FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

Master Thesis

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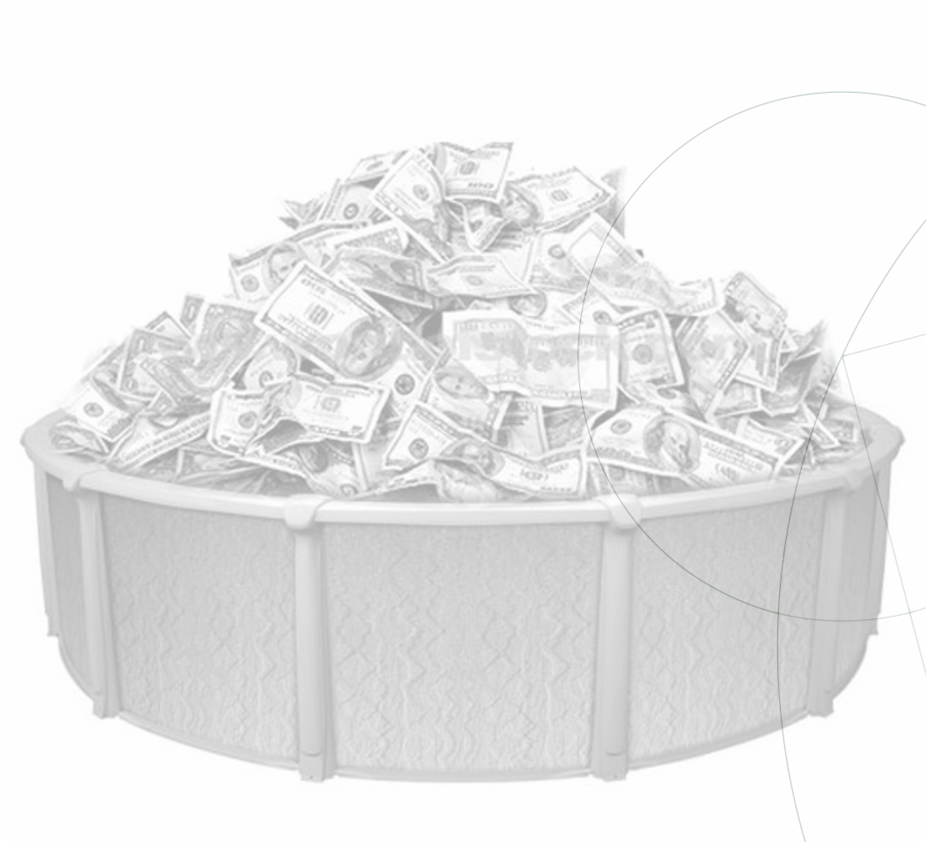
Bond – Callable Mortgage Bond

An Investigation of Prepayment Modelling



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Abstract

The *fair* price of a callable mortgage bond is modelled from an arbitrage-free pricing point of view. The pricing of callable bonds evolves around two aspects of complexity: (1) modelling the dynamic behaviour of market interest rates, and (2) modelling the prepayment behaviour of mortgagors. The latter is the primary focus of this thesis. For that reason the short rate is assumed to follow the tractable Vasicek model. Nonetheless it is explained how more sophisticated term structure models can be incorporated through the Cheyette model. The central part of the thesis consists of a thorough investigation of two different prepayment models, one originating from the academic world and the other from Danske Bank, i.e. the 'real' world. The models are compared on their ability to predict prepayment behaviour, fit observed market quotes and match the rate sensitivity induced by the market. It is concluded that the model originating from the academic world is preferable in various aspects. However, there is always room for improvement. Therefore possible ways in which one may extend the models are discussed. All prices are derived using finite difference methods.

Contents

Introduction

As the title indicates this thesis deals with callable mortgage bonds. More specifically, we attempt to model the *fair* price of a callable mortgage bond from an arbitrage-free pricing point of view. The pricing of callable bonds evolves around two aspects of complexity: (1) modelling the dynamic behaviour of market interest rates, and (2) modelling the prepayment behaviour of mortgagors. The latter is the primary focus of this thesis. A callable mortgage bond is a claim on the cash flows from a pool of mortgages. The investors receive all payments (principal plus interest) made by mortgagors in the pool (less some servicing fee). Hence the pricing of a callable bond is basically equivalent to valuing the cash flows from the underlying pool of mortgages. However, as explained in detail in chapter 1 the valuation of a callable bond is by no means an easy task. This is due to the fact that each mortgagor has a call option (hence the name), which enables the mortgagor to prepay the loan at par value. Thus, in order to determine a pool's cash flow we need to model the prepayment behaviour of the entire pool.

We have attempted on creating a thesis, which does not dwell on standard theory and for that reason we assume that the reader is confident with standard concepts of discrete- and continuous-time finance as well a finite difference methods for solving partial differential equations.[1] The core of the thesis is a thorough investigation of first a prepayment model of the academic world [Stanton (1995)] and second a prepayment model of the 'real' world [Andreasen (2011)]. These two models are then compared in various aspects and ideas concerning further analysis and extensions of the models are discussed. The definitive hope is to present a prepayment model which is capable of creating a good fit to observed market prices given an observed interest rate path.

Before introducing the specific structure of the thesis we feel that it is important to note that a very large amount of workload has been put into countless hours of programming - without this effort the resulting thesis would not have been possible. Any reader who attempts on recreating the results presented here will immediately become aware of this fact. [2]

[1] The basic idea of finite difference methods is outlined in section A.5 in the appendix.

[2] Throughout the thesis we have used the statistical open source program `R` and we are more than happy to share any specific code if the reader is interested.

The Structure of The Thesis

The thesis is structured in the following way: chapter 1 explains the need for prepayment modelling and formulates the difficulties of the problem at hand both literal and mathematical. We briefly present the idea behind the *empirical* prepayment model of [Schwartz and Torous (1989)]. However, as discussed in the chapter one might argue that empirical models have a lack of explanatory power. Next we introduce the mathematical machinery needed in order to solve the pricing problem. The machinery is based on the well-known Feynman-Kac result, which implies that we may solve a boundary value problem, instead of computing conditional expectations. Hence (given a prepayment model) the price of a callable mortgage bond can be computed by solving a boundary value problem, which is done numerically by finite difference methods. We end chapter 1 by presenting the Danish mortgage bond data, which has been used throughout the thesis.

In chapter 2 a toy model is fitted for the market short rate. The word *toy* model is used because the fitting of a sophisticated short rate model could very well be the subject of an entire thesis on its own right. As already mentioned the focus of this thesis is the modelling of prepayment behaviour. Hence, for our purpose it is sufficient with a simple and tractable short rate model, which allows the prepayment models presented in chapter 3 and 4 to fit historically observed market quotes. More precisely, the short rate is assumed to follow a Vasicek model.

The central part of the thesis is found in chapter 3, 4 and 5. Chapter 3 consists of a detailed examination of the *structural* prepayment model presented in [Stanton (1995)]. All of the results presented in [Stanton (1995)] have been recreated in order to obtain a deeper understanding of the model. Furthermore we have adjusted the computational approach of [Stanton (1995)] in order to be consistent with the Danish bond market legislation. At the end of the chapter we estimate key parameters of the model based on observed prepayment rates from Danish mortgage bonds.

In chapter 4 we present the actual prepayment model used by *Danske Bank*. The chapter is based on the minimalistic presentation in [Andreasen (2011)]. Due to the lack of information in [Andreasen (2011)] the main part of the chapter evolves around explaining and examinating the analysis which leads to the final prepayment model. During the analysis we encounter some harsh assumptions, which are made in [Andreasen (2011)] in order to simplify the analysis. We test the validity of some of these assumptions, but we do not attempt on changing the model, due to the fact that we wish to see how it performs compared to the model from [Stanton (1995)]. Given the resulting prepayment model, we finish the chapter by explaining how the actual procedure of pricing a callable mortgage bond is performed under the assumption of Danske Bank prepayment model.

A comparison of the prepayment models presented in chapter 3 and 4 is carried out in chapter 5. Here the two models are compared on their ability to predict prepayment behaviour, fit observed bond prices and match the rate sensitivity induced by the market. Furthermore we discuss their explanatory power.

General Assumptions

Unless noted otherwise the assumptions stated here are assumed throughout the thesis. We consider an economy, where the uncertainty is summarized by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ carrying a Wiener process $(W(t))_{t\geq 0}$. Here Ω denotes the state space, $\mathcal{F}$ is a σ-algebra representing measurable events, $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a filtration, which we may think of as a flow of information, and at last P denotes a probability measure. We may think of the probability measure P as the real-world measure. Our only source of uncertainty originates from the Wiener process (possibly multidimensional) and we therefore assume that the filtration is defined by

$$\mathcal{F}_t = \mathcal{F}_t^W \quad \text{for} \quad t \geq 0,$$

where $\mathcal{F}_t^W$ is the natural filtration of the Wiener process, i.e. $\mathcal{F}_t^W = \sigma\{W(s) : 0 \leq s \leq t\}$.

All functions in the thesis are defined with respect to such a filtered probability space and they are assumed to be *sufficiently well-behaved*. By this we mean that they posses the continuity, differentiability, integrability, etc. needed in the situation at hand.

We assume that the market consists only of non-defaultable securities and a money market defined through a short rate process.[3] As always all securities are priced as expected discounted values under an equivalent martingale measure, which we denote by Q. Such a measure is known to exist when the market is free of arbitrage, which we assume it to be. Furthermore we assume that the market is complete, which implies that the equivalent martingale measure is unique.

The legitimacy of the above assumptions is definitely questionable. However, from a mathematical modelling point of view 'tractability beats reality!'

[3]The assumption that the market does not contain defaultable securities is in general not true when looking at a market for callable mortgages bonds. But the assumption may be justified by the following two arguments: 1) we are looking at the market for Danish mortgages bonds, where default is a very rare event (in case of defaults then the investors cash flow are covered by the Danish Mortgage Bank) and 2) the possible credit risk might be incorporated into the short rate process through a harder discounting.

Chapter 1

Prepayment Modelling and Danish Mortgage Bond Data

1.1 Why the Need for Prepayment Modelling?

The need for prepayment modelling comes from the quest of pricing callable mortgage bonds. Unlike non-callable bonds, callable bonds have an embedded call option. A callable mortgage bond gives its owner a share in the cash flow from a pool of mortgages. The call option gives each mortgagor in the pool the option to *prepay* (pay back) the loan at par value. In the Danish mortgage bond market the decision to exercise the prepayment option has to be announced two months before the next coupon date. The embedded call option implies that the future cash flow of a callable bond is dependent on whether or not the prepayment option is exercised and if it is exercised then at what time and by how many of the mortgagors in the pool. The prepayment decisions made by the mortgagors therefore affect both the amount and timing of the cash flow received by the investor. The need for modelling prepayment behaviour is therefore crucial in order to determine the expected future cash flow of the bond. For that reason, determining the fair price of a callable bond becomes a much more delicate task compared to that of its plain and simple relative - the non-callable bond.

In order to illustrate the difficulties from at mathematical modelling point of view let us consider a callable mortgage bond with maturity T, discrete coupon dates $T_1, \ldots, T_n = T$ and payment Y_i on T_i. For simplicity we assume that the bond is backed by a single loan. Then Y_i equals either (i) the standard coupon payment, if no prepayment has occurred (ii) the remaining principal on the loan in the case of prepayment (iii) zero, if prepayment has already happen. From the theory of arbitrage pricing we know that the arbitrage-free price of a callable mortgage bond can be found by taking the expected discounted cash flow under an equivalent martingale measure, Q, conditioned on the information available in the market. Hence the fair time-t price of the callable mortgage bond, $V(t)$ is given by

$$\begin{aligned} V(t) &= E\left[\sum_{i=1}^{n} e^{\int_t^{T_i} r(s)ds} Y_i \mid \mathcal{F}_t\right] \\ &= \sum_{i=1}^{n} E\left[e^{\int_t^{T_i} r(s)ds} Y_i \mid \mathcal{F}_t\right], \end{aligned} \tag{1.1}$$

where $E[\,\cdot \mid \mathcal{F}_t]$ is a conditional expectation under Q. Here $(r(t))_{t\geq 0}$ is the market short rate process, which we assume solves a stochastic differential equation (SDE) given by

$$\begin{aligned} dr(s) &= \mu(s, r(s))ds + \sigma(s, r(s))dW(s) \\ r(t) &= r. \end{aligned} \tag{1.2}$$

for $s > t$ and where $\mu : [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$.

Under the assumption that a callable mortgage bond is an interest-rate derivative and under sufficient regularity conditions on μ and σ[1] we may replace the conditioning of $\mathcal{F}_t$ with $r(t) = r$ in (1.1). This will be assumed throughout the thesis.

Because of the stochastic nature of the cash flow and the fact that it is strongly dependent on the short rate, due to the interest rate incentive (explained below), it is not clear how the expression given by (1.1) should be computed. However, modelling the prepayment behaviour and applying a PDE-approach[2] based on the well-known Feynman-kac result, enables us to evaluate the expression using a finite difference scheme. Do not worry, this should all become clearer further into the thesis.

There may be many reasons why an individual mortgagor chooses to prepay and we do not expect a prepayment model to capture the full scope of the complexity of this decision making. However there are some reasons which we would expect to be common among many individual mortgagors whereas other reasons may be specific to the individual mortgagor. Assume for example that the level of market interest rates decrease, such that new mortgage loans, at lower coupon rates than before, become available. We expect such a scenario to affect the prepayment decisions of a large group of mortgagors. On the other hand, the scenario of a mortgagor being offered a new job abroad will most likely only affect the prepayment decision of the mortgagor in question. The obvious point to be made here is that we would like prepayment models to contain the most frequent factors across mortgagors concerning prepayment

[1] See section A.1 in the appendix for a short discussion on when the solution (1.2) is Markovian.
[2] PDE is short for partial differential equation.

decisions. We call such factors *refinancing incentives*. The incentive that we expect to affect prepayment behaviour the most is the interest rate incentive, which was mentioned in the example above.

Besides containing the most influential refinancing incentives, the prepayment model should of course, also aim towards being consistent with empirical findings, such as for example the *burnout* effect. The burnout effect can be explained in the following way: given that a large proportion of the mortgage pool has already prepayed it is less likely that those remaining in the pool choose to prepay at future interest rate levels. In other words, if a pool of mortgagors has already experienced much prepayment we expect this prepayment behaviour to 'burn out' over time, so to speak. In figure 1.1 the top plot shows movements in the 15-year swap rate over time, while the bottom plot shows the average observed quarterly prepayment rates for twenty Danish mortgage bonds. The decrease in observed prepayment rates from late 2012 and forward, in spite of the fact that the swap rate continues to stay at a low level, could be interpreted as evidence of a burnout effect in the market. Figure 1.2 also shows sign of a burnout effect, this time reflected through market prices. The upper left plot shows the 15-year swap rate path over time. The lower left plot shows changes in market quotes over time of the Danish mortgage bond `NDA 5 1 01JUL2035`, and the plot to the right shows the same market quotes, but this time plotted against the 15-year swap rate. The 'cloud' on the right plot (and equivalently the increase in quotes on the bottom left plot) corresponding to market prices around 2012-2014 could be interpreted as evidence of a burnout effect in the market and may be explained from market forces. I.e. a burnout effect implies an increase in the market price, due to the fact that the mortgagors left in the pool are less likely to prepay - hence an investor is willing to pay a higher price for such a bond.

Notice the extreme movements in the bond price in figure 1.2 illustrated by the bottom 'cloud' of prices in the right plot (and equivalently the large drop in quotes on the bottom left plot) corresponding to the time period around 2008-2009. The large drop in prices may be explained by the financial crisis. In 2008 there was a serious risk that the housing market were to collapse due to the financial crisis. This would lead to many defaults, and naturally this risk led to a drop in the value of the bonds. In this thesis we do not attempt to model irregular movements in the market quotes due to financial crises, such as those occurring around 2008-2009. For that reason we often exclude this time period when plotting model produced bond values and observed market quotes later on.

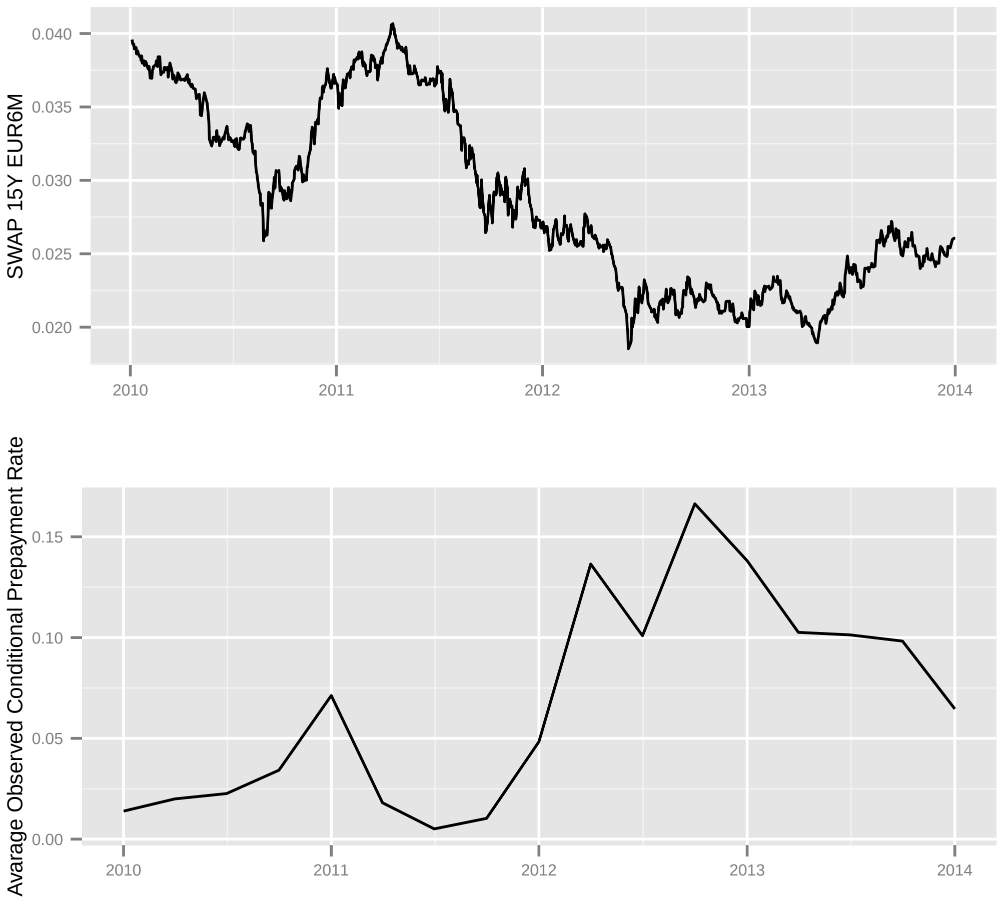


Figure 1.1: Possible evidence of a burnout effect. The top plot shows movements in the 15-year swap rate over time. The bottom plot shows the average observed conditional quarterly prepayment rates for twenty Danish mortgage bonds. The decrease in observed prepayment rates from late 2012 and forward, in spite of the fact that the swap rate continues to stay at a low level, could be interpreted as evidence of a burnout effect in the market.

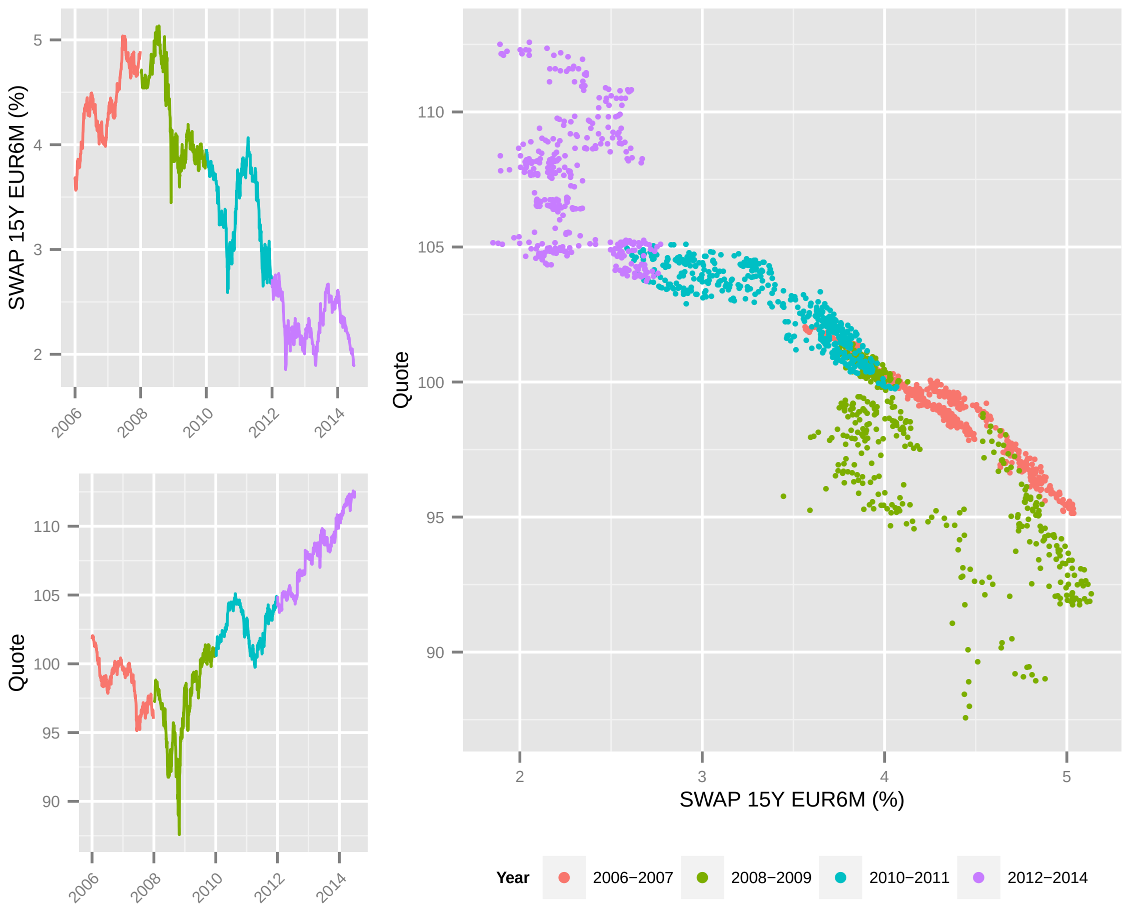


Figure 1.2: Illustrating two things: 1) Possible evidence of a burnout effect. 2) The effects of the financial crisis. The upper left plot shows the 15-year swap rate over time. The lower left plot shows the market quotes of `NDA 5 1 01JUL2035`. The plot to the right shows the same market quotes, but now plotted against the 15-year swap rate. The colouring is made to distinguish the connection over time between the plots.

Empirical findings, such as a burnout effect, implies the existence of *suboptimal behaviour* among mortgagors. As we shall see later, incorporating suboptimal behaviour in a prepayment model is central if the model outcome is to be consistent with empirical findings. Below we briefly present a classic prepayment model as an example of how one may incorporate some of the above mentioned factors into a specific prepayment model, while in chapter 3 we fully investigate the prepayment model presented in [Stanton (1995)].

1.2 An Example of a Classic Prepayment Model

[Schwartz and Torous (1989)] is a classic example of what we would call an *empirical* or *reduced form* model, in the field of prepayment modelling. It models prepayment behaviour through regression analysis based on historical prepayment rates. The *prepayment function*[3] is therefore a function of regression coefficients and explanatory variables. In [Schwartz and Torous (1989)] the prepayment function of a single mortgagor is defined as the instantaneous rate of prepayment conditional on the mortgage not having been prepaid already. More specifically, the prepayment function, p, is assumed to follow a *Cox regression model*[4]

$$p(t \mid v, \theta) = p_0(t \mid \gamma, p) e^{\beta^T v}, \tag{1.3}$$

where $\beta = (\beta_1, \ldots, \beta_n)^T$ and $v = (v_1, \ldots, v_n)^T$ are vectors of regression coefficients and explanatory variables, respectively, and $\theta = (\gamma, p, \beta_1, \ldots, \beta_n)$ is a vector of prepayment parameters. The function p_0 is to be thought of as a base line hazard rate and is chosen as a log-logistic hazard rate, see equation (6) in [Schwartz and Torous (1989)].

In [Schwartz and Torous (1989)] the explanatory variables are chosen in ways such that they ought to capture interest rate incentives, further acceleration of prepayment for sufficiently small market rates, changes in the pool factor[5] and seasonality, cf. equation (8) – (11) in [Schwartz and Torous (1989)]. The interest rate incentive has already been introduced above. The current pool factor could possibly help capturing burnout effects and seasonality refers to the idea that the prepayment behaviour of mortgagors differ at different seasons of the year. After specifying the explanatory variables the next step is to estimate the parameter vector, θ, based upon historical prepayment rates. With the estimated values at hand the prepayment function is fully specified and aimed at matching observed prepayment rates.

[3] We do not present a formal definition of a prepayment function, since the exact meaning of the word may vary in different papers. But in general a prepayment function of a single mortgagor provides the probability (or intensity) of prepayment occurring within a certain time interval, given no prepayment before. Or if the prepayment function refers to the prepayment behaviour of a entire pool it provides the conditional proportion of the pool, which is expected to prepay.

[4] See chapter 4 in [Kalbfleisch and Prentice (2002)] for an introduction to the class of Cox regression models.

[5] Pool factor refers to the percentage of initial principal left in the pool.

The approach presented in [Schwartz and Torous (1989)] is immediately quite appealing, due to the fact that we start out with a prepayment function in which we may include any factor which we suspect may affect prepayment behaviour. This, seemingly, enables us to capture multiple aspects of prepayment behaviour. And at the same time the estimated parameter values are able to explain in which way prepayment is dependent on the specific covariates.

However it is important to note that such a prepayment function is only capable of capturing multiple aspects of *historical* prepayment behaviour. Furthermore, the model is not based on any 'a priori' assumptions, such as for example how prepayments and the market interest rate level depend on each other[6]. The fact that empirical models, such as the one presented in [Schwartz and Torous (1989)], do not have model-based assumptions and are aimed at fitting past prepayment rates imply that they will always lag behind shifts in the structure of the mortgage market. As argued in [Stanton (1995)] it is therefore not clear how the model from [Schwartz and Torous (1989)] would perform in a different economic environment. In chapter 3, we introduce the prepayment model presented in [Stanton (1995)]. This model is what we would call a *structural* model, since it is based on a option-theoretic framework and it consistently links valuation and prepayment. But before going into detail with this model we shall look at how one may price a non-callable bond and hereafter a callable bond under the assumption of optimal behaviour and a frictionless market.

1.3 Valuation of a Non-callable Mortgage Bond

It is clear that there exists a relation between the pricing of callable and non-callable bonds, since the cash flows are identical up to the point of prepayment. We therefore begin by pricing a non-callable bond.

Definition 1.1. *Consider the Danish non-callable mortgage bond with maturity T, coupon rate q and face value K. The cash flow for the holder of the bond is as follows*

- *Consider the fixed points in time $T_1, \ldots, T_n = T$. These are to interpreted as the coupon dates. The Danish mortgage bonds have quarterly payments so that $dt = T_{i+1} - T_i = \frac{1}{4}$. Hence the number of coupons are given by $n = 4T$.*

- *On each coupon date the holder of the bond receives the amount C, which is determined by the annuity formula such that*

$$C = K \frac{q \cdot dt}{1 - (1 + q \cdot dt)^{-n}}. \tag{1.4}$$

[6] In theory the estimate of the parameter determining the dependence structure between the prepayment function and the short rate could be negative, meaning that a lower level of the market interest rates would affect prepayment negatively. A scenario which we would think of as absurd on beforehand.

Note that the non-callable mortgage bond can be considered as a portfolio of n zero coupon bonds (ZCB)[7] with face value C and maturities $T_1, \ldots, T_n$, respectively. Valuing a non-callable bond is then just a task of valuing ZCBs since the time $t < T_1$ value is given by

$$P(t,r) = E_{t,r}\left[\sum_{i=1}^{n} Ce^{-\int_t^{T_i} r(s)ds}\right] = C\sum_{i=1}^{n} E_{t,r}\left[e^{-\int_t^{T_i} r(s)ds}\right],$$

where $E_{t,r}$ denotes an expectation under Q conditioned on $r(t) = r$. If one assumes that the short rate for example follows a Vasicek model, then there exists closed form expressions for $E_{t,r}\left[e^{-\int_t^{T_i} r(s)ds}\right]$ for $i = 1, \ldots, n$.

We however present another approach to the pricing of a non-callable bond. This is a PDE-approach, which is also used when valuing callable bonds. Consider a bond with maturity T and n coupons at dates $T_1 < T_2 < \ldots T_n = T$, where the payment size at T_i is denoted by C_i – in the case of an annuity structured bond we have $C_1 = C_2 = \ldots = C_n = C$.

Remember that if $Z(t,r)$ is the arbitrage-free price at time t of a ZCB with maturity $T^\star$ and face value $C^\star$, then by Feynman-Kac Z solves the boundary value problem (BVP)

$$\begin{aligned} \frac{\partial Z(t,r)}{\partial t} + \mu(t,r)\frac{\partial Z(t,r)}{\partial r} + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 Z(t,r)}{\partial r^2} - rZ(t,r) &= 0 \\ Z(T^\star, r) &= C^*. \end{aligned} \tag{1.5}$$

where the short rate process $r(s)$ is given by (1.2).

Define $Z_i(t,r)$ for $t \le T_i$ and $i = 1, \ldots, n$ as the time t value of the payment at time T_i plus future payments. Note that $Z_n(t,r)$ is then simply the time t price of a ZCB with maturity T_n and payment C_n, which implies $Z_n(t,r)$ solves the BVP (1.5) with $T^\star = T_n$ and $C^* = C_n$. By the definition of Z_i it follows that

$$Z_i(T_i, r) = C_i + Z_{i+1}(T_i, r)$$

for $i < n$. The time-t value of Z_i for $t < T_i$ is therefore given by

$$Z_i(t,r) = (C_i + Z_{i+1}(T_i, r))\, E_{t,r}\left[e^{-\int_t^{T_i} r(s)ds}\right]$$

[7] A ZCB with maturity T and face value K is the simplest of all bonds. It pays K at maturity to the holder.

and it then follows that Z_i solves the BVP

$$\begin{aligned} \frac{\partial Z_i(t,r)}{\partial t} + \mu(t,r)\frac{\partial Z_i(t,r)}{\partial r} + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 Z_i(t,r)}{\partial r^2} - rZ_i(t,r) = 0 \\ Z_i(T_i,r) = C_i + Z_{i+1}(T_i,r). \end{aligned} \tag{1.6}$$

To sum up the PDE-approach shows us that we can value a non-callable mortgage bond by solving a multiple of BVPs. The value of the total cash flow of the bond at time zero is given by $P(0,r) = Z^1(0,r)$. The real benefit is that we do not require closed-form expressions for the expected discounting. Instead we can solve the BVPs using a finite difference scheme. I.e. we discretize the continuous variables $(t,r) \in [0,T] \times \mathbb{R}$ by introducing a two-dimensional grid of discrete points (t_i, r_k) for $i = 0, 1, \ldots, N_t$ and $k = 0, 1, \ldots, N_r$, with equidistant step sizes denoted by Δt and Δr. We set $t_0 = 0$, $t_{N_t} = T$ and r_0 and r_{N_r} are chosen, such that the range of plausible short rate values is covered. More specifically we may solve the problem backwards in time using a Crank-Nicolson scheme, see section A.5 in the appendix.

Assume the short rate to follow a Vasicek process given by

$$dr(t) = 0.093(0.031 - r(t))dt + 0.012dW(t) \tag{1.7}$$

where $W(t)$ is a Wiener process under the pricing measure Q. The parameter values equal the estimates found in chapter 2. Under this assumption, figure 1.3 shows the numerical solution and the true[8] solution of a non-callable mortgage bond.

In figure 1.3 we see the typical negative dependence between the price of a non-callable bond and the level of market interest rates, which in general is known to exists - a dependence, which may be explained by market forces.

In subsection 1.4 we value a callable mortgage bond under simplifying assumptions. Beforehand we expect the value of a callable bond to be similar to the value of a non-callable bond for high rates, since there is no short rate incentive to prepay. On the other hand, we expect the value of a callable bond to be below that of a non-callable bond for small values of the short rate due to the refinancing incentive. From a market equilibrium point of view the investor needs to be compensated (through a lower price) for the prepayment risk. From an arbitrage pricing point of view the price lowers because the relatively large expected coupon payments are replaced by an immediate payoff of the principal.

[8] Here 'true' refers to the closed-form solution.

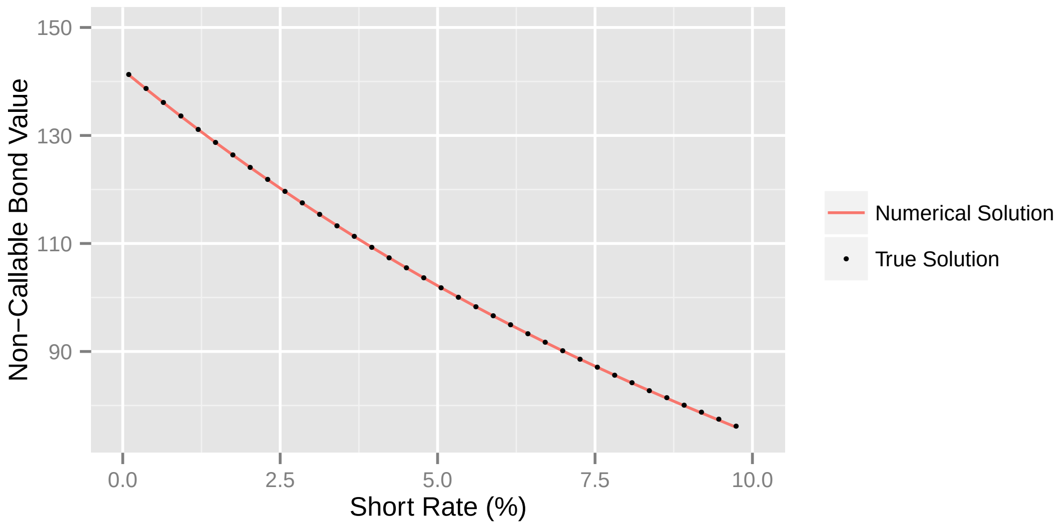


Figure 1.3: The price of a 30-year non-callable mortgage bond with a face value of 100 and a coupon rate of 4 percent. The short rate follows a Vasicek process, specified in (1.7).

1.4 Valuation of a Callable Mortgage Bond - A Naive Approach

We model the value of a callable mortgage bond with coupon rate q as a non-callable bond with coupon rate q, but with a call option written on that bond with a strike equal to par value[9]. This embedded option will in general create a concave dependence between the price of callable bond and the market short rate for low rates, cf. figure 1.5.

Consider a callable mortgage bond with maturity T, coupon rate q and coupon dates $T_1, \dots, T_n = T$, which pays the coupon C (given by (1.4)) at each coupon date, conditioned on no prepayment. Assume that the mortgagor makes prepayment decisions on the coupon dates.[10]

A naive approach to the modelling of prepayment behaviour would be to assume that there are no transaction costs associated with prepaying and that the mortgagor chooses to prepay at time T_i if the expected future value of the loan is greater than the remaining principal at time T_i plus the coupon - i.e. we assume that the mortgagor behaves *optimally* in the sense that the mortgagor wishes to minimize his or her expected mortgage liability. These assumptions simplify the modelling aspect a great deal. But as we shall see later we are forced to remove such simplifying assumptions in order to obtain a model, which produces an empirical plausible outcome.

Define $V_i(t, r)$ for $t \leq T_i$ and $i = 1, \dots, n$ as the time t value of the payment at time T_i plus future payments, assuming no prepayments have been made before time T_i. Then $V_n(t, r)$ is simply the time t

[9] Possibly plus transaction costs.

[10] In later chapters we substitute this assumption with the assumption that a mortgagor make prepayment decisions two months before the next coupon date. This is done in order to be consistent with Danish legislation.

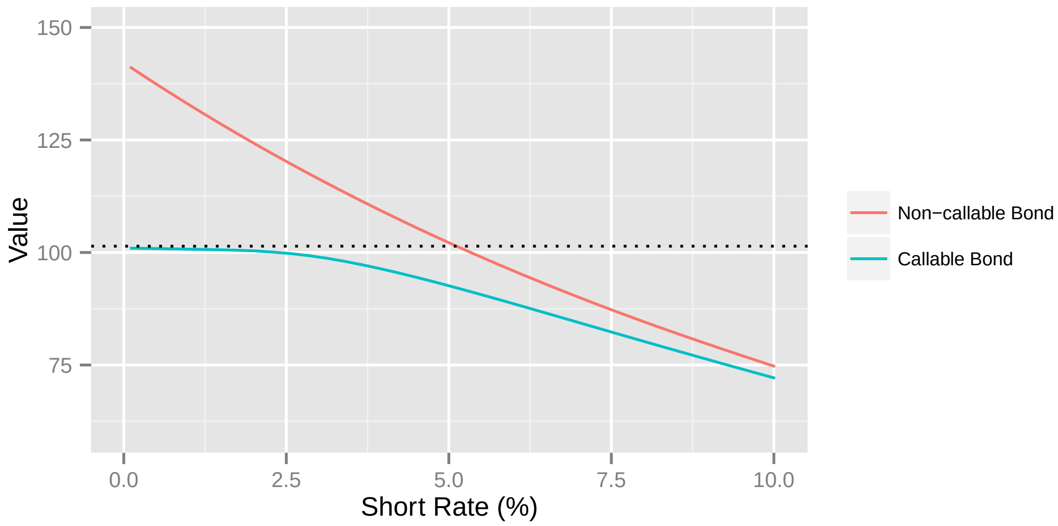


Figure 1.4: The time-zero price of a 30-year non-callable and callable mortgage bond with a face value of 100 and a coupon rate of 4 percent. The short rate follows a Vasicek process, specified in (1.7). The horizontal dotted line shows that the price of the callable bond is bounded by par plus interest.

price of a ZCB with maturity T_n and payment C, which implies that $V_n(t,r)$ solves the BVP (1.5) with $T^\star = T_n$ and $C^* = C$.

The mortgagor will either prepay or continue with the current bond at each coupon date. The optimal borrower chooses the option that cost him the least. Therefore at time T_i it is optimal to prepay if

$$V_{i+1}(T_i, r) > F_{T_i},$$

where F_{T_i} is the remaining principal after coupon i has been paid. If the mortgagor decides to prepay, then he or she has to pay $F_{T_i} + C$ at time T_i. If not then the value is equal to $V_{i+1}(T_i, r) + C$ at time T_i. Therefore

$$V_i(T_i, r) = \min\left(V_{i+1}(T_i, r), F_{T_i}\right) + C$$

and $V_i(t, r)$ is simply the solution to the BVP on $[0, T_i] \times \mathbb{R}$ given by

$$\begin{aligned} 0 &= \frac{\partial V_i(t,r)}{\partial t} + \mu(t,r)\frac{\partial V_i(t,r)}{\partial r} + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 V_i(t,r)}{\partial r^2} - rV_i(t,r) \\ V_i(T_i, r) &= \min\left(V_{i+1}(T_i, r), F_{T_i}\right) + C. \end{aligned}$$

In figure 1.4 the time-zero price of a callable and non-callable bond is plotted against the short rate. We see that the naive approach to the modelling of prepayment behaviour captures the idea that the value of a callable and non-callable bond approach each other for high rates, while the value of the callable bond should be below that of a non-callable bond for low rates. But as indicated by the horizontal dotted line the price of the callable bond is bounded by par plus interest payment. A bound which is obviously

Figure 1.5: Illustrating market quotes well above par value and the concave dependence between the price and the market short rate for low rates relative to the coupon rate. Quotes above par imply non-optimal behaviour. The concave dependence is caused by the embedded call option. Furthermore the cloud of quotes indicates that the bond value is not only depend on market level of interest rates. Market quotes for the 30-year callable Danish mortgage bond `RD 5 01JAN2038` per 100 DKK face value plotted against the short rate. The short rate is implied by the 15-year swap rate, cf. chapter 2.

caused by the assumed optimal prepayment behaviour of the mortgagor in a frictionless market. This bound is often violated in the market, which indicates that mortgagors do not behave optimally - see figure 1.5, which illustrates how the quoted market price of the callable Danish mortgage bond, `RD 5 01JAN2038`, rises well above par plus interest.

Furthermore recall that the cash flow of a mortgage bond comes from a pool of mortgagors, meaning that we need to model, not only the prepayment behaviour of a single mortgagor, but the prepayment behaviour of an entire pool of mortgagors. By saying that the concave graph in figure 1.4 is the price of a mortgage bond we have therefore implicitly assumed the existence of a representative mortgagor - i.e that all mortgagors in the pool are identical. A consequence of this is that all mortgagors would eventually choose to prepay simultaneously if it became optimal to do so. Such pool behaviour does obviously not match empirical findings - cf. the bottom plot in figure 1.1. In chapter 3 we go into detail with a prepayment model, which removes the simplifying assumptions made here by implementing market frictions. These frictions prevent mortgagors from behaving optimally, which imply that the unrealistic bound on callable bond prices are removed. In addition it is assumed that mortgagors are heterogeneous in order to prevent unrealistic prepayment scenarios such as the one where all mortgagors prepay simultaneously.

