PHSX 461: HW07

William Jardee

November 4, 2021

3.7

a) *Suppose that $f(x)$ and $g(x)$ are two eigenfunctions of an operator $\hat{Q}$, with the same eigenvalue q. Show than any linear combination of f and g is itself an eigenfunction of $\hat{Q}$, with eigenvalue q.*

$$\hat{Q}\,|f\rangle = q\,|f\rangle \quad , \quad \hat{Q}\,|g\rangle = q\,|g\rangle$$
$$|\alpha\rangle = a\,|f\rangle + b\,|b\rangle$$

$$\begin{aligned}\hat{Q}\,|\alpha\rangle &= \hat{Q}[a\,|f\rangle + b\,|g\rangle] \\ &= \hat{Q}(a\,|f\rangle) + \hat{Q}(b\,|g\rangle) \\ &= a\hat{Q}\,|f\rangle + b\hat{Q}\,|g\rangle \\ &= aq\,|f\rangle + bq\,|g\rangle \\ &= q[a\,|f\rangle + b\,|g\rangle] = q\,|\alpha\rangle\end{aligned}$$

b) *Check that $f(x) = exp(x)$ and $g(x) = exp(-x)$ are eigenfunctions of the operator d^2/dx^2, with the same eigenvalue. Construct two linear combinations of f and g that are orthogonal eigenfunctions on the interval $(-1, 1)$.*

$$\hat{Q}\,|f\rangle \Rightarrow \frac{\partial^2}{\partial x^2}e^x = e^x \rightarrow q = 1$$
$$\hat{Q}\,|g\rangle \Rightarrow \frac{\partial^2}{\partial x^2}e^{-x} = (-1)^2e^{-x} = e^{-x} \rightarrow q = 1$$

$$|\alpha\rangle = a\,|f\rangle + b\,|g\rangle \quad , \quad |\beta\rangle = c\,|f\rangle + d\,|g\rangle$$

For orthogonality, we need $\langle\alpha|\beta\rangle = \int \alpha(x)^*\beta(x)\,\mathrm{d}x = 0$

$$0 = \int_{-1}^{1} (a^* f^*(x) + b^* g^*(x))^* (cf(x) + bg(x))\,\mathrm{d}x$$

$$0 = \int_{-1}^{1} (ae^{x} + be^{-x})(ce^{x} + de^{-x})\,\mathrm{d}x$$

$$0 = \int_{-1}^{1} [ace^{2x} + ad + bc + bde^{-2x}]\,\mathrm{d}x$$

$$0 = \frac{ac}{2}e^{2x} + [ad+bc]x - \frac{bd}{2}e^{-2x}\bigg|_{-1}^{1}$$

$$0 = \frac{ac}{2}e^{2} + [ad+bc] - \frac{bd}{2}e^{-2} - \left[\frac{ac}{2}e^{-2} - [ad+bc] - \frac{bd}{2}e^{2}\right]$$

$$0 = e^{2}\left[\frac{ac}{2} + \frac{bd}{2}\right] + e^{-2}\left[\frac{-bd}{2} + \frac{-ac}{2}\right] + 2[ad+bc]$$

The easiest solution to this is if we let $ac = -bd$. Two solutions to this question yield the linear combination:

$$|\alpha\rangle = |f\rangle + |g\rangle \quad , \quad |\beta\rangle = |f\rangle - |g\rangle$$

$$|\alpha\rangle = -\,|f\rangle + |g\rangle \quad , \quad |\beta\rangle = |f\rangle + |g\rangle$$

3.9

a) *Cite a Hamiltonian from Chapter 2 (other than the harmonic oscillator) that has only a discrete spectrum.*

b) *Cite a Hamiltonian from Chapter 2 (other than the free particle) that has only a continuous spectrum.*

c) *Cite a Hamiltonian from Chapter 2 (other than the finite square well) that has both a discrete and a continuous part to its spectrum.*

