Linear Algebra Homework 4

William Jardee

March 2020

1. Prove Theorem 4.21(c)"

"If $A \sim B$ and $B \sim C$, then $A \sim C$"

Proof:

Let A, B, and C be $n \times n$ matrices. If $A \sim B$ and $B \sim C$, then by definition of similar matrices: $A = P^{-1}BP$ and $B = Q^{-1}CQ$.By substitution: $A = P^{-1}Q^{-1}CPQ$. It is evident that $(P^{-1}Q^{-1})(PQ) = I$, so $(P^{-1}Q^{-1})$ and (PQ) are inverses. Renaming PQ as S, $A = S^{-1}CS$. By definition of similar matrices, $A \sim C$.

"Quack" ■

2. Prove Theorem 4.22(b)

"Let A and B be $n \times n$ matrices with $A \sim B$, then A is invertible if and only if B is invertible."

Proof:

Let A and B be $n \times n$ matrices such that $A \sim B$. By definition of similar matrices, $A = P^{-1}BP$. If we assume A is invertible, it follows that

$$A^{-1} = (P^{-1}BP)^{-1} = PB^{-1}P^{-1}$$

So in order for A to be invertible, then B must be invertible. The equality proves if and only if relationship.

"Quack" ■

3. Prove Theorem 4.22(f)

"Let A and B be $n \times n$ matrices with $A \sim B$, then $A^m \sim B^m$ for all integers $m \geq 0$."

Proof:

Let A and B be $n \times n$ matrices such that $A \sim B$. This is be a proof by induction. Let us first establish our base case.

Base Case: By definition of similar matrices, $A = P^{-1}BP$. Then $A^2 = AA = P^{-1}BPP^{-1}BP = P^{-1}BBP = P^{-1}B^2P$. So $A^2 \sim B^2$. The case holds for $m = 2$.

Inductive Step: Let us assume that $A^k \sim B^k$ for some integer $k \geq 2$. Then $A^{k+1} = AA^k = P^{-1}BPP^{-1}B^kP = P^{-1}BB^kP = P^{-1}B^{k+1}P$. So if $A^k \sim B^k$, then $A^{k+1} \sim B^{k+1}$ for all $k \geq 2$.

To account for $m = 1$, this is our original assumption, so this must be true. For any $n \times n$ matrix Q, $Q^0 = I$, so $A^0 = I = P^{-1}P = P^{-1}IP = P^{-1}B^0P$. So $A^0 \sim B^0$. The statement can then be generalized to $A^m \sim B^m$ for all integers $m \geq 0$.

"Quack" ■

4. Let A and B be $n \times n$ matrices, each with n distinct eigenvalues. Prove that A and B have the same eigenvectors if and only if $AB = BA$.

   Proof:

   Let A and B be two $n \times n$ matrices with n distinct eigenvalues. We know that A and B are diagonalizable since their are the same number of eigenvalues as the number of dimensions. So, $A = P^{-1}\Lambda P$ and $B = Q^{-1}\Lambda Q$. So $AB = P^{-1}\Lambda PQ^{-1}\Lambda Q$. If $P = Q$, that is to say they have the same eigenvectors, then $AB = Q^{-1}\Lambda QP^{-1}\Lambda P = BA$. If instead of assuming $P = Q$, we assume $AB = BA$, then the following holds. $AB = P^{-1}\Lambda PQ^{-1}\Lambda Q = BA = Q^{-1}\Lambda QP^{-1}\Lambda P$. We see that this is equivalent to substituting P in for Q, and Q for P, so $P = Q$. Both directions hold, so an if and only if relationship exists between the two vectors being diagonalizable and their eigenvectors being equal.

