PHSX 425, HW 07

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Question 1:

a) We are given

$$\rho(\mathbf{r},t) = \rho(\mathbf{r},0)e^{-\sigma t/\epsilon_0}$$

Thus,

$$\rho(\mathbf{r},0) = \frac{Q}{\frac{4}{3}\pi a^3}$$

$$\rho(\mathbf{r},t) = \frac{3Q}{4\pi a^3}e^{-\sigma t/\epsilon_0} \tag{1}$$

b) We can use Guass's Law to calculate the **E**

$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}\int_V \rho\, dV$$

$$E(4\pi r^2) = \frac{3Q}{4\pi a^3}\frac{4\pi r^3}{3\varepsilon_0}e^{-\sigma t/\varepsilon_0}$$

$$\mathbf{E}(\mathbf{r},t) = \left\{ \begin{array}{ll} \frac{Qr}{\varepsilon_0 4\pi a^3}e^{-\sigma t/\varepsilon_0}\hat{r} & r \leq a \\ \frac{Q}{\varepsilon_0 4\pi a^2}e^{-\sigma t/\varepsilon_0}\hat{r} & r \geq a \end{array} \right\} \text{[1]} \tag{2}$$

Using the fact that $\mathbf{J} = \sigma\mathbf{E}$:

$$\mathbf{J}(\mathbf{r},t) = \left\{ \begin{array}{ll} \frac{Q\sigma r}{\varepsilon_0 4\pi a^3}e^{-\sigma t/\varepsilon_0}\hat{r} & r \leq a \\ \frac{Q\sigma}{\varepsilon_0 4\pi a^2}e^{-\sigma t/\varepsilon_0}\hat{r} & r \geq a \end{array} \right\} \tag{3}$$

Finally, using $\nabla\times\mathbf{B} = \mu_0\Big(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\Big)$

$$\nabla\times\mathbf{B} = \left\{ \begin{array}{ll} \mu_0\Big(\frac{Q\sigma r}{\varepsilon_0 4\pi a^3}e^{-\sigma t/\varepsilon_0} + \varepsilon_0\frac{Qr}{\varepsilon_0 4\pi a^3}\frac{-\sigma}{\varepsilon_0}e^{-\sigma t/\varepsilon_0}\Big)\hat{r} & r \leq a \\ \mu_0\Big(\frac{Q\sigma}{\varepsilon_0 4\pi a^2}e^{-\sigma t/\varepsilon_0} + \varepsilon_0\frac{Q}{\varepsilon_0 4\pi a^2}\frac{-\sigma}{\varepsilon_0}e^{-\sigma t/\varepsilon_0}\Big)\hat{r} & r \geq a \end{array} \right\}$$

[1] Sorry that these aren't boxed. I tried to do that boxing that I have done in the past. However, with doing a piecewise it wasn't playing nice. If I do LaTeX solutions in the future I will get that figured out.

$$\nabla \times \mathbf{B} = 0 \tag{4}$$

We already know from Maxwell's equations that $\nabla \cdot \mathbf{B} = 0$. Putting these two together, we know that

$$\mathbf{B}(\mathbf{r}, t) = 0$$

c) **Checking Maxwell's Equations:**

- For $x \leq a$: $\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{Qr}{4\pi a^3 \varepsilon_0} e^{-\sigma t/\varepsilon_0}\right) = \frac{1}{r^2}\frac{3r^2}{4\pi a^3 \varepsilon_0} e^{-\sigma t/\varepsilon_0} = \frac{\rho}{\varepsilon_0}$

  For $x > a$: $\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial t}\left(\frac{Q}{4\pi\varepsilon_0}\right) = 0$
- $\nabla \cdot \mathbf{B} = \frac{1}{r\sin\theta}\frac{\partial \mathbf{B}_\phi}{\partial \phi} = 0$
- $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0$
- For $x \leq a$: $\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \mu_0\left(\frac{Q\sigma r}{\varepsilon_0 4\pi a^3} e^{-\sigma t/\varepsilon_0} + \varepsilon_0\left(\frac{-\sigma}{\varepsilon_0}\frac{Qr}{\varepsilon_0 4\pi a^3} e^{-\sigma t/\varepsilon_0}\right)\right) = 0$

  For $x < a$: $\nabla \times \mathbf{B} = \mu_0\left(\frac{Q\sigma}{\varepsilon_0 4\pi a^2} e^{-\sigma t/\varepsilon_0} + \varepsilon_0\left(\frac{-\sigma}{\varepsilon_0}\frac{Q}{\varepsilon_0 4\pi a^2} e^{-\sigma t/\varepsilon_0}\right)\right) = 0$

So, it seems that all our equations are consistent with Maxwell's Equations.

Question 2:

a) We know that $\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H})$. So, if $\mathbf{H} = 0$

$$\mathbf{B} = M\hat{z}$$

b) We can use the equation that $\mathbf{J} = en\mathbf{v}$; where e is the charge of an electron, n is number volume-density, and $\mathbf{v}$ is the average velocity. Then (remembering Eq. 3)

$$\mathbf{J} = en\mathbf{v}$$

$$\mathbf{v} = \frac{\mathbf{J}}{ne} = \left\{ \begin{array}{ll} \frac{Q\sigma r}{\varepsilon_0 4\pi a^3 en} e^{-\sigma t/\varepsilon_0}\hat{r} & r \leq a \\ \frac{Q\sigma}{\varepsilon_0 4\pi a^2 en} e^{-\sigma t/\varepsilon_0}\hat{r} & r \geq a \end{array} \right\} \tag{5}$$

c) Putting together Eq. 3 and Eq. 5 and recognizing that $\mathbf{v} \times \mathbf{B} = -|\mathbf{v}||\mathbf{B}|\sin(\theta)\hat{\phi}$

$$\mathbf{J} = \sigma \left\{ \begin{array}{ll} \frac{Q\sigma r}{\varepsilon_0 4\pi a^3} e^{-\sigma t/\varepsilon_0}\left(\hat{r} - \frac{M\sin(\theta)\hat{\phi}}{ne}\right) & r \leq a \\ \frac{Q\sigma}{\varepsilon_0 4\pi a^2} e^{-\sigma t/\varepsilon_0}\left(\hat{r} - \frac{M\sin(\theta)\hat{\phi}}{ne}\right) & r \geq a \end{array} \right\} \tag{6}$$