3.13

Show that

$$\langle x\rangle = \int \Phi^* \left(i\hbar \frac{\partial}{\partial p} \right) \Phi \, \mathrm{d}p$$

Let us start by defining what Φ is:

$$\Phi = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar} \Psi(x,t)$$

So, let's plug this in and just run the calculus:

$$\int \Phi^* \left(i\hbar \frac{\partial}{\partial p} \right) \Phi \, \mathrm{d}p$$

$$= \frac{i\hbar}{2\pi\hbar} \int \left[\int e^{ipx/\hbar} \Psi \, \mathrm{d}x \right]^* \frac{\partial}{\partial p} \left[\int e^{ipx'/\hbar} \Psi \, \mathrm{d}x' \right]$$

$$= \frac{i}{2\pi} \int \left[\int e^{ipx/\hbar} \Psi^* \, \mathrm{d}x \right] \left[\int \frac{-ix'}{\hbar} e^{ipx'/\hbar} \Psi \, \mathrm{d}x' \right]$$

Since these three variables, p, x, x', are all independent, they can be moved in and out of each other's integrals.

$$= \frac{1}{2\pi\hbar} \int e^{-i(\frac{p}{\hbar}(x-x')} \Psi^*(x,t) x' \Psi(x',t) \, \mathrm{d}p \, \mathrm{d}x \, \mathrm{d}x'$$

$$= \frac{\hbar}{\hbar} \delta\left[\frac{p}{\hbar}(x-x') \right] \int \Psi^*(x,t) x' \Psi(x',t) \, \mathrm{d}x \, \mathrm{d}x' \Big|_{-\infty}^{\infty}$$

$$= \int \Psi^*(x,t) x \Psi(x,t) \, \mathrm{d}x = \langle x\rangle$$

3.26

Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle, |2\rangle, |3\rangle$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle, \qquad |\beta\rangle = i|1\rangle + 2|3\rangle$$

a) *Construct $\langle\alpha|$ and $\langle\beta|$ (in terms of the dual basis $\langle 1|, \langle 2|, \langle 3|$).*

We can think of $|\alpha\rangle$ and $|\beta\rangle$ as wavefunctions, since the mechanics here are the same (orthonormal basis, linear combinations of states, ...). So, pulling that analog:

$$|\Psi\rangle = \sum c_n |f_n\rangle \Rightarrow \langle\Psi| = \sum c_n^* \langle f_n|$$

and thus:

$$\langle\alpha| = -i\langle 1| - 2\langle 2| + i\langle 3| \quad , \quad \langle\beta| = -i\langle 1| + 2\langle 3|$$

b) *Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$, and confirm that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.*

$$\begin{aligned}\langle\alpha|\beta\rangle &= (-i\langle 1| - 2\langle 2| + i\langle 3|)(i|1\rangle + 2|3\rangle) \\ &= -i^2\langle 1|1\rangle + 2i\langle 3|3\rangle \\ &= 1 + 2i\end{aligned}$$

$$\begin{aligned}\langle\beta|\alpha\rangle &= (-i\langle 1| + 2\langle 3|)(i|1\rangle - 2|2\rangle - i|3\rangle) \\ &= -i^2\langle 1|1\rangle - 2i\langle 3|3\rangle \\ &= 1 - 2i\end{aligned}$$

$$\langle\alpha|\beta\rangle^* = (1+2i)^* = 1 - 2i = \langle\beta|\alpha\rangle$$

c) *Final all nine matrix elements fo the operator $\hat{A} = |\alpha\rangle\langle\beta|$, in this basis, and construct the matrix A. Is it hermitian?*

$$\hat{Q} \Rightarrow Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots \\ Q_{21} & Q_{22} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}, \text{ where } Q_{mn} = \langle e_m | \hat{Q} | e_n \rangle$$

We will use this to calculate the A_{13} element, then point out the pattern and fill out the rest of the matrix.