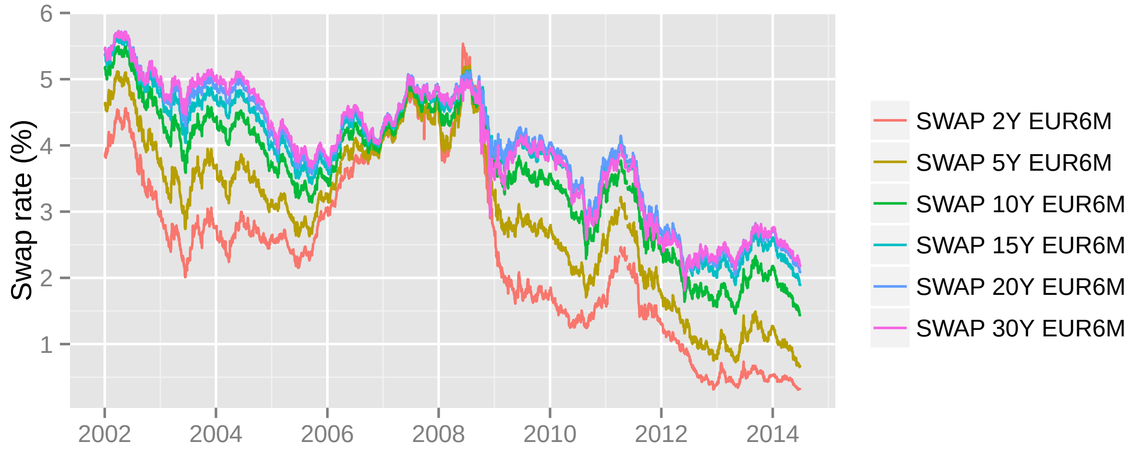


Figure 1.6: Swap rate levels over time

1.5 Danish Mortgage Bond Data

In this section we present all the data used for this thesis. It consists of three main parts, which can be divided into swap rate data, market quotes for callable mortgage bonds and prepayment data for these bonds. For now we will simply present the data and save the analysis for later.

Swap Rate Data

An interest rate swap is a simple interest rate derivative. It is basically a scheme where you exchange a payment stream at a fixed rate of interest, known as the **swap rate**, for a payment at a floating rate.

As an example we have daily observations of `SWAP 15Y EUR6M` between 2002-01-02 and 2014-07-01 (*yyyy-mm-dd*) of the **swap rates**. Here the floating rate is the LIBOR rate and the payment exchanges occurs every 6 months and matures in 15 years. Furthermore we have data on swap rates with other maturities; `SWAP 2Y EUR6M`, `SWAP 5Y EUR6M`, `SWAP 10Y EUR6M`, `SWAP 20Y EUR6M` and `SWAP 30Y EUR6M`. Figure 1.6 shows swap rates over time.

Mortgage Bond Quotes

The data set contains observed market quotes on twenty Danish callable mortgage bonds. Each of them with a maturity of 30 years. We had access to more callable mortgage bonds, but have chosen to focus on some of the most liquid bonds between 2006-01-02 and 2014-07-01. If a bond was not traded at some day, then we do not have a quote that day for that specific bond. Hence each quote in this thesis is a true traded market quote. For details on the bonds, see table 1.1.

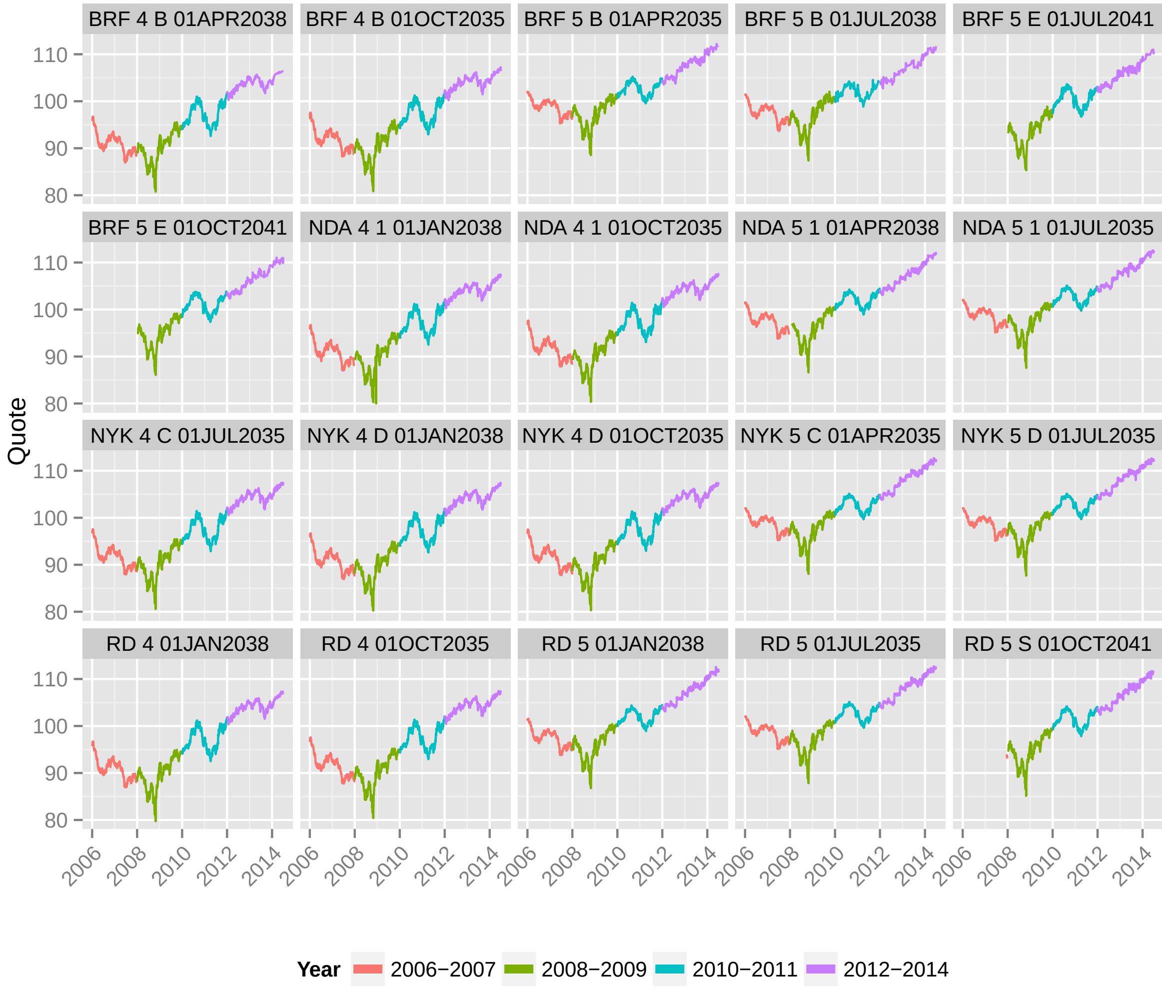


Figure 1.7: Market quotes over time

Figure 1.7 illustrates how the quotes of the bonds have developed over time, and figure 1.8 shows the dependency between `SWAP 15Y EUR6M` and the market quotes.

Prepayment data

For each bond we also have prepayment information. This is quarterly data, which tells us how much of the remaining principal is being prepaid on the coupon dates. The prepayment data are seen in figure 1.9.

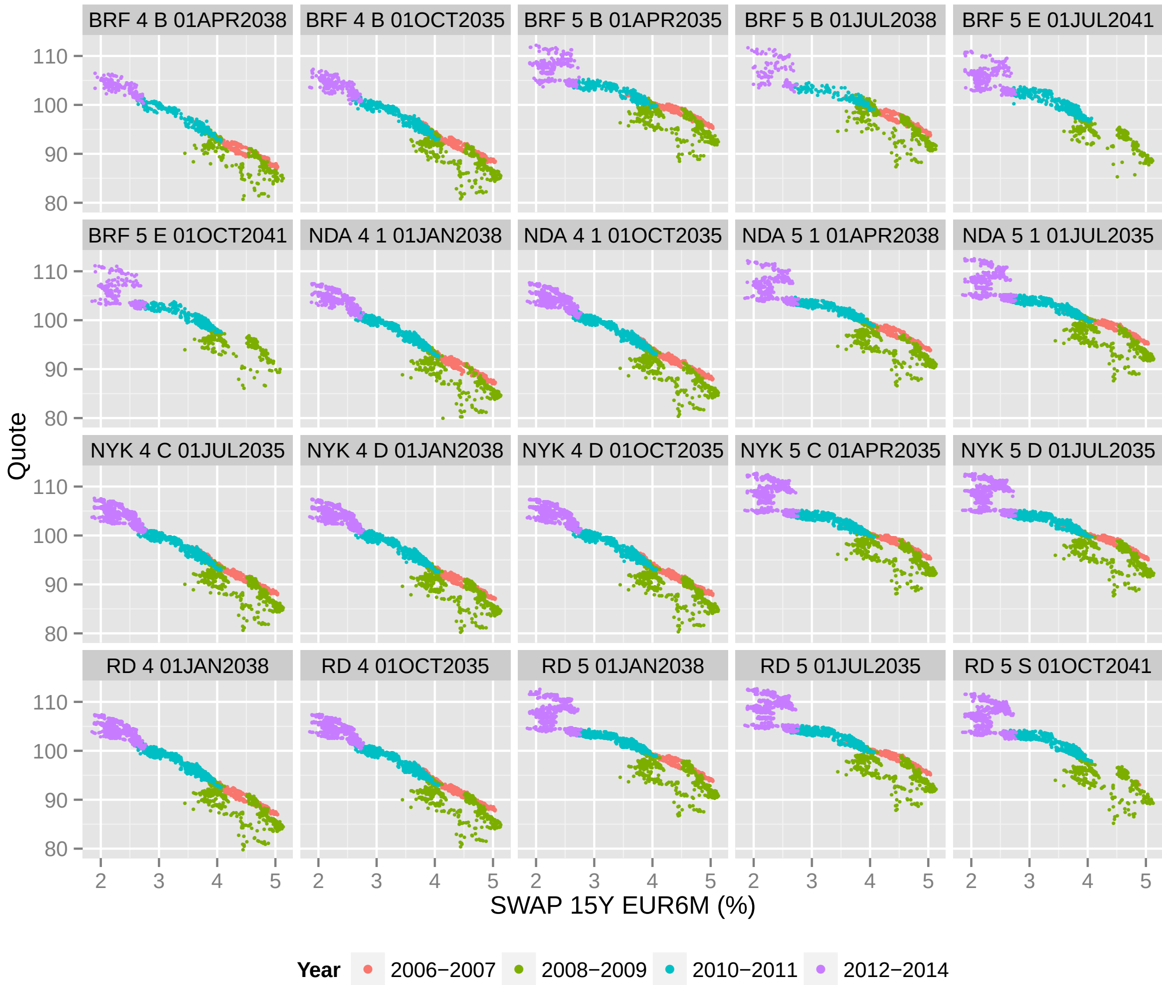


Figure 1.8: Market quotes plotted against the `SWAP 15Y EUR6M`.

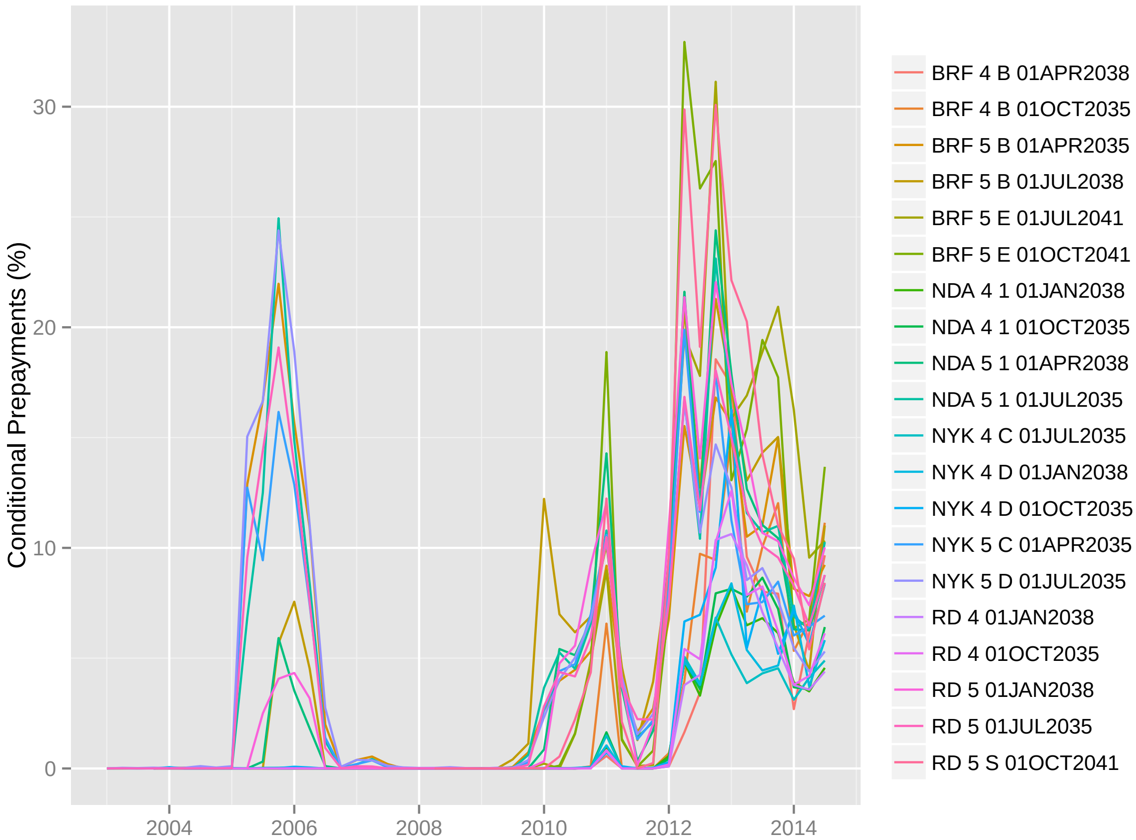


Figure 1.9: Realized conditional prepayments over time. The prepayments are made on the coupon dates only – hence the lines are simply there to make the plot easier to read.

ISIN	Name	Coupon	Issue date	Maturity date
DK0009361461	BRF 4 B 01APR2038	4%	2005-03-01	2038-04-01
DK0009358830	BRF 4 B 01OCT2035	4%	2003-06-11	2035-10-01
DK0009356545	BRF 5 B 01APR2035	5%	2002-03-01	2035-04-01
DK0009360570	BRF 5 B 01JUL2038	5%	2005-01-17	2038-07-01
DK0009366858	BRF 5 E 01JUL2041	5%	2007-12-20	2041-07-01
DK0009366429	BRF 5 E 01OCT2041	5%	2007-12-20	2041-10-01
DK0002015023	NDA 4 1 01JAN2038	4%	2005-03-01	2038-01-01
DK0002012780	NDA 4 1 01OCT2035	4%	2003-06-16	2035-10-01
DK0002014216	NDA 5 1 01APR2038	5%	2005-01-18	2038-04-01
DK0002011386	NDA 5 1 01JUL2035	5%	2002-03-07	2035-07-01
DK0004715505	NYK 4 C 01JUL2035	4%	2003-06-12	2035-07-01
DK0009761645	NYK 4 D 01JAN2038	4%	2005-05-25	2038-01-01
DK0009757296	NYK 4 D 01OCT2035	4%	2003-06-16	2035-10-01
DK0004714458	NYK 5 C 01APR2035	5%	2002-08-26	2035-04-01
DK0009753469	NYK 5 D 01JUL2035	5%	2002-04-25	2035-07-01
DK0009274300	RD 4 01JAN2038	4%	2005-06-01	2038-01-01
DK0009270233	RD 4 01OCT2035	4%	2003-06-12	2035-10-01
DK0009272874	RD 5 01JAN2038	5%	2005-01-14	2038-01-01
DK0009269227	RD 5 01JUL2035	5%	2002-06-24	2035-07-01
DK0009280380	RD 5 S 01OCT2041	5%	2007-12-07	2041-10-01

Table 1.1: Bond details. All with 30 year maturity

Chapter 2

A Toy Model for The Short Rate

As mentioned earlier, the pricing of callable mortgage bonds involves two layers of complexity: (1) modelling the dynamic behaviour of the market short rate, and (2) modelling the prepayment behaviour of mortgagors. It has already been clarified that the focus of this thesis is the modelling of prepayment behaviour. However as pointed out by (1) a model for the dynamic behaviour of the short rate is necessary.

Due to the above this chapter aims towards deriving a toy model for the market short rate. The word *toy* model is used because the fitting of a 'reasonable' short rate model could very well be the subject of an entire thesis on its own right. For the purpose of this thesis it is sufficient with a simple and tractable short rate model, that allows the prepayment models presented in chapter 3 and 4 to fit historically observed market quotes. For that reason we aim at fitting a Vasicek model for the short rate. However, the Vasicek model will most likely not contain the variability needed in order to capture the actual complexity of the market. We therefore end this chapter by introducing the Cheyette model and thereby suggesting how one may extend the term structure modelling aspect of this thesis.

Two of the main troubling aspects that one is faced with when attempting to fit a short rate model are 1) the risk-free aspect and 2) the time aspect. The first aspect refers to the fact that most pricing theory uses the assumption of a universal risk-free interest rate. This is problematic when dealing with reality, since no such interest rate exist in the real market (or at least it is not observable). We therefore prefer to refer to this interest rate as the short rate rather than the risk-free rate, since the observations to which it is fitted will most likely contain some risk.[1]. The second aspect refers to the fact that the short rate, by definition, is instantaneous, but that observed interest rates are not.

[1]This however does not mean that the pricing theory developed using the assumption of a risk-free short rate is lost. Recall that the standard pricing machinery applies to a setup with a risky short rate, cf. the intensity setup presented in [Lando (1998)]

2.1 The Vasicek Model

Let $r(t)$ denote the instantaneous market short rate at time t, which is defined by

$$r(t) = f(t,t) \quad \text{for} \quad t \in [0,T]. \tag{2.1}$$

Here $f(t,T)$ is the instantaneous forward rate with maturity T, contracted at t given by

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T},$$

where $p(t,T)$ is time-t price of a ZCB with maturity T.

We say that a stochastic process $(r(t))_{t\geq 0}$ follows a Vasicek model under real-world measure P if and only if it solves the affine SDE given by

$$dr(t) = \kappa(\theta^P - r(t))dt + \sigma dW^P(t), \quad r(0) = r_0 \tag{2.2}$$

where r_0, κ, θ^P and σ are constants and $W^P(t)$ is a Wiener process under P. Note that from (2.2) we see that it is mean-reverting around θ^P with κ being the speed of reversion. A process solving (2.2) may also be referred to as an Ornstein/Uhlenbeck process or a mean-reverting Gaussian process.

Given that the market is free of arbitrage we know that an equivalent martingale measure, Q, exists. Fix T and define Q by the Radon/Nikodym derivative

$$\frac{dQ}{dP} = e^{-\int_0^T \lambda(s)\mathrm{d}W^P(s) - \frac{1}{2}\int_0^T \lambda(s)\mathrm{d}s} \quad \text{on } \mathcal{F}_T.$$

It then follows, from the Girsanov Theorem (see Theorem 11.3 in [Björk (2003)]) that the dynamics of $(r(t))_{t\geq 0}$ under Q are given by

$$dr(t) = \kappa\left(\left(\theta^P - \frac{\sigma\lambda(t)}{\kappa}\right) - r(t)\right)dt + \sigma dW(t)$$

where $W(t)$ is a Wiener process under Q and $dW^P(t) = -\lambda(t)dt + dW(t)$.[2] We assume that the market price of risk $\lambda(t) = \lambda$ is constant. A constant λ implies that the short rate follows a Vasicek model under

[2]One has to check that $L(t) = e^{-\int_0^t \lambda(s)\mathrm{d}W^P(s) - \frac{1}{2}\int_0^t \lambda(s)\mathrm{d}s}$ is in fact a likelihoodprocess, i.e. $E^P[L(T)] = 1$, which is true if the Novikov Condition is satisfied, see Lemma 11.5 in [Björk (2003)]. With $\lambda(t) = \lambda$ it is obviously satisfied.

Q given by

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t), \quad r(0) = r_0, \tag{2.3}$$

where $\theta = \theta^P - \frac{\sigma\lambda}{\kappa}$.

As already mentioned the Vasicek model is a tractable model, as an example of this we note that the value of a ZCB has a simple closed-form expression.

Proposition 2.1. *In the Vasicek model the price of a ZCB with maturity T is given by*

$$p(t,T) = e^{A(t,T)-r(t)B(t,T)},$$

where

$$B(t,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \tag{2.4}$$

$$A(t,T) = \frac{(B(t,T)-(T-t))(\kappa^2\theta - \frac{1}{2}\sigma^2)}{\kappa^2} - \frac{\sigma^2 B^2(t,T)}{4\kappa}. \tag{2.5}$$

To prove proposition 2.1 one first needs to show that in an affine term structure model the functions $B(t,T)$ and $A(t,T)$ must solve two boundary value problems, cf. proposition 22.2 in [Björk (2003)]. The one for $B(t,T)$ being a Ricatti equation, which is independent of $A(t,T)$. After solving the Ricatti equation you may insert $B(t,T)$ into the equation for $A(t,T)$ and integrate. A proof is carried out in [Björk (2003)].

2.1.1 Estimation of the Short Rate Parameters κ, θ and σ

In the following we present how estimates for the parameters κ, θ and σ have been computed based on swap rate data.

Estimating the Short Rate Parameters κ and σ using Swap Rate Data

In section A.2, in the appendix, it is explained how to estimate the P-parameters of a Vasicek model using maximum likelihood estimation (MLE) *given* observations of the interest rate in question. However, as already mentioned the short rate is unobservable - hence we can not directly use the approach described in the appendix and therefore we have to be creative. Roughly speaking, the 'creative' idea is the following: the assumption of a Vasicek model for the short rate implies that the ZCB yields follow a Vasicek model, which then in the end implies that the 15-year swap rate follows a Vasicek model (the last implication only holds approximately). Given that the 15-year swap rate follows a Vasicek model we are able to estimate the P-parameters of the 15-year swap rate process using MLE based on observations of the 15-year swap rates. Given these estimates we can obtain estimates for the short rate process. The exact procedure is outlined in the following.

Before presenting the estimation procedure we note that it does not necessarily have to be the *15-year* swap rate, which is used in the procedure. The reason for computing estimates for the short rate based on the 15-year swap rate (opposed to using one of the other swap rates in our data set) is that it, in general, is well correlated with the observed bond prices. I.e. when plotting observed bond prices against the 15-year swap rate we recognise a dependence structure, which seems possible to model.[3]

We begin by specifying what we mean by a zero-coupon yield and a swap rate.

Definition 2.2. *The continuously compounded zero coupon yield $y(t,T)$ is given by*[4]

$$p(t,T) = e^{-y(t,T)(T-t)}. \tag{2.6}$$

$y(t,T)$ is the constant rate of interest which will give the same value to $p(t,T)$ as the value given by the market.

Proposition 2.3. *Let swap(t,n) denote the time t level of the fixed rate in a swap contract (i.e. the swap rate) with n payment exchanges. Assume that the contract includes interest exchanges at date $T_1+t,\dots,T_n+t$, where $T_1-t = T_{i+1}-T_i = \delta$. The swap rate is then given by*

$$swap(t,n) = \frac{1-P(t,T_n+t)}{\delta\sum_{i=1}^{n} P(t,T_i+t)}. \tag{2.7}$$

From equation (2.6) and (2.7) we note that there is a link between the swap rate and zero-coupon yields, since swap(t,n) is a function of $y(t,T_1+t),\dots,y(t,T_n+t)$ through the ZCB price. For that reason we take a closer look at $y(t,T+t)$ for some fixed value of T. By the use of Proposition 2.1 and by taking the logarithm on both sides of (2.6) we see that

$$\begin{aligned} & A(t,T+t) - r(t)B(t,T+t) = -y(t,T+t)T \\ \Leftrightarrow \qquad & y(t,T+t) = \frac{1}{T}\left(r(t)B(t,T+t) - A(t,T+t)\right) \end{aligned}$$

From equation (2.4) and (2.5) it is easily seen that $A(t,T+t) = A(0,T)$ and $B(t,T+t) = B(0,T)$. By

[3]This could be explained by the fact that a 30-year callable mortgage bonds have expected lives, which is much shorter than 30 years, due to the potential prepayments.

[4]See section 20.3.3 in [Björk (2003)].

Itô it therefore follows that

$$\begin{aligned}
dy(t, T+t) &= \frac{1}{T} B(0,T) dr(t) \\
&= \frac{1}{T} B(0,T) \kappa(\theta^P - r(t)) dt + \frac{1}{T} B(0,T) \sigma dW^P(t) \\
&= \kappa \left(\frac{1}{T} \left(B(0,T)\theta^P - A(0,T) \right) - y(t, T+t) \right) dt + \frac{1}{T} B(0,T) \sigma dW^P(t) \\
&= \kappa \left(\theta_y(T) - y(t, T+t) \right) dt + \sigma_y(T) dW^P(t)
\end{aligned}$$

where $\theta_y(T) = \frac{1}{T}\left(B(0,T)\theta^P - A(0,T)\right)$ and $\sigma_y(T) = \frac{1}{T}B(0,T)\sigma$. Hence the process $y(t, T+t)$ is also follows a Vasicek model.

Assume that we have observed swap$(t, 1), \ldots$,swap(t, n) in the market. By equation (2.7) we can derive implied values of $P(t, T_1 + t), \ldots, P(t, T_n + t)$ using forward substitution (also known as bootstrapping), since

$$\begin{aligned}
&\text{swap}(t,1) = \frac{1 - P(t, T_1 + t)}{\delta P(t, T_1 + t)} \\
\Leftrightarrow \quad &P(t, T_1 + t) = \frac{1}{\text{swap}(t,1)\delta + 1}
\end{aligned}$$

and

$$\begin{aligned}
&\text{swap}(t,n) = \frac{1 - P(t, T_n + t)}{\delta \sum_{i=1}^{n} P(t, T_i + t)} \\
\Leftrightarrow \quad &P(t, T_n + t) = \frac{1 - \text{swap}(t,n)\delta \sum_{i=1}^{n-1} P(t, T_i + t)}{\text{swap}(t,n)\delta + 1}
\end{aligned}$$

All in all the bootstrapping procedure results in the zero-coupon yields $y(t, T_1 + t), \ldots, y(t, T_n + t)$ implied by market swap rates. In other words, given the swap rate curve on a specific date we can compute the corresponding yield curve on that date. However, our data set only contains observations for swap(t, i) for $i \in \{4, 10, 20, 30, 40, 60\}$ with $\delta = \frac{1}{2}$. In order to overcome this problem and obtain swap(t, i) for $i = 1, .., 60$ we have simply inter - and extrapolated.[5] Figure 2.1 illustrates the inter- and extrapolation together with the derived $y(t, n)$ for different values of n.

Define

$$\overline{y}(t, n) = \frac{1}{n} \sum_{i=1}^{n} y(t, T_i + t) \tag{2.8}$$

[5] More specifically, the function `spline` in the statistical program `R` have been used for the inter - and extrapolation. This is a cubic spline.

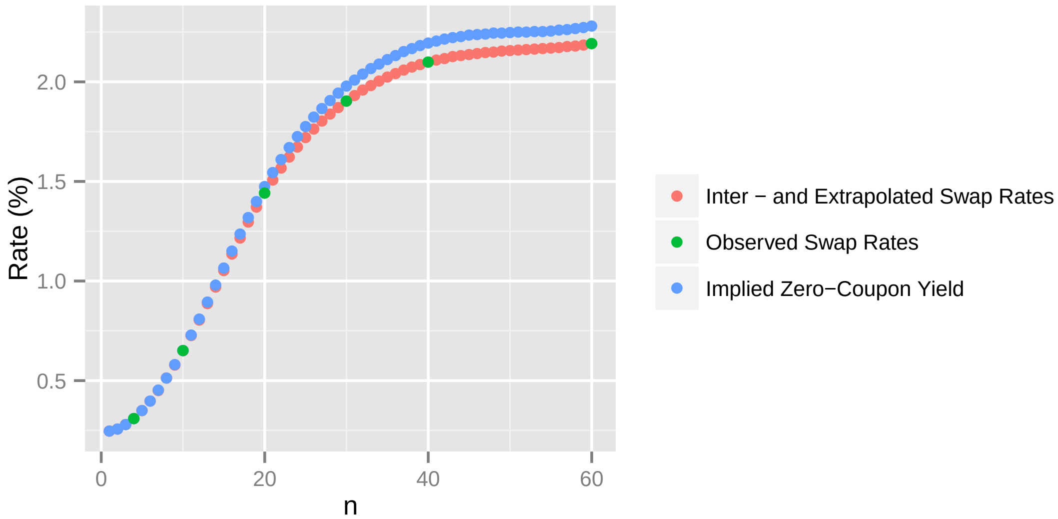


Figure 2.1: The swap curve and derived zero-coupon yield curve as they would have looked on 2014-07-01. The swap curve is found by inter - and extrapolating between the observed values of swap(t, i) for $i \in \{4, 10, 20, 30, 40, 60\}$. The yield curve is derived from the observed swap rates by bootstrapping. The x-axis corresponds to the value of n in swap(t, n) and $y(t, T_n + t)$.

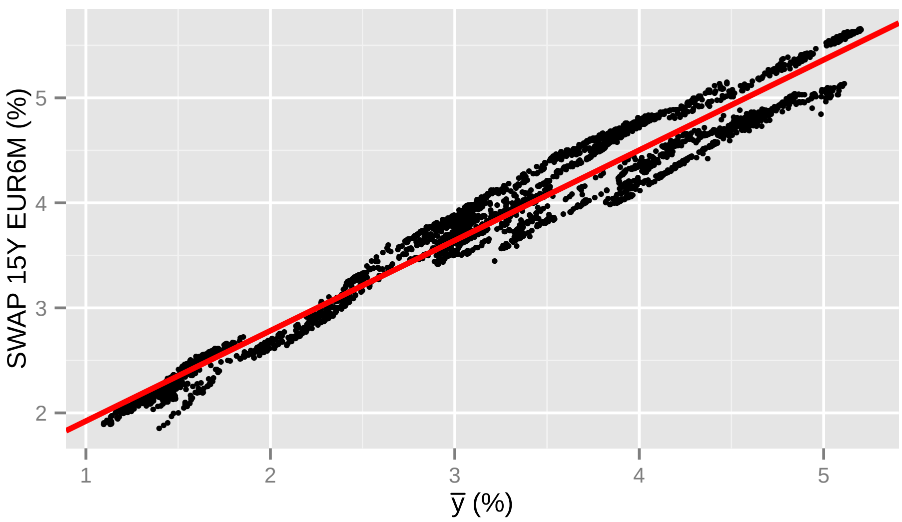


Figure 2.2: Observations of the `SWAP 15Y EUR6M` plotted against $\overline{y}(t, 30)$, which is defined by (2.8).

In figure 2.2 we have plotted observations of the `SWAP 15Y EUR6M` against $\overline{y}(t,30)$. Note that there seems to be a linear relation between swap$(t,30)$ and $\overline{y}(t,30)$. For that reason we may use the following linear approximation

$$\text{swap}(t,30) \approx a + b\overline{y}(t,30) \tag{2.9}$$

where $a = 0.011$ and $b = 0.86$[6]. Using (2.9) we can therefore approximate the dynamics of the 15-year swap rate by

$$\begin{aligned} d\,\text{swap}(t,30) &\approx b\overline{y}(t,30) \\ &= b\frac{1}{30}\sum_{i=1}^{30} dy(t,T_i+t) \\ &= \kappa\left(b\frac{1}{30}\sum_{i=1}^{30}\theta_y(T_i) - b\frac{1}{30}\sum_{i=1}^{3} 0y(t,T_i+t)\right)dt + b\frac{1}{30}\sum_{i=1}^{30}\sigma_y(T_i)dW^P(t) \\ &\approx \kappa\left(\theta_{\text{swap}}(30) - \text{swap}(t,30)\right)dt + \sigma_{\text{swap}}(30)dW^P(t) \end{aligned}$$

where $\theta_{\text{swap}}(30) = a + b\frac{1}{30}\sum_{i=1}^{30}\theta_y(T_i)$ and $\sigma_{\text{swap}}(30) = b\frac{1}{30}\sum_{i=1}^{30}\sigma_y(T_i)$.

Accepting the approximations implies that the 15-year swap rate can be described by a Vasicek model as well. Hence we may estimate the P-parameters of the 15-year swap rate by MLE using observations of the 15-year swap rate. More specifically, the observations being used are swap$(t_1,30),\ldots,$swap$(t_N,30)$ with $\delta = \frac{1}{2}$ and $N = 3135$. Here $t_1 = 0$ represents 2002-01-02. Each t_i is denoted in years from time 0 and t_N represents 2014-07-01. We choose to optimize the log-likelihood function numerically. Numerical optimizers are often quite sensitive to the chosen start guesses for the parameters. For that reason it seems natural to choose the closed-form expressions for the MLEs as the start guess.[7]

Note that the estimation process is under the P-measure and yields estimates of κ, $\theta_{\text{swap}}(30)$ and $\sigma_{\text{swap}}(30)$. Given an estimate of κ an estimate of σ is then found by translating back from the estimate of $\sigma_{\text{swap}}(30)$ in the following way

$$\begin{aligned} \sigma_{\text{swap}}(30) &= \sigma b\frac{1}{30}\sum_{i=1}^{30}\frac{1}{T_i}B(0,T_i) \\ \Leftrightarrow \qquad \sigma &= \frac{\sigma_{\text{swap}}(30)}{b\frac{1}{30}\sum_{i=1}^{30}\frac{1}{T_i}B(0,T_i)}. \end{aligned} \tag{2.10}$$

Because the estimation occurred under the P-measure it did not yield an estimate of θ (the short rate

[6] a and b have been calculated using linear regression.

[7] Under the assumption of equidistant observations a closed-form solution to the MLEs exists.

mean reversion level under Q). Below we show how one may overcome this problem in order to obtain an estimate of θ.

Estimating the Short Rate Mean Reversion Level under Q

We are now left with the estimation of the mean reversion level for the short rate under Q. The estimation of θ is based on the following proposition.

Proposition 2.4. *Assume that the limit of $y(t,T)$ for $T \to \infty$ exists. Assuming that the short rate follows a Vasicek model we then have that*

$$\lim_{T\to\infty} y(t,T) = \theta - \frac{\frac{1}{2}\sigma^2}{\kappa^2}.$$

Proof. Using proposition 2.1 this proof is an easy exercise in L'Hôpital's rule.

$$\begin{aligned}
\lim_{T\to\infty} y(t,T) &= \lim_{T\to\infty} -\frac{\log(p(t,T)}{T-t} \\
&= \lim_{T\to\infty} \frac{r(t)B(t,T) - A(t,T)}{T-t} \\
&= \lim_{T\to\infty} r(t)B_T(t,T) - A_T(t,T) \\
&= \theta - \frac{\frac{1}{2}\sigma^2}{\kappa^2},
\end{aligned}$$

where $B_T(t,T)$ and $A_T(t,T)$ are the derivatives with respect to T. □

Recall that by the use of bootstrapping we can derive implied zero coupon yield levels based on swap rate data. For a large n we then use proposition 2.4 to state that

$$\begin{aligned}
& y(t, T_n + t) \approx \theta - \frac{\sigma^2}{2\kappa^2} \\
\Leftrightarrow \quad & \theta \approx y(t, T_n + t) + \frac{\sigma^2}{2\kappa^2}.
\end{aligned} \tag{2.11}$$

Hence we have derived a method for estimating the mean reversion level under Q. Some readers might be sceptical when looking at (2.11), since the right hand side depends on t. However, we assume that for $n = 60$ it holds that $y(t, T_n + t) = y(t, 30 + t) \approx \lim_{T\to\infty} y(t, T + t)$, which is constant in t. Since the market price of risk is assumed to be constant, we have simply chosen to estimate θ based on the latest observed swap curve[8]. The resulting estimates for the short rate parameters κ, θ and σ are found in table 2.1.[9]

[8] I.e. the curve derived at 2014-07-01, which was illustrated in figure 2.1

[9] Note that we do not present any confidence intervals for the estimates in table 2.1. The reason for this is that it is not clear how such should be computed. For example: our estimate of $\hat{\theta}$ is derived by the use of swap(t,i) for

$\hat{\kappa}$	$\hat{\theta}$	$\hat{\sigma}$
0.0929	0.0305	0.0116

Table 2.1: Estimates for the short rate assuming that it follows a Vasicek model.

2.1.2 The Implied Short Rate Levels

Some might accept the fact that the short rate is unobservable, but in this thesis we do not - or at least we wish to observe the values implicitly through the swap rates. We have agreed on a model, where the parameters from table 2.1 governs the movement of the short rate. I.e.

$$dr(t) = 0.093(0.031 - r(t))dt + 0.012dW(t) \tag{2.12}$$

In order to estimate the parameters κ and σ we used observation of the swap rates $\text{swap}(t_i, 30)$ for $i = 1, \ldots, 3135$. These can also be used to derive the implied short rate levels $r(t_1), \ldots, r(t_{3135})$ by equation (2.7)

$$\begin{aligned}\text{swap}(t_i, 30) &= \frac{1 - P(t_i, 15 + t_i)}{\frac{1}{2}\sum_{j=1}^{30} P(t_i, \frac{1}{2}j + t_i)} \\ &= \frac{1 - e^{A(0,15) - B(0,15)r(t_i)}}{\frac{1}{2}\sum_{j=1}^{30} e^{A(0,\frac{1}{2}j) - B(0,\frac{1}{2}j)r(t_i)}}\end{aligned}$$

We are simply left with 3135 equations, where each one of them only have a single unknown. The final implied path of the short rate is plotted together with observations of the 15-year swap rate in figure 2.3. We note that the general level of the resulting path of the short rate is quite high. This indicates that our resulting short rate model, given by (2.12) is not a trustworthy model for the *true* short rate - hence the name *toy model*. Nevertheless, as we shall see in later chapters the short rate model is sufficient for the purpose of this thesis. In section 2.2 we introduce the Cheyette model in order to suggest how one may extend the term structure modelling aspect of this thesis.

2.2 Extension - The Cheyette Framework

In this section we briefly introduce the idea behind the Cheyette framework and illustrate that the Vasicek model is contained within this framework by computing the price of a non-callable, using a three-dimensional finite difference scheme. The reason for a *brief* introduction is the fact that the analysis' carried out in this thesis are all based on the toy model given by (2.12). Hence we do not actually implement the Cheyette model, when computing prices of callable mortgage bonds in chapter 5 - this

$i \in \{4, 10, 20, 30, 40, 60\}$. Obtaining a confidence interval by the use of Monte Carlo simulation would therefore require a simulation of all the swap levels simultanously. This is not a straight forward task since these are obviously correalated.

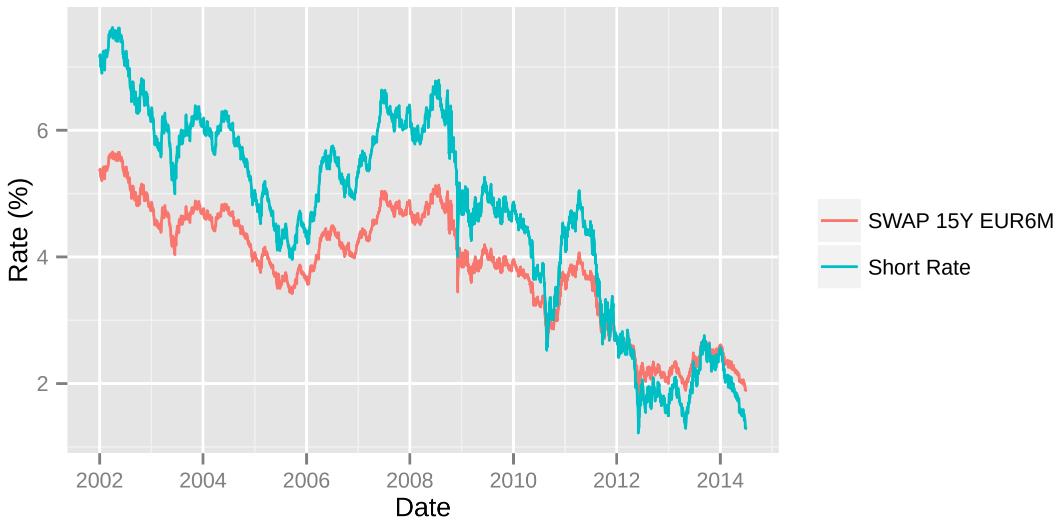


Figure 2.3: The observed path of swap$(t, 30)$ together with the implied of the short rate path. We note that the general level of the resulting path of the short rate is quite high. This indicates that our resulting short rate model, given by (2.12) is not a trustworthy model for the *true* short rate - hence the name *toy model*.

would only really make sense if we fitted a more complicated interest rate model. Thus this section is purely an example of how more sophisticated interest rate models may be incorporated into the pricing frame-work of the prepayment models presented in chapter 3 and 4.

