PHSX 462: HW03

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Griffiths 7.11

Let's start by just writing down the unperturbed solution:

$$|\Psi_0\rangle = \sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)\sin\left(\frac{n_z\pi}{a}z\right) \qquad\qquad E = \frac{\hbar^2\pi^2}{2ma^2}\left(n_x^2+n_y^2+n_z^2\right)$$

With a perturbation of

$$H' = a^3V_0\delta\left(x-\frac{a}{4}\right)\delta\left(y-\frac{a}{2}\right)\delta\left(z-\frac{3a}{4}\right)$$

Doing this for the ground state:

$$\begin{aligned}
E_1^1 &= \left\langle \Psi^0 \middle| H' \middle| \Psi^0 \right\rangle \\
&= \int_0^a \sin^2\left(\frac{n_x\pi}{a}x\right)\sin^2\left(\frac{n_y\pi}{a}y\right)\sin^2\left(\frac{n_z\pi}{a}z\right)a^3V_0\delta\left(x-\frac{a}{4}\right)\delta\left(y-\frac{a}{2}\right)\delta\left(z-\frac{3a}{4}\right)\mathrm{d}V\,\Bigg|_{n_x=n_y=n_z=1} \\
&= 8\sin^2\left(\frac{pi}{a}\frac{a}{4}\right)\sin^2\left(\frac{\pi}{a}\frac{a}{2}\right)\sin^2\left(\frac{\pi}{a}\frac{3a}{4}\right)V_0 \\
&= 2V_0
\end{aligned}$$

$$\boxed{E_1^1 = 2V_0}$$

Let's encode the triply degenerate second energy state as:

$$|1\rangle \to (n_x=2, n_y=1, n_z=1) \qquad |2\rangle \to (n_x=1, n_y=2, n_z=1) \qquad |3\rangle \to (n_x=1, n_y=1, n_z=2)$$

Using the equation: $W_{ij} = \langle\Psi_i|H'|\Psi_j\rangle$:

$$\begin{aligned}
W_{11} &= \int_0^a \sin^2\left(\frac{2\pi}{a}x\right)\sin^2\left(\frac{\pi}{a}y\right)\sin^2\left(\frac{\pi}{a}z\right)\left(\frac{2}{a}\right)^3 a^3V_0\delta\cdots\mathrm{d}V \\
&= 8V_0\sin^2\left(\frac{2\pi}{a}\frac{a}{4}\right)\sin^2\left(\frac{\pi}{a}\frac{a}{2}\right)\sin^2\left(\frac{\pi}{a}\frac{3a}{4}\right) \\
&= 8V_0(1)^2(1)^2\left(\frac{1}{\sqrt{2}}\right)^2 \\
&= 4V_0
\end{aligned}$$

Recognizing this pattern and following it:

$$\begin{aligned}
W_{12} &= 8V_0 \sin\left(\frac{2\pi}{a}\frac{a}{4}\right)\sin\left(\frac{\pi}{a}\frac{a}{4}\right)\sin\left(\frac{2\pi}{a}\frac{a}{2}\right)\sin\left(\frac{\pi}{a}\frac{a}{2}\right)\sin^2\left(\frac{\pi}{a}\frac{3\pi}{4}\right)\\
&= 8V_0(1)\left(\frac{1}{\sqrt{2}}\right)(0)(1)\left(\frac{1}{2}\right) = 0\\
W_{13} &= 8V_0 \sin\left(\frac{2\pi}{a}\frac{a}{4}\right)\sin\left(\frac{\pi}{a}\frac{a}{4}\right)\sin^2\left(\frac{\pi}{a}\frac{a}{2}\right)\sin\left(\frac{2\pi}{a}\frac{3a}{4}\right)\sin\left(\frac{\pi}{a}\frac{3a}{4}\right)\\
&= 8V_0(1)\left(\frac{1}{\sqrt{2}}\right)(1)(-1)\left(\frac{1}{\sqrt{2}}\right) = -4V_0\\
W_{23} &= 8V_0 \sin^2\left(\frac{\pi}{a}\frac{a}{4}\right)\sin\left(\frac{2\pi}{a}\frac{a}{2}\right)\sin\left(\frac{\pi}{a}\frac{a}{2}\right)\sin\left(\frac{2\pi}{a}\frac{3a}{4}\right)\sin\left(\frac{\pi}{a}\frac{3a}{4}\right)\\
&= 8V_0\left(\frac{1}{\sqrt{2}}\right)^2(0)(1)(-1)\left(\frac{1}{\sqrt{2}}\right)^2 = 0\\
W_{22} &= 8V_0 \sin^2\left(\frac{\pi}{a}\frac{a}{4}\right)\sin^2\left(\frac{2\pi}{a}\frac{a}{2}\right)\sin^2\left(\frac{\pi}{a}\frac{3a}{4}\right)\\
&= 8V_0\left(\frac{1}{\sqrt{2}}\right)^2(0)^2\left(\frac{1}{\sqrt{2}}\right)^2 = 0\\
W_{33} &= 8V_0 \sin^2\left(\frac{\pi}{a}\frac{a}{4}\right)\sin^2\left(\frac{\pi}{a}\frac{a}{2}\right)\sin^2\left(\frac{2\pi}{a}\frac{3\pi}{4}\right)\\
&= 8V_0\left(\frac{1}{\sqrt{2}}\right)^2(1)^2\left(\frac{1}{\sqrt{2}}\right)^2 = 4V_0
\end{aligned}$$

