PHSX 462: HW05

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Griffiths 5.1

a) Let's start with what we are provided:

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 \qquad \vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

then, you know, do the thing:

$$\begin{aligned} \vec{R} &= \frac{(m_1(\vec{r} + \vec{r}_2) + m_2\vec{r}_2)}{m_1 + m_2} \\ &= \vec{r}_2 + \frac{m_1}{m_2 + m_1}\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{\mu}{m_2}\vec{r} \end{aligned} \qquad \begin{aligned} \vec{R} &= \frac{(m_2(\vec{r}_1 - \vec{r}) + \vec{r}_1 m_1)}{m_1 + m_2} \\ &= \vec{r}_1 - \frac{m_2}{m_1 + m_2}\vec{r} \\ \vec{r}_1 &= \vec{R} + \frac{\mu}{m_1}\vec{r} \end{aligned} \qquad \checkmark$$

$$\begin{aligned} \nabla_{\vec{r}_1} &= (\nabla_{\vec{r}_1} \cdot \vec{R})\nabla_{\vec{R}} + (\nabla_{\vec{r}_1} \cdot \vec{r})\nabla_{\vec{r}} \\ &= \left(\frac{m_1}{m_1 + m_2}\right)\nabla_{\vec{R}} + \nabla_{\vec{r}} \\ &= \frac{\mu}{m_2}\nabla_{\vec{R}} + \nabla_{\vec{r}} \end{aligned} \qquad \begin{aligned} \nabla_{\vec{r}_2} &= (\nabla_{\vec{r}_2} \cdot \vec{R})\nabla_{\vec{R}} + (\nabla_{\vec{r}_2} \cdot \vec{r})\nabla_{\vec{r}} \\ &= \left(\frac{m_2}{m_1 + m_2}\right)\nabla_{\vec{R}} - \nabla_{\vec{r}} \\ &= \frac{\mu}{m_1}\nabla_{\vec{R}} - \nabla_{\vec{r}} \end{aligned} \qquad \checkmark$$

b)

$$\begin{aligned} (E - V)\Psi &= -\frac{\hbar^2}{2m_1}(\nabla_1)^2\Psi - \frac{\hbar^2}{2m_2}(\nabla_2)^2\Psi \\ &= -\frac{\hbar^2}{2m_1}\left(\frac{\mu}{m_2}\nabla_R + \nabla_r\right)^2\Psi - \frac{\hbar^2}{2m_2}\left(\frac{\mu}{m_1}\nabla_R - \nabla_r\right)^2\Psi \\ &= -\frac{\hbar^2}{2m_1}\left(\left(\frac{\mu}{m_2}\right)^2\nabla_R^2 + \frac{\mu}{m_2}\nabla_R\nabla_r + \nabla_r^2\right)\Psi \\ &\qquad - \frac{\hbar^2}{2m_2}\left(\left(\frac{\mu}{m_1}\right)^2\nabla_R^2 - \frac{\mu}{m_1}\nabla_R\nabla_r + \nabla_r^2\right)\Psi \\ &= \left(\left[-\frac{\hbar^2}{2m_1}\left(\frac{\mu}{m_2}\right)^2 - \frac{\hbar^2}{2m_2}\left(\frac{\mu}{m_1}\right)^2\right]\nabla_R^2 + \left[-\frac{\hbar^2}{2m_1} - \frac{\hbar^2}{2m_2}\right]\nabla_r^2\right)\Psi \\ &= -\frac{\hbar^2}{2}\left[\frac{m_1 + m_2}{(m_1 + m_2)^2}\nabla_R^2 + \frac{m_1 + m_2}{m_1 m_2}\nabla_r^2\right]\Psi \\ &= -\frac{\hbar^2}{2(m_1 + m_2)}\nabla_R^2\Psi - \frac{\hbar^2}{2\mu}\nabla_r^2\Psi \qquad \checkmark \end{aligned}$$

c) Not quite sure what they want us to do exactly here, I think it is just a statement about what the next step to solving would be.

