PHSX 462: HW04

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Griffiths 7.19

a) This is pretty straightforward, if we realize that $\vec{L}$ and $\vec{S}$ act on different basis, so they will commute. I will also do the general calculation for only one component of L, as the other two have an identical derivation.

$$
\begin{aligned}
\left[\vec{L}\cdot\vec{S},\vec{L}\right] &= (\vec{L}\cdot\vec{S})\vec{L} - \vec{L}(\vec{L}\cdot\vec{S}) \\
&= (L_xS_x + L_yS_y + L_zS_z)\vec{L} - \vec{L}(L_xS_x + L_yS_y + L_zS_z) \\
&= (L_xS_xL_x + L_yS_yL_x + L_zS_zL_x - L_xL_xS_x - L_xL_yS_y - L_xL_zS_z)\hat{x} + \cdots \\
&= (L_x^2S_x + L_yL_xS_y + L_zL_xS_z - L_x^2S_x - L_xL_yS_y - L_xL_zS_z)\hat{x} + \cdots \\
&= \left[(L_yL_x - L_xL_y)S_y + (L_zL_x - L_xL_z)S_z\right]\hat{x} + \cdots \\
&= \left[-i\hbar L_zS_y + i\hbar L_yS_z\right]\hat{x} + \left[i\hbar L_xS_z - i\hbar L_zS_x\right]\hat{y} + \left[i\hbar L_xS_y - i\hbar L_yS_x\right]\hat{z} \\
&= i\hbar\vec{L}\times\vec{S}
\end{aligned}
$$

b) This is pretty much an identical calculation as part **a)**, but I have copied it for completeness.

$$
\begin{aligned}
\left[\vec{L}\cdot\vec{S},\vec{S}\right] &= (\vec{L}\cdot\vec{S})\vec{S} - \vec{S}(\vec{L}\cdot\vec{S}) \\
&= (L_xS_x + L_yS_y + L_zS_z)\vec{S} - \vec{S}(L_xS_x + L_yS_y + L_zS_z) \\
&= (L_xS_xS_x + L_yS_yS_x + L_zS_zS_x - S_xL_xS_x - S_xL_yS_y - S_xL_zS_z)\hat{x} + \cdots \\
&= (S_x^2L_x + S_yS_xL_y + S_zS_xL_z - S_x^2L_x - S_xS_yL_y - S_xS_zL_z)\hat{x} + \cdots \\
&= \left[(S_yS_x - S_xS_y)L_y + (S_zS_x - S_xS_z)L_z\right]\hat{x} + \cdots \\
&= \left[-i\hbar S_zL_y + i\hbar S_yL_z\right]\hat{x} + \left[i\hbar S_xL_z - i\hbar S_zL_x\right]\hat{y} + \left[i\hbar S_xL_y - i\hbar S_yL_x\right]\hat{z} \\
&= i\hbar\vec{S}\times\vec{L}
\end{aligned}
$$

c) This one is easy, let's just use the results from the previous two parts.

$$
\begin{aligned}
\left[\vec{L}\cdot\vec{S},\vec{J}\right] &= \left[\vec{L}\cdot\vec{S},\vec{L}+\vec{S}\right] \\
&= \left[\vec{L}\cdot\vec{S},\vec{L}\right] + \left[\vec{L}\cdot\vec{S},\vec{S}\right] \\
&= i\hbar\vec{L}\times\vec{S} + i\hbar\vec{S}\times\vec{L} \\
&= i\hbar\vec{L}\times\vec{S} - i\hbar\vec{L}\times\vec{S} = 0
\end{aligned}
$$

d) For this one we have to realize that L^2 commutes with itself, J^2 and S^2.

$$\begin{aligned}\left[\vec{L}\cdot\vec{S}, L^2\right] &= \frac{1}{2}\left[J^2 - L^2 - S^2, L^2\right] \\ &= \frac{1}{2}\left(\left[J^2, L^2\right] - \left[L^2, L^2\right] - \left[S^2, L^2\right]\right) \\ &= 0\end{aligned}$$

e) Nothing new here

$$\begin{aligned}\left[\vec{L}\cdot\vec{S}, S^2\right] &= \frac{1}{2}\left[J^2 - L^2 - S^2, S^2\right] \\ &= \frac{1}{2}\left(\left[J^2, S^2\right] - \left[L^2, S^2\right] - \left[S^2, S^2\right]\right) \\ &= 0\end{aligned}$$

f) And, again!