Putting these all together into matrix form:

$$W = 4V_0\begin{bmatrix} 1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{bmatrix}$$

The changes to the energies are the eigenstates of this matrix:

$$\begin{aligned}
\begin{vmatrix} 4V_0 & 0 & -4V_0\\ 0 & 0 & 0\\ -4V_0 & 0 & 4V_0\end{vmatrix} &= -4V_0\begin{vmatrix} 0 & -\lambda\\ -4V_0 & 0\end{vmatrix} - 0\begin{vmatrix} 4V_0-\lambda & 0\\ -4V_0 & 0\end{vmatrix} + (4V_0-\lambda)\begin{vmatrix} 4V_0-\lambda & 0\\ 0 & -\lambda\end{vmatrix}\\
&= -4V_0(-4V_0\lambda) + (4V_0-\lambda)(4V_0-\lambda)\lambda) = 0\\
x\lambda &= 8V_0, 0, 0
\end{aligned}$$

$$\boxed{E_2^1 = 8V_0, 0, 0}$$

Griffiths 7.37

a) To do this approximation, let's remember the binomial expansion, specifically for $n=-1$:

$$(1+\alpha)^{-1} = 1 - \alpha + \frac{-1(-2)}{2!}\alpha^2 + \frac{-1(-2)(-3)}{3!}\alpha^3 + \cdots$$

$$\begin{aligned}
H &= \frac{e^2}{4\pi\varepsilon_0 R}\left[1 - \frac{1}{1-\frac{x_1}{R}} - \frac{1}{1-\frac{x_2}{R}} + \frac{1}{1+\frac{-x_1+x_2}{R}}\right] \\
&= \frac{e^2}{4\pi\varepsilon_0 R}\left[1 - \left(1 + \frac{x_1}{R} + \frac{x_1^2}{R^2} + \cdots\right) - \left(1 - \frac{x_2}{R} + \frac{x_2^2}{R^2} + \cdots\right) + \left(1 - \frac{-x_1+x_2}{R}\right.\right. \\
&\qquad \left.\left. + \frac{(-x_1+x_2)^2}{R^2} + \cdots\right)\right] \\
&\approx \frac{e^2}{4\pi\varepsilon_0 R}\left[-\frac{x_1}{R} - \frac{x_1^2}{R^2} + \frac{x_2}{R} - \frac{x_2^2}{R} + \frac{x_1}{R} - \frac{x_2}{R} + \frac{x_1^2}{R^2} + \frac{x_2^2}{R^2} - \frac{2x_1x_2}{R^2}\right] \\
&= \frac{e^2}{4\pi\varepsilon_0 R}\left[\frac{-2x_1x_2}{R^2}\right] = \boxed{\frac{-e^2x_1x_2}{2\pi\varepsilon_0 R^3}}
\end{aligned}$$

b) Let's reverse engineer this:

$$\begin{aligned}
H &= \left[\frac{1}{2m}\left(\frac{1}{\sqrt{2}}(p_1+p_2)\right)^2 + \frac{1}{2}\left(k - \frac{e^2}{2\pi\varepsilon_0 R^3}\right)\left(\frac{1}{\sqrt{2}}(x_1+x_2)\right)^2\right] + \\
&\qquad \left[\frac{1}{2m}\left(\frac{1}{\sqrt{2}}(p_1-p_2)\right)^2 + \frac{1}{2}\left(k + \frac{e^2}{2\pi\varepsilon_0 R^3}\right)\left(\frac{1}{\sqrt{2}}(x_1-x_2)\right)^2\right] \\
&= \left[\frac{1}{4m}\left(p_1^2 + 2p_1p_2 + p_2\right) + \frac{1}{4}\left(k - \frac{e^2}{2\pi\varepsilon_0 R^3}\right)\left(x_1 + 2x_1x_2 + x_2^2\right)\right] + \\
&\qquad \left[\frac{1}{4m}\left(p_1^2 - 2p_1p_2 + p_2\right)2 + \frac{1}{4}\left(k + \frac{e^2}{2\pi\varepsilon_0 R^3}\right)\left(x_1 - 2x_1x_2 + x_2^2\right)\right] \\
&= \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}k(x_1^2 + x_2^2) - \frac{e^2x_1x_2}{2\pi\varepsilon_0 R^2} \qquad \checkmark
\end{aligned}$$

c) *We are skipping this part*

d) Let's start by remembering the general solution for a quantum harmonic oscillator:

$$|\Psi_0\rangle \to \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \qquad\qquad \hat{a}_+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)$$

For brevities sake, and looking back at the answer from part c ($\Delta V \approx -\frac{\hbar}{8m^2\omega_0^3}\left(\frac{e^2}{2\pi\varepsilon_0}\right)^2\frac{1}{R^6}$), we will only be calculating the first non-zero correction.
In the first order correction, we have a product of integrals that look something like

$$\int e^{x^2} x \,\mathrm{d}x\,.$$

This is obviously odd function, and thus the integral will be zero. This will be case for the mixture of $(n_1 = 0, n_2 = 0), (n_1 = 1, n_2 = 0)$, and $(n_1 = 0, n_2 = 1)$. So, let's just straight to the next largest element: $(n_1 = 1, n_2 = 1)$.

$$\begin{aligned}
\langle \Psi^0_{1,1} | H' | \Psi^0_0 \rangle &= \int \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-m\omega}{2\hbar}[x_1^2+x_2^2]} \left(\frac{-e^2 x_1 x_2}{2\pi\varepsilon_0 R^3}\right) (\hat{a_+})^1_1 (\hat{a_+})^1_2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-m\omega}{2\hbar}[x_1^2+x_2^2]} \\
&= \left(\frac{m\omega}{\pi\hbar}\right) \left(\frac{-e^2}{2\pi\varepsilon_0 R^3}\right) \left[\int e^{\frac{-m\omega}{2\hbar}x^2} x \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right) e^{\frac{-m\omega}{2\hbar}x^2}\right]^2 \\
&= \left(\frac{m^2\omega^2}{2\pi\hbar^2}\right) \left(\frac{-e^2}{2\pi\varepsilon_0 R^3}\right) \left[\int e^{\frac{-m\omega}{\hbar}x^2} x^2 - \cancelto{0}{e^{\frac{-m\omega}{\hbar}x^2} \left(\frac{\hbar}{m\omega}\right) \left(-\frac{m\omega}{\pi} x\right)}\right]^2 \\
&= \left(\frac{m^2\omega^2}{2\pi\hbar^2}\right) \left(\frac{-e^2}{2\pi\varepsilon_0 R^3}\right) \left[\sqrt{\pi} \left(\frac{2!}{1!}\right) \left(\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\right)^3\right]^2
\end{aligned}$$

The energy for this new state is double the energy of one QHO in the first excited state: $E_{1,1} = 3\hbar\omega$. So:

$$E_0^1 = \frac{\left(\frac{e^2\hbar}{R^3\pi m\omega 4\varepsilon_0}\right)^2}{\hbar\omega - 3\hbar\omega} \boxed{= -\frac{\hbar}{8m^2\omega^3}\left(\frac{e^2}{2\pi\varepsilon_0}\right)^2 \frac{1}{R^6}}$$

Griffiths 7.45

This question was very tedious and had a lot of integration. I will be cutting a couple corners to help my sanity since I am typing this up, but I will try to justify the jumps.