   "Quack" ■

5. $$x' = -0.8x + 0.4y$$
   $$y' = 0.4x - 0.2y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0.8 & 0.4 \\ 0.4 & -0.2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.8 & 0.4 \\ 0.4 & -0.2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Char Eq:
$(-0.8 - \lambda)(-0.2 - \lambda) - (0.4)(0.4) = 0$
$(0.16 + \lambda + \lambda^2) - 0.16 = 0$
$\lambda^2 + \lambda = 0 = \lambda(\lambda + 1)$
$\lambda = 0, \qquad \lambda = -1$
$\lambda = 0 : -0.8x + 0.4y = 0 \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$\lambda = -1 : -x = -0.8x + 0.4y \to \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
General Solution: $\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{0t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
Since we are asking about activity at ∞, we can disregard the e^{0t} term.

Instead we can just focus on the solution with respect to that other eigenvector.

Plugging in the initial condition of $\begin{bmatrix} 10 \\ 15 \end{bmatrix}$: $C_2 = \frac{35}{4}$, so the solution as we approach infinity looks like: $\frac{35}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, or just the line $\begin{bmatrix} 1 \\ 2 \end{bmatrix} t$.

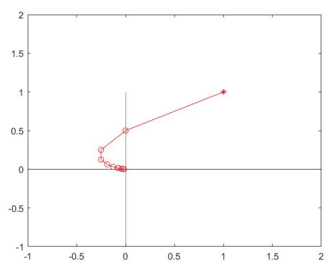
6. $A = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.5 \end{bmatrix}$

   (a) $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$

   $\mathbf{x}_2 = \begin{bmatrix} -0.125 \\ 0.125 \end{bmatrix}$

   $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0.125 \end{bmatrix}$

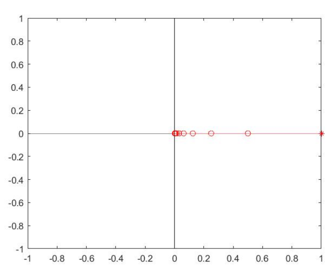


   (b) $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   $\mathbf{x}_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$

   $\mathbf{x}_2 = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$

   $\mathbf{x}_3 = \begin{bmatrix} 0.125 \\ 0 \end{bmatrix}$



(c) $\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.5 \end{bmatrix}$

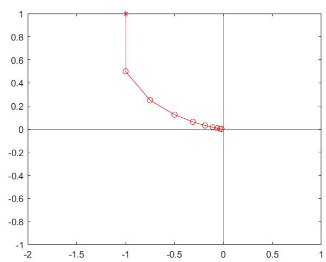
$(0.5 - \lambda)^2 = 0$
$\lambda = 0.5$
eigenspace $= \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})$

So the general solution is $\mathbf{x}_0 = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, or for $\mathbf{x}_k$,

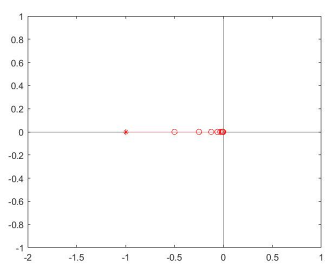
$\mathbf{x}_k = C_1 0.5^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 0.5^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

As the $\lim_{k\to\infty} \mathbf{x}^k \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all C_1, C_2, x_0. So the origin is an attractor.

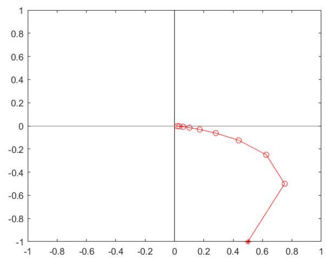
(d) for $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$:



for $x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$:



for $x = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$:



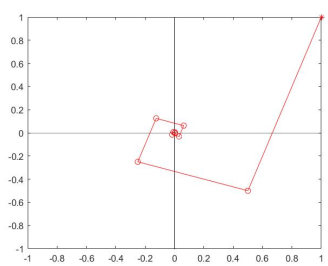
7. $A = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}$

(a) $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathbf{x}_1 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$

$\mathbf{x}_2 = \begin{bmatrix} -0.25 \\ 0.25 \end{bmatrix}$

$\mathbf{x}_3 = \begin{bmatrix} 0.125 \\ -0.125 \end{bmatrix}$

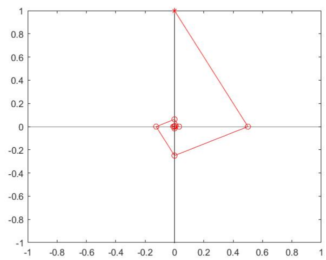


(b) $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathbf{x}_1 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}$