In 1992, Heath, Jarrow, and Morton (HJM) standardized a valuation approach for interest rate derivatives. The HJM framework, is a general model environment where the state variable is now the entire forward rate curve, which is determined only by its volatility term. This is opposed to, for example, the Vasicek model, which only has the short rate, $r(t)$, as state variable. The HJM framework incorporates many previously developed approaches, such as the well-known interest rate models of Ho and Lee (1986), Vasicek (1977) and Hull and White (1990).

The HJM setup suffers because the short rate is non-Markovian in general. I.e. the workload increases immensely, when we have to condition on the entire history instead of just the current situation. However, by restricting the structure of the forward rate volatility function we are able to obtain a Markovian short rate - this is basically the idea of the Cheyette model. Hence the Cheyette model allows the application of standard valuation concepts, such as for example the beloved Feynman-Kac result.

For a more detailed presentation of the HJM framwork and the Cheyette model, see section A.3 and A.4 in the appendix. Here we also show how the Vasicek model is embedded in the Cheyette model.

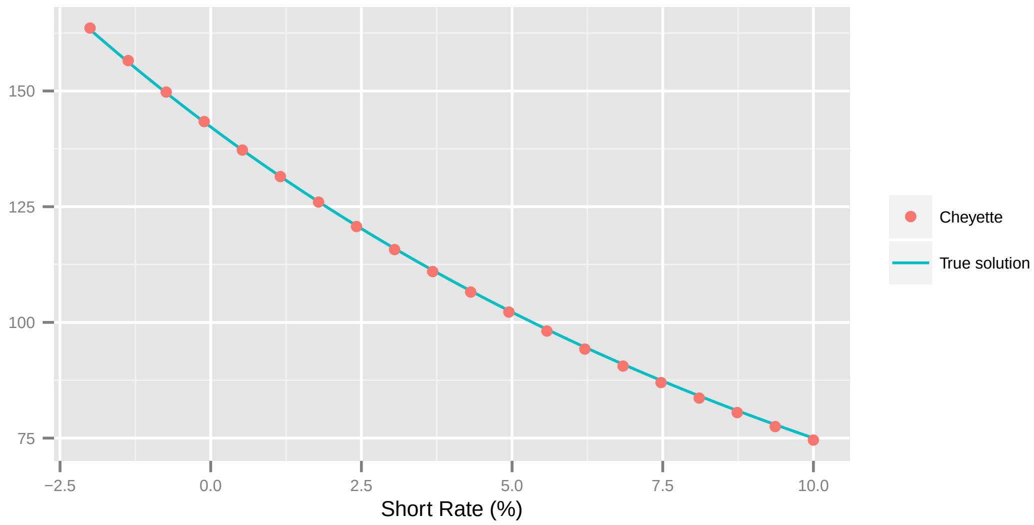


Figure 2.4: The time-zero value of a non-callable bond with maturity T =30 and coupon 4%. The values are derived using both the Cheyette model with $\kappa(t) = \kappa$ and $\eta(t) = \sigma$ (cf. section A.4 in the appendix) and the closed-form solution. It is assumed that the short rate follows a Vasicek model with parameters given by the estimates found in chapter 2.

Figure 2.4 illustrates the price of a non-callable bond computed in two different ways under the assumption that the short rate follows a Vasicek model: 1) using the closed-from expression given by Proposition 2.1 and 2) using a three-dimensional finite difference scheme. We end this section by pointing out that it is by no means a trivial task to implement a three-dimensional finite difference scheme. An introduction to three-dimensional finite difference schemes is found in section A.5 in the appendix.

Chapter 3

A Structural Prepayment Model

In this chapter we look into a prepayment model, which was put forward by Richard Stanton in [Stanton (1995)]. Below we introduce the setup of the model before focusing on recreating most of the results presented in [Stanton (1995)]. Recreating the results will help us gain a deeper knowledge of how one may choose to model prepayment behaviour in a structural way - consistently linking valuation and prepayment. The paper is a continuation of an option-theoretic framework presented in a series of earlier papers. It extends this already developed framework by implementing further frictions into the modelling framework. As we shall see later these new frictions make room for an endogenously produced burnout effect and they also remove unrealistic bounds on the mortgage prices (cf. figure 1.4 and 1.5).

In addition, to what is being presented in [Stanton (1995)], we actually explain how the figures presented here have been computed - an aspect which seems to be long forgotten in the world of research papers. A central part of the model is keeping track of changes in the transaction cost distribution over time, since mortgagors leave the pool over time due to prepayment[1]. Information on the transaction cost distribution at different times is essential in order for the model to predict prepayment behaviour and in order to compute the value of a callable mortgage bond. Furthermore we estimate key parameters of the model based on observed prepayment rates from Danish mortgage bonds.

3.1 Model Assumptions

In the model borrowers can choose to prepay for two reasons: 1) for interest rate reasons and 2) due to exogenous reasons, such as getting divorced, job relocation, etc..

[1]Heterogeneity between mortgagors is modelled through different transaction costs.

The Mortgagor's Optimization Problem

It is assumed that mortgagors act rational in the sense that a mortgagor wishes to minimize his or her mortgage liability, V^l, which at time t is given by

$$V^l(t) = B(t) - O^l(t). \tag{3.1}$$

Here $B(t)$ is the value of non-callable bond with the same maturity and coupon rate as the loan and $O^l(t)$ is the value of the prepayment option to the mortgagor. Since the value of the non-callable bond $B(t)$ is independent on the prepayment option, minimizing (3.1) corresponds to maximizing the value of the prepayment option.

The prepayment option is modelled as a call option on $B(t)$ with the time-varying strike $F(t)(1 + X)$, where $F(t)$ denotes the remaining value of the principal at time t and X denotes the transaction costs associated with prepayment for the mortgagor[2]. Given the transaction costs, the option value is solely dependent on the time (through $F(t)$ and $B(t)$) and the market short rate (through $B(t)$).

The assumption that mortgagors behave rational implies prepayment for interest rate reasons. This is seen by the fact that movements in the short rate changes the option value and thereby affect potential prepayment decisions. For example, a sufficiently large drop in the short rate will put the prepayment option in the money and therefore imply prepayments due to interest rate reasons.

Exogenous Prepayment and Suboptimal Behaviour

Mortgagors can also choose to prepay due to exogenous reasons. This is modelled through a constant intensity λ. This means that the probability of prepaying because of exogenous reasons within a time interval of length dt, given that no prepayment has occurred before, is given by $1 - e^{-\lambda dt}$.

In order for the model to produce empirically plausible outcomes it needs to incorporate suboptimal behaviour. This is achieved through the two following assumptions:

(1) It is assumed that mortgagors within a pool are heterogeneous. This is done explicitly through the transaction costs. More specifically, it is assumed that the transaction costs within a mortgage pool are beta-distributed with shape parameters α and β. The β-distribution is chosen because of its many possible shapes on $(0, 1)$.

[2]Note that transaction costs are assumed to be proportional to the remaining principal. This is in general not true as mentioned in [Stanton (1995)], but it is assumed in order to be able to predict prepayments without needing to know the remaining principal on each mortgage in the pool. This information is simply not available in the market.

(2) Mortgagors are assumed to make prepayment decisions at discrete times opposed to being able to reevaluate their prepayment decisions constantly. This is modelled through a Poisson process with constant intensity ρ. I.e. the time length between decisions points is exponentially distributed with mean $1/\rho$. At first sight this feature might seem strange. But it is in fact a very sensible assumption, since it captures the idea that a real-life mortgagor does not reevaluate his or her prepayment decision constantly.

These two features create the possibility for burnout behaviour without the need of an exogenous burnout factor. Furthermore the restriction, given by (2), removes the unrealistic bound on callable mortgage bond prices. This is caused by the fact that mortgagors might not exercise their prepayment option even if they find it optimal to do so - i.e. they behave non-optimally.

To sum up, the model assumes that we have an economy in which mortgagors are rational: they wish to find an optimal prepayment strategy. The mortgagor chooses a strategy that maximizes the value of the prepayment option and thereby minimizes the mortgage liability. However, this strategy has to be found through a maze of obstacles, such as transaction costs and lack of prepayment decisions, which may prevent the mortgagors from behaving optimally. For that reason the framework of [Stanton (1995)] is sometimes referred to as *bounded rationality.*

Before moving on to the valuation of a callable mortgage bond we note that, given the time, the value of the prepayment option is a function of transaction costs and the short rate. Therefore, given the short rate level there exists a critical transaction cost level $X^\star(t)$ such that if $X \leq X^\star(t)$ then a mortgagor with transaction costs given by X will optimally choose to prepay. Thereby not said that the mortgagor decides to actually prepay. Equivalently, given transaction cost X, there exists a critical short rate level $r^\star(t)$ such that if $r(t) \leq r^\star(t)$ then mortgagor with transaction costs given by X optimally choose to prepay. The optimal prepayment strategy determines the value of the prepayment option, which then gives the value of the mortgage liability - i.e. the mortgage loan. In the following section we explain how this, in the end, is used to determine the price of a callable mortgage bond - the price of a callable mortgage bond being different from the mortgage liability, since the investor does not receive the transaction costs.

3.2 Valuation of a Mortage-Backed Security

Recall that the cash flow of a callable mortgage bond comes from a pool of mortgages. The cash flow of a bond is therefore determined, not only by the prepayment behaviour of a single mortgagor, but by the prepayment behaviour of all mortgagors in the pool. However, in this section we focus on the valuation of a single mortgage and the valuation of a security backed by this mortgage - hence a mortage-backed security. The extension from one mortgage to a pool of mortgages is presented in section 3.5. Before moving on to the specific valuation method we note that the cash flow accruing from the mortgagor differs from the cash flow received by the investor du to the transactions costs. We denote the value of the bond to the investor by $V^a(t)$. The final goal is to compute the asset value, V^a, oppose to the liability value, V^l, given by equation (3.1). But as described in [Stanton (1995)] these two need to be computed simultaneously, since the optimal prepayment strategy, which determines the expected cash flow of the mortgage-backed security, is determined as part of the liability valuation.

We are now ready to present how the asset and liability values are computed. Note that the following procedure differs from the one presented in [Stanton (1995)]. This is because the mortgagor has to announce prepayments two months in advance, due to Danish legislation. Furthermore [Stanton (1995)] models the cash flow as a contiunous payment stream, while we model it as discrete coupon payments.

Assume that the market short rate solves the SDE given by

$$\begin{aligned} dr(s) &= \mu(s, r(s))ds + \sigma(s, r(s))dW(s) \\ r(0) &= r_0 \end{aligned}$$

Let $T_1, \ldots, T_n$ denote the coupon dates for a callable mortgage bond with a coupon rate q. Remember that $T_1 = T_{i+1} - T_i = dt = \frac{1}{4}$, when time is measured in years. Let C denote the coupon size (in case of no prepayments) given by (1.4).

Let $\tilde{T}_i = T_i - \frac{1}{6}$ denote the announcement dates – i.e. the dates two months before the coupon dates. Recall that this is the dates where the mortgagor has to decide whether or not he or she wishes to prepay at the upcoming coupon date. Therefore we wish to evaluate the bond values on these dates rather than on the coupon dates.

Define $V_i^l(t, r)$ for $t \leq \tilde{T}_i$ and $i = 1, \ldots, n$ as the time t value of the payment at time T_i plus future liability, assuming no prepayment has been made before time T_i. Note that $V_n^l(t, r)$ is simply the time t price of a ZCB with maturity T_n and payment C, which implies that $V_n^l(t, r)$ solves the boundary value problem (1.5) with $T^\star = T_n$ and $C^* = C$.

The mortgagor will either prepay or continue with the current bond at each coupon date. By asspumption, the mortgagor chooses the option which cost him or her the least. Therefore at time $\tilde{T}_i$ it is optimal to prepay at time T_i if

$$V_{i+1}^l(T_i, r) > F_{T_i}(1 + X)$$

where F_{T_i} is the remaining principal when coupon i has been paid. Since the mortgagor has to decide two months in advance, he or she choose to prepay if

$$\begin{aligned} & E_{\tilde{T}_i,r}\left[e^{-\int_{\tilde{T}_i}^{T_i} r(s)ds} V_{i+1}^l(T_i, r)\right] > E_{\tilde{T}_i,r}\left[e^{-\int_{\tilde{T}_i}^{T_i} r(s)ds} F_{T_i}(1 + X)\right] \\ \Leftrightarrow \quad & V_{i+1}^l(\tilde{T}_i, r) > d(r)F_{T_i}(1 + X) \end{aligned} \tag{3.2}$$

where

$$d(r) = E_{\tilde{T}_i,r}\left[e^{-\int_{\tilde{T}_i}^{T_i} r(s)ds}\right] = E_{0,r}\left[e^{-\int_0^{\frac{1}{6}} r(s)ds}\right] \tag{3.3}$$

If the mortgagor decides to prepay, then he or she has to pay $F_{T_i}(1+X)+C$ at time T_i. If the mortagagor decides not to prepay then the value is equal to $V_{i+1}^l(T_i, r) + C$ at time T_i.

Therefore if equation (3.2) is true we set the mortgage liability at time $\tilde{T}_i$ equal to

$$V_i^l(\tilde{T}_i, r) = (1 - P_r)V_{i+1}^l(\tilde{T}_i, r) + P_r d(r)F_{T_i}(1 + X) + d(r)C$$

where P_r is the probability of prepaying if it optimal to do so. This is given by

$$P_r = 1 - e^{-(\rho+\lambda)dt}.$$

On the other hand, if equation (3.2) is not true, such that prepayment only occurs for exogenous reasons, the mortgage liability is set equal to

$$V_i^l(\tilde{T}_i, r) = (1 - P_e)V_{i+1}^l(\tilde{T}_i, r) + P_e d(r)F(T_i)(1 + X) + d(r)C$$

where

$$P_e = 1 - e^{-\lambda dt}$$

is the probability of prepaying for exogenous reasons.

Due to these observations, we conclude that $V_i^l(t,r)$ solves the boundary value problem on $[0,\tilde{T}_i] \times \mathbb{R}$ given by

$$0 = \frac{\partial V_i^l(t,r)}{\partial t} + \mu(t,r)\frac{\partial V_i^l(t,r)}{\partial r} + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 V_i^l(t,r)}{\partial r^2} - rV_i^l(t,r)$$
$$V_i^l(\tilde{T}_i,r) = \begin{cases} (1-P_r)V_{i+1}^l(\tilde{T}_i,r) + P_r d(r)F_{T_i}(1+X) + d(r)C & \text{if} \quad V_{i+1}^l(\tilde{T}_i,r) > d(r)F_{T_i}(1+X) \\ (1-P_e)V_{i+1}^l(\tilde{T}_i,r) + P_e d(r)F_{T_i}(1+X) + d(r)C & \text{otherwise.} \end{cases} \tag{3.4}$$

We have now determined the stragegy of the mortgagor and can derive the asset values $V^a(t)$ in a similar way. Define $V_i^a(t,r)$ for $t \le \tilde{T}_i$ and $i = 1,\dots,n$ as the time t value of the payment at time T_i plus future payments recieved by the investor, assuming no prepayments have been made before time T_i.

Similarly $V_n^a(t,r)$ is simply the time t price of a ZCB with maturity T_n and payment C, which implies that $V_n^l(t,r)$ solves the boundary value problem (1.5) with $T^\star = T_n$ and $C^* = C$. Due to the prepayment strategy of the mortgagor then $V_i^a(t,r)$ solves the boundary value problem on $[0,\tilde{T}_i] \times \mathbb{R}$ given by

$$0 = \frac{\partial V_i^a(t,r)}{\partial t} + \mu(t,r)\frac{\partial V_i^a(t,r)}{\partial r} + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 V_i^a(t,r)}{\partial r^2} - rV_i^a(t,r)$$
$$V_i^a(\tilde{T}_i,r) = \begin{cases} (1-P_r)V_{i+1}^a(\tilde{T}_i,r) + P_r d(r)F_{T_i} + d(r)C & \text{if} \quad V_{i+1}^l(\tilde{T}_i,r) > d(r)F_{T_i}(1+X) \\ (1-P_e)V_{i+1}^a(\tilde{T}_i,r) + P_e d(r)F_{T_i} + d(r)C & \text{otherwise.} \end{cases} \tag{3.5}$$

From (3.5) we explicitly see that the asset and liability values need to be calculated simultaneously. Furthermore we note that the asset value is always less than the mortgagor's liability, since the transaction costs paid by the mortgagor is not received by the investor. We derive $V_i^a(\tilde{T}_i, r_k)$ for $i = 1,\dots,n$ and $k = 1,\dots,N_r$ by solving (3.4) and (3.5) using a Crank-Nicolson scheme, cf. A.5 in the appendix.

In figure 3.1 the price of a mortgage-backed security, V^a, is plotted against the short rate for different values of the parameter ρ. We see that the asset value is decreasing in ρ. This is because a higher ρ enables the mortgagor to better follow his or her optimal prepayment strategy. Thereby increasing the value of the prepayment option.

In figure 3.2 the price of a mortgage-backed security, V^a, is plotted against the short rate for different values of transaction costs. The parameter ρ is chosen very high, such that there is practically no delay in exercising of the prepayment option, when it is optimal to do so. We assume that no prepayment occurs for exogenous reasons, hence $\lambda = 0$. We see that the price is increasing in transaction costs. This is due to the fact that the higher the transaction costs the less likely it is that the mortgagor prepays - hence the less valuable is the option. Note that in both figure 3.1 and 3.2 the price rises well above par plus interest for small rates. This happens when the mortgagor is unable to prepay even though it may be optimal to do so, because of transaction costs and lack of prepayment decisions.

In figure 3.3 the parameter ρ is chosen, such that the average time between prepayment decisions is 2 years, i.e. $\rho = \frac{1}{2}$. This is the reason why the price exceeds par even with zero transaction costs - opposed to what we saw in figure 3.2, where ρ was chosen to be very high. Furthermore we note that the maximum price, corresponding to the highest chosen transaction cost, is lowered from figure 3.2 to figure 3.3. This is caused by the fact that the parameter λ is no longer zero in figure 3.3. When the probability of prepayment for exogenous reasons is non-zero it lowers the price of the security, when it is above par, and rises it, when it is below par. This is because when prepayment occurs the investor receives par value. The effect of a non-zero λ is illustrated in figure 3.4.

It is important to note that the effect of a non-zero λ for prices under par is troubling, when applying the model to the Danish mortgage bond market. As mentioned above a non-zero λ increases the price of the security, when the price is below par. This is because when prepayment occurs the investor receives par value in the model. However, in Denmark a mortgagor may choose to prepay in two different ways: 1) By exercising the prepayment option, i.e. paying par value. 2) By buying the corresponding bond in the market at market price. The second point implies that the investor does not receive par if the loan is prepayed when the value is below par, hence a non-zero λ should not increase the value, when it is below par.

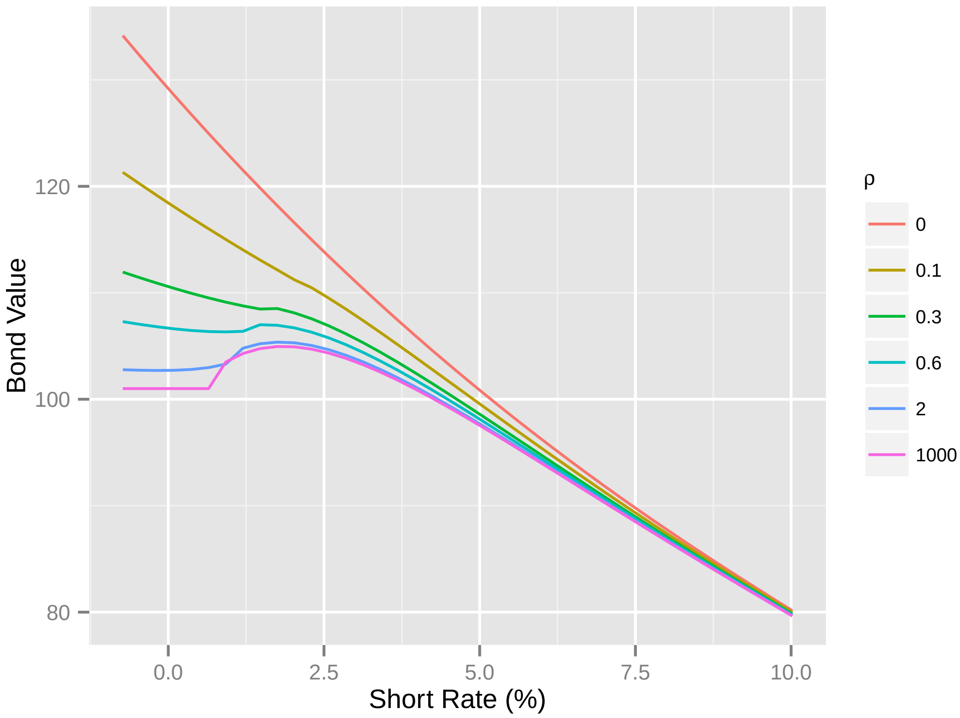


Figure 3.1: The price of a mortgage-backed security for different values of the parameter ρ. The parameter ρ governs how often a mortgagor is able to reevaluate his or her prepayment decision. $\rho = 0$ and $\rho \to \infty$ corresponds to no and constant reevaluation of the prepayment decision, respectively. More specifically, we have plotted the time-zero price of a security backed by a 30-year mortgage with a face value of 100 and a coupon rate of 4%. The parameter λ is set to equal to 0.05 and transaction cost is set to 25% of the remaining principal. The short rate follows a Vasicek model under Q with the parameters given by the estimates found in chapter 2.

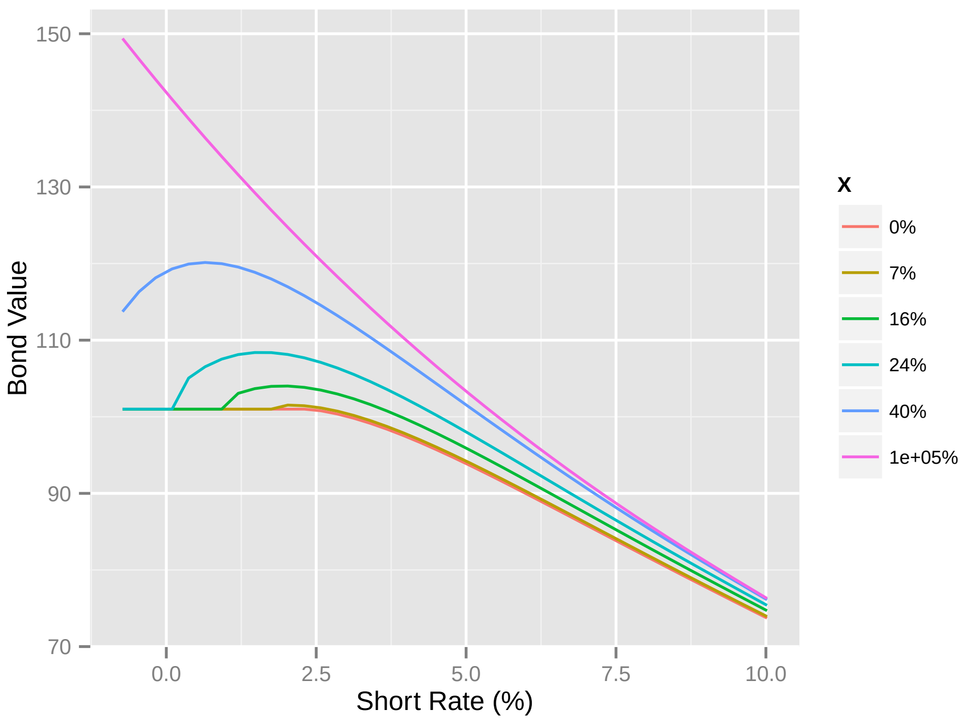


Figure 3.2: The price of a mortgage-backed security for different values of transaction costs with almost no lack of prepayment decision - ρ is set equal to 1000, which implies an average time between decision points of approximately 9 hours. The higher the transaction costs the less likely it is that a mortgagor chooses to prepay. More specifically, we have plotted the time-zero price of a security backed by a 30-year mortgage with a face value of 100 and a coupon rate of 4%. The parameter λ is set to zero. The short rate follows a Vasicek model under Q with the parameters given by the estimates found in chapter 2.

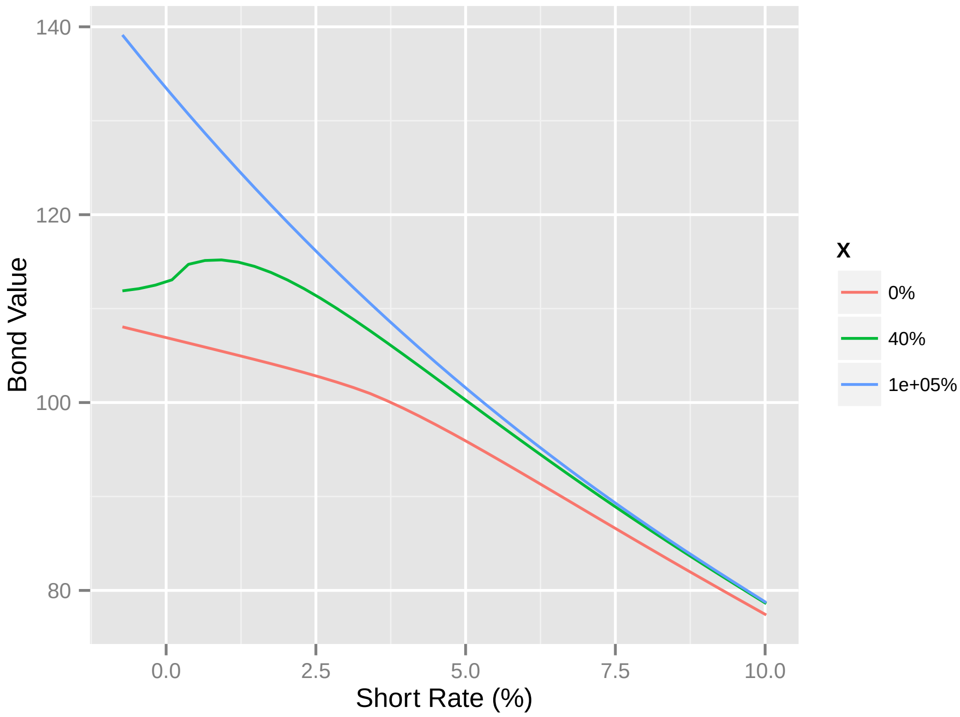


Figure 3.3: The price of a mortgage-backed security for different values of transaction costs with lack of prepayment decision - ρ is set equal to 0.5, which implies an average time between decision points of 2 years. More specifically, we have plotted the time-zero price of a security backed by a 30-year mortgage with a face value of 100 and a coupon rate of 4%. The parameter λ is set to 0.05. The short rate follows a Vasicek model under Q with the parameters given by the estimates found in chapter 2.

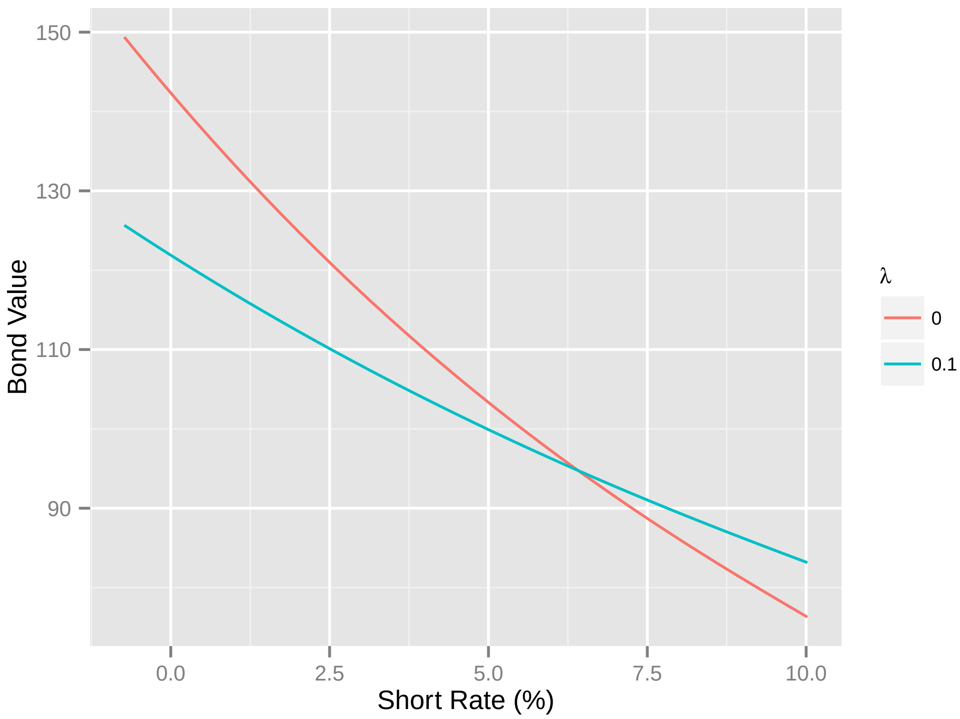


Figure 3.4: The price of a mortgage-backed security for different values of λ. The parameter λ governs the probability of prepayment occurring for exogenous reasons. A λ of non-zero lowers the time-zero price of the security above par and rises it below par. This is because when prepayment occurs the investor receives par value. More specifically, we have plotted the price of a security backed by a 30-year mortgage with a face value of 100 and a coupon rate of 4%. The parameter ρ is set equal to 0.5 and X is set equal to 1000. The short rate follows a Vasicek model under Q with the parameters given by the estimates found in chapter 2

3.3 The Prepayment Behaviour

From the pricing method of a mortgage-backed security outlined in the previous section we know that the prepayment probability of a single mortgagor at time t is given by P_r if it is optimal for the mortgagor to prepay and by P_e otherwise. Loosely speaking the prepayment proportion of an entire pool is then computed by taking a weighted average of the different mortgagors' prepayment probabilities. In subsection 3.3.2 we explain the full depths of computing the prepayment probability for a pool. This is done in order to simulate different types of prepayment behaviour, which may be generated by the model. But first we illustrate how the parameter ρ affects prepayment behaviour over time, given a constant level of mortgagors in the pool, who finds it optimal to prepay.

3.3.1 Illustrating How the Parameter ρ Governs Prepayment Behaviour Over Time

Recall that mortgagors are assumed to make prepayment decisions at discrete times opposed to being able to reevaluate their prepayment decisions constantly. This is modelled through a Poisson process with a constant intensity ρ. I.e. the time length between decisions points is exponentially distributed with mean $1/\rho$. In order to illustrate how ρ governs prepayment behaviour over time we compute annualized conditional prepayment proportions on a quarterly basis for different values of ρ. We assume that there are no exogenous prepayments and that the short rate remains, for 5 years, at a level where fifty percent of the mortgagors currently in the pool find it optimal to prepay. For notational simplicity we introduce the notion of a representative mortgagor. The prepayment probability of the representative mortgagor then corresponds to the proportion of the pool, which we expect to prepay. I.e. the probability of the representative mortgagor deciding to prepay on a given decision date is fifty percent.

Let τ denote the time of prepayment for the representative mortgagor. We are then interested in computing the conditional probability $P(\tau \leq t + dt \mid \tau > t)$. By the law of total probability it follows that

$$
\begin{aligned}
&P(\tau \leq t + dt \mid \tau > t) \\
&= \sum_{k=0}^{\infty} P(\tau \leq t + dt \mid \tau > t\ ,\ N(t + dt) - N(t) = k) P(N(t + dt) - N(t) = k) \\
&= \sum_{k=0}^{\infty} (1 - P(\tau > t + dt \mid \tau > t\ ,\ N(t + dt) - N(t) = k)) P(N(t + dt) - N(t) = k),
\end{aligned}
\tag{3.6}
$$

where $N(t)$ is a Poisson process denoting the number of decision dates until time t.

Recall that in this example there is a fifty percent chance of prepayment at each decision date. Hence the conditional probability under the sum in (3.6) must equal $1/2^k$. To see this note that the probability of not deciding to prepay on a given decision date is 1/2 - hence the probability of not deciding to prepay on k decision dates, is $1/2^k$. Furthermore we know that $N(t + dt) - N(t)$ is Poisson distributed with parameter ρdt. We may therefore write

$$P(\tau \leq t + dt \mid \tau > t) = \sum_{k=0}^{\infty} \left(1 - \frac{1}{2^k}\right) \frac{(\rho dt)^k e^{-\rho dt}}{k!} \tag{3.7}$$

$$= 1 - e^{-\frac{1}{2}\rho dt}, \tag{3.8}$$

where the second equality is easily shown by using that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

The probability of prepayment within the next year at time zero of the 5 year period is therefore given by $1 - e^{-\frac{1}{2}\rho}$. If, for example, $\rho = 1$ we expect approximately 39 percent of the pool to prepay within the next year, at time zero. In order to compute the conditional prepayment proportion after n quarters we need to take into account the proportion of the pool, which we expect to prepay within the first n quarters. This proportion is given by $1 - e^{-\frac{1}{2}\frac{\rho}{4}n}$. The expected proportion left in the pool after n quarters is therefore $e^{-\frac{1}{2}\frac{\rho}{4}n}$. A final expression for the annualized conditional prepayment proportion after n quarters is for that reason given by

$$\left(1 - e^{-\frac{1}{2}\rho}\right) e^{-\frac{1}{2}\frac{\rho}{4}n}. \tag{3.9}$$

In figure 3.5 the prepayment proportions have been plotted for different values of ρ. For large values of ρ we therefore see that every mortgagor who finds it optimal to prepay, does so very quickly, while for lower values of ρ prepayments continue over a longer period of time. A ρ of zero corresponds to no prepayment for rational reasons. A ρ going towards infinity implies that mortgagors reevaluate their prepayment decisions constantly, and hence all mortgagors will decide to prepay instantly in the limit.

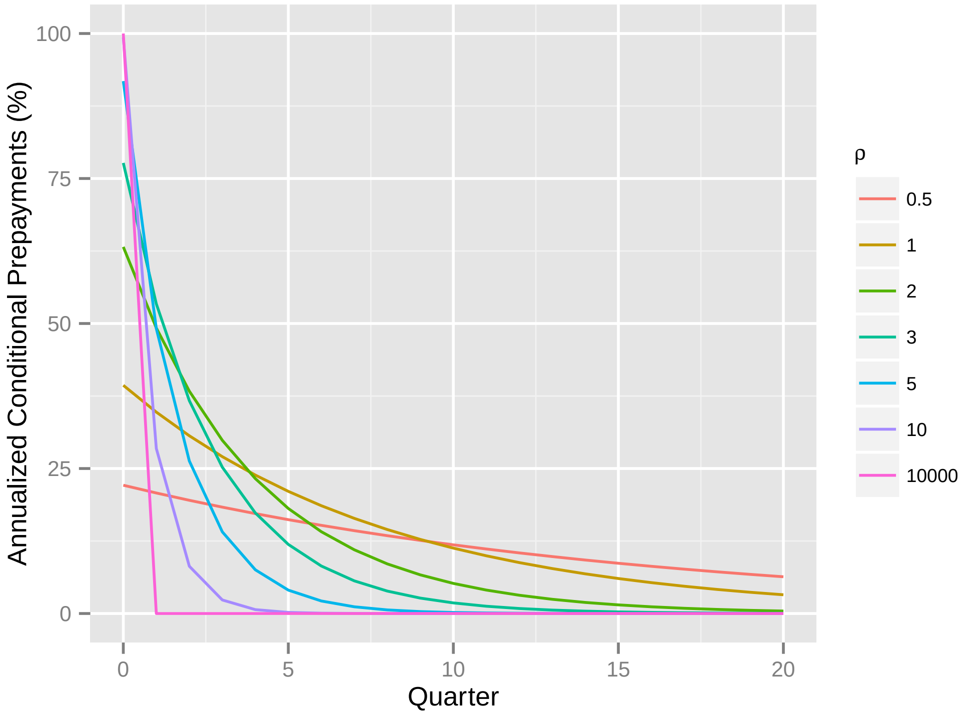


Figure 3.5: The annualized conditional prepayment proportion evolving over time for different values of the parameter ρ. The parameter ρ governs how often a mortgagor is able to reevaluate his or her prepayment decision. $\rho = 0$ and $\rho \to \infty$ corresponds to no and constant reevaluation of the prepayment decision, respectively. It is assumed that there are no exogenous prepayments and that the short rate remains, for 5 years, at a level where 50 percent of the mortgagors currently in the pool find it optimal to prepay.

3.3.2 Changes in The Transaction Cost Distribution and Different Types of Prepayment Behaviour

The initial transaction cost distribution is governed by the parameters α and β, but it changes over time, due to the fact that mortgagors in the pool gradually decide to prepay. In this subsection we explain how to keep track of the changes in the transaction cost distribution over time. This is done through investigation of different types of prepayment behaviour that may be produced by the model. Given a path for the market short rate we look at three hypothetical pools. The set of model parameters governing the prepayment behaviour are α and β and the intensities ρ and λ. We denote the vector of parameters by $\theta = (\alpha, \beta, \rho, \lambda)$. We look at three mortgage pools A, B and C, with their prepayment behaviour specified the parameter sets

$$\theta_A = (0.5, 0.5, 2, 0.05) \tag{3.10}$$

$$\theta_B = (0.5, 0.5, 0.3, 0.05) \tag{3.11}$$

$$\theta_C = (0.5, 4, 0.3, 0.05). \tag{3.12}$$

Note that the chosen parameter values imply that mortgagors in pool A take an average of six months to prepay while those in pool B and C take three years and four months. Furthermore the initial transaction cost distributions are the same for pool A and B, while mortgagors in pool C are in general faced with lower transaction costs.

Imagine that we are standing at time $t > T_1$ (i.e. somewhere after the first coupon date) and wish to determine the model predicted conditional prepayment proportion of a pool at each coupon date before t – given the market short rate. For this we need the distribution of the transaction cost levels in the pool, which changes on each coupon date. Define k so that T_k denotes the latest coupon date before t. For $s \leq T_1$ the initial distribution of transaction cost is simply given by the $\beta(\alpha, \beta)$-distribution.

Below we describe how one may obtain these expected prepayments using a *continuous* initial transaction cost distribution - opposed to the procedure described in the section *Determining the Expected Prepayment Level* in [Stanton (1995)], where a numerical approximation to the underlying 'true' cost distribution is being used. We choose to present the computation of expected prepayments using the 'true' continuous distribution, because it gives a richer illustration of how the transaction cost distribution changes over time, see figure 3.6 and 3.7. But this approach is solely used for illustrative purposes – when it comes to the actual computations in order to determine expected prepayments, we take advantage of the numerical approximation used in [Stanton (1995)]. This is due to the fact that it brings down the computational

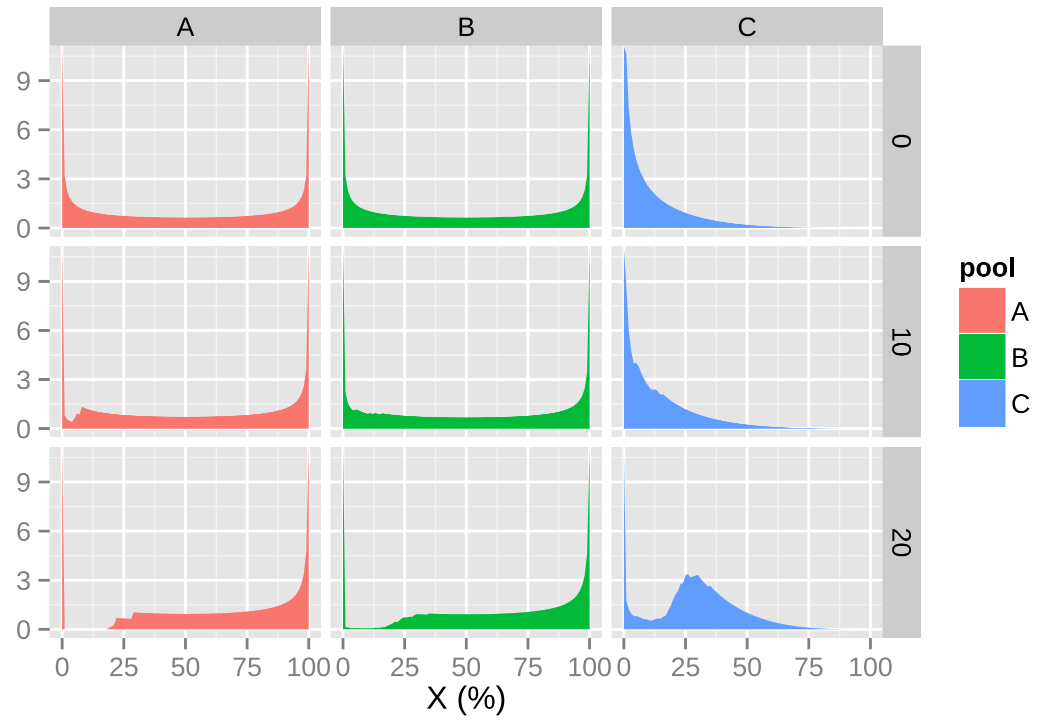


Figure 3.6: Illustrating how the conditional transaction cost density changes over time. Each column corresponds to a pool type denoted A, B and C. Each row corresponds to a timepoint in years. It shows how the mortgagors left in the pool have relatively high transaction costs compared to the intial pool. The densities are obtained by observing the simulated short rate in figure 3.8.

burden associated with calculating the critical cost levels.