a) We can recall that the ground state - eigenstate for the Bohr Hamiltonian is:

$$\Psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

The first order correction to the energy is then:

$$\begin{aligned}
E_0^1 &= \langle 1\,0\,0|eE_{\text{ext}} r\cos(\theta)|1\,0\,0\rangle \\
&= eE_{\text{ext}} \int \frac{1}{\pi a^3} e^{-2r/a} r\cos(\theta) r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{eE_{ext}}{\pi a^3} \int r^3 e^{-2r/a}\,\mathrm{d}r \int \frac{1}{2}\sin(2\theta)\,\mathrm{d}\theta \int \mathrm{d}\phi \\
&= \frac{eE_{ext}}{\pi a^3} \int r^3 e^{-2r/a}\,\mathrm{d}r\,(0) \int \mathrm{d}\phi = 0 \qquad \checkmark
\end{aligned}$$

b) Looking up the $n = 2$ states:

$$\Psi_{200} = \frac{1}{\sqrt{2\pi a}} \frac{1}{2a}\left(1 - \frac{r}{2a}\right) e^{-r/2a} \qquad \Psi_{211} = -\frac{1}{8a^2}\sqrt{\frac{1}{a\pi}} r e^{-r/2a}\sin(\theta) e^{i\phi}$$

$$\Psi_{210} = \frac{1}{4a}\sqrt{\frac{1}{2a\pi}} r e^{-r/2a}\cos(\theta) \qquad \Psi_{21-1} = \frac{1}{8a^2}\sqrt{\frac{1}{a\pi}} r e^{-r/2a}\sin(\theta) e^{-i\phi}$$

Before we start this integration drill; recognize that

$$\int_0^{2\pi} e^{i\phi}\,\mathrm{d}\phi = \int_0^{2\pi} e^{-i\phi}\,\mathrm{d}\phi = \int_0^{2\pi} e^{2i\phi}\,\mathrm{d}\phi = 0$$

So, any integrals that have any one of these terms will be automatically evaluated to zero. I will also be doing all zero terms first.

$$\begin{aligned}
\langle\Psi_{200}|H'|\Psi_{200}\rangle &= eE_{\text{ext}} \int \left[\frac{1}{\sqrt{2\pi a}}\frac{1}{2a}\right]^2 r\cos(\theta)\left(1 - \frac{r}{2a}\right)^2 e^{-r/a} r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&\longrightarrow \int_0^{\pi} \frac{1}{2}\sin(\theta)\,\mathrm{d}\theta = 0
\end{aligned}$$

$$\begin{aligned}
\langle\Psi_{200}|H'|\Psi_{21-1}\rangle &= eE_{\text{ext}} \int \left[\frac{1}{\sqrt{2\pi a}\frac{1}{2a}}\right]\left(1 - \frac{r}{2a}\right) e^{-r/2a} \frac{1}{8a^2}\sqrt{\frac{1}{a\pi}} r e^{-r/2a} \\
&\quad \sin(\theta) e^{-i\phi} r\cos(\theta) r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&\longrightarrow \int_0^{2\pi} e^{-i\phi}\,\mathrm{d}\phi = 0
\end{aligned}$$

$$\begin{aligned}
\langle\Psi_{211}|H'|\Psi_{211}\rangle &= eE_{\text{ext}} \int \left[\frac{1}{8a^2}\sqrt{\frac{1}{a\pi}}\right]^2 r^2 e^{-r/a}\sin^2(\theta) e^{2i\phi} r\cos(\theta) r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&\longrightarrow \int_0^{2\pi} e^{2i\phi}\,\mathrm{d}\phi = 0
\end{aligned}$$

$$\langle \Psi_{211}|H'|\Psi_{210}\rangle = -eE_{\text{ext}} \int \frac{1}{8a^2}\sqrt{\frac{1}{a\pi}} re^{-r/2a}\sin(\theta)e^{i\phi}\frac{1}{4a}\sqrt{\frac{1}{2a\pi}}re^{-r/2a}\cos(\theta)r\cos(\theta)r^2$$
$$\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\longrightarrow \quad \int_0^{2\pi} e^{i\phi}\,\mathrm{d}\phi = 0$$

