

PHSX 461: HW07

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3.7

- a) Suppose that $f(x)$ and $g(x)$ are two eigenfunctions of an operator \hat{Q} , with the same eigenvalue q . Show that any linear combination of f and g is itself an eigenfunction of \hat{Q} , with eigenvalue q .

$$\begin{aligned}\hat{Q}|f\rangle &= q|f\rangle \quad , \quad \hat{Q}|g\rangle = q|g\rangle \\ |\alpha\rangle &= a|f\rangle + b|g\rangle\end{aligned}$$

$$\begin{aligned}\hat{Q}|\alpha\rangle &= \hat{Q}[a|f\rangle + b|g\rangle] \\ &= \hat{Q}(a|f\rangle) + \hat{Q}(b|g\rangle) \\ &= a\hat{Q}|f\rangle + b\hat{Q}|g\rangle \\ &= aq|f\rangle + bq|g\rangle \\ &= q[a|f\rangle + b|g\rangle] = q|\alpha\rangle\end{aligned}$$

- b) Check that $f(x) = \exp(x)$ and $g(x) = \exp(-x)$ are eigenfunctions of the operator d^2/dx^2 , with the same eigenvalue. Construct two linear combinations of f and g that are orthogonal eigenfunctions on the interval $(-1, 1)$.

$$\begin{aligned}\hat{Q}|f\rangle &\Rightarrow \frac{\partial^2}{\partial x^2}e^x = e^x \rightarrow q = 1 \\ \hat{Q}|g\rangle &\Rightarrow \frac{\partial^2}{\partial x^2}e^{-x} = (-1)^2e^{-x} = e^{-x} \rightarrow q = 1\end{aligned}$$

$$|\alpha\rangle = a|f\rangle + b|g\rangle \quad , \quad |\beta\rangle = c|f\rangle + d|g\rangle$$

For orthogonality, we need $\langle\alpha|\beta\rangle = \int \alpha(x)^*\beta(x) dx = 0$

$$0 = \int_{-1}^1 (a^*f^*(x) + b^*g^*(x))^*(cf(x) + dg(x)) dx$$

$$0 = \int_{-1}^1 (ae^x + be^{-x})(ce^x + de^{-x}) dx$$

$$0 = \int_{-1}^1 [ace^{2x} + ad + bc + bde^{-2x}] dx$$

$$0 = \frac{ac}{2}e^{2x} + [ad + bc]x - \frac{bd}{2}e^{-2x} \Big|_{-1}^1$$

$$0 = \frac{ac}{2}e^2 + [ad + bc] - \frac{bd}{2}e^{-2} - \left[\frac{ac}{2}e^{-2} - [ad + bc] - \frac{bd}{2}e^2 \right]$$

$$0 = e^2 \left[\frac{ac}{2} + \frac{bd}{2} \right] + e^{-2} \left[\frac{-bd}{2} + \frac{-ac}{2} \right] + 2[ad + bc]$$

The easiest solution to this is if we let $ac = -bd$. Two solutions to this question yield the linear combination:

$$|\alpha\rangle = |f\rangle + |g\rangle \quad , \quad |\beta\rangle = |f\rangle - |g\rangle$$

$$|\alpha\rangle = -|f\rangle + |g\rangle \quad , \quad |\beta\rangle = |f\rangle + |g\rangle$$

3.9

- a) *Cite a Hamiltonian from Chapter 2 (other than the harmonic oscillator) that has only a discrete spectrum.*
- b) *Cite a Hamiltonian from Chapter 2 (other than the free particle) that has only a continuous spectrum.*
- c) *Cite a Hamiltonian from Chapter 2 (other than the finite square well) that has both a discrete and a continuous part to its spectrum.*

3.13

Show that

$$\langle x \rangle = \int \Phi^* \left(i\hbar \frac{\partial}{\partial p} \right) \Phi \, dp$$

Let us start by defining what Φ is:

$$\Phi = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar} \Psi(x, t) \, dx$$