$$A_{13} = \langle 1|\alpha\rangle \langle \beta|3\rangle$$

$$= \langle 1| (i\,|1\rangle - 2\,|2\rangle - i\,|3\rangle)(-i\,\langle 1| + 2\,\langle 3|)\,|3\rangle$$

$$= (i)(2) = 2i$$

So, an element A_{mn} can be found by taking the product of the mth coefficient from $|\alpha\rangle$ and the nth coefficient form $\langle\beta|$. Using this method:

$$\hat{A} = \begin{pmatrix} (i)(-i) & (i)(0) & (i)(2) \\ (-2)(-i) & (-2)(0) & (-2)(2) \\ (-i)(-i) & (-i)(0) & (-i)(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}$$

To test if this is hermitian, $\hat{A}^\dagger = \hat{A}$

$$\hat{A}^\dagger = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}^{*T} = \begin{pmatrix} 1 & -2i & -1 \\ 0 & 0 & 0 \\ -2i & -4 & -2i \end{pmatrix} \neq \hat{A}$$

So, A is **not** hermitian.

Question 5.

Prove that the momentum operator, $\hat{p}$ is Hermitian.
Hint: *you will need to assume that any functions you use are normalizable. You may also use the results from the previous homework assignment.*

We want to show that $\hat{p}^\dagger = \hat{p}$. To do this, first we can identify that a hermitian operator in momentum space is also a hermitian operator in position space; so, we will be doing this problem with Φ, instead of Ψ.

$$\langle \Phi | \hat{p}^\dagger | \Phi \rangle$$

$$= \langle \hat{p}\Phi | \Phi \rangle$$

$$= \int (\hat{p}\Phi)^* \Phi \, \mathrm{d}p$$

$$= \int (p\Phi)^* \Phi \, \mathrm{d}p$$

$$= \int \Phi^* p \Phi \, \mathrm{d}p$$

$$= \int \Phi^* \hat{p} \Phi \, \mathrm{d}p$$

$$= \langle \Phi | \hat{p} | \Phi \rangle \to \hat{p}^\dagger = \hat{p}$$

3.33

An operator $\hat{A}$, representing observable A, has two (normalized) eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable B, has two (normalized) eigenstates ϕ_1 and ϕ_2, with eigenvalues b_1 and b_2. The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5, \qquad \psi_2 = (4\phi_1 - 3\phi_2)/5$$

a) *Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately after this measurement?*

The wavefunction has been observed and thus has been collapsed to ψ_1. So, the state is ϕ_1

b) *If B is now measure, what are the possible results, and what are their probabilities?*

Since we are completely in state ψ_1, the probabilities will only be decedents of that equation. There will be a $(\frac{3}{5})^2 = \frac{9}{25} = 35\%$ chance for b_1 and a $(\frac{4}{5})^2 = \frac{16}{25} = 65\%$ chance for b_2.

c) *Right after the measurement of B, A is measured again. What is the probability of getting a_1? (Note that the answer would be quite different if I had told you the outcome of the B measurement.*

If we are asking for the probability of a_1, we are thinking about ψ_1:

$$\begin{aligned}\psi_1^2 &= \frac{9\phi_1^2 + 24\phi_1\phi_2 + 16\phi_2^2}{25} \\ &= \frac{3^4 + 23^2 4^2 + 4^4}{5^4} = 1\end{aligned}$$

To do a quick check to make sure that we don't need to renormalize:

$$\begin{aligned}\psi_2^2 &= \frac{16\phi_1^2 - 24\phi_1\phi_2 + 9\phi_2^2}{25} \\ &= \frac{3^2 4^2 - 23^2 4^2 + 3^2 4^2}{5^4} = 0\end{aligned}$$

So, it would seem that there is a 100% chance that we get back a_1.

But what about that hint? What would it be then? Well, if we say that $B = b_1$, $\phi_1 = 1$ and $\phi_2 = 0$. Thus, $\psi_1^2 = \frac{9}{25} = 35\%$ and $\psi_2^2 = \frac{16}{25} = \frac{16}{25} = 65\%$. This is a very different answer from what we got, so the "100%" is consistent with the hint.