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2(\Psi_R\Psi_r) - \frac{\hbar^2}{2\mu}\nabla_r^2(\Psi_R\Psi_r) + V(\Psi_R\Psi_r) = (E_R + E_r)(\Psi_R\Psi_r)$$

$$\Psi_r\left(-\frac{\hbar^2}{2(m_1+m_2)}\right)\nabla_R^2\Psi_R + \Psi_r\left(-\frac{\hbar^2}{2\mu}\right)\nabla_r^2\Psi_r + V(\Psi_R\Psi_r) = (E_R + E_r)(\Psi_R\Psi_r)$$

Griffiths 5.2

a) I did a lot of this in python, so I will be just stating the results:

$$m_e \to 13.6056 \text{ eV}$$

$$\mu \to 13.5983 \text{ eV}$$

$$\boxed{\% \text{ error} = 0.055\%}$$

b)

$$\text{hydrogen} \to -1.8886\text{eV}$$
$$\text{deutromium} \to -1.8892\text{eV}$$

where the $\mu_{deu} = \dfrac{m_e(m_p + m_n)}{m_e + m_n + m_p}$. Using the wavelength equation $\lambda = \dfrac{hc}{E}$

$$\boxed{\Delta\lambda = 0.1788 \text{ nm}}$$

c) with $\dfrac{m_e(m_n + m_e)}{2m_e + m_n}$

$$\boxed{E_{pos} = -13.5983 \text{ eV}}$$

d) with $\dfrac{m_\mu m_p}{m_\mu + m_p}$

$$E_{0,\mu} = 2528.51 \text{ eV}$$

$$\Delta E = 1896.38 \text{ eV} \to \boxed{\lambda = 0.6338 \text{ nm}}$$

Griffiths 5.4

a)

$$\begin{aligned}|\Psi_a(\vec{r_1})\Psi_b(\vec{r_2})|^2 &= \langle\Psi_a\Psi_b|\Psi_a\Psi_b\rangle \\ &= \langle\Psi_a|\Psi_a\rangle\langle\Psi_b|\Psi_b\rangle \\ &= 1\end{aligned}$$

Since there are two orthogonal terms, $\left|\frac{1}{A}\Psi_\pm\right|^2 = 1+1=2$. So:

$$\boxed{A = \frac{1}{\sqrt{2}}}$$

b) this one we are going to be a little more clear with:

$$\begin{aligned}1 &= A^2\int\left[\Psi_a^1\Psi_a^2 + \Psi_a^2\Psi^1\right]^*\left[\Psi_a^1\Psi_a^2 + \Psi_a^2\Psi^1\right] \\ &= A^2\int\left[2\Psi_a^1\Psi_a^2\right]^*\left[2\Psi_a^1\Psi_a^2\right] \\ &= 4A^2\int\left(\Psi_a^1\right)^*\Psi_a^1\int\left(\Psi_a^2\right)^*\Psi_a^2 \\ &= 4A^2\end{aligned}$$

$$\boxed{A = \frac{1}{2}}$$

Question 4

a) The energy of the system can be described as $E_n = E_a + E_b$. Since these particles are distinguishable, the can be in the same state:

$$E_0 = \frac{3}{2}\hbar\omega + \frac{3}{2}\hbar\omega \boxed{= 3\hbar\omega}$$

b) There are two setups that we can do:

$$\psi_1 = (\hat{a}_+\psi_0(x_1))(\psi_0(x_2)) \qquad\qquad \psi_1 = (\psi_0(x_1))(\hat{a}_+\psi_0(x_2))$$

$$\begin{aligned} E_1 &= \left[\frac{1}{2}\hbar\omega\right]4 + \left[\frac{3}{2}\hbar\omega\right]2 \\ &= 2\hbar\omega + 3\hbar\omega \\ &\boxed{= 5\hbar\omega} \end{aligned}$$

c) Only one of the states need to be excited:

$$\begin{array}{ll} \psi_{100}(x_1)\,\psi_{000}(x_2) & \psi_{000}(x_1)\,\psi_{100}(x_2) \\ \psi_{010}(x_1)\,\psi_{000}(x_2) & \psi_{000}(x_1)\,\psi_{010}(x_2) \\ \psi_{001}(x_1)\,\psi_{000}(x_2) & \psi_{000}(x_1)\,\psi_{001}(x_2) \end{array}$$