$$\begin{aligned}\left[\vec{L}\cdot\vec{S}, J^2\right] &= \frac{1}{2}\left[J^2 - L^2 - S^2, J^2\right] \\ &= \frac{1}{2}\left(\left[J^2, J^2\right] - \left[L^2, J^2\right] - \left[S^2, J^2\right]\right) \\ &= 0\end{aligned}$$

Griffiths 4.22 (c) and (d)

c) These are the solutions found in the first part of the problem, and will be useful for later derivations.

$$
\begin{aligned}
[L_z, x] &= i\hbar y & [L_z, y] &= -i\hbar x & [L_z, z] &= 0 \\
[L_z, p_x] &= i\hbar p_y & [L_z, p_y] &= -i\hbar p_x & [L_z, p_z] &= 0
\end{aligned}
$$

First showing that r^2 commutes with L_z:

$$
\begin{aligned}
\left[L_z, r^2\right] &= \left[L_z, x^2 + y^2 + z^2\right] \\
&= \left[L_z, x^2\right] + \left[L_z, y^2\right] + \left[L_z, z^2\right] \\
&= [L_z, x \cdot x] + [L_z, y \cdot y] + [L_z, z \cdot z] \\
&= [L_z, x]\, x + x\, [L_z, x] + [L_z, y]\, y + y\, [L_z, y] + [L_z, z]\, z + z\, [L_z, z] \\
&= i\hbar yx + xi\hbar y + (-i\hbar x)y + (-i\hbar x)y + 0 + 0 \\
&= 0
\end{aligned}
$$

Now, showing that p^2 commutes with L_z:

$$
\begin{aligned}
\left[L_z, p^2\right] &= \left[L_z, p_x^2\right] + \left[L_z, p_y^2\right] + \left[L_z, p_z^2\right] \\
&= [L_z, p_x]\, p_x + p_x\, [L_z, p_x] + [L_z, p_y]\, p_y + p_y\, [L_z, p_y] + [L_z, p_z]\, p_z + p_z\, [L_z, p_z] \\
&= (i\hbar p_y)p_x + p_x(i\hbar p_y) + (-i\hbar p_x)p_y + p_y(-i\hbar p_x) + 0 + 0 \\
&= 0
\end{aligned}
$$

d) Taking a look at the Hamiltonian, remembering that in the last part we already showed that the elements of L commute with p^2:

$$
[H, L_x] = \left[\frac{p^2}{2m} + V, L_x\right] = \frac{1}{2m}\left[p^2, L_x\right] + [V, L_x] = 0 + [V(r), L_x]
$$

Now, we just need to show that $V(r)$ commutes with both r and r^2.

$$
\begin{aligned}
\left[L_x, \sqrt{x^2+y^2+z^2}\right] &= L_x\sqrt{x^2+y^2+z^2} + \sqrt{x^2+y^2+z^2}\,L_x \\
&= (yp_z - zp_y)\sqrt{x^2+y^2+z^2} - \sqrt{x^2+y^2+z^2}(yp_z - zp_y) \\
&= -y\frac{i\hbar}{2\sqrt{x^2+y^2+z^2}}2z + y\sqrt{x^2+y^2+z^2}p_z + z\frac{i\hbar}{2\sqrt{x^2+y^2+z^2}}2y \\
&\qquad - z\sqrt{x^2+y^2+z^2}p_y - \sqrt{x^2+y^2+z^2}(yp_z - zp_y) \\
&= y\sqrt{x^2+y^2+z^2}p_z - y\sqrt{x^2+y^2+z^2}p_z - z\sqrt{x^2+y^2+z^2}p_y \\
&\qquad + z\sqrt{x^2+y^2+z^2}p_y \\
&= 0
\end{aligned}
$$

Imposing the transitive law, we can say that

$$
\left(\left[L^2, L_x\right] = 0\right) \wedge \left([L_x, r] = 0\right) \longrightarrow \left[L^2, r\right] = 0
$$

Since the eigenstates of $V(r)$ can be broken down into the eigenstates of r or r^2, and L^2 and L_x commute with r and r^2, then they both commute with $V(r)$. The proof is nearly identical for L_y and L_z.