$$\langle \Psi_{211}|H'|\Psi_{21-1}\rangle = -eE_{\text{ext}} \int \left(\frac{1}{8a^2}\sqrt{\frac{1}{a\pi}}\right)^2 r^2e^{-r/a}\sin^2(\theta)r\cos(\theta)r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\longrightarrow \quad \int_0^{\pi} \sin^3(\theta)\cos(\theta)\,\mathrm{d}\theta = \frac{1}{4}\sin^4(\theta)\Big|_0^{\pi} = 0$$

$$\langle \Psi_{210}|H'|\Psi_{210}\rangle = -eE_{\text{ext}} \int \left(\frac{1}{4a}\sqrt{\frac{1}{2a\pi}}\right)^2 r^2e^{-r/a}\cos^2(\theta)r\cos(\theta)r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\longrightarrow \quad \int_0^{\pi} \cos^3(\theta)\sin(\theta)\,\mathrm{d}\theta = -\frac{1}{4}\cos^4(\theta)\Big|_0^{\pi} = -\frac{1}{4}+\frac{1}{4} = 0$$

$$\langle \Psi_{210}|H'|\Psi_{21-1}\rangle = eE_{\text{ext}} \int \left(\frac{1}{4a}\sqrt{\frac{1}{2a\pi}}\right) re^{-r/2a}\cos(\theta)\frac{1}{8a^2}\sqrt{\frac{1}{a\pi}}re^{-r/2a}\sin(\theta)e^{i\phi}r\cos(\theta)r^2$$
$$\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\longrightarrow \quad \int_0^{2\pi} e^{i\phi}\,\mathrm{d}\phi = 0$$

$$\langle \Psi_{21-1}|H'|\Psi_{21-1}\rangle = eE_{\text{ext}} \int \left(\frac{1}{8a^2}\sqrt{\frac{1}{a\pi}}\right)^2 r^2e^{-r/a}\sin^2(\theta)e^{-2i\phi}r\cos(\theta)r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\longrightarrow \quad \int_0^{2\pi} e^{-2i\phi}\,\mathrm{d}\phi = 0$$

Now, for the only non-zero element. For this we will need to recall

$$\int_0^{\infty} x^n e^{-x/a}\,\mathrm{d}x = n!a^{n+1}$$

$$\langle \Psi_{200}|H'|\Psi_{210}\rangle = eE_{\text{ext}} \int \left[\frac{1}{\sqrt{2\pi a}}\frac{1}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a}\right]\left[\frac{1}{4a^2}\sqrt{\frac{1}{2\pi a}}re^{-r/2a}\right] r\cos(\theta)r^2$$
$$\begin{aligned}
&\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= 2\pi eE_{\text{ext}}\left(\frac{1}{\sqrt{2\pi a}}\frac{1}{2a}\right)\left(\frac{1}{4a^2}\sqrt{\frac{1}{2a\pi}}\right)\int\left(1-\frac{r}{2a}\right)e^{-r/a}r^4\,\mathrm{d}r\int\cos^2(\theta)\sin(\theta)\,\mathrm{d}\theta \\
&= -eE_{\text{ext}}\left(\frac{1}{8a^4}\right)\left[4!-\frac{5!}{2}\right]a^5\frac{1}{3}\cos^3(\theta)\Big|_0^{\pi} \\
&= -3aeE_{\text{ext}}
\end{aligned}$$

So, our perturbation matrix is:

$$W = (-3aeE_{\text{ext}}) \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding the eigenvalues of this matrix:

$$\begin{aligned}
\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} &= \lambda \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} \\
&= \lambda \begin{vmatrix} 0 & -\lambda \\ 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \\
&= \lambda(\lambda - \lambda^3) = \lambda^2(1 - \lambda^2) = 0 \\
&\longrightarrow \lambda = 0 \qquad \lambda = \pm 1
\end{aligned}$$

