are sharp and violates the heisenberg uncertainty

c) using the basic rule of quantum numbers $\ell < n$

so $\boxed{n \geq 3}$

3. $\langle r \rangle = \int_0^\infty \psi^* r \psi \, dr \qquad \psi = R_{10} Y_0^0 = \left(\frac{1}{a_0}\right)^{3/2} 2 e^{-r/a_0} \frac{1}{2\sqrt{\pi}}$

$$= \int_0^\infty \left(\frac{1}{a_0^3 \pi}\right)^{3/2} e^{-r/a_0} r \left(\frac{1}{a_0^3\pi}\right)^{1/2} e^{-r/a_0} dr$$

$$= \frac{1}{a_0^3\pi} \int_0^\infty r e^{-2r/a_0} dr$$

$$= \frac{1}{a_0^3\pi}\left[-\frac{a_0}{2} e^{-2r/a_0}\Big|_0^\infty + \int_0^\infty \frac{a_0}{2} e^{-2r/a_0} dr\right]$$

$$= \frac{1}{a_0^3\pi}\left[\frac{a_0}{2} - \left(\frac{a_0}{2}\right)^2 e^{-2r/a_0}\Big|_0^\infty\right]$$

$$= \frac{1}{a_0^3\pi}\left[\frac{a_0}{2} + \left(\frac{a_0}{2}\right)^2\right] =$$

Disregard. Very wrong.

So don't turn in scratch work. :(

(2)

3. $\langle r \rangle = \int_0^\infty \psi^* r \psi \, d\tau$ $\qquad \psi \Rightarrow R_{10} Y_0^0 = \left(\frac{1}{a_0}\right)^{3/2} 2 e^{-r/a_0} \frac{1}{2\pi}$

+4

$$= \int_0^\infty R_{10}^* \, r R_{10} \, r^2 dr \underbrace{\int Y_0^0 \sin\theta \, d\theta \, d\phi}_{\text{Convention says this is 1}}$$

$$= \int_0^\infty \left(\frac{1}{a_0}\right)^{3/2} 2 e^{-r/a_0} \, r^3 \left(\frac{1}{a_0}\right)^{3/2} 2 e^{-r/a_0} \, dr$$

$$= \left(\frac{4}{a_0^3}\right) \int_0^\infty r^3 e^{-2r/a_0} \, dr$$

$$= \left(\frac{4}{a_0^3}\right)\left(r^3 \left(-\frac{a_0}{2}\right) e^{-2r/a_0} + \left(\frac{a_0}{2}\right) \int_0^\infty 3r^2 \left(e^{-2r/a_0}\right) dr \right)$$

$$= \left(\frac{4}{a_0^3}\right)\left(r^3 \left(\frac{-a_0}{2}\right) e^{-2r/a_0} \Big|_0^\infty - \left(\frac{a_0}{2}\right)^2 3r^2 e^{-2r/a_0} \Big|_0^\infty + \int_0^\infty 6r \left(\frac{a_0}{2}\right)^2 e^{-2r/a_0} \, dr \right)$$

$$= \left(\frac{4}{a_0^3}\right)\left(r^3 \left(\frac{-a_0}{2}\right) e^{-2r/a_0} \Big|_0^\infty - \left(\frac{a_0}{2}\right)^2 3r^2 e^{-2r/a_0} \Big|_0^\infty - \left(\frac{a_0}{3}\right)^3 6r e^{-2r/a_0} + \int 6 \left(\frac{a_0}{2}\right)^3 e^{-2r/a_0} \, dr \right]$$

$$= \left(\frac{4}{a_0^3}\right)\left(r^3 \left(-\frac{a_0}{2}\right) e^{-2r/a_0} - \left(\frac{a_0}{2}\right)^2 3r^2 e^{-2r/a_0} - \left(\frac{a_0}{2}\right)^3 6r e^{-2r/a_0} - 6\left(\frac{a_0}{2}\right)^4 e^{-2r/a_0} \right) \Big|_0^\infty$$

$$= \left(\frac{4}{a_0^3}\right)\left(r^3 \left(\frac{-a_0}{2}\right) - (3r^2)\left(\frac{a_0}{2}\right)^2 - \left(\frac{a_0}{2}\right)^3 (6r) - 6\left(\frac{a_0}{2}\right)^4 \right) e^{-2r/a_0} \Big|_0^\infty$$

At this point I will use a solution from last homework and say $\lim \to \infty \Rightarrow 0$.

$$= -\left(\frac{4}{a_0^3}\right)\left(0 \left(-\frac{a_0}{2}\right) - 0 \left(\frac{a_0}{2}\right)^2 - \left(\frac{a_0}{2}\right)(0) - 6\left(\frac{a_0}{2}\right)^4 \right)$$

$$= \frac{3}{2} a_0 = \frac{3}{2 \cdot 1} a_0$$ for Hydrogen, $Z = 1$

You should include Z from the start.

$$\boxed{\langle r \rangle = \frac{3}{2Z} a_0}$$