So, let's plug this in and just run the calculus:

$$\begin{aligned} & \int \Phi^* \left(i\hbar \frac{\partial}{\partial p} \right) \Phi \, dp \\ &= \frac{i\hbar}{2\pi\hbar} \int \left[\int e^{ipx/\hbar} \Psi \, dx \right]^* \frac{\partial}{\partial p} \left[\int e^{ipx'/\hbar} \Psi \, dx' \right] \\ &= \frac{i}{2\pi} \int \left[\int e^{ipx/\hbar} \Psi^* \, dx \right] \left[\int \frac{-ix'}{\hbar} e^{ipx'/\hbar} \Psi \, dx' \right] \end{aligned}$$

Since these three variables, p, x, x' , are all independent, they can be moved in and out of each other's integrals.

$$\begin{aligned} &= \frac{1}{2\pi\hbar} \int e^{-i(\frac{p}{\hbar}(x-x'))} \Psi^*(x, t) x' \Psi(x', t) \, dp \, dx \, dx' \\ &= \frac{\hbar}{\hbar} \delta \left[\frac{p}{\hbar}(x - x') \right] \int \Psi^*(x, t) x' \Psi(x', t) \, dx \, dx' \Big|_{-\infty}^{\infty} \\ &= \int \Psi^*(x, t) x \Psi(x, t) \, dx = \langle x \rangle \end{aligned}$$

3.26

Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle, |2\rangle, |3\rangle$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle, \quad |\beta\rangle = i|1\rangle + 2|3\rangle$$

a) Construct $\langle\alpha|$ and $\langle\beta|$ (in terms of the dual basis $\langle 1|, \langle 2|, \langle 3|$).

We can think of $|\alpha\rangle$ and $|\beta\rangle$ as wavefunctions, since the mechanics here are the same (orthonormal basis, linear combinations of states, ...). So, pulling that analog:

$$|\Psi\rangle = \sum c_n |f_n\rangle \Rightarrow \langle\Psi| = \sum c_n^* \langle f_n|$$

and thus:

$$\langle\alpha| = -i\langle 1| - 2\langle 2| + i\langle 3| \quad , \quad \langle\beta| = -i\langle 1| + 2\langle 3|$$

b) Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$, and confirm that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.

$$\begin{aligned} \langle\alpha|\beta\rangle &= (-i\langle 1| - 2\langle 2| + i\langle 3|)(i|1\rangle + 2|3\rangle) \\ &= -i^2\langle 1|1\rangle + 2i\langle 3|3\rangle \\ &= 1 + 2i \end{aligned}$$

$$\begin{aligned} \langle\beta|\alpha\rangle &= (-i\langle 1| + 2\langle 3|)(i|1\rangle - 2|2\rangle - i|3\rangle) \\ &= -i^2\langle 1|1\rangle - 2i\langle 3|3\rangle \\ &= 1 - 2i \end{aligned}$$

$$\langle\alpha|\beta\rangle^* = (1 + 2i)^* = 1 - 2i = \langle\beta|\alpha\rangle$$

c) Find all nine matrix elements for the operator $\hat{A} = |\alpha\rangle\langle\beta|$, in this basis, and construct the matrix A . Is it hermitian?

$$\hat{Q} \Rightarrow Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots \\ Q_{21} & Q_{22} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}, \text{ where } Q_{mn} = \langle e_m | \hat{Q} | e_n \rangle$$

We will use this to calculate the A_{13} element, then point out the pattern and fill out the rest of the matrix.

$$\begin{aligned} A_{13} &= \langle 1 | \alpha \rangle \langle \beta | 3 \rangle \\ &= \langle 1 | (i | 1 \rangle - 2 | 2 \rangle - i | 3 \rangle) (-i \langle 1 | + 2 \langle 3 |) | 3 \rangle \\ &= (i)(2) = 2i \end{aligned}$$