In section 3.1 we described that, given the short rate, there exists a critical transaction cost level $X^{\star}_{T_i}$ such that if $X \leq X^{\star}_{T_i}$ then it is optimal to prepay for a mortgagor with transaction costs given by X. For every time point $T_1, \ldots, T_k$ we compute the corresponding critical transaction cost level. We denote these levels by $X^{\star}_{T_1} \ldots, X^{\star}_{T_k}$, i.e.. the levels, which satisfy $V^{l}_{i+1}(\tilde{T}_i, r(\tilde{T}_i)) = d(r(\tilde{T}_i))F_{T_i}(1+X^{\star}_{T_i})$ for $i = 1, \ldots, k$ where $\tilde{T}_i$ denotes announcement date i and $d(r)$ is given by (3.3).

For every time $0 \leq s \leq T_i$, there is a corresponding 'density' function, denoted by g_{i-1}, and a corresponding 'distribution' function, defined by $G_{i-1}(x) = \int_0^x g_{i-1}(y)\mathrm{d}y$ for $i = 1, \ldots, k+1$. Note that it is only a probability density function and probability distribution function for $i = 1$. The current time t-distribution of the transaction cost levels are then given by g_k.

Given the density g_{i-1} and $X^{\star}_{T_i}$ we compute the subsequent density g_i for $i = 1, \ldots, k$ by

$$g_i(x) = g_{i-1}(x)\left[1 - P_e 1_{\left(x > X^{\star}_{T_i}\right)} - P_r 1_{\left(x \leq X^{\star}_{T_i}\right)}\right]. \tag{3.13}$$

The reason behind equation (3.13) is that the probability of a mortgagor prepaying if his or her transaction cost is below the critical level is P_r, while if his or her transaction cost is above the critical level the probability is P_e.

We are now ready to compute the expected conditional prepayments at each coupon date $T_1, \ldots, T_n$. First, note that $G_i(X^\star_{T_i})$ gives the *actual* proportion of mortgagors who find it optimal to prepay at time T_i. The conditional proportion is simply found by dividing with the number of mortgagors left in the pool at time T_i. The expected conditional prepayment proportion at time T_i for $i = 1, \ldots, k$, $P(T_i)$, is therefore found by

$$P_{T_i} = \left(1 - \frac{G_{i-1}\left(X^\star_{T_i}\right)}{G_{i-1}(1)}\right) P_e \; + \frac{G_{i-1}\left(X^\star_{T_i}\right)}{G_{i-1}(1)} P_r. \tag{3.14}$$

In figure 3.6 and 3.7 it is illustrated how the density and distribution for the transaction costs changes over time for the different pools given the simulated short rate path specified in figure 3.8.

From equation (3.14) we see that the expected conditional prepayment proportion at time T_i is found by taking an weighted average of the probability for prepayment due only to exogenous reason and due to both rational and exogenous reasons. The weights being the expected fractions, conditioned on the size of the pool at the time T_i, that finds it optimal to prepay and that does not find it optimal to prepay.

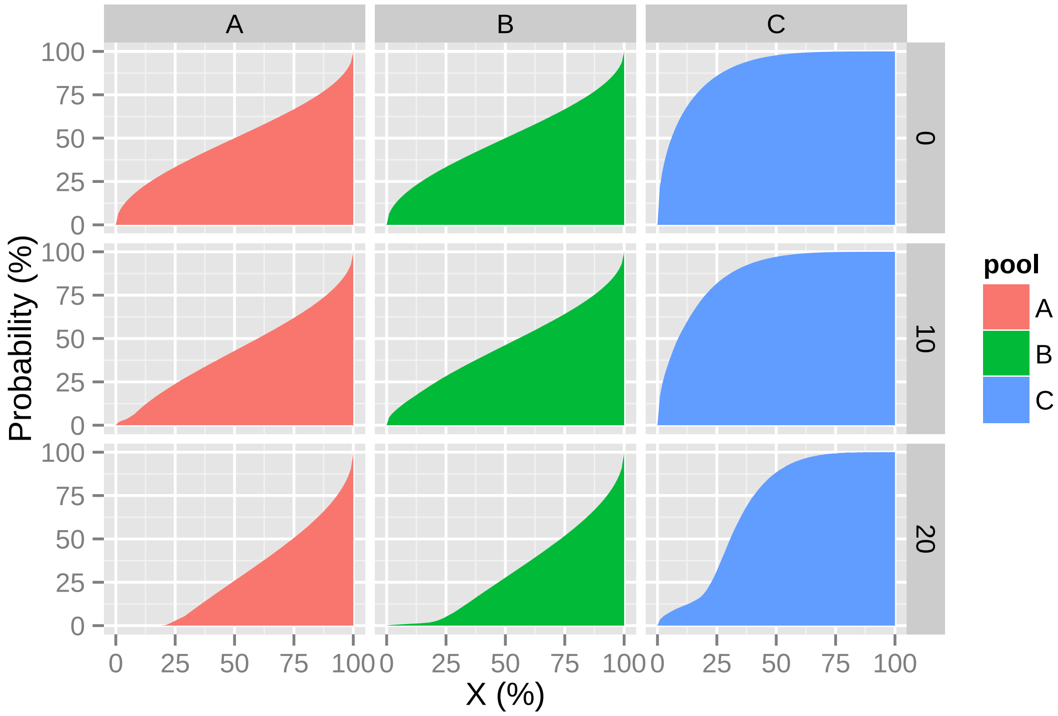


Figure 3.7: Illustrating how the conditional transaction cost distribution changes over time. Each column corresponds to a pool type denoted A, B and C. Each row corresponds to a timepoint in years. The distributions shows how the mortgagors with relatively small transactions costs leaves the pool. The distributions are obtained by observing the simulated short rate in figure 3.8.

A Discrete Initial Transaction Cost Distribution

With a risk of being framed as hypocritical, we shall now, as mentioned earlier, discard the continuous cost distribution in favor of the numerical approximation used in section *Determining the Expected Prepayment Level* in [Stanton (1995)]. This is used for the rest of the thesis and therefore presented below. Computing the critical cost levels is computationally heavy when using a continuous initial cost distribution, since we need to calculate the value of a new mortgage at each time point. Instead we follow [Stanton (1995)] and substitute the true initial cost distribution, G_0, by the discrete approximation

$$\hat{G}_0(x) = \sum_{j=1}^{m} c_{j0} 1_{(X_j < x)}, \tag{3.15}$$

with

$$c_{j0} = \frac{1}{m} \tag{3.16}$$

$$X_j = G_0^{-1}\left(\frac{2j-1}{2m}\right) \tag{3.17}$$

for $j = 1, \ldots, m$. Then the critical cost level at each time point may be obtained from computing the value of m mortgages. This is clearly a faster method as long as m is low relatively to the number of time points. The larger the number of m the closer the numerical approximation is to the true cost distribution, but the greater the computational burden.

More specifically the critical cost levels are found by valuating m mortgages, each with a different transaction cost X_j given by (3.17). Let $V^l_{i+1,j}(\tilde{T}_i, r)$ denote the $V^l_{i+1}(\tilde{T}_i, r)$-value corresponding to X_j. At time $\tilde{T}_i$ we then find the values of the m mortgages corresponding to the 'observed' short rate at time $\tilde{T}_i$ - this is done by interpolation if the 'observed' short rate is not contained in the grid. Next we find the X_j and X_{j+1} satisfying

$$V^l_{i+1,j}(\tilde{T}_i, r(\tilde{T}_i)) > d(r(\tilde{T}_i))F_{T_i}(1 + X_j) \tag{3.18}$$

$$V^l_{i+1,j+1}(\tilde{T}_i, r(\tilde{T}_i)) \leq d(r(\tilde{T}_i))F_{T_i}(1 + X_{j+1})). \tag{3.19}$$

We then know that the critical transaction cost level at time $\tilde{T}_i$, $X^\star_{\tilde{T}_i} \in (X_j, X_{j+1})$ and it is found by interpolation. If there, at time $\tilde{T}_i$, exists no X_j satisfying the inequality (3.18) then the critical cost level is set equal to zero. On the other hand if no X_{j+1} satisfies (3.18) then we set $X^\star_{\tilde{T}_i}$ equal to 1.

Given the critical transaction cost levels $X^\star_{\tilde{T}_i}$ for $i = 1, \ldots, k$, the numerical transaction cost distribution is updated by updating the weights (i.e. the c_{ji}) in the following way

$$\hat{G}_i(x) = \sum_{j=1}^{m} c_{ji} 1_{(X_j < x)} \tag{3.20}$$

where the expected proportion of the pool with transaction cost X_j at time $t > T_i$ is given by

$$c_{ji} = \begin{cases} \frac{c_{ji-1}(1-P_r)}{1-\hat{P}_{T_i}}, & X_j \leq X^\star(t_{i-1}) \\ \frac{c_{ji-1}(1-P_e)}{1-\hat{P}_{T_i}}, & X_j > X^\star(t_{i-1}), \end{cases} \tag{3.21}$$

for $j = 1, \ldots, m$ and $i = 2, \ldots, k$, where $\hat{P}_{T_i}$ denotes the expected conditional prepayment proportion at time T_i given by equation (3.14) using $\hat{G}$ instead of G.

Given the numerical approximation to the initial transaction cost distribution, determined by the cost levels X_j and the weights c_{j1}, cf. (3.15) and (3.16). It is now straight forward to compute the expected prepayment proportions $\hat{P}_{T_1}, \ldots, \hat{P}_{T_k}$ using (3.20), (3.21) and (3.14) with $\hat{G}$ instead of G.

Illustrating Different Types of Prepayment Behaviour

In figure 3.8 quarterly expected conditional prepayment rates for three 30-year mortgage bonds each with a coupon rate of 4% are plotted over the 20-year period $[0, 20]$. The three bonds are issued at time 0 and are backed by three hypothetical pools A, B and C, whose prepayment behaviour is specified by (3.10), (3.11) and (3.12). The top plot shows a simulated path of the short rate under P, assuming that it follows a Vasicek model with parameters $\kappa = 0.02$, $\theta^P = 0.01$, $\sigma = 0.01$ and $\theta = 0.03$.

To begin with note that the prepayment rates in figure 3.8 have a lower bound of approximately 1.25%. This is because we have set $\lambda = 0.05$, and hence the expected proportion of the pool prepaying for exogenous reasons each quarter is given by $1 - e^{-\lambda \cdot 0.25} \approx 0.0125$.

The initial transaction cost distribution is the same for pool A and B. But the average time between prepayment decisions in pool A is six months, while it is three years and four months in pool B. This means that mortgagors in pool A are in general faster at prepaying, which means that we should expect higher prepayment proportions for pool A than for pool B, given little prepayment before. In figure 3.8 this difference in prepayment behaviour is illustrated by the large peaks in pool A. On the other hand, note that there are times where the prepayment rates for pool B are higher than for pool A. This happens for several reasons. First, remember that the mortgage liability is decreasing in ρ (cf. figure 3.1), meaning that there is always a larger proportion of pool B, which finds it optimal to prepay than in pool

A. Second, the main part of mortgagors in pool A who find it optimal to prepay may already have done so. This explains why prepayment rates for pool B are above those for pool A in the last six years or so. Comparing pool B and C we see that the only difference between these two are a higher general level of transaction costs in pool B. Therefore there will always be a larger proportion of the mortgagors in pool C who find it optimal to prepay. This implies that prepayment rates for pool C are always higher than for pool B, which is also what we see in figure 3.8. At last we note that all three pools exhibit burnout out behaviour over time. This is seen in figure 3.8 by the fact that the level of prepayment rates for all three pools show a decreasing tendency over time even though the market short rate continues to stay at a low level at the end of the time period.

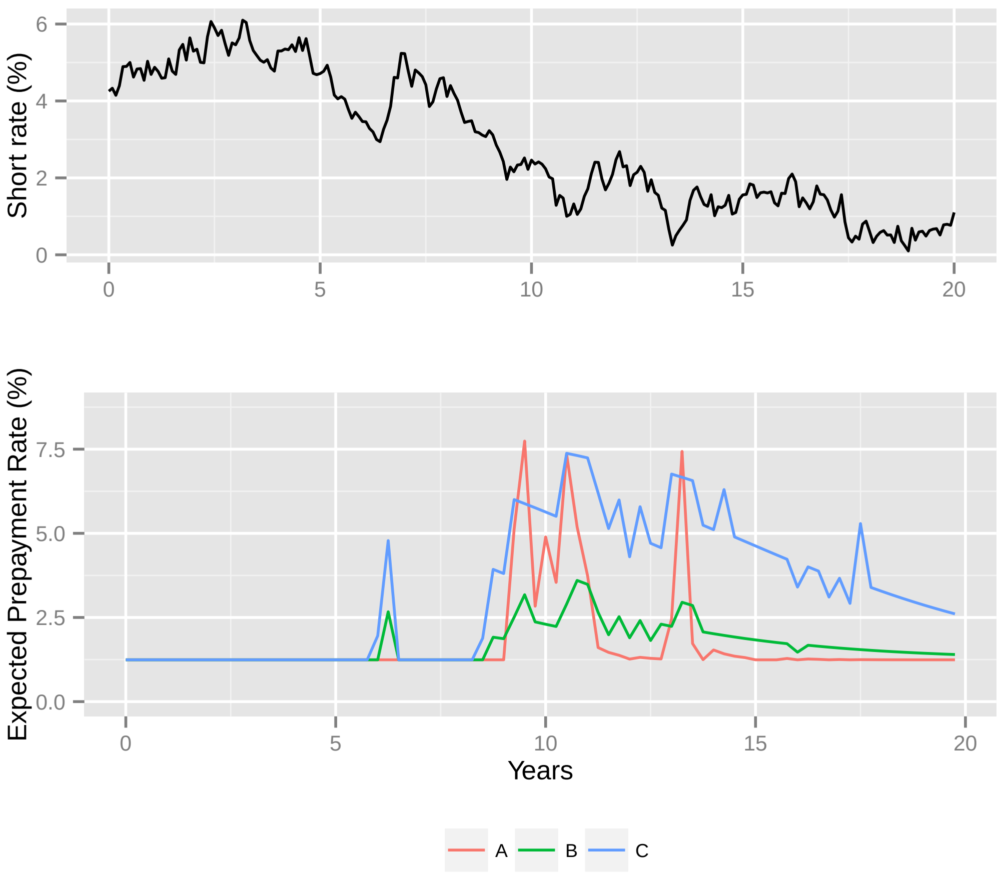


Figure 3.8: The top plot shows a simulated path of the market short rate under P, assuming that it follows a Vasicek model with parameters $\kappa = 0.02$, $\theta^P = 0.01$, $\sigma = 0.01$ and $\theta = 0.03$. The bottom plot shows the expected conditional prepayment proportions for the three hypothetical pools A, B and C. The pools are backed by 30-year mortgages issued at time-zero with a coupon rate of 4 percent. The bottom plot shows how different values of the parameters α, β, ρ and λ imply different types of prepayment behaviour. Furthermore note that all three pools exhibit burnout out behaviour over time. This is seen by the fact that the level of prepayment rates for all three pools show a decreasing tendency over time even though the short rate continues to stay at a low level at the end of the time period.

3.4 Estimating the Prepayment Parameters

In this section we estimate the parameters governing the prepayment behaviour of the model, α, β, ρ and λ, based on observed prepayment rates from Danish mortgage pools. In the estimation process we do not take the maximum likelihood approach, although it is usually preferred to other methods due to the asymptotic efficiency of the resulting estimators. Furthermore it is assumed that the short rate follows a Vasicek model with the estimates found in chapter 2.

The reason for not using MLEs is twofold. First of all the randomness of the prepayment rates produced by the model comes from the underlying short rate. And although we know the distribution of the short rate it is not clear how the resulting prepayment rates will be distributed. But following the approach taken in [Schwartz and Torous (1989)] we should, in principle, be able to come up with a likelihood function, since the prepayment behaviour of a single mortgagor is modelled trough hazard rates. However the heterogeneity of the mortgagors makes it less straight forward compared to [Schwartz and Torous (1989)]. Furthermore, following the approach from [Schwartz and Torous (1989)], the eventual log-likelihood function will become a double sum indexed over pools and the number of mortgagors in each pool[3] with the prepayment function of the single mortgagor as the skeleton. This means that the number of mortgagors, which has prepayed during a specific time period is needed as input in the log-likelihood function - cf. the log-likelihood function presented in [Schwartz and Torous (1989)]. This is a problem because we do not know the specific number of mortgages within a pool, and therefore we do not know how many mortgagors that has prepayed either. In order to avoid this problem we are forced to assume a common principal on all mortgages. Using this assumption we may then determinate the number mortgages, which have been prepayed during the specific time period[4].

However, as noted in both [Stanton (1995)] and [Schwartz and Torous (1989)] the assumption of a common principal on all mortgages is troubling. This is because the MLEs are invariant with respect to the assumed principal, while their asymptotic variances are not. Say, for example, that the remaining principal of a pool at the beginning of a specific time period is 100 and that 50% of the remaining mortgages prepay within that time period. Assuming that the common principal is 10 per mortgage implies that 5 mortgagors have prepayed during the period. On the other hand, assuming that the common principal is 1 per mortgage would imply that 50 mortgagors prepayed during the period. Changing the number of mortgages, which have been prepayed, changes the Hessian matrix of the log-likelihood and thereby the asymptotic variances of the MLEs, which therefore can not be trusted.

[3]Under the assumption that prepayment decisions across time and mortgages are conditionally independent.
[4]This is possible, because information on remaining total principal and prepayment rates are available.

Because of the above problems with the maximum likelihood method the estimation approach taken in [Stanton (1995)] is instead the *generalized method of moments* (GMM), or more precisely the *two-step generalized method of moments* (2SGMM), a method which was developed in [Hansen (1982)]. We shall only use the first step of the 2SGMM and the reason for this is explained later in the section.

The Generalized Method of Moments

As in subsection 3.3.2 we let the parameter vector of interest be denoted by $\theta = (\alpha, \beta, \rho, \lambda)$. Following the notation from [Stanton (1995)] we let w_{ij} denote the proportion of pool i prepaying in quarter[5] j, for $i = 1, .., N$ and $j = 1 \ldots, T$.

The GMM is based on a set of 'moment conditions'. In our case we choose T moment condition, one for each time point of observations. I.e. we need to identifying a vector-valued function of parameters and observations, $g : \mathbb{R}^T \times \mathbb{R}^4 \to \mathbb{R}^T$, which has unconditional expectation of zero

$$E^P[g(w_i, \theta_0)] = 0, \tag{3.22}$$

where $w_i = (w_{i1}, \ldots, w_{iT})^T$, $\theta_0 \in \mathbb{R}^4$ is a vector of true parameter values and E^P denotes the expectation under the real-world measure P. We assume that $T \geq 4$ - hence there are at least as many moment conditions as there are parameters to be estimated. Given observations $\tilde{w}_i = (\tilde{w}_{i1}, \ldots, \tilde{w}_{iT})^T$ for $i = 1, .., N$, then substituting the expectation in (3.22) with its sample average yields that

$$\begin{aligned} \bar{g}(\theta) &\equiv \frac{1}{N}\sum_{i=1}^{N} g(\tilde{w}_i, \theta) \\ &= \left(\frac{1}{N}\sum_{i=1}^{N} g(\tilde{w}_{i1}, \theta), \ldots, \frac{1}{N}\sum_{i=1}^{N} g(\tilde{w}_{iT}, \theta)\right)^T \\ &= 0. \end{aligned} \tag{3.23}$$

Note that if $T > 4$ this implies that no solution to (3.23) exists. We therefore aim at finding θ, which makes $\bar{g}(\theta)$ as close to zero as possible by minimizing the quadratic form $\bar{g}(\theta)^T W \bar{g}(\theta)$, where W is a positive definite $T \times T$ weighting matrix.

The GMM estimator is now defined by

$$\hat{\theta} = \arg\min_{\theta} \bar{g}(\theta)^T W \bar{g}(\theta), \tag{3.24}$$

[5] Since our observed prepayment rates are on a quarterly basis.

Under suitable regularity conditions the GMM estimator is known to be consistent and asymptotic normal.

We now proceed to identifying the function g. Let

$$\bar{w}_{ij}(\theta) = E^P[w_{ij} \mid \mathcal{F}_t, \theta], \tag{3.25}$$

where $\mathcal{F}_t$ denotes the information generated up to and including time t, such as the path of observed short rate values. Note that by the law of iterated expectations we have that

$$E^P[w_{ij} - \bar{w}_{ij}(\theta_0)] = E^P\left[w_{ij} - E^P[w_{ij} \mid \mathcal{F}_t, \theta_0]\right] \tag{3.26}$$

$$= 0. \tag{3.27}$$

We may therefore define

$$g(w_i, \theta) = w_i - \bar{w}_i(\theta), \tag{3.28}$$

where $\bar{w}_i = (\bar{w}_{i1}, \dots, \bar{w}_{iT})^T$.

The Estimation Procedure

As mentioned earlier we only perform the first step of the 2SGMM, which means that the weighting matrix, W, is set equal to the identity matrix. The second step consists of choosing

$$W = \left(\frac{1}{N}\mathbf{E}\left(\hat{\theta}_1\right)^T \mathbf{E}\left(\hat{\theta}_1\right)\right)^{-1}, \tag{3.29}$$

which is done in order to achieve an asymptotically efficient estimator. Here $\hat{\theta}_1$ denotes the first-step estimator and $\mathbf{E}$ is a $T \times T$ matrix given by

$$\mathbf{E}(\theta) = \begin{pmatrix} g_{11} & g_{11} & \cdots & g_{1T} \\ g_{21} & g_{21} & \cdots & g_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N1} & \cdots & g_{NT} \end{pmatrix}, \tag{3.30}$$

where g_{ij} equals the jth entry of the vector $g(w_i, \theta)$ for $i = 1, \dots, N$ and $j = 1, \dots, T$.

The reason for not performing the second step of the 2SGMM is that it requires the number of observed pools to be higher than the number of time points ($N > T$). If this is not the case the matrix $\mathbf{E}\left(\hat{\theta}_1\right)^T \mathbf{E}\left(\hat{\theta}_1\right)$ does not have full rank and is thereby not invertible. Also, given that we know the path of the short rate from the issue date of the earliest issued bond, the bonds should not be issued after the beginning of the observation period. If a bond is issued after the beginning of the observation period this would mean that certain entries in the matrix $\mathbf{E}$ would be left unspecified. Furthermore, if we do not know the path of the short rate before the observation period, it is also a problem if the bond is issued before the beginning of the observation period. This is because we cannot keep track of changes in the transaction cost distribution, unless we know the short rate path. In our data sample, from the Danish mortgage market, it was not possible to find data, which would allow us to use the entire observed time period and at the same time perform step two in the 2SGMM. This was in spite of the fact that we had access to quarterly prepayment rates from 265 pools over a time period of twelve and half years. The reason for this data-challenge stems from the following: when the price of bonds with the current lowest coupon rate on the market go above par, each Danish mortgage bank issue two to three new bonds with a lower coupon. This means that the maximum amount of bonds issued at the same time with the same coupon rate is somewhere around fifteen.

Notice that the above problem exists because we average over pools as in [Stanton (1995)] (opposed to over time) using one moment condition for each quarter of data. The reason for averaging over pools is the following: we know that the prepayment behaviour of two pools may change differently over time, while their average over time might be equal. If we were to average over time the difference in prepayment behaviour between such two pools might therefore not be captured.

The estimation of the parameters α, β, ρ, λ is based on observed prepayment rates from twenty 30-year callable mortgage bonds (see section 1.5) during time periods of up to twelve and a half years[6]. Some of the bonds were issued after the beginning of the period, which implies that certain entries of the matrix $\mathbf{E}$ are left unspecified. For that reason we only average over the number of pools, for which observations exists at each time point.

Note that the objective function, which we seek to minimize, has discrete jumps. This is due to the fact that the transaction cost distribution changes at discrete time points. A numerical optimization method using numerical derivatives is therefore not be preferable. For that reason we have used the Powell algorithm as our weapon of choice. This algorithm does not assume that the obejctive function is differentiable and no derivatives are being used. Due to the fact that the objective function could have

[6]Since the bonds are issued at different times the length of the time period for which we have prepayment data varies from bond to bond.

$\hat{\alpha}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\lambda}$
0.429	2.776	0.766	0.037

Table 3.1: Estimates for the prepayment parameters.

several optimums, we have performed the optimization procedure for a number of different start guesses. We then chose the set of estimates, which resulted in the lowest objective function value. The resulting estimators are found in table 3.1. To readers who attempt to recreate the estimation procedure: 'We wish you good luck!' - it is by no means straight forward.

Confidence intervals for the estimators in table 3.1 may be found by the use of Monte Carlo simulation in the following way: 1) let the already found estimates from table 3.1 play the role of 'true' parameters. 2) Simulate a path for the short rate, r, and compute prepayment rates based on the 'true' parameters (as we did in subsection 3.3.2). This then provides prepayment rates for a set of 'representative' pools[7] (one for each type of mortgage bond). 3) Use the simulated prepayment rates to compute new estimates. 4) Repeat $1) - 3)$ a sufficiently large number of times and look at the resulting distributions of the repeatedly found estimates. We, however, do not provide any confidence intervals for the estimators presented in table 3.1. This is due to fact that it is quite time-consuming to compute a single vector of estimates. We simply did not have the time to repeat this procedure, say ten thousand times.

The found estimate of λ implies that the probability of a mortgagor prepaying within a given year, due to exogenous reasons, is $1 - e^{-\hat{\lambda}} \approx 4\%$. The estimated value of ρ implies that the average time between prepayment decisions is approximately one year and three months. Furthermore, if we assume that it is optimal to prepay for an entire year, the probability of a mortgagor actually prepaying within that year (due to rational reasons) is $1 - e^{-\hat{\rho}} \approx 54\%$. The estimated values of α and β governs the initial transaction cost distribution. The initial density and distribution functions are plotted in figure 3.9. According to the estimates of α and β the expected initial transaction cost level is $\hat{\alpha}/(\hat{\alpha} + \hat{\beta}) \approx 13\%$. Note that this includes both explicit and implicit transaction costs - implicit transaction costs here refers to the inconvenience of going to the mortgage bank, lost working hours, etc.. Note that even though the expected initial transaction cost level produced by our estimates is relatively large, it is much lower than the transaction cost level obtained in [Stanton (1995)].

[7] We are not able to compute simulated prepayment rates for different pools, because possible differences between pools of the same type are not incorporated in the model.

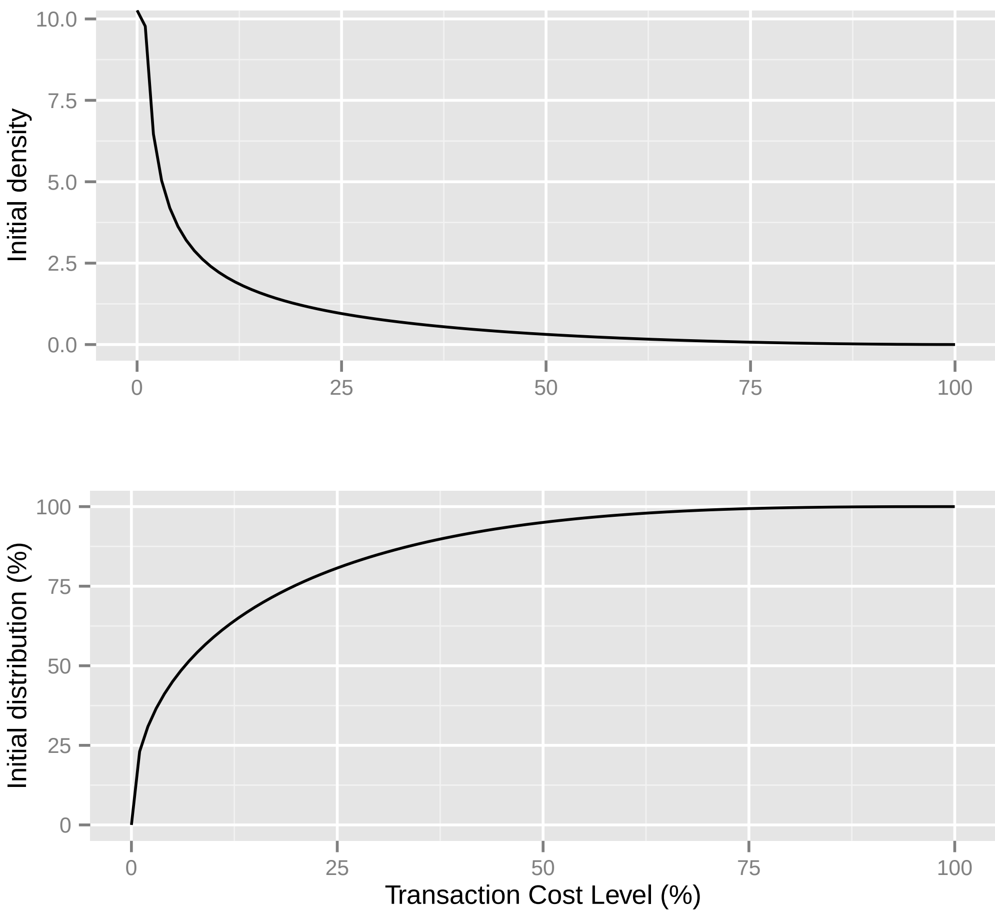


Figure 3.9: The top plot shows the initial transaction cost density function and the bottom plot shows the initial transaction cost distribution function. Both plots uses the estimated values of α and β found in table 3.1.

In figure 3.10 we have plotted the model prepayment rate predictions versus the observed prepayment rate each quarter. More specifically we have computed the model prepayment rates based on the 'implied' path of the market short rate (cf. chapter 2) and the estimates from table 3.1 for each of the twenty pools. We have then taken the average over pools at each time point. Likewise, the observed prepayment rates in figure 3.10 are the average observed prepayment rates.

From figure 3.10 we see that there seems to be two ways in which we might create a better fit. First of all we notice that observed prepayment rates are zero for long periods, but this will never be predicted by the model with a non-zero λ, since model prepayment rates for a period of length dt are bounded below by $1 - e^{-\lambda dt}$. Secondly, the predicted prepayment rates in the time period 2012 - 2013 are too low compared with the observed. A possible option for creating a better fit would therefore be to adjust the estimated values of the parameters ρ and λ. Due to the problem with a non-zero λ explained in the end of section 3.2 and the fact that we want the model to be capable of predicting prepayment rates of zero, we adjust the estimate of λ by setting it equal to zero. In order for the model to produce higher prepayment rates around 2012 - 2013 we adjust the estimate of ρ by setting it equal to $\frac{3}{2}$. The adjusted model predictions are also plotted in figure 3.10.

In figure 3.10 we see that the adjusted model creates a better fit where observed prepayment is zero and it implies a higher maximum prepayment rate around 2012 - 2013. But we also see that the adjusted model's prepayment rates decrease too fast around the end of 2014, opposed to when the model is using the estimates from table 3.1. In other words the endogenously produced burnout effect in the adjusted model is too strong do to the earlier high peaks in prepayment rates. All in all both the original and the adjusted model captures the general tendency in observed prepayment behaviour, but with room for improvements. Increasing the estimated value of β would imply a lower general level of transaction costs and thereby possibly create prepayments at higher short rate levels, such that there might be a better fit around 2006. However, increasing the value of β simply results in higher predicted prepayment rates around 2012 - 2014. In order for the model to capture a wider aspect of prepayment behaviour a possible solution could be to model X, ρ and λ as functions of variables such as time and the interest level, opposed to being constant.

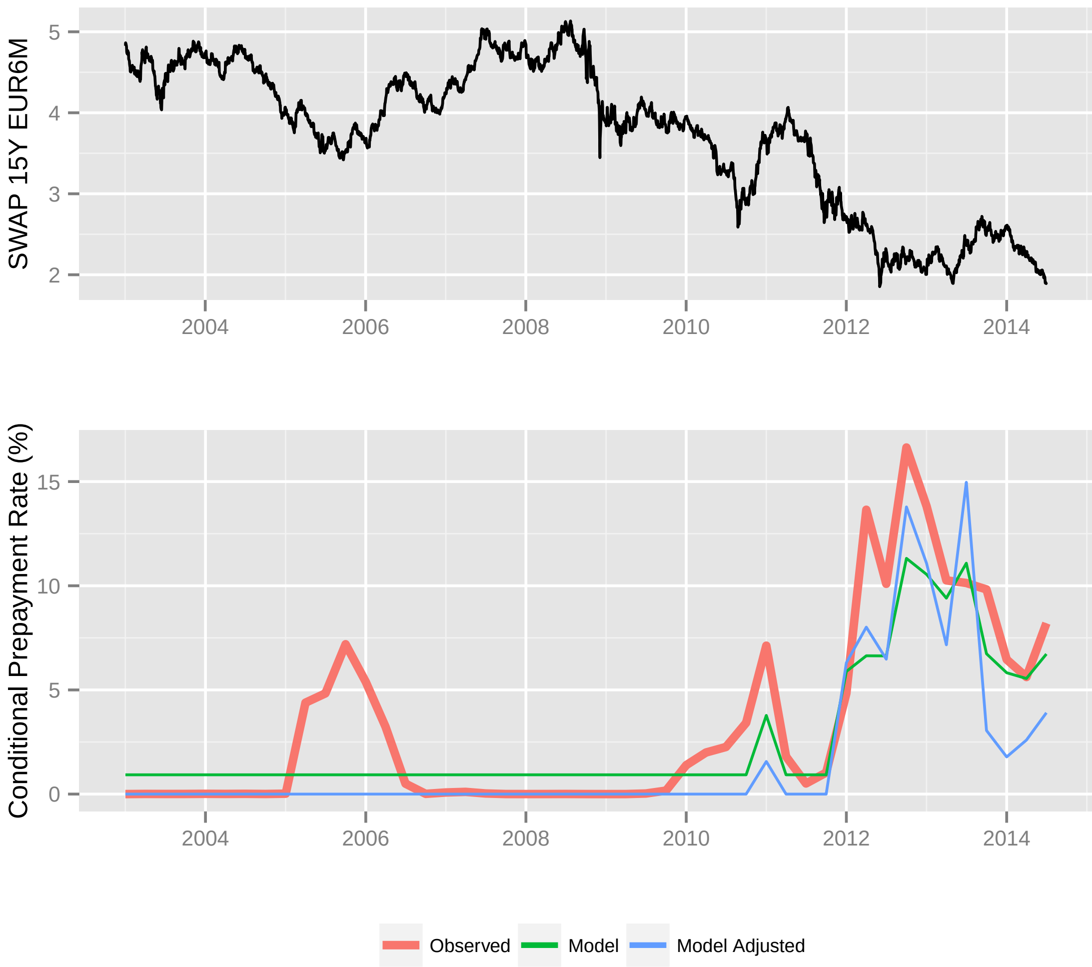


Figure 3.10: The model prepayment rate predictions based on the estimates from table 3.1 and model prepayment rate predictions based on the α and β estimates from table 3.1, but with $\rho = 1.5$ and $\lambda = 0.001$ (i.e. the adjusted model) versus the average observed prepayment rate each quarter. More specifically we have computed the model prepayment rates based on the implied path of the market short rate (cf. chapter 2) for twenty pools. We have then taken the average over pools at each time point. Wquivalently, the observed prepayment rates are the average observed prepayment rates. The change in ρ and λ from model to adjusted model imply a better fit of the peaks around 2012-2014. However, this change in parameter values also creates a worse fit from mid 2013 and forward, due to a too strong endogenously produced burnout effect in the adjusted model.

3.5 Valuation of a Callable Mortgage Bond - Pool Prices

In this section we extend the valuation method of a security backed by a single mortgage (see section 3.2) to the valuation of a security backed by a pool of mortgages, i.e. a callable mortgage bond.

In theory the transition from a single mortgage to a pool of mortgages is straight forward. Given the transaction cost distribution in a pool, the value of a bond is computed as the weighted average of different mortgages in the pool – the weights being those used in the numerical approximation to the transaction cost distribution, cf. section 3.3.2. This procedure makes sense, since the cash flow from a pool is simply the sum of cash flow from the individual mortgagors.

Using this approach it is straight forward to compute the time-zero value of a bond. We simply compute the time-zero values of m securities each backed by a single mortgage with different transaction costs X_j for $j = 1, .., m$ defined by (3.17). We then weigh the time-zero value of each security by $\frac{1}{m}$, cf. (3.16). However, when computing the value of a bond at time $\tilde{T}_i$ (for $i > 1$) one has to remember not to weigh the time-$\tilde{T}_i$ values of the m securities by the initial transaction cost distribution, since the distribution has most likely changed from time zero to time $\tilde{T}_i$, due to prepayments. It is therefore important to keep track of changes in the distribution, cf. section 3.3.2. More specifically, we compute the correct weights using (3.21). Note that the bond value is therefore dependent on the entire path of the short rate from time zero until the time in question. Or at least dependent on the short rate levels on past announcement dates, assuming that these levels are representive for the mortgagors decisions at hand. One could argue that the rate levels before the exact announcement dates also have an effect.

In chapter 5 we compare the resulting model prices with observed market quotes.

Chapter 4

The Danske Bank Prepayment Model

The main part of this chapter consists of an investigation and derivation of the Danske Bank Prepayment Model (DBPM). A model, which is presented in [Andreasen (2011)]. The starting point is a simple prepayment function consisting of a few intuitive prepayment parameters. This prepayment function is examined through a *minimal mortgage* model (2M-model) - the *minimal* in the 2M-model referring to the simplifying assumptions and approximations, which are made in order to gain a more straight forward analysis of the given prepayment function. In addition to examining the prepayment function through the 2M-model we have taken a closer look at the validity of the assumptions and approximations made in the 2M-model.

It is important to note that the entire purpose of the analysis in section 4.1 - 4.3 is to arrive at the resulting prepayment model, which is summed up by equation (4.30). Given the prepayment model, section 4.4 explains how the actual procedure of pricing a callable mortgage bond is carried out within the setup of DBPM. Furthermore we illustrate how resulting bond prices of the DBPM depend on the parameter s. We postpone an analysis of the prepayment behaviour predicted by DBPM until chapter 5, where it is compared to the prepayment model presented in chapter 3.

4.1 The Basic Prepayment Model

Consider a callable Danish mortgage bond with payments $T_1, \ldots, T_n = T$. In [Andreasen (2011)] it is claimed that a sensible approach to prepayment behaviour is to model the *proportional* and *conditional* prepayment of a mortgage pool for a discrete coupon period $[T_i, T_{i+1})$ for $i = 0, 1, \ldots, n-1$, as

$$P_{T_i} = P(V(T_i)) = s\varphi(V(T_i) - 1), \tag{4.1}$$

where $V(T_i)$ is the bond price *per unit* notional at time T_i, φ is a distribution function and $s \in [0, 1)$ is a constant, which determines the maximal prepayment per period.

First of all we note that the prepayment function is increasing in $V(t) - 1$. And since the bond price is per unit notional this reflects the natural idea that there should be a positive dependence between prepayment and a larger positive difference in bond price and face value. This characteristic simply means that mortgagors are more likely to prepay the larger the difference is between bond price and principal value. The more precise structure of this positive dependence is then determined by the choice of distribution function, meaning that the prepayment function is not restricted to only one possible dependence structure. Note that the model captures prepayment for interest rate reasons, because of the negative dependence between the value of the bond and the short rate. One may therefore agree that this is a somewhat sensible approach to the modelling of prepayment behaviour (although it is very basic) assuming that we request a prepayment function with a few intuitive parameters.

4.1.1 Continuous Version of the Prepayment Function

We wish to derive a continuous-time version of the prepayment function given by (4.1), since this enables us to analyse the prepayment function in a continuous setup.