So, there are <u>three</u> energy levels:

$$\boxed{E_2, E_2 \pm 3aeE_{\text{ext}}}$$

Question 4

$$\left\langle \Psi_n \middle| \Psi_n \right\rangle$$

$$\left(\left\langle \Psi_n^0\right| + \lambda \left\langle \Psi_n^1\right|\right)\left(\lambda\left|\Psi_n^1\right\rangle + \left|\Psi_n^0\right\rangle\right)$$

$$\lambda\left\langle \Psi_n^0 \middle| \Psi_n^1\right\rangle + \left\langle \Psi_n^0 \middle| \Psi_n^0\right\rangle + \lambda^2\left\langle \Psi_n^1 \middle| \Psi_n^1\right\rangle + \lambda\left\langle \Psi_n^1 \middle| \Psi_n^0\right\rangle$$

$$1 + \lambda\left\langle \Psi_n^0\right|\left(\sum_{m\neq n} c_m^{(n)}\left|\Psi_m^0\right\rangle\right) + \lambda\left(\sum_{m\neq n}\left(c_m^{(n)}\right)^*\left\langle \Psi_m^0\right|\right)\left|\Psi_n^0\right\rangle + O(\lambda^2)$$

$$1 + 0 + 0 + O(\lambda^2)$$

$$1 + O(\lambda^2)$$

Question 5

a) The first thing we need to identify is the degenerate "W" matrix. Since the first two states are degenerate:

$$W = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}$$

Any, by observation, the eigenvalues are:

$$\lambda = \pm\varepsilon$$

solving for the eigenvectors:

$$\begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -\varepsilon & \varepsilon \\ \varepsilon & -\varepsilon \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\varepsilon\alpha + \varepsilon\beta = 0 \qquad -\varepsilon\alpha + \varepsilon\beta = 0$$

$$\alpha = -\beta \qquad \alpha = \beta$$

Putting these into two normalized vectors:

$$\boxed{\begin{aligned} |+\rangle &= \tfrac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \\ |-\rangle &= \tfrac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right) \end{aligned}}$$

b) At this point (with question 3) we are beating a dead horse, so let's go!

$$\begin{aligned}
\langle +|H|+\rangle &= \frac{1}{\sqrt{2}}\left(\langle 1| + \langle 2|\right) H \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \\
&= \frac{1}{2}\left(E_0 + 2\varepsilon + E_0\right) = E_0 + \varepsilon \\
\langle +|H|-\rangle &= \frac{1}{\sqrt{2}}\left(\langle 1| + \langle 2|\right) H \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right) \\
&= \frac{1}{2}\left(E_0 + \varepsilon - \varepsilon - E_0\right) = 0 \\
\langle +|H|3\rangle &= \frac{1}{\sqrt{2}}\left(\langle 1| + \langle 2|\right) H\, |3\rangle \\
&= \frac{1}{\sqrt{2}}\left(\delta + 0\right) = \frac{\delta}{\sqrt{2}} \\
\langle -|H|-\rangle &= \frac{1}{\sqrt{2}}\left(\langle 1| - \langle 2|\right) H \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right) \\
&= \frac{1}{2}\left(E_0 - 2\varepsilon + E_0\right) = E_0 - \varepsilon \\
\langle -|H|3\rangle &= \frac{1}{\sqrt{2}}\left(\langle 1| - \langle 2|\right) H\, |3\rangle \\
&= \frac{1}{\sqrt{2}}\left(\delta + 0\right) = \frac{\delta}{\sqrt{2}} \\
\langle 3|H|3\rangle &= E_1
\end{aligned}$$

Putting this into matrix form:

$$\boxed{H_{|+\rangle,|-\rangle,|3\rangle} = \begin{bmatrix} E_0 + \varepsilon & 0 & \frac{\delta}{\sqrt{2}} \\ 0 & E_0 - \varepsilon & \frac{\delta}{\sqrt{2}} \\ \frac{\delta}{\sqrt{2}} & \frac{\delta}{\sqrt{2}} & E_1 \end{bmatrix}}$$

c) Using $\sum_{m\neq n} \frac{\left\langle \Psi_m^0 \middle| H \middle| \Psi_n^0 \right\rangle}{E_n^0 - E_m^0} \Psi_m^0$:

$$\Psi_+^1 = \frac{\left\langle \Psi_-^0 \middle| H \middle| \Psi_+^0 \right\rangle}{E_+^0 - E_-^0} \Psi_-^0 + \frac{\left\langle \Psi_3^0 \middle| H \middle| \Psi_+^0 \right\rangle}{E_+^0 - E_3^0} \Psi_3^0 = \frac{\delta}{\sqrt{2}\left(E_0 + \varepsilon - E_1\right)} \left|3\right\rangle$$

$$\boxed{\Psi_1^1 = \frac{\delta}{\sqrt{2}\left(E_0 + \varepsilon - E_1\right)} \left|3\right\rangle}$$