So, an element A_{mn} can be found by taking the product of the m th coefficient from $|\alpha\rangle$ and the n th coefficient from $\langle\beta|$. Using this method:

$$\hat{A} = \begin{pmatrix} (i)(-i) & (i)(0) & (i)(2) \\ (-2)(-i) & (-2)(0) & (-2)(2) \\ (-i)(-i) & (-i)(0) & (-i)(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}$$

To test if this is hermitian, $\hat{A}^\dagger = \hat{A}$

$$\hat{A}^\dagger = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}^{*T} = \begin{pmatrix} 1 & -2i & -1 \\ 0 & 0 & 0 \\ -2i & -4 & -2i \end{pmatrix} \neq \hat{A}$$

So, A is **not** hermitian.

Question 5.

Prove that the momentum operator, \hat{p} is Hermitian.

Hint: *you will need to assume that any functions you use are normalizable. You may also use the results from the previous homework assignment.*

We want to show that $\hat{p}^\dagger = \hat{p}$. To do this, first we can identify that a hermitian operator in momentum space is also a hermitian operator in position space; so, we will be doing this problem with Φ , instead of Ψ .

$$\begin{aligned} & \langle \Phi | \hat{p}^\dagger | \Phi \rangle \\ &= \langle \hat{p} \Phi | \Phi \rangle \\ &= \int (\hat{p} \Phi)^* \Phi \, dp \\ &= \int (p \Phi)^* \Phi \, dp \\ &= \int \Phi^* p \Phi \, dp \\ &= \int \Phi^* \hat{p} \Phi \, dp \\ &= \langle \Phi | \hat{p} | \Phi \rangle \rightarrow \hat{p}^\dagger = \hat{p} \end{aligned}$$

3.33

An operator \hat{A} , representing observable A , has two (normalized) eigenstates ψ_1 and ψ_2 , with eigenvalues a_1 and a_2 , respectively. Operator \hat{B} , representing observable B , has two (normalized) eigenstates ϕ_1 and ϕ_2 , with eigenvalues b_1 and b_2 . The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5, \quad \psi_2 = (4\phi_1 - 3\phi_2)/5$$

- a) *Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately after this measurement)?*

The wavefunction has been observed and thus has been collapsed to ψ_1 . So, the state is ϕ_1

- b) *If B is now measure, what are the possible results, and what are their probabilities?*

Since we are completely in state ψ_1 , the probabilities will only be decedents of that equation. There will be a $(\frac{3}{5})^2 = \frac{9}{25} = 35\%$ chance for b_1 and a $(\frac{4}{5})^2 = \frac{16}{25} = 65\%$ chance for b_2 .

- c) *Right after the measurement of B , A is measured again. What is the probability of getting a_1 ? (Note that the answer would be quite different if I had told you the outcome of the B measurement.*

If we are asking for the probability of a_1 , we are thinking about ψ_1 :

$$\begin{aligned} \psi_1^2 &= \frac{9\phi_1^2 + 24\phi_1\phi_2 + 16\phi_2^2}{25} \\ &= \frac{3^4 + 23^2 4^2 + 4^4}{5^4} = 1 \end{aligned}$$

To do a quick check to make sure that we don't need to renormalize:

$$\begin{aligned} \psi_2^2 &= \frac{16\phi_1^2 - 24\phi_1\phi_2 + 9\phi_2^2}{25} \\ &= \frac{3^2 4^2 - 23^2 4^2 + 3^2 4^2}{5^4} = 0 \end{aligned}$$

So, it would seem that there is a 100% chance that we get back a_1 .

But what about that hint? What would it be then? Well, if we say that $B = b_1$, $\phi_1 = 1$ and $\phi_2 = 0$. Thus, $\psi_1^2 = \frac{9}{25} = 35\%$ and $\psi_2^2 = \frac{16}{25} = \frac{16}{25} = 65\%$. This is a very different answer from what we got, so the “100%” is consistent with the hint.