As mentioned in the above the prepayment is proportional and conditional, meaning that P_{T_i} is the *proportion* of mortgagors expected to prepay in the period $[T_i, T_{i+1})$ *conditioned* on the number of mortgagors which has already prepayed in earlier periods. We may therefore write the outstanding notional at time T_i as

$$\begin{aligned} N(T_{i+1}) &= N(T_i) - N(T_i)P_{T_i} \\ &= N(T_i) - N(T_i)s\varphi(V(T_i) - 1) \end{aligned} \tag{4.2}$$

From (4.2) it follows that after n coupon periods the outstanding notional is given by

$$\begin{aligned}N(T_n) =& N(T_{n-1}) - N(T_{n-1})P_{T_{n-1}} \\ =& N(T_{n-1})(1 - P_{T_{n-1}}) \\ =& N(T_{n-2})(1 - P_{T_{n-2}})(1 - P_{T_{n-1}}) \\ &\vdots \\ =& N(T_0)\prod_{k=0}^{n-1}(1 - P_{T_k}).\end{aligned}$$

And we may rewrite the outstanding notional in the following way

$$\begin{aligned}N(T_n) &= N(T_0)\prod_{k=0}^{n-1}(1 - P_{T_k}) \\ &= N(T_0)e^{\sum_{k=0}^{n-1}\log(1-P_{T_k})} \\ &= N(T_0)e^{\sum_{k=0}^{n-1}\log(1-s\varphi(V(T_k)-1))}.\end{aligned}$$

Now let $s = 1 - e^{-\alpha\Delta t}$ for some positive constant α and $\Delta t = T_{k+1} - T_k$ to obtain

$$\begin{aligned}N(T_n) &= N(T_0)e^{\sum_{k=0}^{n-1}\log(1-(1-e^{-\alpha\Delta t})\varphi(V(T_k)-1))} \\ &= N(T_0)e^{\sum_{k=0}^{n-1}\frac{\log(1-(1-e^{-\alpha\Delta t})\varphi(V(T_k)-1))}{\Delta t}\Delta t} \\ &= N(T_0)e^{\sum_{k=0}^{n-1}\frac{f(\Delta t)}{\Delta t}\Delta t},\end{aligned} \tag{4.3}$$

where $f_k(\Delta t) = \log(1 - (1 - e^{-\alpha\Delta t})\varphi(V(T_k) - 1))$, with $f_k(0) = 0$.

To obtain a continuous-time version we let Δt converge to zero and note that

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{f_k(\Delta t)}{\Delta t} &= \lim_{\Delta t\to 0}\frac{f_k(\Delta t) - f(0)}{\Delta t} \\ &= \frac{\mathrm{d}}{\mathrm{d}\Delta t}f_k(\Delta t)\bigg|_{\Delta t=0} \\ &= -\alpha\varphi(V(T_k) - 1).\end{aligned}$$

Furthermore we see that the sum in the exponential function in (4.3) is a Riemann sum and therefore by letting Δt converge to zero the sum becomes an integral in the limit. In the end this gives us a continuous-time version of the outstanding notional, which is given by

$$N(t) = N(0)e^{-\int_0^t \alpha\varphi(V(u)-1)\mathrm{d}u}. \tag{4.4}$$

From (4.4) we see that we may think of $\alpha\varphi(V(u)-1)$ as a prepayment *rate* or *intensity*. In the rest of the thesis we shall denote this prepayment rate by π, such that

$$\pi(V) = \alpha\varphi(V-1). \tag{4.5}$$

A continuous-time version of the prepayment function is therefore given by

$$\tilde{P}(t) = 1 - e^{-\int_t^{t+dt} \pi(V(u))\mathrm{d}u}. \tag{4.6}$$

The expression given by (4.6) gives the proportion of the remaining mortgagors in the pool that prepay within the time interval $(t, t+dt)$. If for example $1 - e^{-\int_t^{t+1} \pi(V(u))\mathrm{d}u} = 0.5$, for all t, then fifty percent of the remaining mortgagors would prepay each year - assuming that t is measured in years.

4.2 The Minimal Mortgage Model

I this section we present the 2M-model, which proposes to valuate a callable mortgage bond by solving a simplified version of the standard continuous-time pricing PDE. It is argued that under certain assumptions we may set the partial derivative of the bond value, with respect to time, equal to zero. Therefore the pricing PDE becomes a pricing ODE. This creates the possibility for a more straight forward analysis of the proposed prepayment model. At times we may refer to the 2M-model as the ODE-method, while valuation of the bond using the pricing PDE will be denoted as the PDE-method.

4.2.1 Deriving the Pricing PDE

First of all, note that if we assume that a bond is paying a continuous coupon rate q and a scheduled repayment of $\beta(t)$ per unit notional at time t, then the notional is continuously reduced according to

$$N(t) = N(0)e^{-\int_0^t (\pi(V(s))+\beta(s))ds}. \tag{4.7}$$

We seek to approximate the annuity structure by a continuous version of the annuity loan. Hence the repayment rate $\beta(t)$, has to have a certain form. The payment rate without prepayments, $N(t)(q+\beta(t))$ has to be constant for all $t \in [0,T]$. We therefore need to determine $\beta(t)$ such that this is fulfilled.

Consider the annuity with a yearly coupon rate q and maturity T. Let n denote the payments per year so there is nT payments in total. Let $T_1, \ldots, T_{nT} = T$ denote the payment dates and note the interest rate per paymenst equals q/n. From [Lando and Poulsen (2006)] we have the following formula for the discrete deduction per unit notional at coupon date T_k for $k = 1, \dots, nT$

$$d(T_k) = \frac{q/n}{1-(1+q/n)^{-n(T-T_k)}}\left(1 - \frac{q}{n}\frac{1-(1+q/n)^{-n(T-T_k+1/n)}}{q/n}\right),$$

But since β is the yearly repayment rate we need to multiply with n on both sides such that

$$nd(T_k) = \frac{q}{1-(1+q/n)^{-n(T-T_k)}}(1+q/n)^{-n(T-T_k+1/n)}.$$

Now by letting n converge to infinity we obtain the continuous repayment rate

$$\beta(t) = \frac{q}{1-e^{-q(T-t)}}e^{-q(T-t)}. \tag{4.8}$$

Note that $\beta(t) \to \infty$ when $t \to T$, which implies that $N(t) \to 0$ for $t \to T$.

The cash flow of a callable Danish mortgage bond with maturity T and coupon q can now be approximated by a continous payment stream including interest payments, repayments and prepayments. I.e. we have that the arbitrage-free time-t price of such a bond is given by the expected discounted payment stream

$$\begin{aligned} V(t,r) &= E_{t,r}\left[\int_t^T e^{-\int_t^u r(s)\mathrm{d}s}N(u)(q+\beta(u)+\pi(V(u,r(u))))\mathrm{d}u\right] \\ &= E_{t,r}\left[\int_t^T e^{-\int_t^u r(s)\mathrm{d}s}e^{-\int_t^u(\pi(V(s,r(s)))+\beta(s))\mathrm{d}s}(q+\beta(u)+\pi(V(u,r(u))))\mathrm{d}u\right], \end{aligned} \tag{4.9}$$

The second equality comes from the fact that the value of the bond is per unit notional, meaning that the notional at time t equals one. Therefore we have that

$$\begin{aligned} N(u) &= e^{-\int_0^u(\pi(V(s,r(s)))+\beta(s))\mathrm{d}s} \\ &= N(t)e^{-\int_t^u(\pi(V(s,r(s)))+\beta(s))\mathrm{d}s} \\ &= e^{-\int_t^u(\pi(V(s,r(s)))+\beta(s))\mathrm{d}s}. \end{aligned}$$

Assume now that under Q the short rate solves the SDE given by

$$\begin{aligned} dr(u) &= \mu(u,r(u))dt + \sigma(u,r(u))dW(u) \\ r(t) &= r \end{aligned} \tag{4.10}$$

for $u > t$ and where $\mu : [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$.

By Feynman-Kac it follows that the bond value then solves the following non-linear PDE

$$\begin{aligned}
&\frac{\partial V}{\partial t}(t,r)+\mu(t,r)\frac{\partial V}{\partial r}(t,r)+\frac{1}{2}\sigma^2(t,r)\frac{\partial^2 V}{\partial r^2}(t,r)-(r+\beta(t)+\pi(V(t,r)))V(t,r)\\
&\qquad+(q+\beta(t)+\pi(V(t,r)))=0 \qquad (4.11)\\
&V(T,r)=0,
\end{aligned}$$

where the non-linearity comes from the prepayment rate π, since it contains the expression $\varphi(V(t,r)-1)$.

4.2.2 Perpetual Bonds

In order to simplify the analysis of the prepayment function given by (4.6), we briefly introduce the concept of a perpetual bond.

First note that if $\beta(t)=\beta$ is constant then the bond is no longer structered as a continuous annuity loan. Considering (4.11) as a problem on $[0,\infty)\times\mathbb{R}$ without a boundary, then forms a PDE for a perpetual bond, where the the interest payments, q, and repayments, β, continues for ever.

In this subsection we assume that the market short rate r is constant and that there are no prepayments. The time-zero value of the perpetual bond per unit notional is then

$$\begin{aligned}
V_P &= \int_0^\infty e^{-rt}N(t)(q+\beta)\mathrm{d}t\\
&= \int_0^\infty e^{-(r+\beta)t}(q+\beta)\mathrm{d}t\\
&= \frac{q+\beta}{r+\beta}.
\end{aligned}$$

Furthermore the duration is given by

$$\begin{aligned}
D_P &= -\frac{\partial V_P/\partial r}{V_P}\\
&= -\left(-\frac{q+\beta}{(r+\beta)^2}\frac{r+\beta}{q+\beta}\right)\\
&= \frac{1}{r+\beta}.
\end{aligned}$$

We compare the duration of the perpetual bond with that of an annuity with maturity T. The time-zero value of an annuity bond with a continuous payment stream c is given by

$$\begin{aligned}V_A &= \int_0^T e^{-rt} c \, \mathrm{d}t \\ &= \frac{c}{r}\left(1 - e^{-rT}\right)\end{aligned}$$

and the duration is therefore

$$\begin{aligned}D_A &= -\frac{\partial V_A / \partial r}{V_A} \\ &= \frac{1}{V_A}\left(\frac{c}{r^2}\left(1 - e^{-rT}\right) - \frac{c}{r} T e^{-rT}\right) \\ &= \frac{1}{r} - \frac{T e^{-rT}}{1 - e^{-rT}}.\end{aligned}$$

We see that $D_P = D_A$ when

$$\beta = \left[\frac{1}{r} - \frac{T e^{-rT}}{1 - e^{-rT}}\right]^{-1} - r.$$

This means that by choosing β such that

$$\beta = \left[\frac{1}{q} - \frac{T e^{-qT}}{1 - e^{-qT}}\right]^{-1} - q. \tag{4.12}$$

we match the duration of the perpetual bond with that of an annuity, at par ($V_P = 1$ for $q = r$). And as we shall see in the next section this observation turns out to be central in the 2M-model's valuation method of a mortgage bond.

4.2.3 From PDE to ODE

We now return to the assumption that the market short rate is described by a stochastic process, which solves the SDE given by (4.10). In the 2M-model it is claimed that for a constant β, time-independent prepayment rate and time-independent functions $\mu(r)$, $\sigma(r)$, the time-derivative of the mortgage bond price is zero. Under these assumptions the argument goes as follows:

Basically we notice that under the assumption that the drift and volatility of the short rate and the prepayment rate are time-independent the only time-dependent factor in the valuation of the bond is the repayment rate, β. In subsection 4.2.2 it was argued that choosing β according to equation 4.12 we no longer deal with a standard mortgage bond, since its cash flows do not equal that of an annuity. However the constant-β-bond's duration will match that of an annuity with maturity T at par. For that reason the analysis of the prepayment model is continued under the assumption that bond value is time-independent.

Hence the PDE in (4.11) is now reduced to an ODE given by

$$\mu(r)\frac{\partial V}{\partial r}(r) + \frac{1}{2}\sigma^2(r)\frac{\partial^2 V}{\partial r^2}(r) - (r + \beta + \pi(V(r)))V(r) + (q + \beta + \pi(V(r))) = 0. \tag{4.13}$$

Assume for example that $q = 6\%$, then choosing $\beta = 0.033$ implies that the solution to (4.13) matches the rate sensitivity of a 30-year annuity around short rate levels of 6 percent. The core of 2M-model is that the valuation of the mortgage bond is performed by solving the ODE given by (4.13) instead of solving the PDE from (4.11).

Note that although the bond value is assumed to be time-independent, the time-aspect is not completely removed, due to the fact that the parameter β depends on T. However, the arguments presented above are by no means bulletproof. For example, the argument that the perpetual bond's duration matches that of an annuity with maturity T at par, is based on the assumption of a constant market short rate, while now the short rate is assumed to be stochastic. For that reason the line of arguments is not consistent. Furthermore the assumptions about the prepayment rate and the drift and volatility of the short rate being time-independent are not true in general. We should therefore think of the 2M-models valuation method as being only approximately true.

Below we show how one may numerically solve the ODE given by (4.13). In order to test how well the numerical solution to (4.13) approximates a numerical solution to original boundary value problem given by (4.11) we demonstrate how to solve (4.11) numerically in subsection 4.2.5. We then compare the two numerical solutions under different short rate models.

4.2.4 Solving the Non-Linear ODE

From now on and forward it will be assumed that φ is the uniform distribution function

$$\begin{aligned}\varphi(x) &= \begin{cases} 0 & , \quad x < a \\ \frac{x-a}{b-a} & , \quad a \leq x \leq b \\ 1 & , \quad x > b, \end{cases} \\ &= \min\left(1, \max\left(\frac{x-a}{b-a}, 0\right)\right)\end{aligned}$$

for $-\infty < a < b < \infty$.

Assuming that φ is the uniform distribution function means that if for example $a = 0$ is chosen, then no prepayment occurs if $V-1$ is negative. Furthermore we automatically assume that the positive dependence between prepayment and the value of $V-1$ is linear for $V-1 \in [a,b]$. This will in general not be true[1]. But as we will see in later sections the assumption of a uniform distribution simplifies the mathematical analysis a great deal.

We solve (4.13) numerically using a finite difference scheme. But since there is no time dimension the structure of the scheme becomes different from what we present in chapter A.5, although the basic idea of finite difference remains the same. Due to the missing time dimension the approach here will be to perform the numerical scheme presented below over and over until a given convergence level is reached. Because the numerical scheme for solving (4.13) is slightly different from what is being presented in chapter A.5, we briefly go through the scheme below.

From equation (4.13) and by the use of theorem A.5 we know that

$$\mu(r_k)\delta_r(k) + \frac{1}{2}\sigma^2(r_k)\delta_{rr}(k) - (r_k + \beta + \pi(V(r_k)))\, V(r_k) \approx -\left(q + \beta + \pi(V(r_k))\right). \tag{4.14}$$

for $k = 0, \dots, N_r$, where $N_r + 1$ is the number of steps in the discretization and δ_r and δ_{rr} are given by (A.22)-(A.26).

We may write equation 4.14 in matrix form as

$$\bar{\mathcal{A}}V(\mathbf{r}) \approx g(V(\mathbf{r})), \tag{4.15}$$

where

$$\bar{\mathcal{A}}V(\mathbf{r}) = \underbrace{\begin{pmatrix} \mu(r_0)\delta_r(0) + \frac{1}{2}\sigma^2(r_0)\delta_{rr}(0) - (q+\beta+\pi(V(r_0))) \\ \vdots \\ \mu(r_{N_r})\delta_r(N_r) + \frac{1}{2}\sigma^2(r_{N_r})\delta_{rr}(N_r) - (q+\beta+\pi(V(r_{N_r}))) \end{pmatrix}}_{(N_r+1)\times(N_r+1)} V(\mathbf{r}).$$

and

$$\begin{aligned} V(\mathbf{r}) &= (V(r_0), \dots, V(r_{N_r}))^T \\ g(V(\mathbf{r})) &= (-(q+\beta+\alpha\varphi(V(r_0)-1)), \dots, -(q+\beta+\alpha\varphi(V(r_{N_r})-1)))^T \end{aligned}$$

[1]E.g. the presence of burnout makes the prices increase, while the prepayments decrease.

Since $\delta_r(k)$ and $\delta_{rr}(k)$ are row vectors (by equation (A.22)-(A.26)) equation (4.15) forms a tridiagonal matrix system.

Note that the non-linearity of (4.15) in the bond price causes problems, unless the value of $\varphi(V(r_k)-1)$ is known for $k=0,\ldots,N_r$. This problem is overcome in the following way; assume that we have a set of values $V^h(\mathbf{r}) = (V^h(r_0)\ \ldots\ V^h(r_{N_r}))^T$. Then we can solve the linear tridiagonal system

$$\begin{pmatrix} \mu(r_0)\delta_r(0) + \frac{1}{2}\sigma^2(r_0)\delta_{rr}(0) - \big(q+\beta+\pi(V^h(r_0))\big) \\ \vdots \\ \mu(r_{N_r})\delta_r(N_r) + \frac{1}{2}\sigma^2(r_{N_r})\delta_{rr}(N_r) - \big(q+\beta+\pi(V^h(r_{N_r}))\big) \end{pmatrix} V^{h+1}(\mathbf{r}) = g(V^h(\mathbf{r}))$$

for V^{h+1}. Now use this updated V^{h+1} to solve the ODE for V^{h+2} and continue the procedure until convergence is reached. We then end up with the value of the mortgage bond in all the grid points and a proper start guess of $V(r_0),\ldots,V(r_{N_r})$ is the only requirement. Inspired by subsection 4.2.2 the chosen start guess in our algorithm is $V^0(\mathbf{r}) = \left(\frac{q+\beta}{r_0+\beta},\ldots,\frac{q+\beta}{r_{N_r}+\beta}\right)$.

4.2.5 Solving the Non-Linear PDE

Before proceeding with the analysis of the prepayment function we wish to evaluate the cost of going from a PDE to ODE. We compare the solutions derived by solving the PDE (continuous annuity structure) with the ODE, where β is chosen to match the duration of the bond at par.

We also solve the boundary value problem (4.11) using a finite difference scheme. Note that since (4.11) is non-linear in the bond price it is not wise to go straight ahead and follow the so-called θ-method (see chapter A.5) step by step. Instead we perform a mixture of the implicit and explicit method in order to avoid troubles with the non-linearity. Let us write the PDE from (4.11) as

$$\frac{\partial V}{\partial t}(t,r) + \mu(t,r)\frac{\partial V}{\partial r}(t,r) + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 V}{\partial r^2}(t,r) + G(V(t,r)) + H(V(t,r)) = 0,$$

where $H(V(t,r)) = \pi(V(t,r))(1-V(t,r))$ and $G(t,r,V(t,r)) = -(r+\beta(t))V(t,r)$ are collections of all the non-linear and linear terms in V, respectively.

From Theorem A.5 it follows that

$$\begin{aligned}\frac{V(t_{i+1},r_k)-V(t_i,r_k)}{\Delta t} &+ \mu(t_i,r_k)\delta_r V(t_i,r_k) + \frac{1}{2}\sigma^2(t_i,r_k)\delta_{rr}V(t_i,r_k) \\ &+ G(t_i,r_k,V(t_i,r_k)) + H(V(t_{i+1},r_k)) + q + \beta(t_i) \approx 0,\end{aligned} \tag{4.16}$$

where δ_r and δ_{rr} are difference operators given by Theorem A.5.

Note that valuing V at time t_i in the linear terms and at time t_{i+1} in the non-linear term, means that we go implicit in the linear terms and explicit in the non-linear term, when calculating backwards in time. This is done in order to overcome the non-linearity and still preserve stability from the implicit method - opposed to going explicit in all terms and thereby obtaining a less stable scheme, cf. chapter A.5. In equation (4.16) the non-linear term causes no trouble since it is evaluated at time t_{i+1} and it is therefore known. Equation (4.16) can now be solved in the usual finite difference manner.

4.2.6 The ODE-method Versus The PDE-Method

In order to examine the magnitude of loss in precision, when assuming that the time-derivative of the bond price is zero, we now proceed with a comparison of the numerical solution to the ODE given by (4.13) and the numerical solution to the PDE from (4.11). We have performed the comparison for different values of s and under two different models for the short rate under Q. The first being a Vasicek model, which is in line with the assumptions made in the 2M-model, i.e. the drift and volatility are time-independent. And the second being a simple Hull-White model, which implies that the mean-reversion is now assumed to be a deterministic function of time. The assumption of a Hull-White model for the short rate implies that the 2M-model assumption of a time-independent drift is no longer valid. Therefore we expect the loss of precision in the numerical solution to the ODE to be larger when the short rate follows a Hull-White model opposed to a Vasicek model.

In figure 4.1 the numerical solution to the ODE given by (4.13) is plotted against the short rate together with the time-zero solution to the PDE from (4.11). The four plots corresponds to different values of s and the short rate is assumed to follow a Vasicek model. Note for $s = 0$ the closed-form solution for a T-year annuity may be found using the well-known closed-form solution for the price of a ZCB in the Vasicek model, cf. Proposition 2.1.

From what we see in figure 4.1 it is clear that some precision is lost when the time dimension is removed. But when we introduce prepayments the loss of precision only seems significant for high rate values,

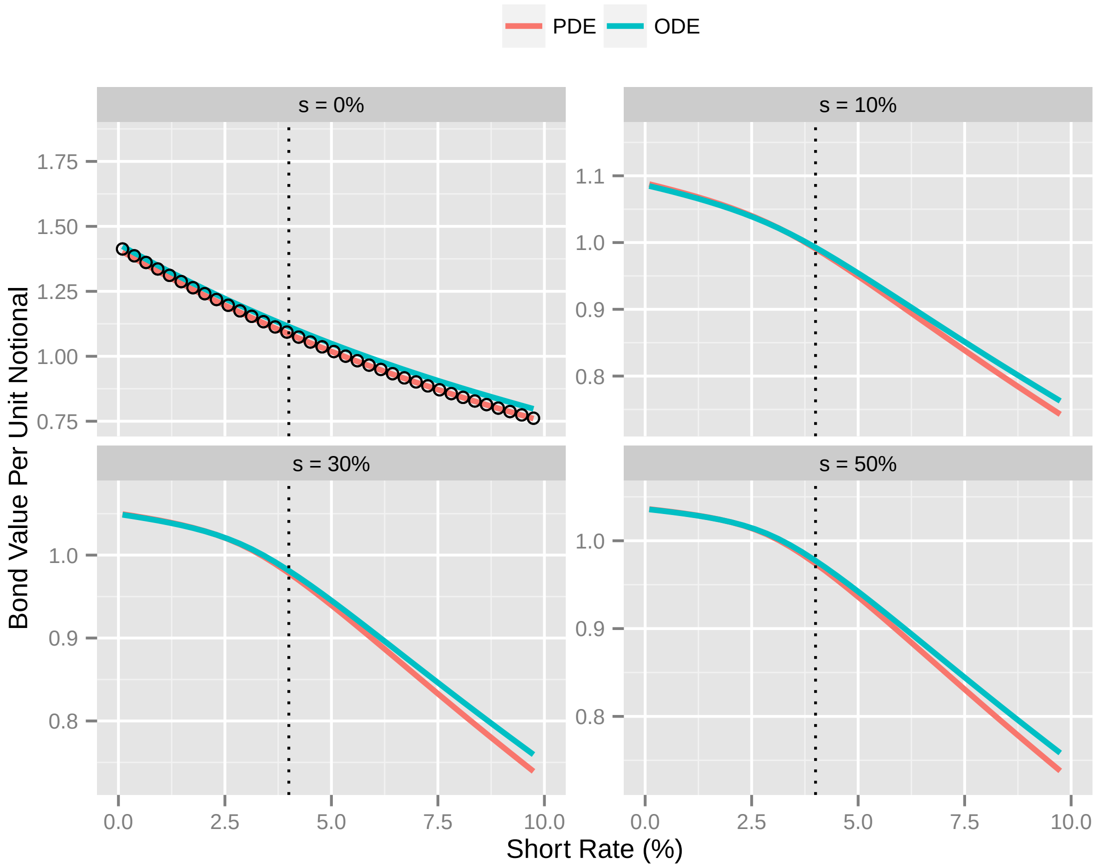


Figure 4.1: Comparison of the ODE and PDE methods, when the short rate follows a Vasicek model under Q. The short rate parameters are given by the estimates found in chapter 2. The points are the true solution and the vertical dotted lines are where $r = q$, which illustrate where the slope of the curves should be similar. The remaining parameter values are $T = 30$, $q = 4\%$, $a = 0$, $b = 0.1$.

while for rates less than the coupon rate the two numerical solutions seem to agree. This is good news for the 2M-model, since prepayment modelling focuses on the low rate area. However, the closeness of the two numerical solutions for low rates is probably caused by the fact that the assumptions made in the 2M-model, i.e. that drift and volatilty for the short rate is zero, are satisfied in the Vasicek model.

In order to test the outcome of the 2M-model when these assumptions are no longer valid we have assumed that the short rate follows a simple Hull-White model. This means that the mean-reversion is now assumed to be a deterministic function of time. More precisely, we assume that the short rate process under Q solves the SDE

$$dr(t) = \kappa(\theta(t) - r(t))dt + \sigma dW(t),$$

where κ, σ are constant and $\theta(t)$ is a deterministic function of time.

In figure 4.2 and 4.3 we see the comparison between the ODE and PDE methods, assuming that the short rate follows a Hull-White model. The difference between the two plots is that we have used a different mean-reversion functions in the two plots. The mean-reversion functions are found in figure 4.4, where the one to the left is used in figure 4.2 and the one to the right is used in figure 4.3. Note that value of the two different mean-reversion functions are the same at time zero. This way they correspond to different future scenarios for the mean-reversion given the same starting point.

In figure 4.2 and 4.3 the two numerical solutions are in general further from each other than in figure 4.1. This does not come as a surprise, since the assumption that the drift of the short rate is time-independent is now violated. The pattern drawn here seems to be that the more the mean-reversion changes through time the larger the loss of precision in the solution to the ODE is. But it should be noted that the disagreement between the two numerical solutions in figure 4.2 and 4.3 are sensitive to the choice of parameter values. Both a smaller value of κ a and larger value of q seems to significantly improve the match between the two numerical solutions. Furthermore we present no arguments in regards to whether or not the chosen time-dependent mean-reversion rates are realistic. Nonetheless we have shown - in a purely illustrative sense - that the 2M-model framework might come at a cost, when the underlying assumptions are not fulfilled.

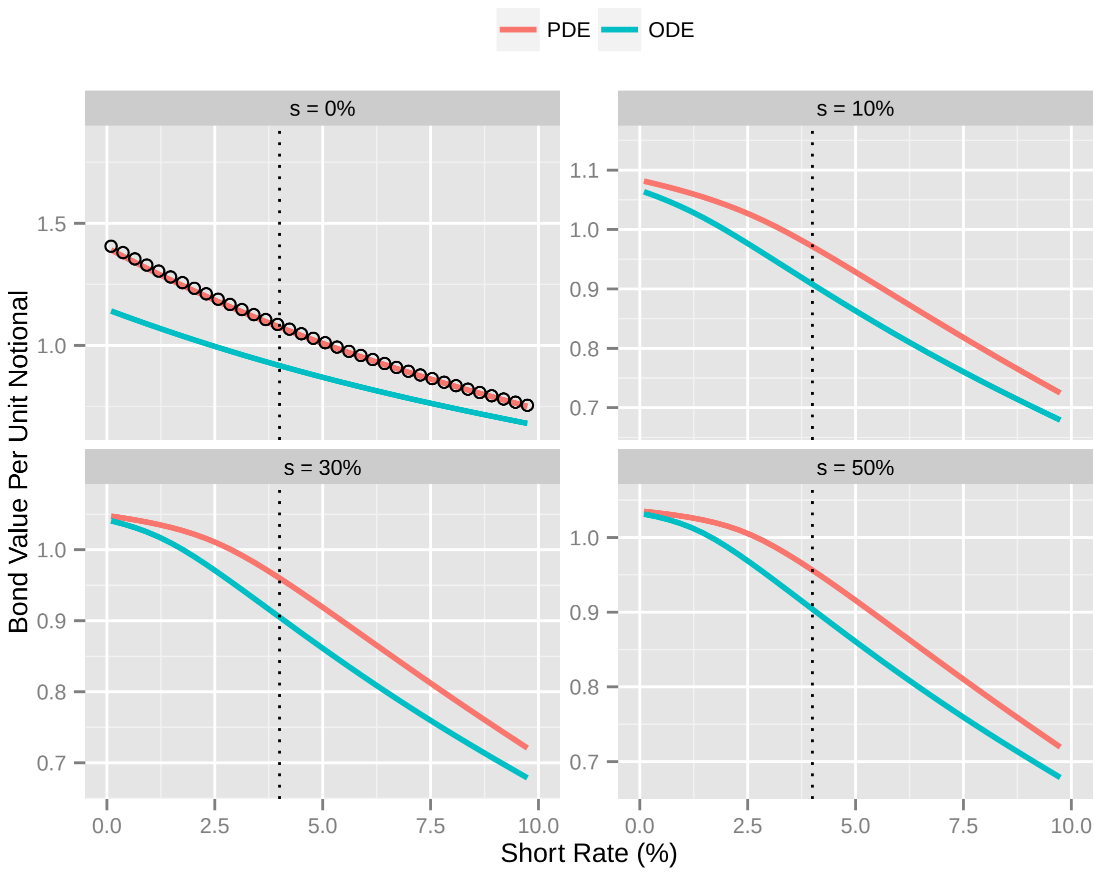


Figure 4.2: Comparison of the ODE and PDE methods, when the short rate follows a Hull-White model under Q with the parameters for κ and σ given by the estimates in chapter 2, while the mean-reversion is illustrated by the left plot in figure 4.4. The points are the true solution and the vertical dotted lines are where $r = q$, which illustrate where the slope of the curves should be similar. The remaining parameter values are $T = 30$, $q = 4\%$, $a = 0$, $b = 0.1$.

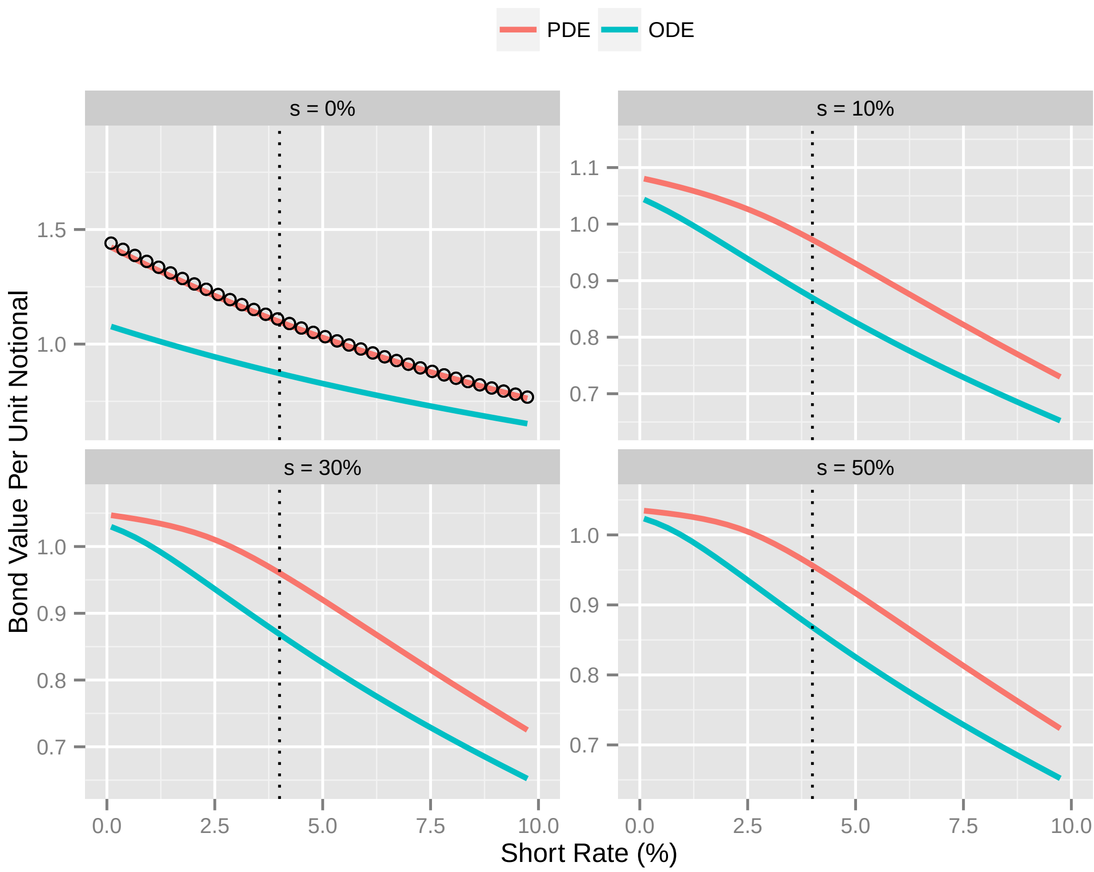


Figure 4.3: Comparison of the ODE and PDE methods, when the short rate follows a Hull-White model under Q with the parameters for κ and σ found in chapter 2, while the mean-reversion is illustrated by the right plot in figure 4.4. The points are the true solution and the vertical dotted lines are where $r = q$, which illustrate where the slope of the curves should be similar. The remaining parameter values are $T = 30$, $q = 4\%$, $a = 0$, $b = 0.1$.

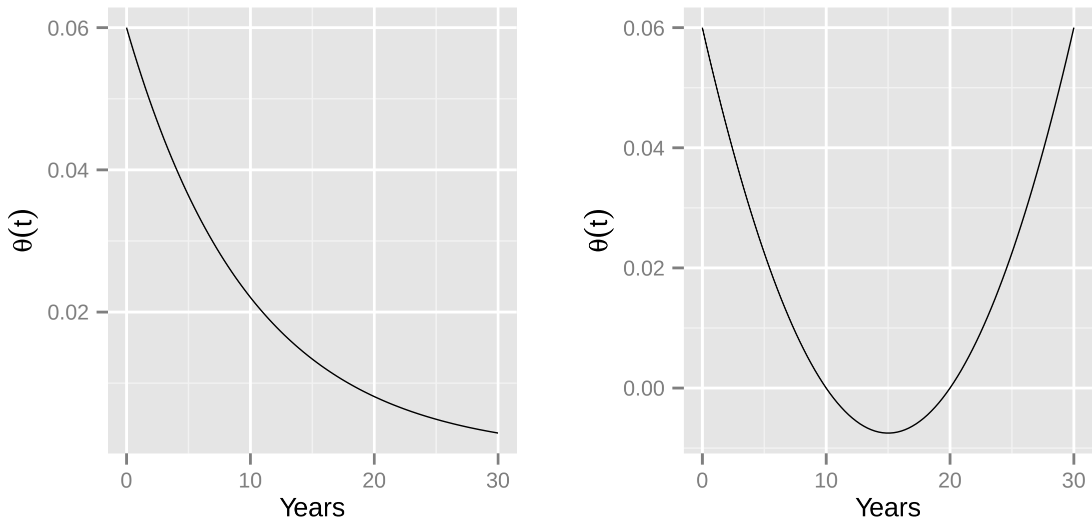


Figure 4.4: Different time-dependent mean-reversion rates plotted over 30 years. The left plot illustrates the mean-reversion function used for figure 4.2. The right plot illustrates the mean-reversion function used for figure 4.3.

4.3 Refining the Prepayment Model

The 2M-model proposes that one may value a mortgage bond by numerically solving the ODE given by equation (4.13). In this section we will examine the prepayment model trough the 2M-model in order to refine it and thereby also improve the valuation method for the mortgage bond[2]. The refining is done in order to avoid undesirable positive convexity in the bond price for low short rates and furthermore to obtain a delta[3] for the bond, which matches empirical findings for low short rates.

4.3.1 The Intrinsic Value

In broad strokes the idea is to solve equation (4.11) under simplifying assumptions, which make the partial derivatives in the equation disappear (both with respect to time and short rate). We denote the solution to this simplified equation, given by (4.17), as *the intrinsic value*. We shall use the intrinsic value as a supporting tool in the analysis of the prepayment function. The intrinsic value and bond value are compared, which yields that if we fix the parameter b in a specific way we ensure a smooth delta for low rates and thereby avoid the possibility of unwanted convexity. Furthermore, for delta to be comparable with empirical findings the fixing of b suggests that we should let the parameter α become a function of the short rate. The reason for the workings of these changes will become clear in the following analysis.

The simplifying assumptions are: β is constant and the drift and volatility of the short rate are zero. Following the same argument as in subsection 4.2.3 this implies that the bond value differentiated with respect to time equals zero. Equation (4.11) therefore leads to the following implicit equation for the bond value

$$0 = (q + \beta + \pi(V(r))) - (r + \beta + \pi(V(r)))V(r). \tag{4.17}$$

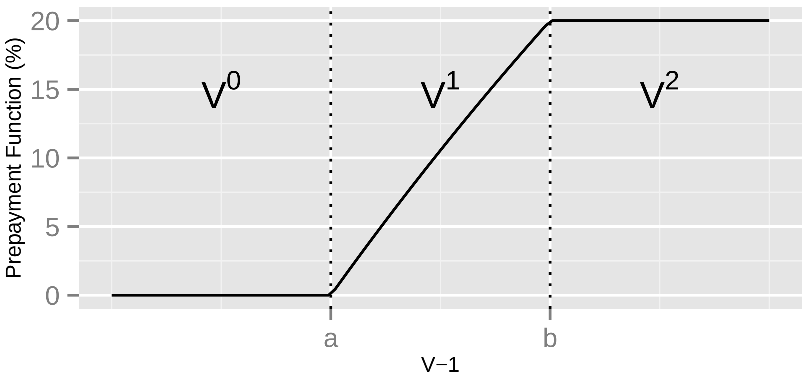
As mentioned in subsection 4.2.4 we assume that φ is the uniform distribution function with support on the interval (a, b). And we assume that $a = 0$, i.e.

$$\varphi(V) = \varphi(V(r) - 1) = \begin{cases} 0 & V(r) \leq 1 & \text{(4.18a)} \\ \dfrac{V(r) - 1}{b} & 1 < V(r) < b + 1 & \text{(4.18b)} \\ 1 & V(r) \geq b + 1 & \text{(4.18c)} \end{cases}$$

Note that we have three different scenarios depending on where on the distribution function the solution to equation (4.17) sits. Let V^0, V^1 and V^2 denote the solution to equation (4.17) in each of the three

[2] The actual valuation method performed by Danske Bank is presented in section 4.4

[3] The derivative of the bond value with respect to the short rate



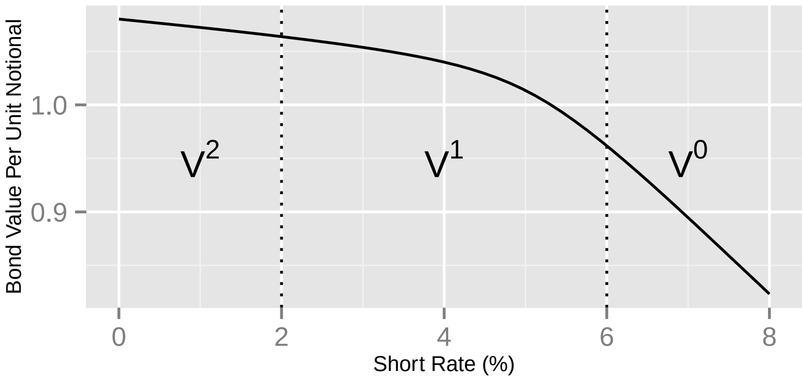


Figure 4.5: The top figure shows the prepayment function together with the areas corresponding to the solutions V^0, V^1 and V^2. The bottom figure illustrates how V^0, V^1 and V^2 corresponds to high, medium and low short rate values, respectively.

scenarios (4.18a), (4.18a) and (4.18c), respectively. The situation is sketched in figure 4.5. Assuming that V is independent of time yields the prepayment function

$$\tilde{P}(V) = 1 - e^{-\int_t^{t+dt} \alpha\varphi(V-1)\mathrm{d}u} = 1 - e^{-\alpha\varphi(V-1)dt}, \tag{4.19}$$

In the first and the third case equation 4.17 gives us that

$$V(r) \leq 1 \Rightarrow \varphi(V) = 0 \Rightarrow V(r) = \frac{q+\beta}{r+\beta} \equiv V^0(r). \tag{4.20}$$

$$V(r) \geq b+1 \Rightarrow \varphi(V(r)-1) = 1 \Rightarrow V(r) = \frac{q+\beta+\alpha}{r+\beta+\alpha} \equiv V^2(r).$$

In the second case $V(r) \in (1, b+1) \Rightarrow \varphi(V(r)) = \frac{V(r)-1}{b}$ and equation 4.17 becomes the following quadratic equation

$$0 = -\tilde{\alpha}V(r)^2 - (r(t) + \beta - 2\tilde{\alpha})V(r) + (q + \beta - \tilde{\alpha}),$$

where $\tilde{\alpha} = \frac{\alpha}{b}$. The solution to this equation is given by

$$\begin{aligned}
V(r) &\overset{(V\geq 0)}{=} \frac{2\tilde{\alpha} - (r+\beta) + \left[(r+\beta-2\tilde{\alpha})^2 + 4\tilde{\alpha}(q+\beta-\tilde{\alpha})\right]^{\frac{1}{2}}}{-2\tilde{\alpha}} \\
&= \frac{2\tilde{\alpha} - (r+\beta) \pm \left[(r+\beta)^2 + 4\tilde{\alpha}(q-r)\right]^{\frac{1}{2}}}{2\tilde{\alpha}} \\
&\equiv V^1(r)
\end{aligned}$$

We now know that the solution to equation (4.17) is given by either V^0, V^1 or V^2 and we may write the joined solution as

$$V(r) = \min\left(V^0(r), \max\left(V^1(r), V^2(r)\right)\right). \tag{4.21}$$

4.3.2 First Refinement: Fixing the parameter b

In figure 4.6 the intrinsic value and the bond value are plotted against the short rate. The parameter values used in figure 4.6 equals those used in [Andreasen (2011)]. Note that there is an indicator function on the volatility. This simply implies that if the short rate becomes negative it will move towards becoming positive again without any randomness.