Question 5

a)

$$1 = A^2 \int \left[2\sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\sin\left(\frac{\pi}{a}x_3\right) + 3\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right)\sin\left(\frac{5\pi}{a}x_3\right)\right]^*$$

$$\times \left[2\sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\sin\left(\frac{\pi}{a}x_3\right) + 3\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right)\sin\left(\frac{5\pi}{a}x_3\right)\right]$$

recognizing that none of the states between the two state, so the cross terms are zero

$$= A^2 \int \left[4\sin^2\left(\frac{2\pi}{a}x_1\right)\sin^2\left(\frac{\pi}{a}x_2\right)\sin^2\left(\frac{\pi}{a}x_3\right) + 9\sin^2\left(\frac{\pi}{a}x_1\right)\sin^2\left(\frac{2\pi}{a}x_2\right)\sin^2\left(\frac{5\pi}{a}x_3\right)\right]$$

$$\text{remembering that } \int_0^{n\pi} \sin^2(x) = \frac{x}{2}$$

$$= A^2\left[4\left(\frac{a}{2}\right)^3 + 9\left(\frac{a}{2}\right)^3\right]$$

$$\boxed{A = \frac{1}{\sqrt{13}}\left(\frac{2}{a}\right)^{3/2}}$$

This makes sense, since the second part is the normal coefficient, and the $\sqrt{13}$ accounts for normalizing the two different wavefunctions.

b) The wavefunction of interest is the second one, with the coefficient of 3. So, the probability of having E_3 with this energy is:

$$\boxed{\left(\frac{3}{\sqrt{13}}\right)^2 = \frac{9}{13}}$$

c) Independent of the energy, the average x value is the center of the well. So, we have a $\frac{4}{13}$ chance for $\frac{a}{2}$ and a $\frac{9}{13}$ chance for $\frac{a}{2}$, so $\boxed{\langle x\rangle = \frac{a}{2}}$.

d) We need to find the expectation value of each the energies independently, then add them all together:

$$\left|\frac{4}{13}\begin{bmatrix}4\\1\\1\end{bmatrix} + \frac{9}{13}\begin{bmatrix}1\\4\\25\end{bmatrix}\right| = \frac{1}{13}\left|\begin{bmatrix}25\\40\\229\end{bmatrix}\right| = \frac{294}{13}$$

$$\boxed{\langle E\rangle \approx \frac{22.62\,\hbar^2\pi^2}{2ma^2}}$$

Griffiths 5.8

b) I will do the next part, with the fermion, then remove all the negatives to take the permanent.

$$\frac{1}{\sqrt{3!}}\begin{vmatrix} |\Psi_a\rangle_1 & |\Psi_b\rangle_1 & |\Psi_c\rangle_1 \\ |\Psi_a\rangle_2 & |\Psi_b\rangle_2 & |\Psi_c\rangle_2 \\ |\Psi_a\rangle_3 & |\Psi_b\rangle_3 & |\Psi_c\rangle_3 \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}}\left[|\Psi_a\rangle_1 \begin{vmatrix} |\Psi_b\rangle_2 & |\Psi_c\rangle_2 \\ |\Psi_b\rangle_3 & |\Psi_c\rangle_3 \end{vmatrix} - |\Psi_a\rangle_2 \begin{vmatrix} |\Psi_b\rangle_1 & |\Psi_c\rangle_1 \\ |\Psi_b\rangle_3 & |\Psi_c\rangle_3 \end{vmatrix} + |\Psi_a\rangle_3 \begin{vmatrix} |\Psi_b\rangle_1 & |\Psi_c\rangle_1 \\ |\Psi_b\rangle_2 & |\Psi_c\rangle_2 \end{vmatrix}\right]$$

$$= \frac{1}{\sqrt{6}}[|\Psi_a\rangle_1 |\Psi_b\rangle_2 |\Psi_c\rangle_3 - |\Psi_a\rangle_1 |\Psi_c\rangle_2 |\Psi_b\rangle_3 - |\Psi_b\rangle_1 |\Psi_a\rangle_2 |\Psi_c\rangle_3 + |\Psi_c\rangle_1 |\Psi_a\rangle_2 |\Psi_b\rangle_3$$
$$+ |\Psi_b\rangle_1 |\Psi_c\rangle_2 |\Psi_a\rangle_3 - |\Psi_c\rangle_1 |\Psi_b\rangle_2 |\Psi_a\rangle_3]$$

