PHSX 425: HW12

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In class, we developed a calculation technique for working out the transmission and reflection of optical coatings at normal incidence (which we defined as the z-axis

Question 1

The boundary conditions I assumed for the electric field were $E_1 = E_2$ *and* $\partial E_1/\partial z = \partial E_2/\partial z$*. Must I assume anything about materials 1 and 2 for these boundary conditions to be correct?*

Since we are perpendicular to the boundary, then we do not need to take into account the parallel restriction:

$$\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$$

We do, however, need to account for the perpendicular part of the B-field.

$$\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$$

So, this would mean that $\mu_1 = \mu_2$ since the B-field must also be continuous. We can see this through the derivative boundary condition; that the E, and B, field must stay in sync at the boundary.

Question 2

Suppose a substrate ($n = n_0$) is coated with a material $n = n_1$. Both indexes of refraction are real. The thickness d_1 of the coatings is 1/4 wave, i.e. $n_1 k_{vac} d_1 = \pi/2$. Light enters the coating from vacuum ($n_2 = 1$) at normal incidence.

a) *What value of n_1 would result in zero reflection form the coating? In that case, what fraction of the power would be transmitted into the substrate?*

From lecture

$$E_{F,j+1} = \frac{1}{2}\left(1 + \frac{n_j}{n_{j+1}}\right)E'_{F,j} + \frac{1}{2}\left(1 - \frac{n_j}{n_{j+1}}\right)E'_{R,j} \qquad \text{(Equation 5)}$$

$$E_{R,j+1} = \frac{1}{2}\left(1 - \frac{n_j}{n_{j+1}}\right)E'_{F,j} + \frac{1}{2}\left(1 + \frac{n_j}{n_{j+1}}\right)E'_{R,j} \qquad \text{(Equation 6)}$$

$$E'_{F,j} = E_{F,j}e^{-in_j k_{vac} d_j} \qquad \text{(Equation 7)}$$

$$E'_{R,j} = E_{R,j}e^{in_j k_{vac} d_j} \qquad \text{(Equation 8)}$$

If we want zero reflection, then we just set that value (E_{R_2}) equal to zero. From Equation 6:

$$0 = \frac{1}{2}\left(1 - \frac{n_1}{n_2}\right)E'_{F,1} + \frac{1}{2}\left(1 + \frac{n_1}{n_2}\right)E'_{R,1}$$

$$(n_1 - n_2)E'_{F,1} = (n_1 + n_2)E'_{R,1} \qquad (1)$$

Using both Equation 5 and Equation 6 for the interface between the coating and the substrate:

$$E_{F,1} = \frac{1}{2}\left(1 + \frac{n_0}{n_1}\right)E'_{F,0} + \frac{1}{2}\left(1 - \frac{n_0}{n_1}\right)E'_{R,0}$$

$$E_{R,1} = \frac{1}{2}\left(1 - \frac{n_0}{n_1}\right)E'_{F,0} + \frac{1}{2}\left(1 + \frac{n_0}{n_1}\right)E'_{R,0}$$

Recognizing that the substrate can be thought of as infinitely thick we can say that $E'_{R,0}$, and thus the two equations above become:

$$E_{F,1} = \frac{1}{2}\left(1 + \frac{n_0}{n_1}\right)E'_{F,0} \qquad E_{R,1} = \frac{1}{2}\left(1 - \frac{n_0}{n_1}\right)E'_{F,0}$$

$$E_{F,1}\Big(2n_1\frac{1}{n_1+n_0}\Big)E'_{F,0}$$

$$\begin{aligned} E_{R,1} &= \frac{1}{2}\Big(\frac{n_1-n_0}{n_1}\Big)\Big(\frac{2n_1}{n_1+n_0}\Big)E_{F,1} \\ &= \frac{n_1-n_0}{n_1+n_0}E_{F,1} \end{aligned}$$

These two statements of E refer to different sides of the material. Equation 7 and Equation 8 can be used to relate the two:

$$E'_{F,1} = E_{F,1}e^{-in_1k_{vac}d_1} = E_{F,1}e^{-i\pi/2} = -iE_{F,1} \rightarrow E_{F,1} = iE'_{F,1} \tag{2}$$

$$E'_{R,1} = E_{R,1}e^{+in_1k_{vac}d_1} = E_{R,1}e^{+i\pi/2} = iE_{R,1} \rightarrow E_{R,1} = -iE'_{R,1} \tag{3}$$

$$-iE'_{R,1} = i\frac{n_1-n_0}{n_1+n_0}E'_{F,1}$$

$$E'_{R,1} = \frac{n_0-n_1}{n_1+n_0}E'_{F,1}$$

plugging this into the (1):

$$(n_1-n_2)E'_{F,1} = (n_1+n_2)\frac{n_0-n_1}{n_0+n_1}E'_{F,1}$$

$$(n_1-n_2) = (n_1+n_2)\Big(\frac{n_0-n_1}{n_0+n_1}\Big)$$

By simple algebra:

$$2n_1^2 = 2n_2n_0$$

$$\implies \boxed{n_1 = \sqrt{n_0}}$$

$$iE'_{F,1} = \frac{1}{2}\Big(1+\frac{n_0}{n_1}\Big)E'_{F,0} \rightarrow E'_{F,1} = -\frac{i}{2}\Big(1+\frac{n_0}{n_1}\Big)E'_{F,0}$$

$$-iE'_{R,1} = \frac{1}{2}\Big(1-\frac{n_0}{n_1}\Big)E'_{F,0} \rightarrow E'_{R,1} = \frac{i}{2}\Big(1-\frac{n_0}{n_1}\Big)E'_{F,0}$$