The transition from V^1 to V^2 in the intrinsic value may cause positive convexity in the price as depicted on the right in figure 4.6. This is undesirable for more than one reason. From standard theory on callable bonds we know that the bond price may display negative convexity, when the embedded option to prepay is in the money, i.e. for low rates - cf. chapter 1. Also, by getting rid of the kink depicted in figure 4.6 we obtain a smooth delta for low rates and smoothness is always a desirable mathematical property, e.g delta-hedging is not well-defined in the case of figure 4.6. Furthermore, because we wish to use the intrinsic value as an instrument in order to analyse the bond value for low rates, we prefer that the intrinsic value and bond value converge for low rates. In that way we argue that the analysis performed on the intrinsic value can be transferred to the bond value for low rates.

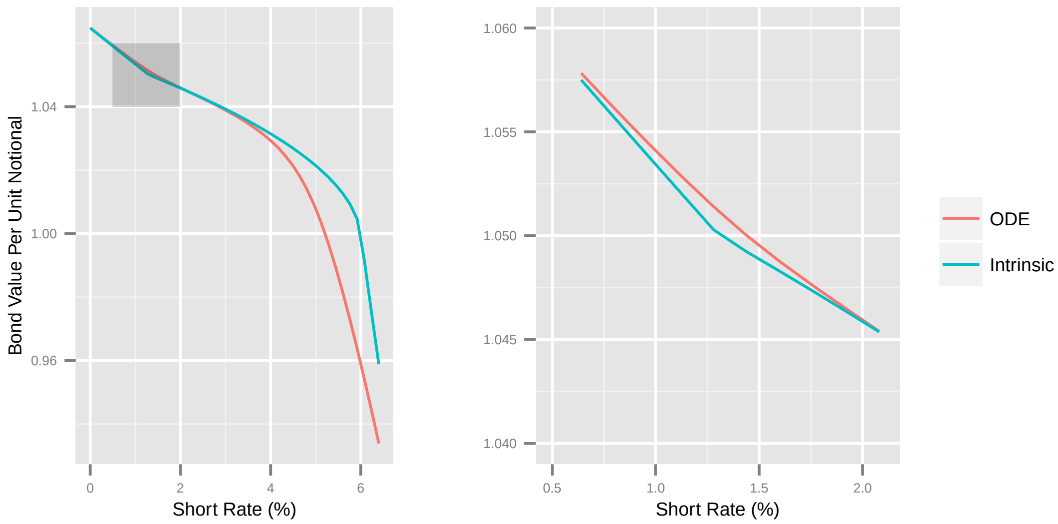


Figure 4.6: Illustrating the unwanted positive convexity in the intrinsic value. In the right plot we have zoomed in on the left plot. The short rate is assumed to follow a Vasicek model under Q with $\mu(r) = 0$ and $\sigma(r) = 0.007 \cdot 1_{r>0}$. The remaining parameter values are $T = 30$, $q = 6\%$, $a = 0$, $b = 0.05$, $s = 20\%$.

The convexity is created when V^2 kicks in, which is when the prepayment is capped. Therefore the convexity issue may be avoided by choosing b sufficiently high, such that V^2 never kicks in - in the V^1-region the slope of the prepayment function is negatively dependent on b and it only depends on the ratio $\frac{\alpha}{b}$, given the value of V.[4] However, this implies that two (or more) different sets of parameter values (s_1, b_1) and (s_2, b_2) give rise to the same prepayment function and thereby the same bond value, if $\frac{\alpha_1}{b_1} = \frac{\alpha_2}{b_2}$, where $\alpha_i = -\log(1 - s_i)/dt$ for $i = 1, 2$. Such a scenario is illustrated in figure 4.7. In order to avoid both the unwanted convexity and a non-unique bond value we fix the parameter b such that V^2 kicks in at exactly $r = 0$.[5] When the parameter b is fixed only the scale parameter s determines the slope of the prepayment function and we thereby avoid a non-unique bond value.

In order to fix the parameter b, such that V^2 kicks in exactly at $r = 0$, we first notice that

$$V^2(0) = \frac{q + \beta + \alpha}{\beta + \alpha} = 1 + \frac{q}{\beta + \alpha} \approx 1 + \frac{q}{\alpha},$$

where the approximation follows from the fact that we typically expect β to be small relative to α.[6]. We then see that

$$\varphi(V^2(0) - 1) \approx \varphi\left(\frac{q}{\alpha}\right) = 1$$

[4] These claims are easily shown by the use of standard calculus.

[5] The fact that V^2 should kick in when $r = 0$ is chosen in [Andreasen (2011)]. But in a market where negative interest rates are reality it might be more appropriate to fix b such that V^2 kicks in at some reasonable negative value of r. However, fixing b, such that V^2 kicks in at $r = 0$ simplifies some of the following analysis.

[6] Recall that $\alpha = -\log(1 - s)/dt$ For a bond with 30 year maturity, $q = 4\%$, quarterly payments and $s = 20\%$ then $\alpha \approx 0.9$ and $\beta \approx 0.04$.

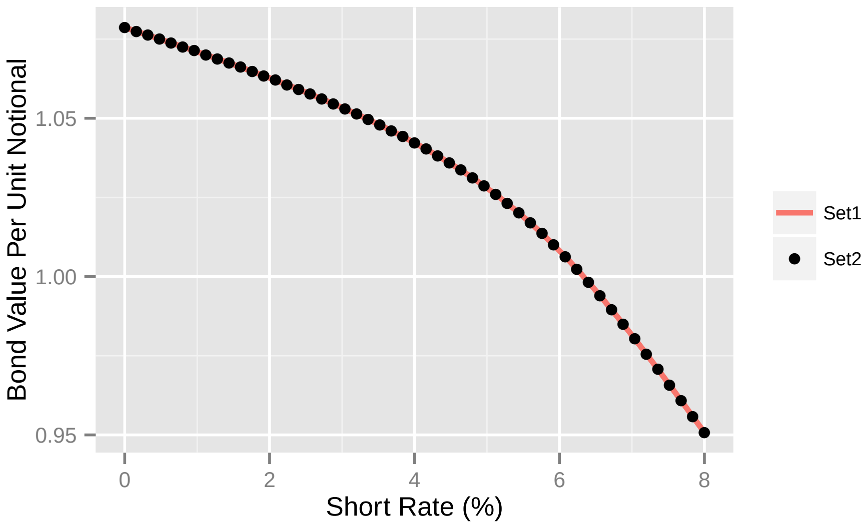


Figure 4.7: Illustrating how different parameter values of s and b give rise to the same bond value. A sufficiently large b (i.e. the V^2 solution never kicks in) implies that prepayment is only dependent on the ratio α/b. Hence different sets of parameter values (s_1, b_1) and (s_2, b_2) give rise to the same prepayment function and thereby the same bond value, if $\frac{\alpha_1}{b_1} = \frac{\alpha_2}{b_2}$, where $\alpha_i = -\log(1-s_i)/dt$ for $i = 1, 2$. The short rate is assumed to follow a Vasicek model under Q with the parameters given by the estimates found in chapter 2. The remaining parameter values are $T = 30$, $q = 6\%$, $a = 0$, $b_1 = 0.1$, $b_2 = 0.4$, $s_1 = 20\%$, $s_2 = 59.04\%$.

if we choose

$$b = \frac{q}{\alpha}. \tag{4.22}$$

In figure 4.8 it is shown how choosing b according to equation (4.22) removes the undesirable convexity.

4.3.3 Second Refinement: Letting the Parameter α Depend on the Short Rate

The second point of this section is to refine the prepayment model, such that the delta of the bond matches empirical findings for low rates. And the strategy is to do so by looking at the delta of the intrinsic value. According to [Sjögren (2011)] there exists empirical evidence that bonds are trading above par with a delta of zero. For that reason we would like the prepayment model to be capable of imposing such a scenario in the resulting bond value.

To begin with we compute the delta of the intrinsic value by differentiating equation (4.17) with respect to the short rate, which gives us that

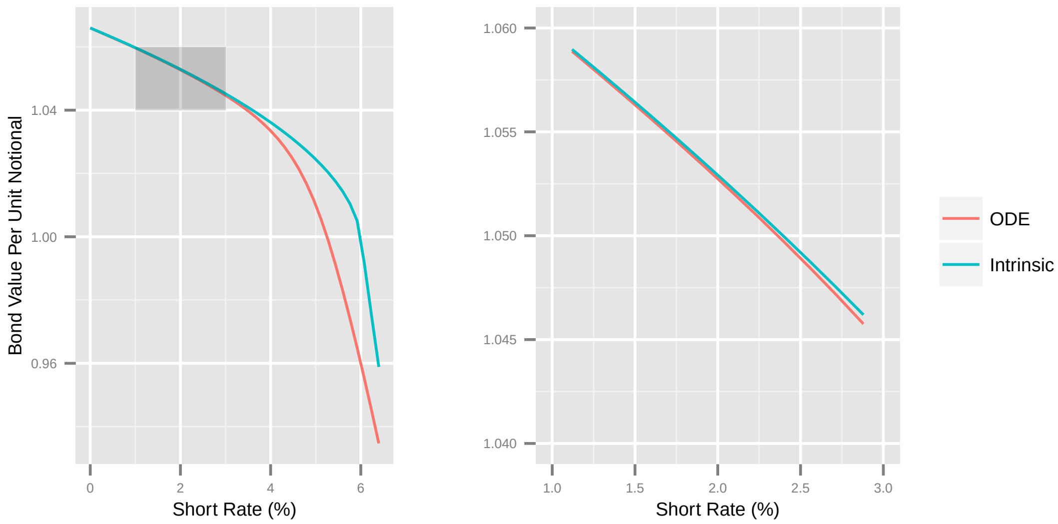


Figure 4.8: The smoothing effect of choosing b according to equation (4.22). In the right plot we have zoomed in on the left plot. The short rate is assumed to follow a Vasicek model under Q with $\mu(r) = 0$ and $\sigma(r) = 0.007 \cdot 1_{r>0}$. The remaining parameter values are $T = 30$, $q = 6\%$, $a = 0$, $b = 0.07$, $s = 20\%$.

$$\Delta = V_r(r) = -\frac{V(r)}{\pi_V(V(r))(V(r)-1) + (r + \beta + \pi(V(r)))}, \tag{4.23}$$

where V_r and π_V denote the partial derivatives with respect to r and V, respectively.

It is easily seen that delta is negative for all r where π_V is well-defined. Even when $V(r) < 1$ we have that $\pi_V(V(r)) = \alpha\varphi_V(V(r)) = 0$. Under the assumption that the intrinsic value and the bond value follow each other for small rates, the negative delta implies that the maximal bond price in the model is obtained when the short rate equals zero[7] and it is given by

$$V(0) = V^2(0) \approx 1 + \frac{q}{\alpha}. \tag{4.24}$$

Furthermore delta corresponding to the maximal price is given by

$$\Delta\big|_{r=0} = -\frac{q + \beta + \alpha}{(\beta + \alpha)^2}. \tag{4.25}$$

[7]This is of course not true if we allow the short rate to become negative.

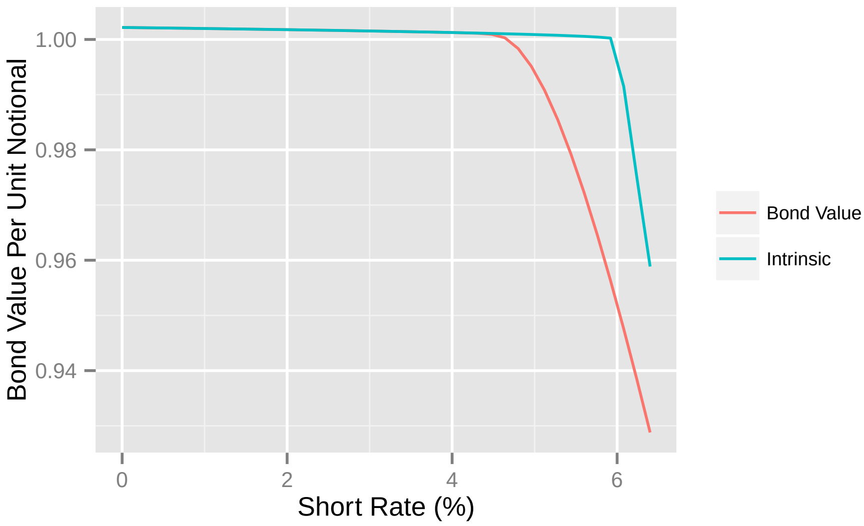


Figure 4.9: The effect of choosing a high s. A high value of s implies a high value of α, which then implies that the delta converges to zero. However, simultaneously the maximum price of the bond converges towards par. We are therefore not able to obtain a delta of zero for prices above par with the current prepayment model. The short rate is assumed to follow a Vasicek model under Q with $\mu(r) = 0$ and $\sigma(r) = 0.007 \cdot 1_{r>0}$. The remaining parameter values are $T = 30$, $q = 6\%$, $a = 0$, $b = 0$, $s = 99.9\%$.

From equation (4.24) and (4.25) it is clear that the choice of α controls both the maximal price and the delta. We may try to push delta towards zero by choosing a high value of s, corresponding to high value of α. However, this approach will move the maximal price towards par - we wish a delta of zero when the price is above par. This situation is illustrated in figure 4.9.

It is clear, by (4.25), that a delta of zero can not be realized for any choice of s except in the limit. Therefore a delta of zero for prices above par is not obtainable within the current prepayment framework. In order for the model to capture such a scenario we need to impose further prepayments for low rates. So far prepayments have only been dependent on the value of the bond and thereby implicitly on the short rate. In the market we would expect to see a higher concentration of prepayments for low rates. An idea would therefore be to intensify the prepayment intensity for low rates. This can be done making the prepayment rate directly dependent on the short rate. For that reason it is proposed that the prepayment model is refined by letting α become a function of the short rate.

More specifically it is suggested in [Andreasen (2011)] that we let $\alpha(r)$ be defined by

$$\alpha(r) = \alpha_0 \left(1 + A \left[\max(r_1 - r, 0)\right]^p\right), \tag{4.26}$$

where $\alpha_0 = -\frac{\log(1-s)}{dt}$, A and $p \geq 1$[8] are constants and $r_1 = kr_0$ where $k > 1$ is constant and $r_0 > 0$ is some low rate. This means that for 'high' rates, $r \geq r_1$, the α-function is constant as before and equal to the base line value α_0. But for 'low' rates, $r < r_1$, the α-function is increasing in decreasing rates with power p. In other words the extra short rate driven prepayments will kick in at r_1 with a power of p.

As argued above, defining α by equation (4.26), obviously creates extra prepayment for low rates. But we still need to make sure that this higher expected prepayment creates the possibility for bond prices above par to have a delta of zero. In the analysis below it is therefore shown how A should be chosen in order to secure a delta of zero at r_0.

Assuming that $\alpha = \alpha(r)$ and differentiating equation (4.17) with respect to r now yields

$$V_r(r) = -\frac{V(r) + \alpha_r(r)\varphi(V(r))(V(r)-1)}{\alpha(r)\varphi_V(V(r))(V(r)-1) + (r + \beta + \alpha(r)\varphi(V(r)))}. \tag{4.27}$$

From equation (4.27) we see that the only way to force $V_r(r_0) = 0$ is by letting

$$\begin{aligned} \alpha_r(r_0) &= -\frac{V(r_0)}{\varphi(V(r_0))(V(r_0)-1)} \\ &= -\frac{1}{\varphi(V(r_0))}\frac{q + \beta + \alpha(r_0)\varphi(V(r_0))}{q - r_0} \\ &\approx -\frac{\alpha(r_0)}{q - r_0}, \end{aligned} \tag{4.28}$$

where the second equality follows from equation (4.17) and the approximation is based on the assumption that $\frac{q+\beta}{\varphi} + \alpha(r_0) \approx \alpha(r_0)$.

From (4.28) and by the use of equation (4.26) we see that

$$\begin{aligned} & -p\alpha_0 A(r_1 - r_0)^{p-1} \approx -\frac{\alpha_0\left(1 + A(r_1 - r)^p\right)}{q - r_0} \\ \Leftrightarrow \quad & A \approx \left(p\frac{q - r_0}{r_1 - r_0} - 1\right)^{-1}(r_1 - r_0)^{-p}. \end{aligned} \tag{4.29}$$

Note that we need $p \geq 1$, since if $p < 1$ we might have that α would decrease when the short rate hits r_1, which is the opposite of what we want.

All in all we end up with the following refined prepayment model: *The conditional proportion of a*

[8]The reason why p has to be greater than or equal to one will be explained later on.

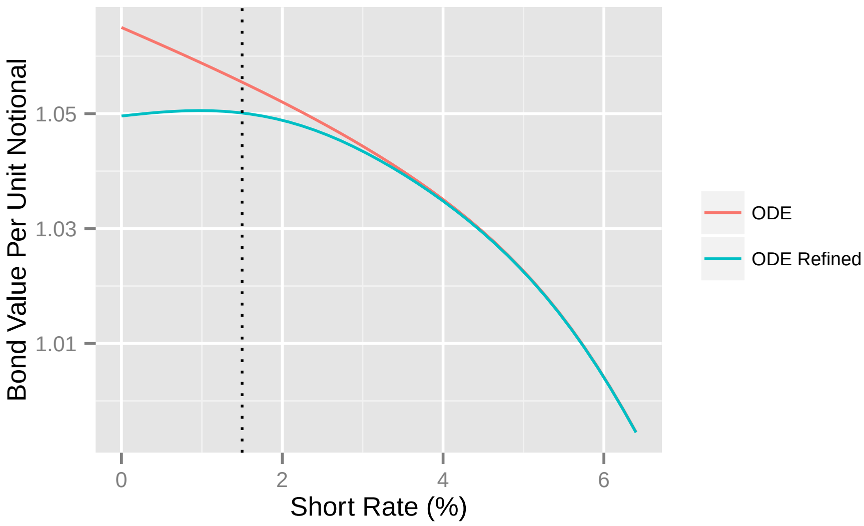


Figure 4.10: The effect of the refined prepayment model. The extra short rate driven prepayments implemented in the refined prepayment model imply a model, which is capable of obtaining a delta of zero for bond prices above par. The vertical dotted line indicates the value of r_0, where the maximum bond value should occur. The short rate is assumed to follow a Vasicek model under Q with estimates found in chapter 2. The remaining parameter values are $T = 30$, $q = 6\%$, $a = 0$, $b = 0.07$, $s = 20\%$.

mortgage pool, which is expected to prepay within a time period of length dt is given by

$$p(r, V) = 1 - e^{-\alpha(r)\varphi(V-1)dt}, \tag{4.30}$$

where $\alpha(r)$ *is defined by the equations* (4.26) *and* (4.29) *with* $\alpha_0 = -\frac{\log(1-s)}{dt}$. φ *is defined as the uniform distribution function with parameters* $a = 0$ *and* $b = \frac{q}{\alpha_0}$. *The parameters* r_0, k *and* p *are set equal to* $\frac{q}{4}$, 2 *and* 2, *respectively.*[9]

In figure 4.10 the bond value is computed using the 2M-model with both the basic and the refined prepayment function. Note that figure 4.10 illustrates how the extra short rate driven prepayments implemented in the refined prepayment model implies a model, which is capable of obtaining a delta of zero for bond prices above par.

4.4 Valuation of a Callable Mortgage Bond

In this section we go into detail with how to implement the prepayment model (4.30) in order to compute prices of callable mortgage bonds. Notice that there is no need for extending the prepayment model from a single mortgagor to a pool of mortgagors. This is because the starting point of DBPM was the prepayment proportion of an entire pool - opposed to the model presented in chapter 3, where the pre-

[9] We do not argue for the chosen values of r_0, k and p, but simply note that these are the values used by Danske Bank.

payment behaviour of a single mortgagor was the starting point. Besides describing the implementation, we illustrate how the parameter s affects the resulting bond price.

The pricing procedure is similar to the one presented in section 3.2. The price is computed using a PDE-approach, which gives us a multiple of boundary value problems to solve. Consider a Danish mortgage bond with coupon rate q, quarterly payments on $T_1, \ldots, T_n$. The size of the payments are constant (in case of no prepayments) due to the annuity structure and denoted denoted C. Assume that the market short rate solves the SDE given by

$$dr(s) = \mu(s, r(s))ds + \sigma(s, r(s))dW(s)$$
$$r(0) = r_0$$

and let $\tilde{T}_i = T_i - \frac{1}{6}$ denote the announcement dates – i.e. the dates two months before the coupon dates. Recall that this is the dates where the mortgagor has to decide whether or not he or she wishes to prepay at the upcoming coupon date. Therefore we wish to evaluate the bond values on these dates rather than on the coupon dates.

Define $V_i(t, r)$ for $t \leq \tilde{T}_i$ and $i = 1, \ldots, n$ as the time t value of the payment at time T_i plus future payments, assuming no prepayments have been made before time T_i. Note that $V_n^l(t, r)$ is simply the time t price of a ZCB with maturity T_n and payment C, which implies that $V_n(t, r)$ solves the boundary value problem (1.5) with $T^\star = T_n$ and $C^* = C$.

Let F_{T_i} denote the remaining principal when coupon i has been paid. Define

$$v_{i+1}^{T_i}(r) = \frac{V_{i+1}(T_i, r)}{F_{T_i}}$$

for $i = 1, \ldots, n-1$ as the bond value per unit notional at time T_i of the future payments after time T_i. We seek the bond value per unit notional at time $\tilde{T}_i$ of the future payments after time T_i. Define this by

$$v_{i+1}^{\tilde{T}_i}(r) = \frac{V_{i+1}(\tilde{T}_i, r)}{d(r)F_{T_i}},$$

where $d(r) = E_{0,r}\left[e^{-\int_0^{\frac{1}{6}} r(s)ds}\right]$. Hence at time $\tilde{T}_i$ the expected proportion of prepayments at time T_i is given by

$$p\left(r, v_{i+1}^{\tilde{T}_i}(r)\right) = 1 - e^{-\alpha(r)\varphi\left(v_{i+1}^{\tilde{T}_i}(r)-1\right)dt}$$

Then $V_i(t,r)$ must be the solution to the boundary value problem on $[0,\tilde{T}_i]\times\mathbb{R}$ given by

$$\begin{aligned} 0 &= \frac{\partial V_i(t,r)}{\partial t} + \mu(t,r)\frac{\partial V_i^l(t,r)}{\partial r} + \frac{1}{2}\sigma^2(t,r)\frac{\partial^2 V_i(t,r)}{\partial r^2} - rV_i(t,r) \\ V_i(\tilde{T}_i,r) &= \left(1-p\left(r,v_{i+1}^{\tilde{T}_i}(r)\right)\right)V_{i+1}(\tilde{T}_i,r) + p\left(r,v_{i+1}^{\tilde{T}_i}(r)\right)d(r)F_{T_i} + d(r)C \end{aligned}$$

All in all we end up with n PDEs, where each one of them can be solved by a finite difference schemes. For example the Crank-Nicolson scheme, which is outlined in section A.5 in the appendix.

Note that the only free parameter in the DBPM is the parameter s, which governs the base line level of prepayment, α_0. For $s=0$ no prepayment will occur in the mortgage pool. For $s\to 1$ prepayment converges towards one, if bond price is above par. In figure 4.11 the value of a callable bond is plotted for different values of s. For $s\to 1$ the shape of the bond price looks similar to what we saw under the assumption of optimal behaviour in section 1.4. In other words, all mortgagors in the pool choose to prepay in the given coupon period.

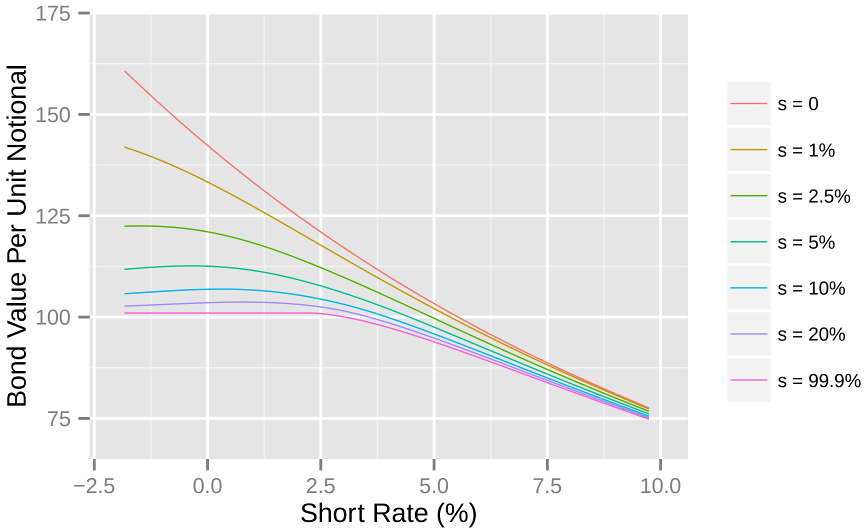


Figure 4.11: The price of a 30-year callable mortgage bond at the first announcement date for different values of the parameter s. The bond has a coupon rate of 4%. For $s = 0$ the price equals that of a non-callable bond, while for $s \to 1$ the per period prepayment proportion converges to one, if bond price is above par. In other words, all mortgagors in the pool chooses to prepay in the given coupon period. The short rate is assumed to follow a Vasicek model under Q with the parameters given by the estimates found in chapter 2.

Chapter 5

A Comparison of the Two Models

In this chapter we compare the prepayment model derived in chapter 4 (i.e. *DBPM*) with the prepayment model derived in chapter 3, which we denote as the *Stanton model*. The models are compared on their ability to predict prepayment behaviour, fit observed bond prices and match the rate sensitivity induced by the market. Furthermore we compare their explanatory power.

5.1 Prepayment Behaviour and Implied s

We begin by evaluating the two models on their ability to predict prepayment rates. We do so by comparing their predictions with observed prepayment rates. The comparison is based on observations from twenty Danish mortgage pools in the period 2010-2014. The reason for choosing this time period is that there is not much action before 2010 - the observed prepayment level is around zero for the most part before 2010.

In figure 5.1 we have plotted the two models' predictions together with observed quarterly prepayment rates. More specifically, for each model we have computed the expected prepayments for each of the twenty pools and then taken the average over pools at each time point. Likewise, the observed prepayments illustrated in figure 5.1 are the average observed prepayment rates.

Notice that two curves for DBPM are plotted in figure 5.1, one labelled *DBPM Basic* and the other labelled *DBPM Implied*. DBPM Basic is equal to DBPM in its original form, i.e. with a constant value of the parameter s. The value of s used in DBPM Basic differs from pool to pool and can be found in table 5.2.[1] Given a value of s, all that is needed in order to compute prepayment predictions at some coupon date T_i is then the short rate value and the bond value (per unit notional) on the corresponding announcement date. For example, consider a bond with a 4 percent coupon rate and $s = 0.2$. Assume

[1] The values of s found in table 5.2 are chosen in order to fit observed prices.

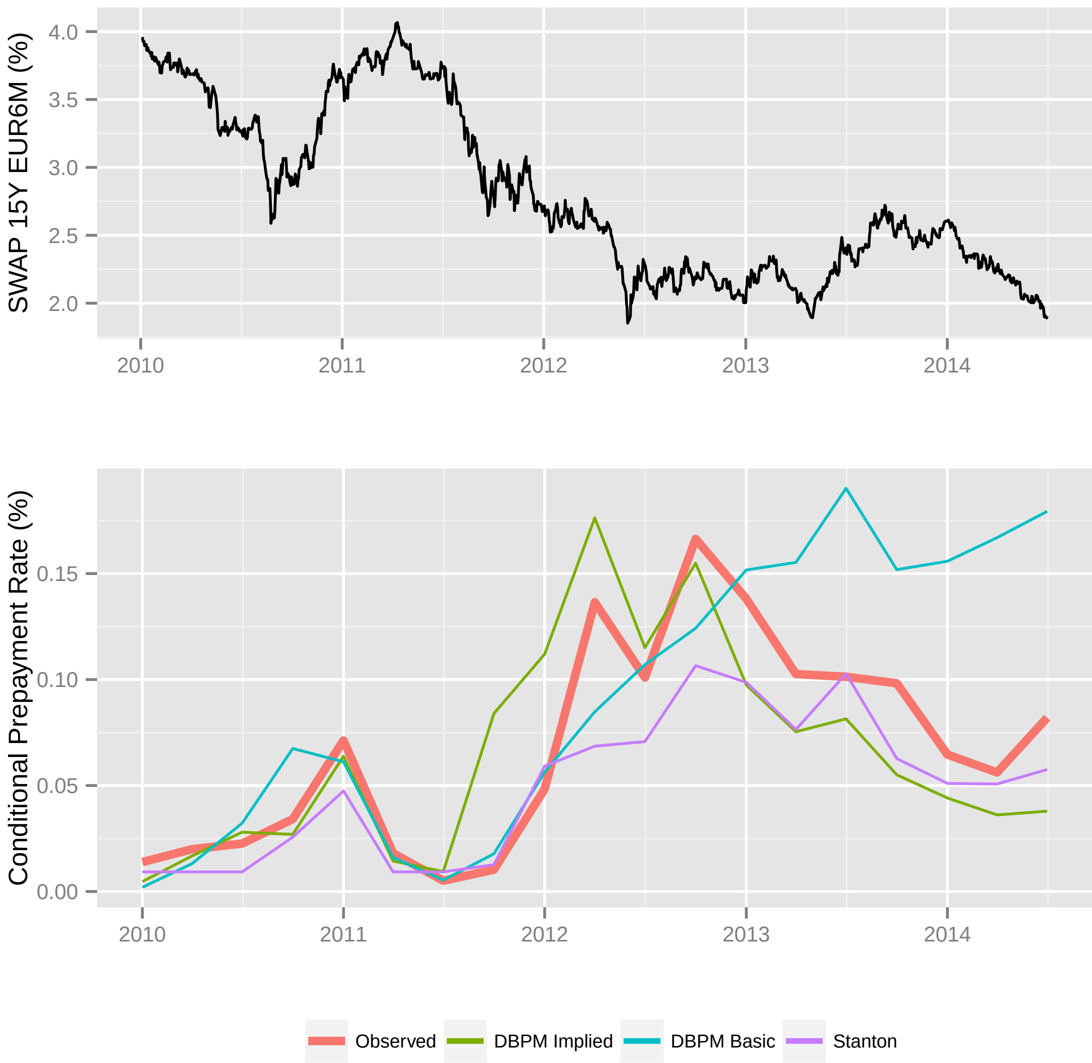


Figure 5.1: Prepayment predictions by the Stanton model and DBPM together with observed quarterly prepayment rates for twenty Danish mortgage bond. More specifically, for each model we have computed the expected prepayments for each of the twenty pools and then taken the average over pools at each time point. Likewise, the plotted observed prepayments are the average observed prepayment rates. Notice that two curves for DBPM are plotted, one labelled *DBPM Basic* and the other labelled *DBPM Implied*. DBPM Basic is for a fixed value of s, while in DBPM Implied the value of s is calibrated to market prices.

that at the announcement date we have $r = 0.03$ and $V = 1.02$. Then the proportion of the remaining pool, which is expected to prepay on a quarterly basis (according to DBPM) is given by

$$p(r, V) = 1 - e^{-\alpha(r)\phi(V-1)\cdot 0.25} \approx 9.5\%. \tag{5.1}$$

Opposed to DBPM Basic the value of s is updated at each time point in DBPM Implied. More specifically, we calculate an implied value of s based on market quotes for the bond in question. I.e. we choose the value if s such that the DBPM bond price matches the price observed in the market. The input used for the resulting expected prepayment rates for both DBPM Basic and Implied, in figure 5.1, are the observed market quotes and 'observed' values of the short rate.[2] For details on how expected prepayment rates are computed in the Stanton model see section 3.4.

When comparing the prepayment rates produced by DBPM Basic and DBPM Implied, one should recall that DBPM in its basic form is a quite simple prepayment model. The expected prepayment proportion is only affected through two channels: 1) the difference in bond price and par, i.e. the larger the difference the higher is the expected prepayment proportion. This implicitly captures the interest-rate incentive. 2) Short rate driven prepayments that kicks in when the short rate drops below some critical level. DBPM Basic is therefore not capable of capturing other factors, such as for example a burnout effect. This is well illustrated by the poor fit of DBPM Basic in figure 5.1 from late 2012 and forward.

In order for DBPM to capture other effects Danske Bank 'cheats' and calibrates the model to market prices through the parameter s. The cheating part is of course the process of calibration seen from a theoretic modelling perspective. However when calibrating the model to market prices, the model becomes capable of capturing effects that are reflected through market prices.

The chain of causality in DBPM Implied can be explained in the following way: In the basic model a drop in the short rate implies an increase in expected prepayments through the function $\alpha(r)$ (if the drop is sufficiently large) and implicitly through the distribution function φ (given that the price rises above par). The latter effect is also present in the implied model. But an increase in the market price implies a lower value of s in the implied model. The decrease in s is needed in order for the implied model to match market prices. A lower value of s has a negative effect on expected prepayment through α_0.[3] Therefore a drop in the short rate does not necessarily imply an increase in expected prepayment - cf. the prepayment rates produced by DBPM Implied in figure 5.1 from late 2012 and forward. Here the

[2] 'Observed' in quotation marks, because the short rate is not actually observed, cf. chapter 2.
[3] Recall that $\alpha_0 = -\log(1 - s)/dt$.

$\hat{\alpha}$	$\hat{\beta}$	$\hat{\rho}$	$\hat{\lambda}$
0.2	4.096	0.351	0.016

Table 5.1: Estimates for the parameters in the Stanton model based only on observed prepayment rates before 2010.

expected prepayment rate decreases though the swap rate stays at a low level. This could be interpreted as evidence of DBPM capturing a burnout effect, when calibrated to market prices.

From figure 5.1 we see that the Stanton model and DBPM Implied fits observed prepayment rates better than DBPM Basic, which is no surprise. If the reader prefers numbers to illustrations this may be concluded from the sums of squared errors, which are given by

$$\text{Stanton Error} = 0.0146 \tag{5.2}$$

$$\text{DBPM Basic Error} = 0.0499 \tag{5.3}$$

$$\text{DBPM Implied Error} = 0.0191. \tag{5.4}$$

The reasonable fit of the Stanton model might be explained from the fact the parameters of the Stanton model, α, β, ρ and λ, were estimated in order to minimize the distance between observed and model prepayment rates. In order to test the significance of this we estimate the parameters of the Stanton model based only on prepayment observations before 2010.[4]

The resulting 'out-of-sample' estimators in the Stanton model are found in table 5.1. Not surprisingly the estimator of ρ[5] is much lower than the original estimator (see table 3.1), due to the fact that not much prepayment occurs before 2010. This implies a worse fit of the Stanton model in the period 2010-2014, which is illustrated in figure 5.2. However, it still performs significantly better than DBPM Basic.

5.2 Comparing Pool Prices

We proceed by comparing prices on callable mortgage bonds produced by DBPM Basic and the Stanton model. Since the two models disagree on predicted prepayment behaviour (see section 5.1) they will of course neither agree on pool prices. It does not make sense to compare preciseness of the Stanton model with that of DBPM Implied, in respect to fitting observed market prices. This is because the value of s

[4]Another idea could be to estimate the parameters based on observations from half of the twenty bonds over the entire time period and then see how the model would perform on the remaining bonds. We also performed this experiment and the resulting estimates were similar to those obtained through the out-of-sample test. For that reason we have not included them.

[5]Recall that the parameter ρ governs the frequency of prepayment decisions.

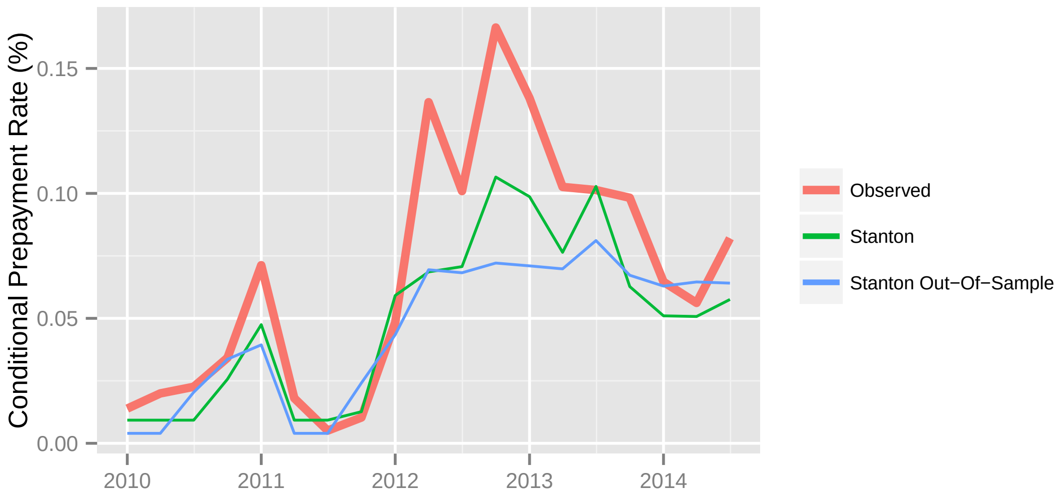


Figure 5.2: Prepayment predictions produced by the Stanton model, when the model parameters are estimated only on observations from before 2010, see table 5.1. The estimator of ρ is much lower than the original estimator, due to the fact that not much prepayment occurs before 2010. This implies a worse fit of the Stanton model in the period 2010-2014. However, it still performs significantly better than DBPM Basic, cf. figure 5.1.

is chosen based on the observed price in DBPM Implied, such that the model price equals the observed price. I.e. DBPM Implied will fit observed quotes perfectly, since they are input in the model. Hence plotting DBPM Implied bond prices together with observed prices is simply useless. However, we can compare the Stanton model and DBPM Implied on rate sensitivity, which is what we attempt in section 5.4.

In figure 5.3 and 5.4 the model prices are plotted together with observed market prices for the bond NDA 4 1 01OCT2035 over time, for both the Stanton model and DBPM Basic. In figure 5.5 and 5.6 we have once again plotted model prices together with observed prices, but now as a function of the short rate. For each observed market price we have computed a corresponding model bond value. More specifically, for each model we have computed the value of a mortgage bond, with specifications equal to the specific bond (i.e. maturity, coupon rate, etc.). This gives us a grid of bond values with 120 columns corresponding to the different announcement dates and with rows corresponding to different short rate values. For dates in between announcement dates we have ignored the discounting and simply pulled out the value from the latest announcement date column (going forward in time) corresponding to the short rate on the date in question. In the Stanton model one has to remember to weigh each column with the corresponding distribution of transaction costs.[6]

[6]This is easier said than done, cf. section 3.5.

The general picture from the figures 5.3 - 5.6 is that the Stanton model succeeds in capturing both the interest-rate aspect, seen by the fitting of the concave relationship between price and short rate, and the burnout tendency, seen for example by the fitting of the 'cloud' in figure 5.5. Equivalently, the fit of the Stanton model over time, illustrated in figure 5.3, is indeed very satisfying.

On the other hand DBPM Basic does not capture the increase in prices from 2012 and forward, which is illustrated in both figure 5.4 and 5.6. This does not come as a surprise, since the increase in prices could be interpreted as evidence of a burnout effect and from section 5.1 we know that DBPM Basic is not capable of capturing such a market effect.

Figure 5.3: The Stanton model prices over time together with observed market prices for the bond `NDA 4 1 01OCT2035`. The model captures both interest-rate incentives and burnout effects, which results in a very good fit.