$\mathbf{x}_2 = \begin{bmatrix} -0.25 \\ 0 \end{bmatrix}$

$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0.125 \end{bmatrix}$



(c) $(-\lambda)^2 + 0.25 = 0$
$\lambda = \sqrt{-0.25} = 0.5i$

$$0.5i \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

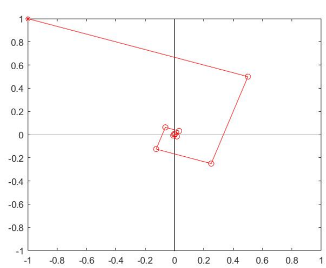
$(0.5i)x = 0.5y \rightarrow ix = y$, so the eigenvalue for $\lambda = 0.5i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$

$(-0.5i)x = 0.5y \rightarrow -ix = y$, so the eigenvalue for $\lambda = -0.5i$ is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

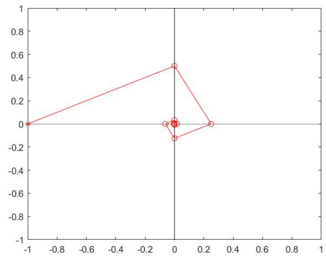
So the general solution is $\mathbf{x}_0 = C_1(0.5i) \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2(-0.5i) \begin{bmatrix} 1 \\ -i \end{bmatrix}$, or for $\mathbf{x}_k$: $\mathbf{x}_k = C_1(0.5i)^k \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2(-0.5i)^k \begin{bmatrix} 0 \\ -i \end{bmatrix}$

As the $\lim_{k\to\infty} \mathbf{x}^k \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all C_1, C_2, x_0. So the origin is an attractor.

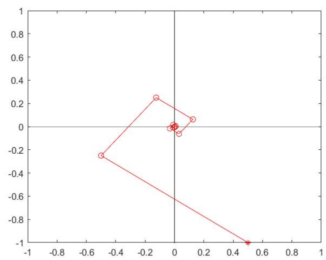
(d) for $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$:



for $x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$:



for $x = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$:



8. $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}$

   We know by theorem 4.30, if we have a Markov matrix, which we do, then there is only one positive eigenvalue, which is 1.

   $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
   $$6 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 2 & 4 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

   Char Eq:
   $$0 = \begin{vmatrix} -3 & 2 & 2 \\ 0 & -4 & 4 \\ 3 & 2 & -6 \end{vmatrix} = \begin{vmatrix} -3 & 2 & 2 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{vmatrix} = \begin{vmatrix} -3 & 2 & 2 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{vmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{3}z \\ z \\ 0 \end{bmatrix}$$

   So the eigenvector, and steady state vector, is $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$.

9. $L = \begin{bmatrix} 1 & 5 & 3 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \end{bmatrix}$

We know there is one positive eigenvalue, and one positive eigenvector, by theorem 4.35.

Char eq: $0 = (1-\lambda)\lambda^2 - 5(\frac{-\lambda}{3}) + 3(\frac{2}{3}) = \lambda^3 - \lambda^2 - \frac{5}{3}\lambda + \frac{2}{3}$

The only positive solution to the above equation is: $\lambda = 2$.

Solving for the eigenvector:

$$2\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$6\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 15 & 9 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$0 = \begin{bmatrix} -3 & 15 & 9 \\ 1 & -6 & 0 \\ 0 & 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & -3 & 9 \\ 1 & -6 & 0 \\ 0 & 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -6 & 0 \\ 0 & 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So the eigenvector looks like: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} 1 \\ \frac{1}{6} \\ \frac{1}{18} \end{bmatrix} \to \begin{bmatrix} 18 \\ 3 \\ 1 \end{bmatrix}$

So the positive eigenpair is $(2, \begin{bmatrix} 18 \\ 3 \\ 1 \end{bmatrix})$.