So, for a bosonic system:

$$\frac{1}{\sqrt{6}}[|\Psi_a\rangle_1 |\Psi_b\rangle_2 |\Psi_c\rangle_3 + |\Psi_a\rangle_1 |\Psi_c\rangle_2 |\Psi_b\rangle_3 + |\Psi_b\rangle_1 |\Psi_a\rangle_2 |\Psi_c\rangle_3 + |\Psi_c\rangle_1 |\Psi_a\rangle_2 |\Psi_b\rangle_3$$
$$+ |\Psi_b\rangle_1 |\Psi_c\rangle_2 |\Psi_a\rangle_3 + |\Psi_c\rangle_1 |\Psi_b\rangle_2 |\Psi_a\rangle_3]$$

c) Stealing the answer from the last part, before changing the signs:

$$\frac{1}{\sqrt{6}}[|\Psi_a\rangle_1 |\Psi_b\rangle_2 |\Psi_c\rangle_3 - |\Psi_a\rangle_1 |\Psi_c\rangle_2 |\Psi_b\rangle_3 - |\Psi_b\rangle_1 |\Psi_a\rangle_2 |\Psi_c\rangle_3 + |\Psi_c\rangle_1 |\Psi_a\rangle_2 |\Psi_b\rangle_3$$
$$+ |\Psi_b\rangle_1 |\Psi_c\rangle_2 |\Psi_a\rangle_3 - |\Psi_c\rangle_1 |\Psi_b\rangle_2 |\Psi_a\rangle_3]$$

Griffiths 5.9

a) I just brute forced these bad bois to find all of them.

For the first energy: $E_0 = 2K$, degeneracy 1

$$\Psi_1(x_1)\Psi_1(x_2)\left|\uparrow\downarrow\right\rangle - \Psi_1(x_1)\Psi_1(x_2)\left|\downarrow\uparrow\right\rangle$$

for the second energy: $E_1 = 5K$, degeneracy 4

$$\Psi_1(x_1)\Psi_2(x_2)\left|\downarrow\downarrow\right\rangle - \Psi_2(x_1)\Psi_1(x_2)\left|\downarrow\downarrow\right\rangle$$

$$(\Psi_1(x_1)\Psi_2(x_2))\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle) - (\Psi_2(x_1)\Psi_1(x_2))\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle)$$

$$\Psi_1(x_1)\Psi_2(x_2)\left|\uparrow\uparrow\right\rangle - \Psi_2(x_1)\Psi_1(x_2)\left|\uparrow\uparrow\right\rangle$$

$$(\Psi_1(x_1)\Psi_2(x_2))\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle) + (\Psi_2(x_1)\Psi_1(x_2))\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)$$

for the third energy: $E_2 = 8k$, degeneracy 1

$$\Psi_2(x_1)\Psi_2(x_2)\left|\uparrow\downarrow\right\rangle - \Psi_2(x_1)\Psi_2(x_2)\left|\downarrow\uparrow\right\rangle$$

for the second energy: $E_3 = 10K$, degeneracy 4

$$\Psi_1(x_1)\Psi_3(x_2)\left|\downarrow\downarrow\right\rangle - \Psi_3(x_1)\Psi_1(x_2)\left|\downarrow\downarrow\right\rangle$$

$$(\Psi_1(x_1)\Psi_3(x_2))\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle) - (\Psi_3(x_1)\Psi_1(x_2))\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle)$$

$$\Psi_1(x_1)\Psi_3(x_2)\left|\uparrow\uparrow\right\rangle - \Psi_3(x_1)\Psi_1(x_2)\left|\uparrow\uparrow\right\rangle$$

$$(\Psi_1(x_1)\Psi_3(x_2))\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle) + (\Psi_3(x_1)\Psi_1(x_2))\frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)$$