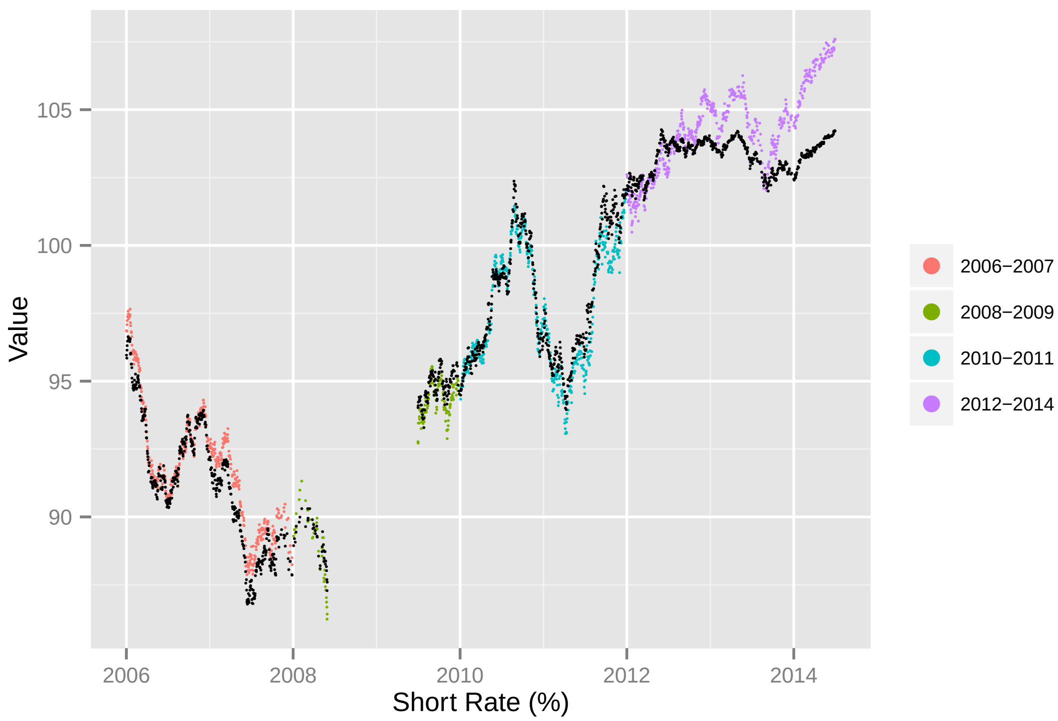


Figure 5.4: DBPM Basic (with a fixed value of s) prices over time together with observed market prices for the bond `NDA 4 1 01OCT2035`. The model captures interest-rate incentives. However it is not capable of capturing other effects, which results in a poor fit from late 2012 and forward.

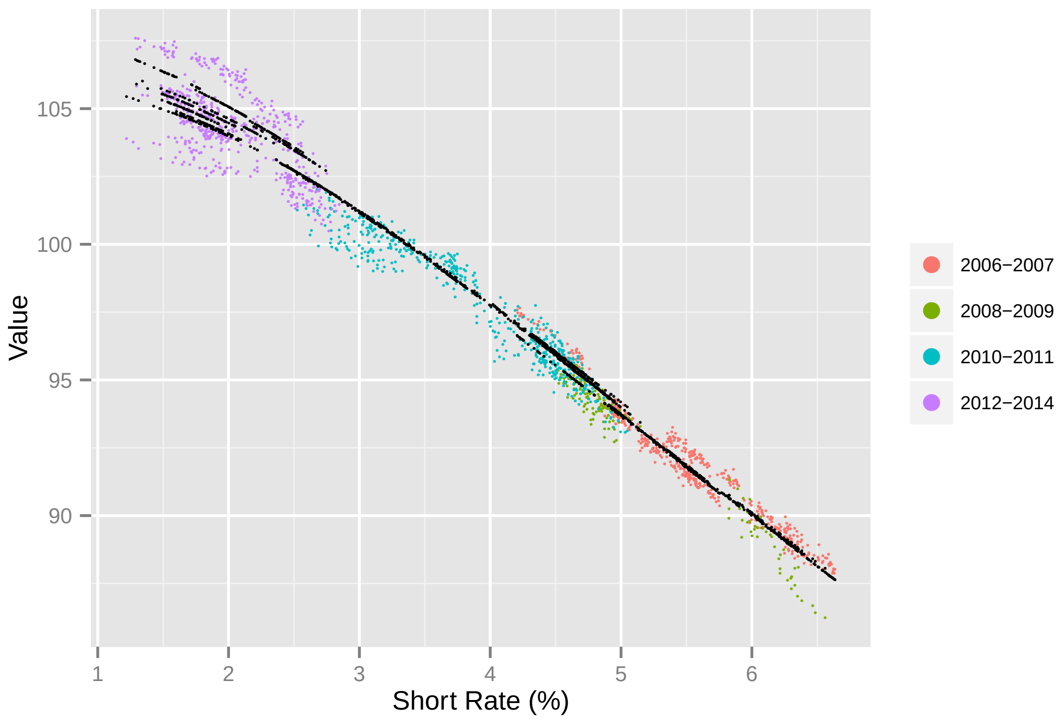


Figure 5.5: The Stanton model prices together with observed market prices plotted against the short rate for the bond `NDA 4 1 01OCT2035`. The model captures both interest-rate incentives and burnout effects, which results in a very good fit.

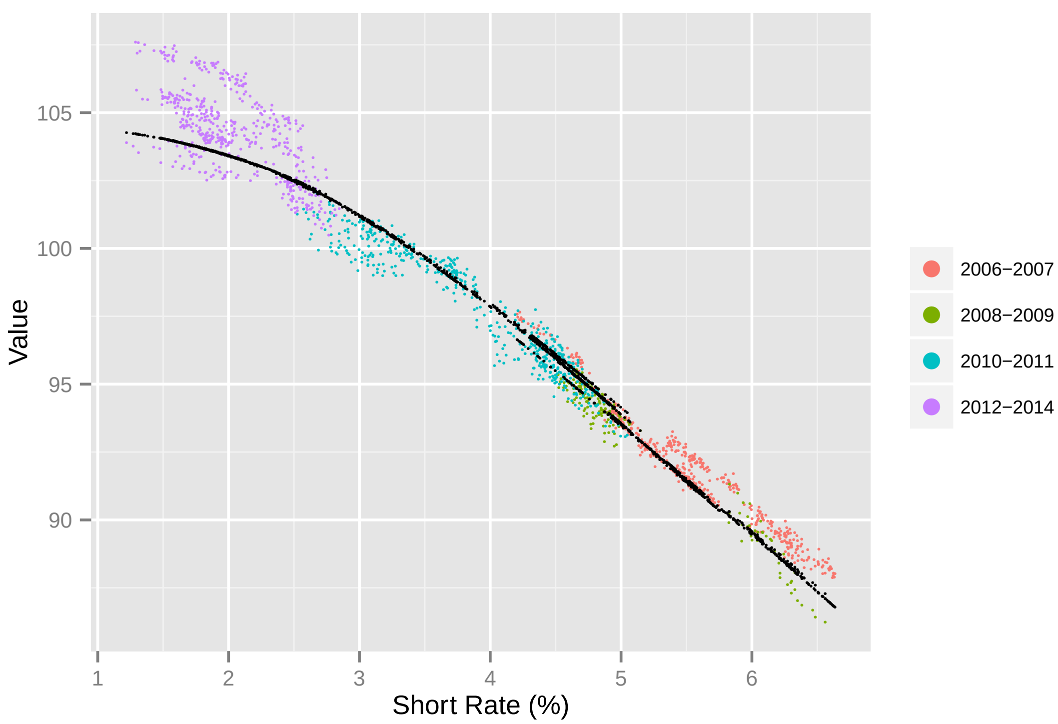


Figure 5.6: DBPM Basic (with a fixed value of s) prices together with observed market prices plotted against the short rate for the bond `NDA 4 1 01OCT2035`. The model captures interest-rate incentives. However it is not capable of capturing other effects, which results in a poor fit from late 2012 and forward.

It is important to note that in order to match the level of observed prices we have added an individual spread to each of the twenty bonds. This is done by using the default intensity framework presented in [Lando (1998)]. More specifically, we have adjusted the short rate process by adding a constant default intensity. The added intensities lie in the range from 0.1 percent to 1 percent. In figure 5.7 and 5.8 we have plotted model prices together with observed market prices against the market short rate for all twenty bonds. Figure 5.7 corresponds to the Stanton model and figure 5.8 corresponds to DBPM Basic. The general picture is equivalent to what we saw in figure 5.3 - 5.6.

Recall that the pricing of callable mortgage bonds involves two layers of complexity: (1) modelling the dynamic behaviour of market interest rates, and (2) modelling the prepayment behaviour of mortgagors. The latter has the most significant effect on resulting prices in low rate areas, since this is where most prepayment activity occurs. In high rate areas the callable bond behaves similar to a non-callable bond, hence the term structure modelling plays the largest role for the resulting prices in high rate areas.

Keeping the above in mind the model fits illustrated in figure 5.7 and 5.8 were obtained in the following way: In DBPM Basic the parameter s is the only free parameter. Hence we fitted the model prices by choosing a value of s and a spread for each bond. In the Stanton model the modelling of prepayment behaviour is determined by the already estimated values of the prepayment parameters, cf. table 3.1. We simply chose spreads in order for the model prices to lie in the same area as observed prices. The chosen s values and spreads for the twenty pool can be seen in table 5.2.

ISIN	Name	s	DBPM Spread	Stanton Spread
DK0009361461	BRF 4 B 01APR2038	15%	0.3%	0.6%
DK0009358830	BRF 4 B 01OCT2035	15%	0.3%	0.6%
DK0009356545	BRF 5 B 01APR2035	18%	0.1%	0.4%
DK0009360570	BRF 5 B 01JUL2038	18%	0.3%	0.5%
DK0009366858	BRF 5 E 01JUL2041	25%	0.6%	1%
DK0009366429	BRF 5 E 01OCT2041	25%	0.5%	0.8%
DK0002015023	NDA 4 1 01JAN2038	15%	0.3%	0.7%
DK0002012780	NDA 4 1 01OCT2035	15%	0.3%	0.6%
DK0002014216	NDA 5 1 01APR2038	18%	0.3%	0.6%
DK0002011386	NDA 5 1 01JUL2035	18%	0.1%	0.4%
DK0004715505	NYK 4 C 01JUL2035	15%	0.1%	0.6%
DK0009761645	NYK 4 D 01JAN2038	15%	0.3%	0.6%
DK0009757296	NYK 4 D 01OCT2035	15%	0.3%	0.6%
DK0004714458	NYK 5 C 01APR2035	18%	0.1%	0.4%
DK0009753469	NYK 5 D 01JUL2035	18%	0.1%	0.4%
DK0009274300	RD 4 01JAN2038	15%	0.3%	0.7%
DK0009270233	RD 4 01OCT2035	15%	0.3%	0.6%
DK0009272874	RD 5 01JAN2038	18%	0.3%	0.5%
DK0009269227	RD 5 01JUL2035	18%	0.1%	0.4%
DK0009280380	RD 5 S 01OCT2041	18%	0.5%	0.7%

Table 5.2: Table showing the values of s and spreads chosen in DBPM and the Stanton model in order to fit observed market prices for twenty Danish mortgage bonds.

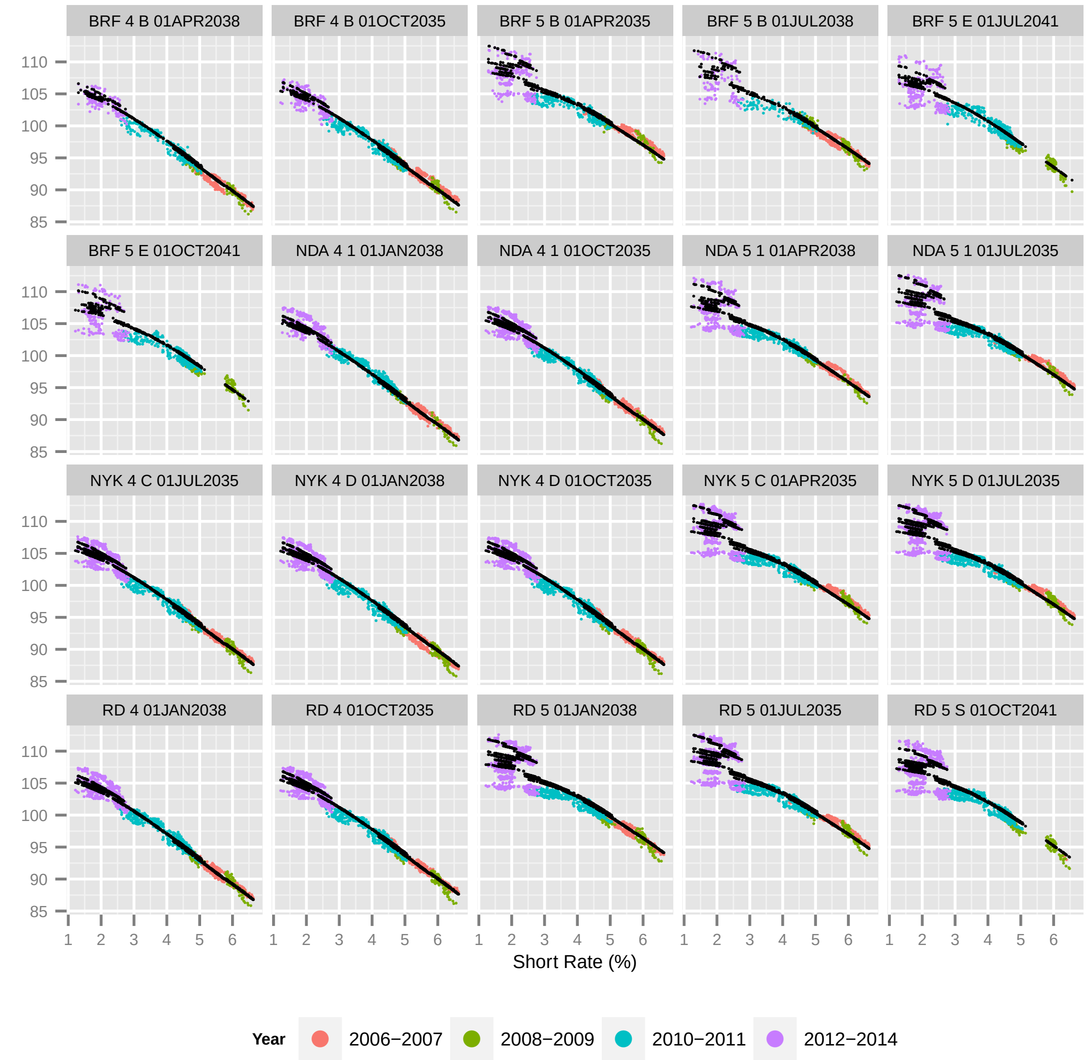


Figure 5.7: The Stanton model prices together with observed market prices plotted against the short rate for twenty Danish mortgage bonds

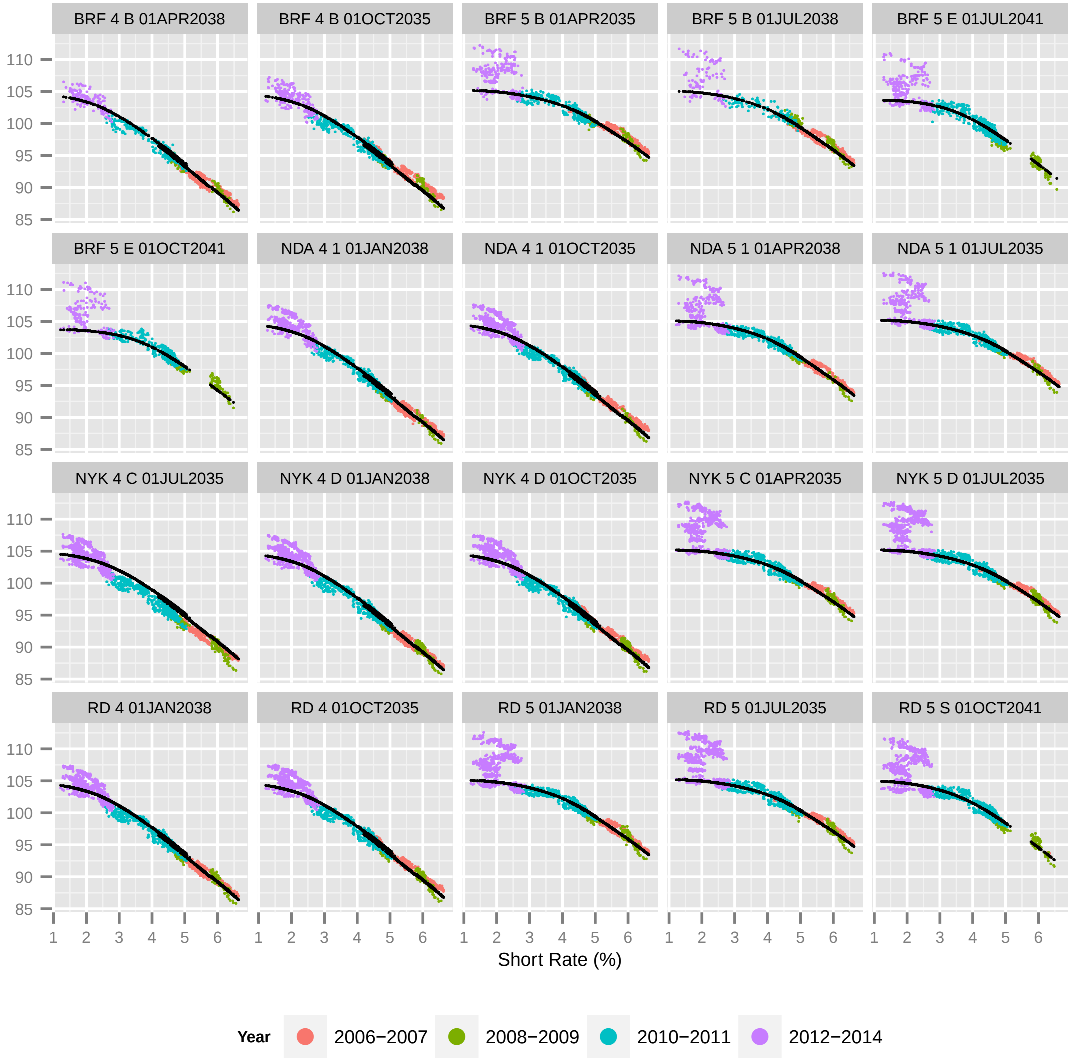


Figure 5.8: DBPM prices together with observed market prices plotted against the short rate for twenty Danish mortgage bonds

5.3 Explanatory Power

By explanatory power we think of how informative the models are – for example, what can be concluded from certain parameter values.

The Danske Bank Prepayment Model

Let us first have a look at DBPM. Recall that the prepayment function is given by

$$p(r,V) = 1 - e^{-\alpha(r)\varphi(V-1)dt},$$

where

$$\alpha(r) = \alpha_0 \left(1 + A \left[\max(r_1 - r, 0)\right]^p\right)$$

and

$$A = \left(p\frac{q - r_0}{r_1 - r_0} - 1\right)^{-1} (r_1 - r_0)^{-p}$$

with $\alpha_0 = -\frac{\log(1-s)}{dt}$. φ is defined as the uniform distribution function with parameters $a = 0$ and $b = \frac{q}{\alpha_0}$. The parameters, set by Danske Bank, for r_0, k and p is equal to $\frac{q}{4}$, 2 and 2, respectively.

$\alpha(r)\varphi(V-1)dt$ is the intensity of prepayments. The greater it is the more prepayment is expected. α is a function of the short rate, where the intensity increases with $\alpha_0 A(r_1 - r)^p$ whenever the short rate is below $r_1 = kr_0 = q/2$. Recall that $r_0 = q/4$ is where the callable bond value is supposed to peak. It is also assumed that the prepayments are linear in the bond values above par, i.e. that φ is the uniform distribution function. These considerations does not give much explanatory power, but it is more of a justification of the prepayment function.

The only parameter not chosen 'a priori' by Danske Bank is s. In the first basic setup, it had the interpretation of the maximum prepayments per coupon date. That is not entirely true any longer though. By now it is somewhat unclear what the implied s denotes. When calibrating the model to market quotes, then the implied s contains all sorts of information. The model does not tell us the reasoning for the current state in the market. We can only say that the model expects many prepayments for a high s and low amount of prepayments if the implied s is relatively small. The models explanatory power is weak.

The Stanton Model

In the Stanton model things are different. The model contains four parameters α, β, ρ and λ. They all suggests information of different aspects about the mortgagors in the initial pool. Furthermore the model is based on the decision making by a single mortgagor, given a level of transaction costs. In Danske

Bank the prepayment proportion are modelled directly by some function that seems to give reasonable empirical results.

The parameters α and β contains information about the initial distribution of transaction cost levels in the pool. If the initial distribution, given by α and β, is right-skewed then one could argue that this implies that the mortgages in the pool are relatively small. The reason being mortgagors are less likely to prepay if the loan size is small.[7] Likewise a left-skewed transaction cost distribution could be interpreted as the mortgagors having large loan sizes.

The explanatory power of ρ and λ are straight forward and already well described, cf. chapter 3. We remind the reader that ρ is a measure of how often the mortgagors consider prepaying. I.e. the average time between the mortgagors decision dates are given by $1/\rho$, which is quite sensible from an explanatory point of view. λ governs the probability of prepaying for exogenous reasons. As mentioned earlier it could be sensible to set λ equal to zero in Denmark, due to the Danish legislation.[8] Hence an estimated value of $\lambda > 0$ indicates that there are some factors in the market which are not captured by the model.

All in all it should be clear that the explanatory power of the Stanton model is superior compared to DBPM. However, one should remember that the explanatory 'conclusions' derived by the Stanton model are only valid to some extent. Nevertheless, it does give an indication of how the prepayments are affected by different aspects of the mortgagors in the pools.

[7]Due to the fact that the transaction costs associated with prepaying are large relative to the loan.

[8]If a mortgagor needs to prepay and the value is below par, then the mortgagor will buy the corresponding bond in the market instead of actually prepaying.

5.4 Rate Sensitivity Performance

In this section we investigate the perfomance of DBPM and the Stanton model regarding rate sensitivity by back testing. More explicitly we see how sensitive the models are to changes in the short rate and compare this to actual changes in the market.

Imagine that you are a Risk Manager and that your company has large investments in mortgage bonds. The value of these investments changes with the level of market interest rates. This might interfere with your desired risk preferences and thereby implies a need for hedging in order to maintain a certain risk profile. Recall that the price of a callable mortgage bond is, in general, a concave function of the underlying interest rate for sufficiently small values of the short rate. The negative convexity of the bond implies that small changes in the underlying rate can effect the duration of the bond substantially. The concave shape thereby creates a need for dynamic hedging - oppose to a static hedge, which would work if the dependence between the bond and underlying rate were of a more linear character. The interest rate risk of a mortgage bond may be hedged taking a short position in the fixed leg of an interest rate swap and hence receiving the floating leg. For example if the market interest rate level increases the value of the mortgage bond would decrease, while the value of the floating leg would increase.

In this thesis we do not perform any specific hedging experiments. However, from the above it is clear that the hedging of interest rate risk calls for a model which attempts to match the market's rate sensitivity. The rate sensitivity analysis carried out below focuses on data from 2012 and forward. The reason for this is that the short level in general is lower in this time period. Hence this is were the modelling of prepayment behaviour has the most significant effect on bond prices. In figure 5.9 the market quotes for eight of the twenty Danish mortgage bonds are shown - simply to get a grasp of some of the data, which will be used in the analysis.

Analysis procedure

We begin by performing the analysis for DBPM (Implied). Assume that we are standing at a specific day and that we wish to derive a rate sensitivity curve for a specific bond (i.e bond value as a function of the short rate). Given a set of bond specifications (maturity, coupon rate, etc..), the first step is to determine the implied s based on the current short rate level and the current market value of the bond. Next we can derive bond values in a set of grid points $(\tilde{T}_i, r_k)$ by the finite difference method outlined in section A.5 in the appendix. At each time $\tilde{T}_i$ we spline between the spatial grid points in order to derive a full curve.

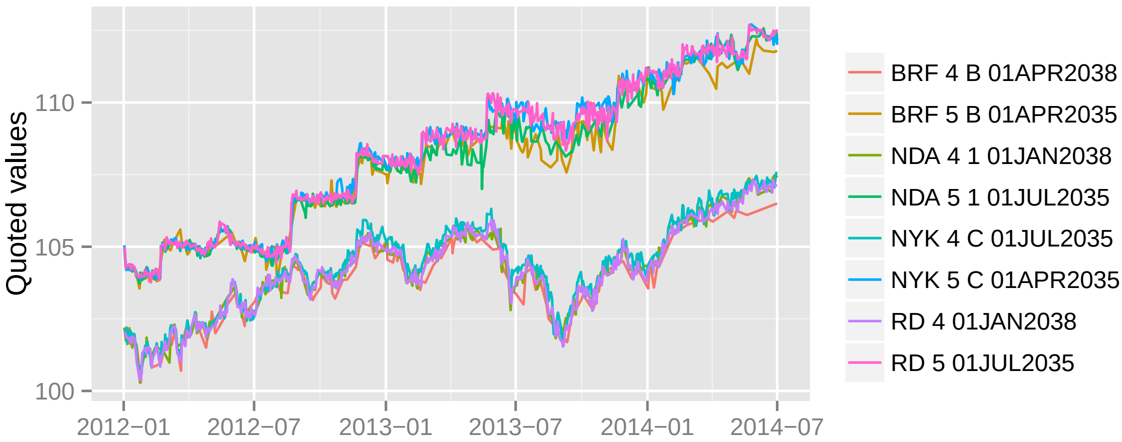


Figure 5.9: Market quotes during the time period where the analysis is being carried out. Note that the prices fall into two groups. The top ones are 5% bonds maturing in 2035 and the bottom one are 4% bonds maturing in 2038

Note that each value of s and each set of bond specifications forms a different rate sensitivity curve. Hence the shape of the curve changes each day based on the implied s. Figure 5.10 shows the curve as it looked on 2012-01-02 for `NDA 5 1 01JUL2035`.

As a Risk Manager you might perform a hedge based on the current rate sensitivity curve. For that reason it is important to investigate the following questions; do future quotes actually develop as the curve predicts? And if so, for how long is the current curve then representable for the bond value development? We have performed back testing in order to answer these questions. For each bond and each day we have derived a rate sensitivity curve. We have then compared model predictions to observed market quotes for different time horizons. More specifically we have compared the model predictions with observations one day later, one week later, one month later, etc.. In order to obtain model predictions one simply has to retrieve the values on the rate sensitivity curve corresponding to the 'future' short rate levels.

Besides the rate sensitivity curve, figure 5.10 also shows the 'future' market quotes and short rate levels. If the model was perfect, then the points would be located on the curve[9]. We investigate the prediction errors, i.e. the differences between the curve and the future observed quotes. These errors have been derived for all the bonds on all possible days after 2012-01-01.

Results

[9]The bond values derived by the model depend on the time difference as well. However, for the time horizons in figure 5.10 the changes, due to change in time, are almost zero. Therefore the picture on figure 5.10 is still valid. For the final back testing analysis we have taken the time differences into account though.

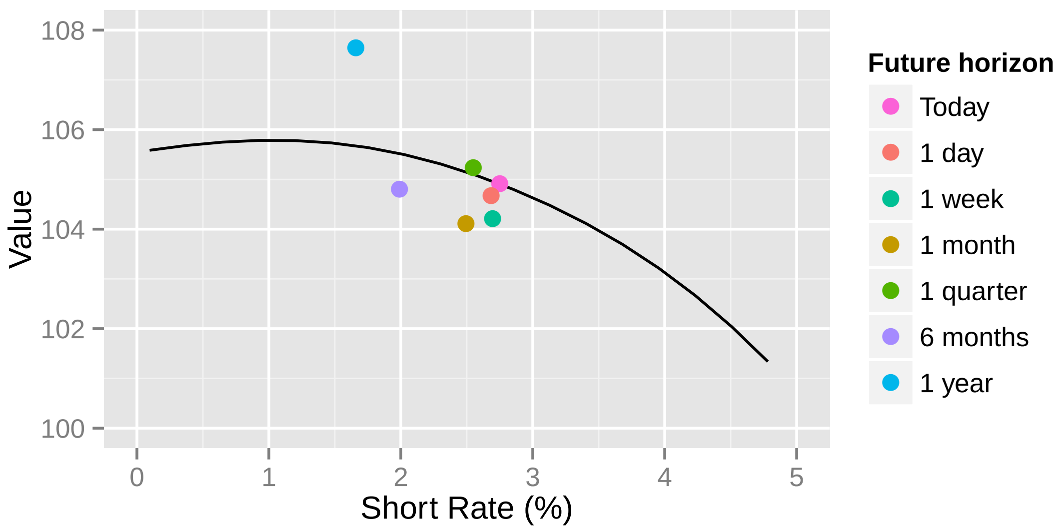


Figure 5.10: Rate sensitivity curve produced by DBPM, as it would have looked on 2012-01-02. The curve is based on `NDA 5 1 01JUL2035`. The short rate level was 2.75%, while the market quote was 104.9. This lead to an implied s equal to 15.9%. The points are the future market quotes and short rate levels.

In figure 5.11 the predicted errors are compared to the changes in the short rate. More specifically, we define the prediction errors as the change in quoted values minus the change in model values. The error is plotted against the change in the short rate level (future level minus 'current' level).
Figure 5.11 shows the resulting plots stratified by the future time horizons. Looking at the top three plots it is difficult to spot any systematic tendencies in the errors. They all seem randomly distributed around 0 for all changes in the short rate. The picture changes in the lower three plots though. Especially when the time horizon is at least 6 months. Here the predicted bond values are too low, since almost all the errors are positive. This might indicate that the model needs to be recalibrated. The red lines indicate the average errors.

Comparison to the Stanton model

We have derived a similar analysis using the Stanton model. This model is not calibrated to the daily observed quotes. Hence we do not expect the model to predict the correct value tomorrow if the value today is off. But we are still able to compare the changes in the quoted values to the changes in the model values.

The problem with DBPM is that it does not capture the jumps in market quotes over time. The Stanton model does however imply jumps in the bond value over time. These occur on the coupon dates due to changes in the distribution of the transaction costs. For illustrative reasons we have performed the rate sensitivity analysis for the Stanton model where we have ignored the future changes in the transaction cost distribution and only taken the 'current' distribution into account. The result is in shown by the

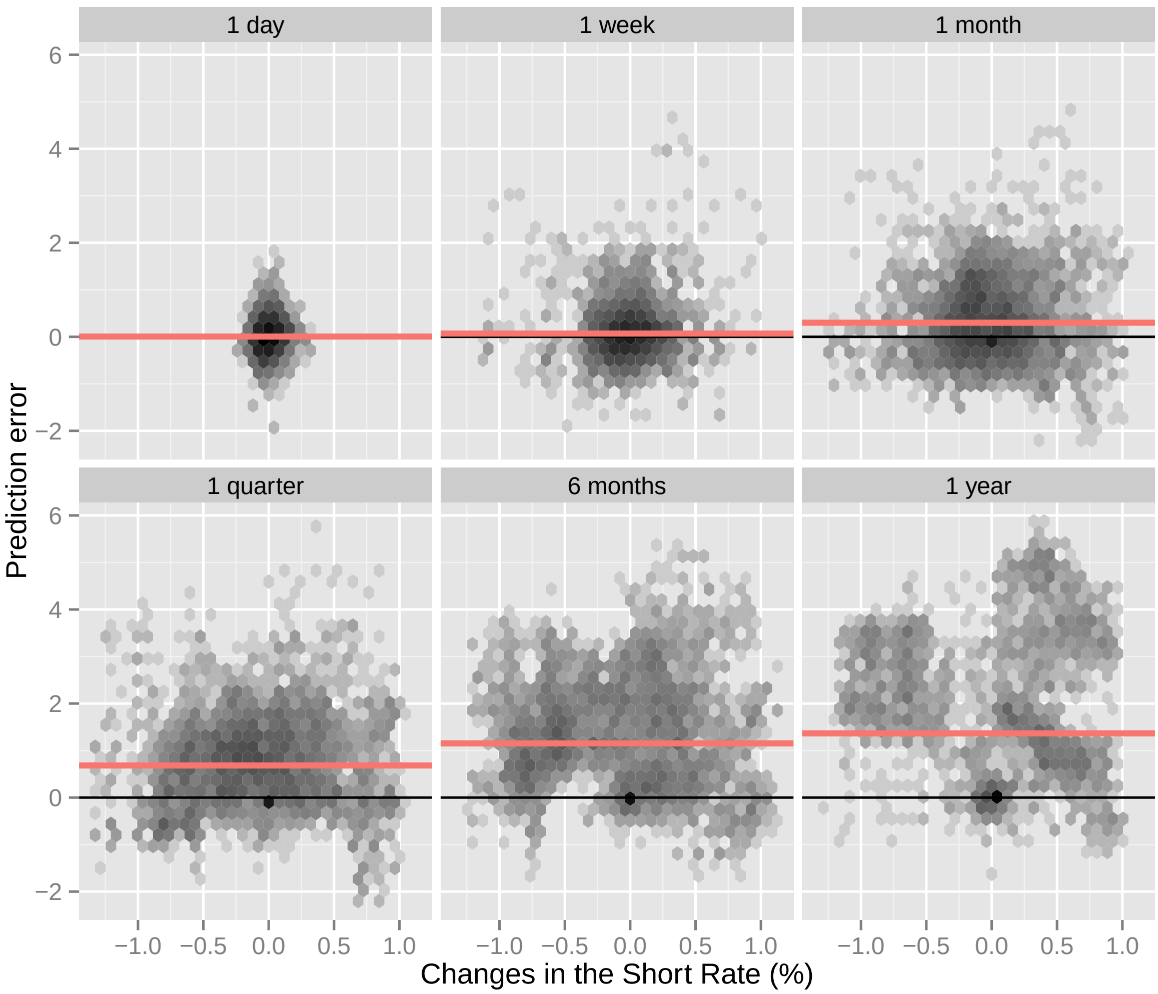


Figure 5.11: Prediction errors (the change in quoted values minus the change in the model values) for DBPM plotted against changes in the short rate levels. The darkness of the hexagons illustrates the number of points in them. The red lines indicate the average errors.

second column in figure 5.12. We see that ignoring the changes in the distribution of the transaction costs lead to a result similar to DBPM. However, when asked for future quotes in the Stanton model, then one should account for the future changes in the transaction cost distribution. The results of the rate sensitivity analysis performed on the Stanton model, taking these changes into account, is found in the third column in figure 5.12. From figure 5.12 it is clear that the prediction errors in general are smaller for the correct implemented Stanton model compared to DBPM.

However before jumping to any conclusion we have to admit, that we have 'cheated' a bit in the third column of figure 5.12. The transaction cost distribution in the Stanton model have been updated based on the 'observed' short rate values. But since the rate sensitivity experiment is based on predicting the unknown future we are of course not allowed to use 'future' 'observed' short rate values. In order to perform a fair test of the Stanton model one therefore should use the future expected short rate values. But then the performance of the model is of course very dependent on the quality of the short rate model. Therefore such an analysis is left out. However, given a 'good' short rate model, we expect the Stanton model to perform better than DBPM if using expected short rate values for updating the transaction cost distribution.

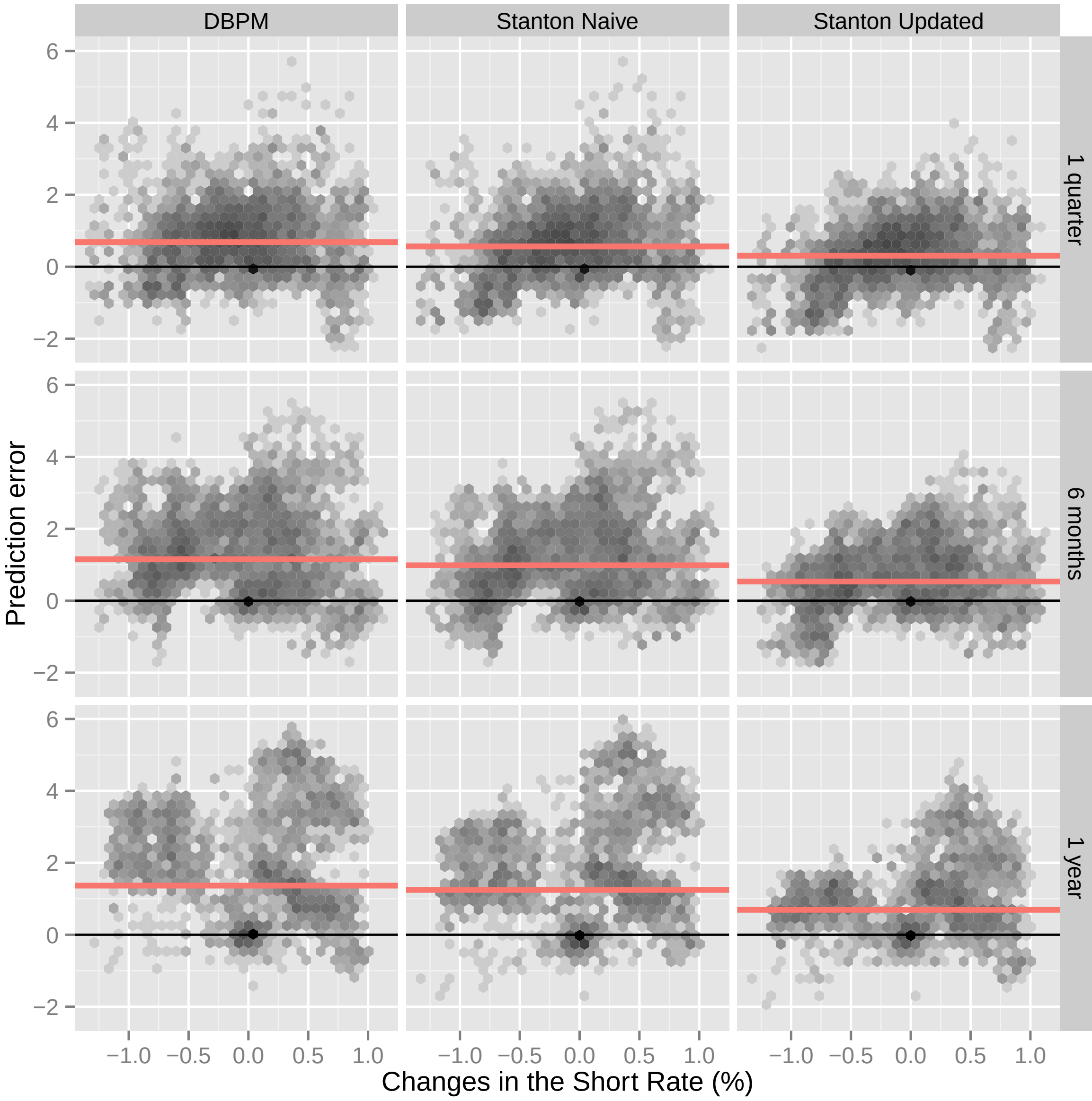


Figure 5.12: The prediction error (the change in quoted values minus the change in the model values derived by the Stanton model) plotted against changes in the short rate levels. The plots to the left are derived by DBPM. The plots in the middle are derived by the Stanton model without taking future changes in the transaction cost into account. The plots to the right are derived by the Stanton model, where the future weights have been used. The rows indicates different time horizons. The red lines indicate the average errors.

Conclusion and Ideas

This part of the thesis serves both as a conclusion and as a platform for ideas on how one might possibly extend the Danske Bank Prepayment model and the Stanton model.

We begin by patting ourselves on the shoulder; 1) we managed to fitted a tractable short rate model under which we could fit market quotes using the presented prepayment models. 2) We were able to implement both the Danske Bank Prepayment Model (in its basic form with a constant value of the parameter s and in its implied version, where s is calibrated to market prices) and the Stanton model in a finite difference setup[10]. 3) A comparison of the models were carried out in order to investigate various aspects of the models. 4) As icing on the cake, we presented a model, purely based on theoretical considerations, that is capable of creating a good fit to observed market prices given an observed interest rate path – not only for one bond, but for all twenty bonds in our data set, cf. figure 5.7. It should be clear by now that we favour the Stanton model. But before discussing possible extensions of the Stanton model, we take a second to discuss what aspects of DBPM that could be interesting to investigate further.

The comparison carried out in chapter 5 showed that the Stanton model in general was preferable to DBPM both from a 'fitting observations point of view' and from a 'conceptual modelling point of view'. One of the main problems of the calibrated version of DBPM is that it lacks explanatory power due to the fact that all market information is contained in the single variable s, cf. section 5.3. An obvious extension to DBPM would therefore be to model the parameter s as a function of some variable opposed to being constant. An analysis could be carried out based on plots of implied s values as a function of time or as a function of the short rate. However when looking at plots for the implied values of s it is not immediately clear how the modelling of s should be approached.

We now turn our focus to the Stanton model. The good fit of the Stanton model might be explained from the fact the parameters were estimated in order to minimize the distance between observed and model prepayment rates for the twenty bonds in question. Furthermore one might criticize the fact that

[10] We even implemented the valuation of a bond using a 3-dimensional grid through the Cheyette-model.

all twenty bonds have coupon rates of either 4% or 5%. Hence, it would be reasonable to ask whether or not the model also would fit observed prices on bonds with different coupon rates? For that reason we plotted the Stanton model prices together with observed prices on twenty two 30-year bonds with coupon rates of 6% and 7% using the same parameter values as earlier. The result was similar to that of figure 5.7.

