PHSX 461: Exam 2

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Question 1

a)

$$\langle x \rangle = \int \phi^*(p)\hat{x}\phi(p)\,\mathrm{d}p$$

$$= \int \phi(p)\Big(i\hbar\frac{\partial}{\partial p}\Big)\phi(p)\,\mathrm{d}p$$

$$\boxed{\langle x \rangle = (i\hbar)\int \phi^*(p)\frac{\partial}{\partial p}\phi(p)\,\mathrm{d}p}$$

$$\langle p \rangle = \int \phi^*(p)\hat{p}\phi(p)\,\mathrm{d}p$$

$$= \int \phi^*(p)\phi(p)p\,\mathrm{d}p$$

$$\boxed{\langle p \rangle = \int |\phi(p)|^2 p\,\mathrm{d}p}$$

b) We need to show that $\hat{p}\,|p_0\rangle = p_0\,|p_0\rangle$ We can do this by a simple proof by contradiction. If we assume that $\hat{p}\,|p_0\rangle \neq p_0\,|p_0\rangle$, or in functional form: $\hat{p}\phi_{p_0}(p) \neq p_0\phi_{p_0}(p)$, then $\int \hat{p}\phi_{p_0}(p) \neq \int p_0\phi_{p_0}(p)$. Looking at the left side:

$$\int \hat{p}\phi_{p_0}(p)\,\mathrm{d}p$$

$$= \int p\delta(p-p_0)\,\mathrm{d}p$$
$$= p_0$$

Now, looking at the right side:

$$\int p_0\phi_{p_0}(p)\,\mathrm{d}p$$

$$= p_0\int \delta(p-p_0)\,\mathrm{d}p$$
$$= p_0$$

But, we said that there were not equal, so we have reached a contradiction and it would thus be logical to say that ϕ_{p_0} is the eigenfunction of $\hat{p}$

c)

$$\langle p_0|p_0\rangle \Longrightarrow \int \phi_{p_0}^*\phi_{p_0}\,\mathrm{d}p$$

$$= \int \delta(p-p_0)\delta(p-p_0)\,\mathrm{d}p$$

$$= \int \delta^2(p-p_0)\,\mathrm{d}p$$

$$= \int \delta(p-p_0)\,\mathrm{d}p = 1$$

This fits the idea that $\langle \mathbf{e}_m|\mathbf{e}_n\rangle = \delta_{mn}$ from dirac orthonormailty. Since p_0 is an eigen value and $|p_0\rangle$ is its eigen state, it is very much **physically recognizable**!

d) transfer over using Fourier transform:

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}}\int \phi(k)e^{i(kx-\frac{\hbar k^2}{2m}t)}\,\mathrm{d}k$$

$$= \frac{1}{\sqrt{2\pi}}\int \phi(p)e^{i(\frac{p}{\hbar}x-\frac{p^2}{2\hbar m}t)}\frac{1}{\hbar}\,\mathrm{d}p$$

$$= \frac{1}{\hbar\sqrt{2\pi}}\int \delta(p-p_0)e^{i(\frac{p}{\hbar}x-\frac{p^2}{2\hbar m}t)}\,\mathrm{d}p$$

$$= \frac{1}{\hbar\sqrt{2\pi}}e^{i(\frac{p_0}{\hbar}x-\frac{p_0^2}{2\hbar m}t)}$$

$$\boxed{= \frac{1}{\hbar\sqrt{2\pi}} \exp\left(i\frac{p_0}{\hbar}x\right) \exp\left(-i\frac{p_0^2}{2\hbar m}t\right)}$$

You can see that it "wiggles" in the x and has the quivalent of $\exp(-iE_n t/\hbar)$ where the $E_0 = p_0^2/em$ ($\hat{T} = \hat{p}^2/2m$).

e) I would expect $\langle p \rangle \Rightarrow \langle p|p_0|p\rangle = p_0$

$$\Rightarrow \int \phi^*_{p_0}(p)\hat{p}\phi(p)_{p_0}\,\mathrm{d}p$$

$$= \int \phi^*_{p_0}(p)\phi(p)_{p_0} p\,\mathrm{d}p$$

$$= \int \delta(p-p_0)\delta(p-p_0)p\,\mathrm{d}p$$

$$= p\Big|_{p_0} = p_0$$

$$\boxed{\langle p \rangle = p_0}$$