$$E_{F,2} = \frac{1}{2}\Big(1+\frac{n_1}{n_2}\Big)\Big(\frac{-i}{2}\Big(1+\frac{n_0}{n_1}\Big)E'_{F,0}\Big) + \frac{1}{2}\Big(1-\frac{n_1}{n_2}\Big)\Big(\frac{i}{2}\Big(1-\frac{n_0}{n_1}\Big)E'_{F,0}\Big)$$

$$= \frac{-i}{4n_1n_2}[(n_2+n_1)(n_1+n_0)+(n_1-n_2)(n_1-n_0)]E'_{F,0}$$

$$= -\frac{i}{2n_1n_2}[n_1^2+n_2n_0]$$

To find the transmission coefficient, we need to find I_T/I_I. We can combine

$$I = \frac{1}{2}\varepsilon v E^2 \quad v = \frac{1}{\sqrt{\varepsilon\mu}} \quad n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

$$I = \frac{1}{2}\frac{n\sqrt{\varepsilon_0\mu_0}}{\mu}E^2$$

$$\frac{I_T}{I_I} = T = \frac{\frac{1}{2}\frac{n_0\sqrt{\varepsilon_0\mu_0}}{\mu'_0}(E'_{F,0})^2}{\frac{1}{2}\frac{n_2\sqrt{\varepsilon_0\mu_0}}{\mu'_2}(E'_{F,2})^2}$$

$$T = \frac{n_0}{n_2}\frac{\mu'_2}{\mu'_2}\Big(\frac{-i}{2n_1n_2}[n_1^2+n_2n_0]\Big)^{-2}$$

$$n_2 = 1 \qquad n_1 = \sqrt{n_0}$$

$$\Rightarrow \frac{n_0}{1}\Big(\frac{1}{2\sqrt{n_0}}[n_0+n_0]\Big)^{-2}$$

$$\boxed{T = n_0(\sqrt{n_0})^{-2} = 1}$$

This makes sense, as nothing is reflected.

b) *Find the reflectivity coefficient for a 1/4 wave coating of* MgF_2 ($n = 1.38$) *on BK7 glass* ($n = 1.52$). *What would be the reflectivity of glass without the coating?*

$$E_{F,2} = \frac{1}{2}\Big(1+\frac{n_1}{n_2}\Big)\frac{i}{2}\Big(1+\frac{n_0}{n_1}\Big)E'_{F,0} + \frac{1}{2}\Big(1-\frac{n_1}{n_2}\Big)\frac{-i}{2}\Big(1-\frac{n_0}{n_1}\Big)E'_{F,0}$$

$$= \frac{i}{4}\Big[\Big(1+\frac{n_1}{n_2}\Big)\Big(1+\frac{n_0}{n_1}\Big) - \Big(1-\frac{n_1}{n_2}\Big)\Big(1-\frac{n_0}{n_1}\Big)\Big]E'_{F,0}$$

$$= \frac{i}{4n_1n_2}[(n_2+n_1)(n_1+n_0)-(n_2-n_1)(n_1-n_0)]E'_{F,0}$$

$$= \frac{i}{4n_1n_2}(2n_0n_2+2n_1^2)E'_{F,0}$$

$$E_{F,2} = \frac{i}{2n_1n_2}(n_0n_2+n_1^2)E'_{F,0} \tag{4}$$

$$E'_{R,2} = \frac{1}{2}\left(1-\frac{n_1}{n_2}\right)\frac{i}{2}\left(1+\frac{n_0}{n_1}\right)E'_{F,0} + \frac{1}{2}\left(1+\frac{n_1}{n_2}\right)\frac{-i}{2}\left(1-\frac{n_0}{n_1}\right)E'_{F,0}$$

$$= \frac{i}{4n_1n_2}[(n_2-n_1)(n_1+n_0)-(n_2+n_1)(n_1-n_0)]E'_{F,0}$$

$$E'_{R,2} = \frac{i}{2n_1n_2}(n_2n_0-n_1^2)E'_{F,0} \tag{5}$$

plugging in (4) into (5):

$$E'_{R,2} = \frac{i}{2n_1n_2}[n_2n_0-n_1^2]\frac{1}{\frac{i}{2n_1n_2}[n_0n_2+n_1^2]}E'_{F,2}$$

$$= \frac{n_2n_0-n_1^2}{n_2n_0+n_1^2}E'_{F,2}$$

Using the equation for the intensity from above, and the fact that $\mu'_0 = \mu_2$ for the boundary conditions:

$$T = \frac{n_0}{n_2}\frac{\mu_2}{\mu'_0}\left[\frac{i}{2n_1n_2}(n_0n_2+n_1^2)\right]^{-2} = \frac{n_0}{n_2}\left[\frac{2n_1n_2}{n_0n_2+n_1^2}\right]^2$$

$$T = n_0n_2\left[\frac{2n_1}{n_0n_2+n_1^2}\right]^2$$

$$R = \frac{n_2}{n_2}\frac{\mu_2}{\mu_2}\left[\frac{n_2n_0-n_1^2}{n_2n_0+n_1^2}\right]^2 = \left[\frac{n_2n_0-n_1^2}{n_2n_0+n_1^2}\right]^2$$

Simply plugging in $n_2 = 1$, $n_1 = 1.38$ and $n_0 = 1.52$

$$\boxed{T = 0.9874} \qquad \boxed{R = 0.0126}$$

If instead we plug in $n_1 = n_2 = 1$, and $n_0 = 1.52$

$$\boxed{T = 0.9574} \qquad \boxed{R = 0.0426}$$