Griffiths 7.20

$$E^1_{fs} = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right) \qquad \text{(Equation 7.68)}$$

$$E^1_r = -\frac{(E_n)^2}{2mc^2}\left(\frac{4n}{l+1/2} - 3\right) \qquad \text{(Equation 7.58)}$$

$$E^1_{so} = \frac{(E_n)^2}{mc^2}\left[\frac{n(j(j+1) - l(l+1) - 3/4)}{l(l+1/2)(l+1)}\right] \qquad \text{(Equation 7.67)}$$

We are trying to derive Equation 7.68 from Equation 7.58 and Equation 7.67. Let us do the case where $j = l + 1/2$ first:

$$\begin{aligned}
E^1_{fs} &= E^1_r + E^1_{so} \\
&= -\frac{(E_n)^2}{mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right] + \frac{(E_n)^2}{mc^2}\left[\frac{n\left[j(j+1) - l(l+1) - \frac{3}{4}\right]}{l(l+1/2)(l+1)}\right] \\
&= \frac{(E_n)^2}{2mc^2}\left[\frac{-4n(l)(l+1) + 2n(j(j+1) - l(l+1) - 3/4)}{l(l+1/2)(l+1))} + 3\right]
\end{aligned}$$

Looking at just the numerator:

$$\begin{aligned}
&-4n\left(j+\frac{1}{2}\right)\left(j+\frac{3}{2}\right) + 2n\left[j(j+1) - \left(j+\frac{1}{2}\right)\left(j+\frac{3}{2}\right) - \frac{3}{4}\right] \\
&= -n\left[4j^2 + 10j + 6\right] \\
&= -2n(2j+3)(j+1) \\
&= -4n\left(j+\frac{3}{2}\right)(j+1)
\end{aligned}$$

At the same time, the numerator becomes:

$$\left(j+\frac{1}{2}\right)(j+1)\left(j+\frac{3}{2}\right)$$

$$\begin{aligned}
E^1_{fs} &= \frac{(E_n)^2}{2mc^2}\left[\frac{-4n\left(j+\frac{3}{2}\right)(j+1)}{\left(j+\frac{1}{2}\right)(j+1)\left(j+\frac{3}{2}\right)} + 3\right] \\
&= \frac{(E_n)^2}{2mc^2}\left[3 - \frac{4n}{j+\frac{1}{2}}\right] \quad \checkmark
\end{aligned}$$

Doing this same math for $j - 1/2$; jumping straight to the part were we analyze the numerator and plugged in j for l:

$$\begin{aligned}
&-4n\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right) + 2n\left[j(j+1) - \left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right) - \frac{3}{4}\right] \\
&= -4n\left(j^2 - \frac{1}{4}\right) + 2n\left[j^2 + j - j^2 + \frac{1}{4} - \frac{3}{4}\right] \\
&= n\left[-4j^2 + 2j\right] \\
&= -4nj\left(j - \frac{1}{2}\right)
\end{aligned}$$

The denominator becomes:

$$\left(j-\frac{1}{2}\right)j\left(j+\frac{1}{2}\right)$$

So:

$$\begin{aligned}
E_{fs}^1 &= \frac{(E_n)^2}{2mc^2}\left[\frac{-4nj\left(j-\frac{1}{2}\right)}{\left(j-\frac{1}{2}\right)j\left(j+\frac{1}{2}\right)}+3\right] \\
&= \frac{(E_n)^2}{2mc^2}\left[3-\frac{4n}{j+\frac{1}{2}}\right] \quad \checkmark
\end{aligned}$$

Griffiths 7.21

First, we need to get the general jumps:

$$E_2 = -\frac{13.6\text{eV}}{4} \qquad\qquad E_3 = -\frac{13.6\text{eV}}{9}$$

Next, we can use Equation 7.68 (from the previous problem) to calculate each of the correction energies:

$$E^1_{fs}(n=2, j=1/2) = -\frac{(13.6\text{eV})^2}{2(4)^2 m_e c^2}\cdot 5 \qquad E^1_{fs}(n=2, j=3/2) = -\frac{(13.6\text{eV})^2}{2(4)^2 m_e c^2}$$

$$E^1_{fs}(n=3, j=1/2) = -\frac{(13.6\text{eV})^2}{2(9)^2 m_e c^2}\cdot 9 \qquad E^1_{fs}(n=3, j=3/2) = -\frac{(13.6\text{eV})^2}{2(9)^2 m_e c^2}\cdot 3$$

$$E^1_{fs}(n=3, j=5/2) = -\frac{(13.6\text{eV})^2}{2(9)^2 m_e c^2}$$

The $n = 2$ level splits into two different energy levels, and the $n = 3$ splits into three different energy levels.

Finally, passing these values to a Python script:

Transition	ΔEnergy	Photon λ
$E_3(j=1/2) \to E_2(j=3/2)$	1.888753eV	656.434nm
$E_3(j=3/2) \to E_2(j=3/2)$	1.888874eV	656.392nm
$E_3(j=5/2) \to E_2(j=3/2)$	1.888914eV	656.378nm
$E_3(j=1/2) \to E_2(j=1/2)$	1.888934eV	656.371nm
$E_3(j=3/2) \to E_2(j=3/2)$	1.889055eV	656.329nm
$E_3(j=5/2) \to E_2(j=3/2)$	1.889095eV	656.315nm

The distance between the lines, going down the list:

$$\boxed{0.042\text{nm, } 0.014\text{nm, } 0.007\text{nm, } 0.042\text{nm, } 0.014\text{nm}}$$

Griffiths 7.24

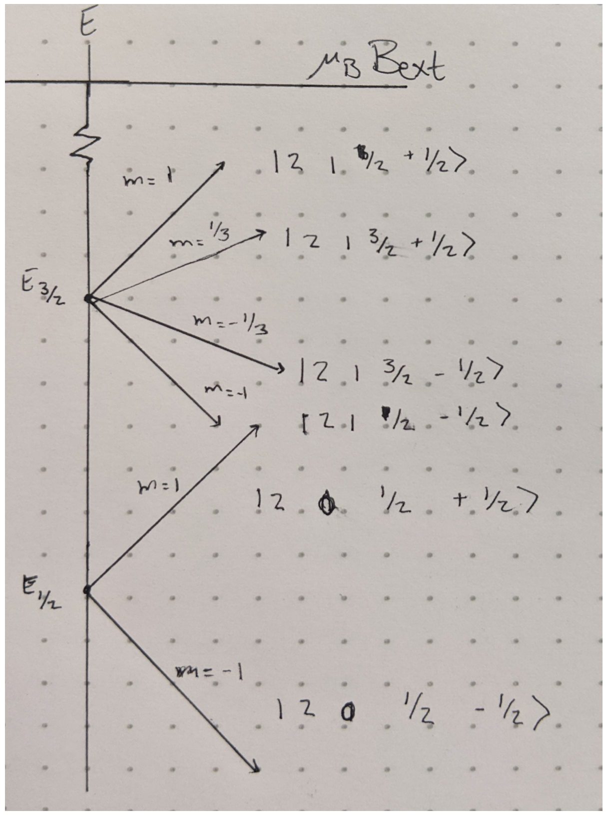

Figure 1: The energy levels, as a function of $\mu_B B_{\text{ext}}$.

Following the derivation used in the book, first we need to find the unperturbed energy (Bohr energy), then we need to find both the fine-structure correction, then the Zeeman correction. Aligning the $\vec{B}_{\text{ext}}$ along the $\hat{z}$ direction:

$$E^0 = -\frac{13.6\text{eV}}{4}$$

$$E^1_{fs} = \frac{(E_n)^2}{2m_e c^2}\left[3 - \frac{4n}{j+\frac{1}{2}}\right] \to \frac{(13.6\text{eV})^2}{2(4)^2 m_e c^2}\left[3 - \frac{8}{j+\frac{1}{2}}\right]$$

$$E^1_z = \frac{e\hbar}{2m} g_j B_{\text{ext}} m_j$$

Where $m_e \approx 0.511\text{MeV}/c^2$, and $g_j = \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right]\left\langle \vec{J} \right\rangle$. So, $g_{1/2} = 2$ and $g_{3/2} = \frac{2}{3}$

Putting this all together, we get the following energies:

$$\boxed{E = \begin{cases} E^0 + E^1_{fs(1/2)} \pm \mu_B B_{\text{ext}} & \text{for } \left|2\,1\,\tfrac{1}{2}\ \pm\tfrac{1}{2}\right\rangle \\ E^0 + E^1_{fs(3/2)} \pm \mu_B \left(\tfrac{1}{3}\right) B_{\text{ext}} & \text{for } \left|2\,1\,\tfrac{3}{2}\ \pm\tfrac{1}{2}\right\rangle \\ E^0 + E^1_{fs(3/2)} \pm \mu_B B_{\text{ext}} & \text{for } \left|2\,1\,\tfrac{3}{2}\ \pm\tfrac{3}{2}\right\rangle \end{cases}}$$

For the graph I have labeled the two energies that get augmented as:

$$E_{1/2} = E^0 + E^1_{fs(1/2)} = -\frac{13.6\text{eV}}{4}\left(1 + 5\frac{13.6\text{eV}}{8m_e c^2}\right), \quad E_{3/2} = E^0 + E^1_{fs(3/2)} = -\frac{13.6\text{eV}}{4}\left(1 + \frac{13.6\text{eV}}{8m_e c^2}\right)$$

The plotting of these energies can then be seen in Figure 1.