We end the thesis by suggesting two possible extensions to the Stanton model: 1) intuitively it does not seem fair that the length between a mortgagors decisions points are constant over time. We expect that mortgagors reevaluate their mortgage situation more often, when the market interest rate level is low (relative to the coupon rate) than when it is high. Therefore an obvious extension would be to model the parameter ρ as a function of the short rate[11]. 2) Information on the loan sizes in mortgage pools are available in the market.[12] And it does not take long to conclude that the probability of prepayment for a single mortgagor is positively correlated to the loan size. We therefore suggest that the parameters α and β are chosen based on the information on the loan sizes.[13] For example if a pool consisted mainly of small loans then α and β should be chosen such that imply a right-skewed cost distribution, cf. section 5.3.

[11] One could for example be inspired by the function $\alpha(r)$ in DBPM.
[12] More precisely, the mortgagors are divided into three groups depending on their loan size.
[13] Recall that the parameters α and β governs the initial transaction cost distribution of a pool.

Appendix A

Appendix

A.1 Unique and Markovian Solution to a Stochastic Differential Equation

The concept of a stochastic differential equation (SDE) is a universal brick in the modelling aspect of financial mathematics and we will therefore briefly touch upon what kind of assumptions one may impose when dealing with these.

Let W be a d-dimensional Wiener process and consider the stochastic differential equation

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) \tag{A.1}$$

where $\mu : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, \infty] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$.

We wish to introduce assumptions on μ and σ, which ensure that for each $x \in \mathbb{R}^n$ there is a unique and Markovian process X satisfying (A.1) with initial condition x. In order to do so we need to define the concepts of a *Lipschitz condition* and a *growth condition.* We say that μ and σ satisfy a Lipschitz condition and a growth condition in x if there exits a constant k, such that for $t \geq 0$ and any $x, y \in \mathbb{R}^n$

Lipschitz condition:

$$\begin{aligned} \| \mu(t,x) - \mu(t,y) \| &\leq k \| x - y \| \\ \| \sigma(t,x) - \sigma(t,y) \| &\leq k \| x - y \| \end{aligned} \tag{A.2}$$

Growth condition:

$$\begin{aligned} \| \mu(t,x) \|^2 &\leq k \left(1+ \| x \|^2\right) \\ \| \sigma(t,x) \|^2 &\leq k \left(1+ \| x \|^2\right) \end{aligned} \tag{A.3}$$

where for any matrix, A, the norm is defined by $\| A \|= tr\left(AA^T\right)^{1/2}$ with tr denoting the trace. This norm is equal to the standard Euclidean norm, when A is vector. We are now ready to state the following proposition

Proposition A.1. *Assume that μ and σ are measurable and satisfy* (A.2) *and* (A.3)*. Then for each $x \in \mathbb{R}^n$ there exits a unique process X satisfying* (A.1)*, which is $\mathcal{F}_t$-adapted, Markovian and has continuous trajectories with initial condition x. Furthermore there exits a constant c such that for $t \in [0,T]$*

$$E\left(\| X(t) \|^2\right) \leq ce^{ct}\left(1+ \| x \|^2\right).$$

For a proof of proposition A.1 see [Karatzas and Shreve (1991)]. Note that proposition A.1 does not cover the well-known case of the Cox-Ingersoll-Ross (CIR) process, since here the diffusion term becomes $\sigma(t,X(t)) = \sigma\sqrt{X(t)}$, which does not satisfy the Lipschitz condition (globally). This is easily seen, since

$$|\sigma\sqrt{x} - \sigma\sqrt{0}| \leq k|x-0| \Leftrightarrow \frac{1}{\sqrt{x}} \leq \frac{k}{\sigma}$$

where the last inequality is obviously not satisfied because $\frac{1}{\sqrt{x}}$ can be made arbitrarily large. However in [Karatzas and Shreve (1991)] a result for 'square root' diffusions like the CIR process is presented in one dimension. For a further discussion on the subject see [Karatzas and Shreve (1991)].

A.2 The Vasicek Model

We present and prove results on the conditional distribution for the short rate in the Vasicek model under the measure Q. We explain how to compute MLEs for the parameters in a Vasicek model.

A.2.1 The Distribution of the Short Rate

Lemma A.2. *Let $h(t)$ be a given deterministic function of time. Fix $s < t$ and define the process X by*

$$X(t) = \int_s^t h(v)dW(v).$$

Then the conditional distribution of X is given by

$$X(t)|\mathcal{F}_s \sim N\left(0, \int_s^t h^2(v)\,dv\right) \tag{A.4}$$

Proof. The goal is to prove that the conditional characteristic function of X is then given by

$$E\left[e^{ikX(t)}|\mathcal{F}_s\right] = e^{-\frac{k^2}{2}\int_s^t h^2(v)\mathrm{d}v}, \quad k \in \mathbb{R},$$

since this is equivalent to proving (A.4).

Consider the process $Y(t) = e^{ikX(t)}$. By the use of Ito's formula the dynamics of Y are given by

$$\begin{aligned} dY(t) &= ikY(t)dX(t) + \frac{(ik)^2}{2}Y(t)(dX(t))^2 \\ &= ikY(t)h(t)dW^P(t) - \frac{k^2}{2}Y(t)h^2(t)dt. \end{aligned}$$

And since $Y(s) = 1$ we may write Y on integral form as

$$Y(t) = 1 - \frac{k^2}{2}\int_s^t Y(v)h^2(v)\mathrm{d}v + ik\int_s^t Y(v)h(v)\mathrm{d}W^P(v).$$

Taking conditional expectations on both sides and using Fubini's Theorem yields that

$$m(t) = 1 - \frac{k^2}{2}\int_s^t h^2(v)m(s)\mathrm{d}v + ikE\left[\int_s^t Y(v)h(v)\mathrm{d}W^P(v)|\mathcal{F}_s\right]. \tag{A.5}$$

where $m(t) = E[Y(t)|\mathcal{F}_s]$. If the process Y lies in $\mathcal{L}^2[s,t]$ [1] then the last term equals zero, cf. proposition 4.7 in [Björk (2003)]. In order for $Y \in \mathcal{L}^2[s,t]$ we need to show

$$\int_s^t E\left[Y(v)^2\right]\mathrm{d}v = \int_s^t E\left[e^{2ikX(v)}\right]\mathrm{d}v < \infty. \tag{A.6}$$

That (A.6) is true is easily seen, since $E\left[e^{2ikX(v)}\right] = e^2E\left[e^{ikX(v)}\right]$, which is obviously bounded because the characteristic function $E\left[e^{ikX(v)}\right]$ is bounded by one. Hence the last term in (A.5) vanishes.

By differentiating on both sides of equation (A.5) we see that

$$\begin{aligned} m'(t) &= -\frac{k^2}{2}h^2(t)m(t) \\ m(s) &= 1. \end{aligned}$$

[1] The space $\mathcal{L}^2[s,t]$ is defined in [Björk (2003)] see definition 4.3.

And the solution to this standard differential equation is given by

$$m(t) = E[Y(t)|\mathcal{F}_s] = e^{-\frac{k^2}{2}\int_s^t h^2(v)\mathrm{d}v}.$$

□

Proposition A.3. *Let u be a positive constant and assume that $r(t)$ is given by* (2.3)*, then*

$$r(t+u)|r(t) \sim N\left(e^{-\kappa u}r(t) + \theta\left(1 - e^{-\kappa u}\right), \frac{\sigma^2\left(1 - e^{-2\kappa u}\right)}{2\kappa}\right)$$

Proof. Consider the process $Z(t) = e^{\kappa t}r(t)$. By the use of Ito's formula the dynamics of Z are given by

$$\begin{aligned} dZ(t) &= \kappa Z(t)dt + e^{\kappa t}dr(t) \\ &= \left(\kappa Z(t) + e^{\kappa t}\kappa(\theta - r(t))\right)dt + e^{\kappa t}\sigma dW(t) \\ &= \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}dW(t). \end{aligned}$$

On integral form we may therefore write

$$Z(t+u) = Z(t) + \kappa\theta\int_t^{t+u} e^{\kappa s}ds + \sigma\int_t^{t+u} e^{\kappa s}dW(s).$$

And since $r(t+u) = e^{-\kappa(t+u)}Z(t+u)$ we have that

$$\begin{aligned} r(t+u) &= e^{-\kappa u}r(t) + \kappa\theta e^{-\kappa(t+u)}\int_t^{t+u} e^{\kappa s}ds + \sigma e^{-\kappa(t+u)}\int_t^{t+u} e^{\kappa s}dW(s) \\ &= e^{-\kappa u}r(t) + \theta\left(1 - e^{-\kappa u}\right) + \sigma e^{-\kappa(t+u)}\int_t^{t+u} e^{\kappa s}dW(s). \end{aligned}$$

By applying lemma A.2 on the last term it is straight forward to see that

$$r(t+u)|r(t) \sim N\left(e^{-\kappa u}r(t) + \theta\left(1 - e^{-\kappa u}\right), \frac{\sigma^2\left(1 - e^{-2\kappa u}\right)}{2\kappa}\right).$$

□

A.2.2 Maximum Likelihood Estimation

Imagine that we have observations of the short rate at the discrete time points $t_0 < t_1 < \ldots < t_n$, and denote the observed short rate at time t_i by $\tilde{r}_{t_i}$ for $i = 0, \ldots, n$. The aim of this subsection is to present how one may obtain Maximum likelihood estimators (MLEs) of the parameters, $\alpha = (\kappa, \theta^P, \sigma)$, in the Vasicek model under P based on observations $\tilde{r}_{t_0}, \ldots, \tilde{r}_{t_n}$.

Formally we need the concept of a parameterized statistical model, which simply consists of a measurable space $(\Omega, \mathcal{F})$, a flow of information generated by a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a family of probability measures $\{P_\alpha : \alpha \in \Theta\}$ defined on the space $(\Omega, \mathcal{F})$, where Θ is a parameter set. The basic idea is that one of the parameters $\alpha \in \Theta$ is true, i.e it governs the true mechanism behind movements in the short rate.

Let $p_n(r(t_1), \ldots, r(t_n)|r(t_0), \kappa, \theta^P, \sigma)$ denote the joined conditional density function for $(r(t_1), \ldots, r(t_n))$. By the use of the Markov property and Bayes theorem for conditional densities it follows that

$$\begin{aligned} p_n(r(t_1), \ldots, r(t_n)|r(t_0), \kappa, \theta^P, \sigma) =& p_{t_n|t_{n-1}}(r(t_n)|r(t_0), r(t_1), \ldots, r(t_{n-1}), \kappa, \theta^P, \sigma) \cdot \\ & p_{t_{n-1}}(r(t_1), \ldots, r(t_{n-1})|r(t_0), \kappa, \theta^P, \sigma) \\ =& \prod_{i=1}^{n} p_{t_i|t_{i-1}}(r_i|r(t_0), \ldots, r(t_{i-1}), \kappa, \theta^P, \sigma) \\ =& \prod_{i=1}^{n} p_{t_i|t_{i-1}}(r(t_i)|r(t_{i-1}), \kappa, \theta^P, \sigma), \end{aligned}$$

where $p_{t_i|t_{i-1}}$ denotes the density of $r(t_i)|r(t_0), \ldots, r(t_{i-1})$ for $i = 1, \ldots, n$. From proposition A.3 we know that $p_{t_i|t_{i-1}}(r(t_i)|r(t_{i-1}), \kappa, \theta^P, \sigma)$ equals the density of a normal distribution with mean and variance given by proposition A.3. The log-likelihood function is therefore given by

$$l(\kappa, \theta^P, \sigma|\tilde{r}_{t_0}, \ldots, \tilde{r}_{t_n}) = -\log \prod_{i=1}^{n} p_{t_i|t_{i-1}}(\tilde{r}_{t_i}|\tilde{r}_{t_{i-1}}, \kappa, \theta^P, \sigma).$$

MLEs for κ, θ^P and σ may now be found by numerical optimization of the log-likelihood.

Under the assumption that observations are observed at equidistant time points it is possible to obtain closed-form expressions for the MLEs in the Vasicek model. Such closed-form formulas may be found in [Tang and Chen (2009)].

A.3 The Heath-Jarrow-Morton Framework

In this section we will present the general setup of the HJM framework, which forms the basis for the Cheyette model. First let the instantanous forward rate $f(t, T)$ be given by

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW^P(t) \tag{A.7}$$

where $(\alpha(t, T))_{t\geq 0}$ and $(\sigma(t, T))_{t\geq 0}$ are adapted processes, satisfying regularity conditions given by (2.2-5) in [Beyna (2013)]. Note that conceptually equation (A.7) is one SDE for each fixed T.

By definition of the forward rate we have that

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}$$

where $P(t,T)$ is the time-t value of a ZCB with maturity T. The short rate is given by

$$r(t) = f(t,t).$$

Hence we implicitly define the dynamics of the bond market by specifying the forward rate curves.

The following theorem shows the main result in the HJM setup under the martingale measure. The result states that the dynamics of the forward rate can be expressed in terms of the volatility only.

Theorem A.4. *Assume that the family of forward rates is given by* (A.7) *and that the induced bond market is complete and free of arbitrage. Then*

$$df(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)dsdt + \sigma(t,T)dW(t)$$

where W is a Wiener process under the risk neutral measure.

Proof. Let $f(t,T)$ be given by (A.7) and assume a complete market with no arbitrage possibilities. This can be translated to the existence of a unique martingale measure, Q. Let the measure Q be given by

$$\frac{dQ}{dP} = e^{-\int_0^T \lambda(s)dW(t) - \frac{1}{2}\int_0^T \lambda(t)^2 dt}$$

on $\mathcal{F}_T$ and assume $E^Q\left[e^{\int_0^T \lambda(s)dW(t) - \frac{1}{2}\int_0^T \lambda(t)^2 dt}\right] = 1$. By the Girsanov Theorem it follows that

$$dW^P(t) = -\lambda(t)dt + dW(t)$$

where $W(t)$ is a Q-Wiener process. Hence

$$df(t,T) = (\alpha(t,T) + \sigma(t,T)\lambda(t)dt + \sigma(t,T)dW(t) \tag{A.8}$$

Now by proposition 20.5 (3) in [Björk (2003)] the ZCB dynamics are given by

$$dP(t) = p(t,T)\left(r(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2\right)dt - p(t,T)\left(\int_t^T \sigma(t,s)ds\right)dW^P(t)$$

and using result 15.6.1 (no arbitrage) in [Björk (2003)] gives us that

$$-\int_t^T \alpha(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 = -\lambda(t)\int_t^T \sigma(t,s)ds \quad P-a.s.,$$

Differentiating with respect to T yields that

$$\alpha(t,T) = \sigma(t,T)\left(\int_t^T \sigma(t,s)ds + \lambda(t)\right) \tag{A.9}$$

Inserting A.9 into (A.8) yields the wanted result. □

On integral form Theorem A.4 states that

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T)\int_s^T \sigma(s,u)duds + \int_0^t \sigma(s,T)dW(s)$$

A.4 The Cheyette Model

Essentially the Cheyette model is a specification of the volatility structure of the instantaneous forward rate that allows for Markovian representation of the dynamics of the short rate. If the forward rate is driven by a single Brownian motion, then the Cheyette model will in general result in a model which is Markovian in two state variables, one being locally deterministic.

For the one-factor HJM framework, it is assumed that the volatility is given by

$$\sigma(t,T) = g(T)h(t) \tag{A.10}$$

where g is a time-dependent positive function and h is a non-negative process. Theorem A.4 "simplifies" to

$$\begin{aligned}
f(t,T) &= f(0,T) + \int_0^t g(T)h(s)\int_s^T g(u)h(s)\,du\,ds + \int_0^t g(T)h(s)dW(s) \\
&= f(0,T) + g(T)\int_0^t h(s)^2\int_s^T g(u)\,du\,ds + g(T)\int_0^t h(s)dW(s) \\
&= f(0,T) + \frac{g(T)}{g(t)}\left(X(t) + \frac{Y(t)}{g(t)}\int_t^T g(s)ds\right)
\end{aligned} \tag{A.11}$$

where X and Y are defined by

$$X(t) = g(t)\left(\int_0^t h(s)^2 \int_s^t g(u)du ds + \int_0^t h(s)dW(s)\right)$$
$$Y(t) = g(t)^2 \int_0^t h(s)^2 ds$$

By Leibniz's rule[2] and Itô it follows that the dynamics are given by

$$\begin{aligned} dX(t) &= \frac{g'(t)}{g(t)}X(t)dt + g(t)d\left(\int_0^t h(s)^2 \int_s^t g(u)duds\right) + g(t)h(t)dW(t) \\ &= \frac{g'(t)}{g(t)}X(t)dt + g(t)\left(\int_0^t d\left(h(s)^2 \int_s^t g(u)du\right) ds + h(t)^2 \int_t^t g(u)du\right) + g(t)h(t)dW(t) \\ &= \frac{g'(t)}{g(t)}X(t)dt + g(t)^2 \int_0^t h(s)^2 ds + g(t)h(t)dW(t) \\ dY(t) &= \left(h(t)^2 g(t)^2 + 2\frac{g'(t)}{g(t)}Y(t)\right) dt \end{aligned}$$

$$\begin{aligned} dX(t) &= \left(\frac{g'(t)}{g(t)}X(t) + Y(t)\right) dt + g(t)h(t)dW(t) \\ dY(t) &= \left(h(t)^2 g(t)^2 + 2\frac{g'(t)}{g(t)}Y(t)\right) dt \end{aligned}$$

Now define the process $\eta(t) = g(t)h(t)$ and the function $\kappa(t) = -\frac{g'(t)}{g(t)}$ and note that

$$dX(t) = (-\kappa(t)X(t) + Y(t))\, dt, + \eta(t)dW(t) \qquad X(0) = 0 \tag{A.12}$$

$$dY(t) = \left(\eta(t)^2 - 2\kappa(t)Y(t)\right) dt, \qquad Y(0) = 0 \tag{A.13}$$

I.e. if $\eta(t) = m(t, X(t), Y(t))$ for some function $m : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^+$, then $(X(t), Y(t))$ is Markovian. Hence the evolution of the instantaneous forward rate curve is Markovian, since any point on the curve can be described by (A.11). Specifically we have that

$$V(t) = F(t, x, y) = E_{t,x,y}\left[e^{-\int_t^T r(s)ds}\psi(X(T), Y(T))\right]$$

leads to the pricing PDE

$$\begin{aligned} \frac{\partial}{\partial t}F(t, x, y) - r(t)F(t, x, y) + (-\kappa(t)x + y)\frac{\partial}{\partial x}F(t, x, y) + \frac{1}{2}\eta(t)^2\frac{\partial^2}{\partial x^2}F(t, x, y) \\ + \left(\eta(t)^2 - 2\kappa(t)y\right)\frac{\partial}{\partial y}F(t, x, y) = 0 \\ F(T, x, y) = \psi(x, y) \end{aligned} \tag{A.14}$$

[2] $\frac{d}{dt}\int_{a(t)}^{b(t)} f(s,t)ds = \int_{a(t)}^{b(t)} \frac{d}{dt}f(s,t)ds + f(b(t),t)b'(t) - f(a(t),t)a'(t)$

where

$$r(t) = f(t,t) = f(0,t) + X(t)$$

The BVP (A.14) can be solved using a finite difference scheme. Such a scheme is outlined in A.5.2. More precisely we use 'The Mitchell scheme', which can be considered the pendant to Crank-Nicolson, but in a two-dimensional PDE.

Vasicek Incorporated in the Cheyette Model

Assume the short rate follows a Vasicek model and recall that

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}$$

After some calculations, one end up with the forward curve

$$f(t,T) = \theta + e^{-\kappa(T-t)}(r(t) - \theta) - \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)^2$$

By Itô's formula

$$df(t,T) = m(t)dt + \sigma e^{-\kappa(T-t)} dW(t)$$

where we do not care about m. Defining the forward rate by

$$df(t,T) = m(t)dt + \sigma(t,T)dW(t)$$

with $\sigma(t,T) = \sigma e^{-\kappa(T-t)}$, then one has implicitly assumed a Vasicek model for the short rate. Note that

$$\sigma(t,T) = \sigma e^{-\kappa(T-t)} = g(T)h(t)$$

with $g(T) = e^{-\kappa T}$ and $h(t) = \sigma e^{\kappa t}$. Hence Vasicek model is contained within in the Cheyette model. Furthermore (A.12) and (A.13) are then given by

$$\begin{aligned} dX(t) &= (Y(t) - \kappa X(t))\, dt + \sigma dW(t) \\ dY(t) &= \left(\sigma^2 - 2\kappa Y(t)\right) dt. \end{aligned}$$

In chapter 2 we use the above to compare the solution of a non-callable mortgage bond using the Cheyette model with the closed form solution under Vasicek - see figure 2.4.

A.5 Finite Difference Methods

The aim of this section is to introduce the concept of finite difference methods and to construct finite difference schemes which can be used to solve partial differential equations. In short we seek for a trustworthy and time efficient PDE solver. The basic idea of finite difference methods is to approximate the partial derivatives with difference quotients also called difference operators. There exists many variants of the methods. In the following we outline the exact methods used in this thesis.

A.5.1 The θ-method

Consider the PDE on $[0,T]\times\mathbb{R}$

$$\begin{aligned}\frac{\partial F}{\partial t}(t,x)+\mathcal{A}F(t,x)+g(t,x)=0\\ F(T,x)=\psi(x),\end{aligned}\tag{A.15}$$

where $g:[0,T]\times\mathbb{R}$ is known and $\mathcal{A}$ is an operator given by

$$\mathcal{A}F(t,x)=\mu(t,x)\frac{\partial F}{\partial x}(t,x)+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x)-r(t,x)F(t,x).$$

The basic idea of finite difference methods is to approximate the partial derivatives in equation (A.15) with difference quotients also known as difference operators. In order to compute difference operators, such as the forward, backward and central differences, which are defined in Theorem A.5, we need to discretize the t-direction and x-direction of (A.15), also known as time and space directions, respectively. We discretize the continuous variables $(t,x)\in[0,T]\times\mathbb{R}$ by introducing a two-dimensional grid of discrete points (t_i,x_k) for $i=0,1,\dots,N_t$ and $k=0,1,\dots,N_x$, where $t_i=i\Delta t$ and $x_{j+1}=x_j+\Delta x$, with Δt and Δx denoting the step size in each direction.

First note that the $F(t_{N_t},x_k)=F(T,x_k)=\psi(x_k)$ is known. The idea is to approximate $F(t_i,x_k)$ in all grid points by computing our way backwards in time through the grid. More specifically, at each time step we shall approximate the true solution $F(t_i,x_k)$ for every value in the space direction, given the approximations at the time step before $F(t_{i+1},x_0),\dots,F(t_{i+1},x_{N_x})$. Theorem A.5 explains which difference operators one may use in order to obtain a consistent scheme - by consistent we mean that the chosen difference operators converge towards the derivatives when the grid size goes to zero.

Theorem A.5. *Assume that $[a,b]$ is a compact set. Let $f\in C^4[a,b]$ and $x-h,x,x+h\in[a,b]$. Then*

$$f'(x)\simeq\frac{f(x+h)-f(x-h)}{2h}\equiv\delta_h[f]'(x)\tag{A.16}$$

and $\delta_h[f]'(x) - f'(x) = \mathcal{O}(h^2)$*. Here* $\delta_h[f]$ *is called the central difference of* f*. Furthermore*

$$f'(x) \simeq \frac{f(x+h) - f(x)}{h} \equiv \triangle_h[f](x), \tag{A.17}$$

$$f'(x) \simeq \frac{f(x) - f(x-h)}{h} \equiv \triangledown_h[f](x) \tag{A.18}$$

where $\triangle_h[f]'(x) - f'(x) = \mathcal{O}(h)$ *and* $\triangledown_h[f]'(x) - f'(x) = \mathcal{O}(h)$*. These approximations are known as the forward and backward differences of* f*. At last we have that*

$$f^{(2)}(x) \simeq \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \equiv \delta_h[f]''(x) \tag{A.19}$$

with $\delta_h[f]''(x) - f''(x) = \mathcal{O}(h^2)$*.*

Proof. We will only prove (A.16), since the proofs of (A.17)-(A.19) are very similar as explained in the end of this proof. From the theory of Taylor expansions we know that the second-order Taylor expansion around x for $f(x+h)$ and $f(x-h)$ can be written as

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f^{(2)}(x)}{2!} + \frac{h^3 f^{(3)}(c_1)}{3!} \tag{A.20}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2 f^{(2)}(x)}{2!} - \frac{h^3 f^{(3)}(c_2)}{3!}. \tag{A.21}$$

where $c_1 \in (x, x+h)$ and $c_2 \in (x-h, x)$. Subtracting equation (A.21) from equation (A.20) yields

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{2}\frac{h^2\left(f^{(3)}(c_1) + f^{(3)}(c_2)\right)}{3!}$$

and since $f^{(3)}$ is continuous on $[a, b]$ the extreme value theorem states that $f^{(3)}$ is bounded on $[a, b]$ and we may therefore write

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \mathcal{O}(h^2).$$

The proof of (A.19) is done in a similar way by adding equation (A.20) and (A.21) and then solving for $f^{(2)}(x)$, while the proofs of (A.17) and (A.18) are done by doing a first-order Taylor expansion around x for $f(x+h)$ and $f(x-h)$ and then solving for $f'(x)$ in each equation.

$\square$

We now proceed to explain the concept of finite difference methods in detail. To begin with we consider the space direction and replace the differential operator $\mathcal{A}$ in (A.15) with the associated difference operator

$$\bar{\mathcal{A}} = \mu\delta_x + \frac{1}{2}\sigma^2\delta_{xx} - r$$

where δ_x and δ_{xx} are difference operators given by either the forward, backward or central differences from theorem A.5. The choice of using either the forward, backward or central difference will lead to different finite difference schemes. Below we present the schemes used in this thesis.

The goal for the finite difference scheme is an equation like $AF(t_i, \mathbf{x}) = BF(t_{i+1}, \mathbf{x})$, where A and B are $(N_x + 1) \times (N_x + 1)$ matrices and

$$F(t, \mathbf{x}) = \begin{pmatrix} F(t, x_0) \\ \vdots \\ F(t, x_{N_x}) \end{pmatrix}$$

Due to the discretization it is convenient to define the rowvectors

$$\delta_x(k) = \underbrace{\frac{1}{2\Delta x}(0, \ldots, 0, -1, 0, 1, 0, \ldots, 0)}_{\text{non-zero in the } (k-1)\text{'th and } (k+1)\text{'th element}} \tag{A.22}$$

$$\delta_{xx}(k) = \underbrace{\frac{1}{\Delta x^2}(0, \ldots, 0, 1, -2, 1, 0, \ldots, 0)}_{\text{non-zero in the } (k-1)\text{'th, } k\text{'th and } (k+1)\text{'th element}} \tag{A.23}$$

for $1 \le k \le N_x - 1$. For $k = 0$ and $k = N_x$ we define the corresponding rowcevtors by

$$\delta_x(0) = \frac{1}{\Delta x}(-1, 1, 0, \ldots, 0) \tag{A.24}$$

$$\delta_x(N_x) = \frac{1}{\Delta x}(0, \ldots, 0, -1, 1) \tag{A.25}$$

$$\delta_{xx}(0) = \delta_{xx}(N_x) = (0, \ldots, 0) \tag{A.26}$$

The above use of vector notation allows us to interpret $\bar{\mathcal{A}}$ as a tridiagonal matrix, since the discretization leads to

$$\bar{\mathcal{A}}F(t, \mathbf{x}) = \underbrace{\begin{pmatrix} \mu(t, x_0)\delta_x(0) + \frac{1}{2}\sigma^2(t, x_0)\delta_{xx}(0) - r(t, x_0) \\ \vdots \\ \mu(t, x_{N_x})\delta_x(N_x) + \frac{1}{2}\sigma^2(t, x_{N_x})\delta_{xx}(N_x) - r(t, x_{N_x}) \end{pmatrix}}_{(N_x+1)\times(N_x+1)} F(t, \mathbf{x})$$

The attentive reader might have noticed that the central differences have been used in (A.22) and (A.23). This is not possible on the boundaries of the spatial grid due to the fact that $F(t, x_0 - \Delta x)$ and $F(t, x_{N_x} + \Delta x)$ are unknown for all t. Instead we assume $F(t, x)$ to be linear in the x direction for $x = x_0$ and $x = x_{N_x}$. Ie. we let $\delta_{xx} = 0$ and use the forward and backward differences for the approximation of the first derivative. Hence we define the rowvectors given by (A.24), (A.25) and (A.26).

In some cases this assumption might cause problems. But in the case where $F(t,x)$ denotes the time t value of a bond and where x denotes the short rate, then it seems reasonable. We simply assume the bond value to be linear in the short rate, whenever the short rate are very low or high.

So far we have dealt with all the derivatives in the x direction and we are left with the time derivative. We approximate this by

$$\frac{\partial F(t_i,\mathbf{x})}{\partial t} \approx \frac{1}{\Delta t}(F(t_{i+1},\mathbf{x}) - F(t_i,\mathbf{x})) \tag{A.27}$$

for $i \in \{0,\ldots,N_t-1\}$. Let $\theta \in [0,1]$ and consider the last approximation by

$$\bar{\mathcal{A}}F(t_i,\mathbf{x}) \approx \theta\bar{\mathcal{A}}F(t_i,\mathbf{x}) + (1-\theta)\bar{\mathcal{A}}F(t_{i+1},\mathbf{x}) \tag{A.28}$$

for $i \in \{0,\ldots,N_t-1\}$. Substituting $\mathcal{A}$ with $\bar{\mathcal{A}}$, replacing the time derivative with (A.27) and using (A.28) then the PDE (A.15) becomes

$$\frac{1}{\Delta t}(F(t_{i+1},\mathbf{x}) - F(t_i,\mathbf{x})) + \theta\bar{\mathcal{A}}F(t_i,\mathbf{x}) + (1-\theta)\bar{\mathcal{A}}F(t_{i+1},\mathbf{x}) + g(t_i,\mathbf{x}) \approx 0$$

where $g(t,\mathbf{x}) = (g(t,x_0),\ldots,g(t,x_{N_x}))^T$. Rearranging terms and we derive at

$$\left(\frac{1}{\Delta t}I_{N_x+1} - \theta\bar{\mathcal{A}}\right)F(t_i,\mathbf{x}) = \left(\frac{1}{\Delta t}I_{N_x+1} + (1-\theta)\bar{\mathcal{A}}\right)F(t_{i+1},\mathbf{x}) + g(t_i,\mathbf{x}) \tag{A.29}$$

where I_n denotes the $n \times n$-identity matrix. Notice that (A.29) forms a tridiagonal matrix system, hence the system can be efficient solved by 'Thomas algorithm' instead of the slower matrix inversion method. The method gives the solution to the PDE in all the grid points (t_i,x_k) and particularly the values at time 0. This is done step by step backwards from the terminal boundary condition $F(T,\mathbf{x}) = (\psi(x_0),\ldots,\psi(x_{N_x})^T$. Therefore the $F(t_{i+1},\mathbf{x})$ is considered known and $F(t_i,\mathbf{x})$ as the unknown in (A.29). The finite difference outlined here is often referred to as the θ-method.

The method is called the explicit method if $\theta = 0$, since the x-derivatives is approximated using the known solution values at t_{i+1}, which is easiest seen by (A.28). We say that a finite difference method is stable if the errors made at one time step do not cause the errors to increase as the computations are continued. The problem is that the explicit method is only stable whenever $0 \leq \Delta t \leq \frac{\Delta x^2}{2}$, cf. (4.9) in [Seydel (2009)].

With $\theta = 1$ the derivatives is approximated using the unknown values at time t_i. There is no simple explicit formula with which the unknown can be obtained one after the other in (A.29). All equations in the system must be considered simultaneously. This is called the implicit method and is unconditionally stable.

Both the explicit and implicit method have accuracy $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$, but this can be improved by the unconditionally stable Crank-Nicolson method, where $\theta = \frac{1}{2}$. This method have accuracy $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$, cf. theorem 4.4 in [Seydel (2009)].

A.5.2 Two-dimensional PDE - The ADI-method

We now consider the two-dimensional PDE on $[0,T] \times \mathbb{R}^2$ by

$$\begin{aligned} \frac{\partial}{\partial t} F(t,x,y) + \mathcal{A}F(t,x,y) + \mathcal{B}F(t,x,y) = 0 \\ F(T,x,y) = \psi(x,y) \end{aligned} \tag{A.30}$$

where $\mathcal{A}$ and $\mathcal{B}$ are operators given by

$$\begin{aligned} \mathcal{A}F(t,x,y) &= \mu(t,x,y)\frac{\partial}{\partial x}F(t,x,y) + \frac{1}{2}\sigma(t,x,y)^2\frac{\partial^2}{\partial x^2}F(t,x,y) - \frac{r(t,x,y)}{2}F(t,x,y) \\ \mathcal{B}F(t,x,y) &= \gamma(t,x,y)\frac{\partial}{\partial y}F(t,x,y) - \frac{r(t,x,y)}{2}F(t,x,y) \end{aligned}$$

The goal is to derive at a numerical method, which is unconditional stable, has second-order accuracy in the time domain and with minimal workload. We start by taking a look at the 'Alternating-Direction-Implicit'-method.

First lets discretize $[0,T] \times \mathbb{R}^2$ by $t_i = i\Delta t$, $x_k = x_0 + k\Delta x$ and $y_j = y_0 + j\Delta y$ for $i = 0,\ldots,N_t$, $k = 0,\ldots,N_x$ and $N_y = 0,\ldots,N_y$. The $\Delta t, \Delta x$ and Δy denotes the equidistant step sizes in the different directions.

The Classic ADI

The main idea behind the ADI method, is that each time step in the algorithm is split in two. Over the first intermediate step we go *explicit* in the y-direction and *implicit* in the x-direction, in the sense explained earlier. In the second intermediate step we go *explicit* in the x-direction and *implicit* in the y-direction.

Let the time derivative be approximated in a similar way as in (A.27). Then the PDE (A.30) becomes

$$\frac{1}{\Delta t}(F(t_{i+1},x,y)-F(t_i,x,y))+\mathcal{A}F(t,x,y)+\mathcal{B}F(t,x,y)=0$$

where $\mathcal{A}F(t,x,y)$ and $\mathcal{B}F(t,x,y)$ is to be evaluated anywhere between t_i and t_{i+1} in the time entry. Hence we can write the PDE as

$$\begin{aligned}0=&\frac{F(t_{i+1},x,y)+F(t_i+{}^{\Delta t}\!/2,x,y)-F(t_i+{}^{\Delta t}\!/2,x,y)-F(t_i,x,y)}{\Delta t}\\&+\mathcal{A}F(t_i+{}^{\Delta t}\!/2,x,y)+\frac{1}{2}\left(\mathcal{B}F(t_{i+1},x,y)+\mathcal{B}F(t_i,x,y)\right)\end{aligned}$$

Rearranging and we have

$$0=\frac{F(t_{i+1},x,y)-F(t_i+{}^{\Delta t}\!/2,x,y)}{\Delta t}+\frac{1}{2}\Big(\mathcal{A}F(t_i+{}^{\Delta t}\!/2,x,y)+\mathcal{B}F(t_{i+1},x,y)\Big) \qquad (A.31)$$

$$+\frac{F(t_i+{}^{\Delta t}\!/2,x,y)-F(t_i,x,y)}{\Delta t}+\frac{1}{2}\Big(\mathcal{A}F(t_i+{}^{\Delta t}\!/2,x,y)+\mathcal{B}F(t_i,x,y)\Big) \qquad (A.32)$$

Now the PDE is satisfied if both (A.31) and (A.32) equals zero, which leads to

$$\begin{aligned}\left(\frac{2}{\Delta t}-\mathcal{A}\right)F(t_i+{}^{\Delta t}\!/2,x,y)&=\left(\frac{2}{\Delta t}+\mathcal{B}\right)F(t_{i+1},x,y)\\\left(\frac{2}{\Delta t}-\mathcal{B}\right)F(t_i,x,y)&=\left(\frac{2}{\Delta t}+\mathcal{A}\right)F(t_i+{}^{\Delta t}\!/2,x,y)\end{aligned} \qquad (A.33)$$

Replacing the differential operator $\mathcal{A}$ with the associated difference operator

$$\bar{\mathcal{A}}=\mu\delta_x+\frac{1}{2}\sigma^2\delta_{xx}-\frac{r}{2}$$

where δ_x and δ_{xx} are given by (A.22),(A.23), (A.24) and (A.25) and with $\delta_{xx}(0)=\delta_{xx}(N_x)=0$. Replace also the differential operator $\mathcal{B}$ with the associated difference operator

$$\bar{\mathcal{B}}=\gamma\delta_y-\frac{r}{2}$$

We have used the five point stencil[3] to approximate the derivative in the y-direction, so that

$$\delta_y(j)=\frac{1}{12\Delta y}(0,\dots,0,-1,8,0,-8,1,0,\dots,0)$$

for $2\le j\le N_y-2$, which is non-zero in the $(j-2)$'th, $(j-1)$'th, $(j+1)$'th and $(j+2)$'th entry. For

[3] yet another difference operator. This one uses 5 points to approximate the derivative opposed to using 2 or 3 respectively.

$j = 1, N_y - 1$ the central differences are used so

$$\delta_y(j) = \underbrace{\frac{1}{2\Delta y}(0, \dots, 0, -1, 0, 1, 0, \dots, 0)}_{\text{non-zero in the } (j-1)\text{'th and } (j+1)\text{'th element}}$$

and on the boundaries $j = 0, N_y$ the forward and backward differences are used. Hence

$$\begin{aligned}\delta_y(0) &= \frac{1}{\Delta y}(-1, 1, 0, \dots, 0) \\ \delta_y(N_y) &= \frac{1}{\Delta y}(0, \dots, 0, -1, 1)\end{aligned}$$

We can now interpret $\bar{\mathcal{A}}F(t, \mathbf{x}, y) \in \mathbb{R}^{N_x+1}$ and $\bar{\mathcal{B}}F(t, x, \mathbf{y}) \in \mathbb{R}^{N_x+1}$ in a matrix sense as earlier. By $\bar{\mathcal{B}}F(t, \mathbf{x}, y_j) \in \mathbb{R}^{N_x+1}$ we think of it as

$$\bar{\mathcal{B}}F(t, \mathbf{x}, y_j) = \begin{pmatrix} \gamma(t, x_0)\delta_y(j)F(t, x_0, \mathbf{y}) - \frac{r(t,x_0,y_j)}{2}F(t, x_0, y_j) \\ \vdots \\ \gamma(t, x_{N_x+1})\delta_y(j)F(t, x_{N_x+1}, \mathbf{y}) - \frac{r(t,x_{N_x+1},y_j)}{2}F(t, x_{N_x+1}, y_j) \end{pmatrix}$$

We do admit that it can get a little confusing at this point, where we feel the need for the kind of tricky notations. One has to be very careful with the implementation of the scheme and make sure to use the correct differences in the different grid points. Similarly we think of $\bar{\mathcal{A}}F(t, x_k, \mathbf{y})$ as a vector of length $N_y + 1$. Replacing the operators in (A.33) and the classic ADI scheme is obtained by

$$\left(\frac{2}{\Delta t}I_{N_x+1} - \bar{\mathcal{A}}\right)F(t_i + {}^{\Delta t}/_2, \mathbf{x}, y_j) = \left(\frac{2}{\Delta t}I_{N_x+1} + \bar{\mathcal{B}}\right)F(t_{i+1}, \mathbf{x}, y_j) \tag{A.34}$$

$$\left(\frac{2}{\Delta t}I_{N_y+1} - \bar{\mathcal{B}}\right)F(t_i, x_k, \mathbf{y}) = \left(\frac{2}{\Delta t}I_{N_y+1} + \bar{\mathcal{A}}\right)F(t_i + {}^{\Delta t}/_2, x_k, \mathbf{y}) \tag{A.35}$$

First (A.34) is solved for each $y_0, \dots, y_{N_y}$. Then one have obtained the values $F(t_i + {}^{\Delta t}/_2, x_k, y_j)$ for all $k = 0, \dots, N_x$ and $j = 0, \dots, N_y$. Next (A.35) is solved for each $x_0, \dots, x_{N_x}$ and the values $F(t_i, x_k, y_j)$ are obtained for all $k = 0, \dots, N_x$ and $j = 0, \dots, N_y$. Notice that the first equation forms a tridiagonal system, while the other equation forms a pentadiagonal system. For an efficient solver of the pentadiagonal system, we have used the one outlined in [Karawia (2005)]. As before the process starts at $t = T$ and ends when we reach $t = 0$.

The Mitchell scheme

The Mitchell scheme looks less symmetrical than the classic ADI split, since the two steps now are given by

$$\left(\frac{1}{\Delta t}I_{N_x+1} - \theta_x\bar{\mathcal{A}}\right)U(\mathbf{x}, y_j) = \left(\frac{1}{\Delta t}I_{N_x+1} + (1-\theta_x)\bar{\mathcal{A}} + \bar{\mathcal{B}}\right)F(t_{i+1}, \mathbf{x}, y_j) \tag{A.36}$$

$$\left(\frac{1}{\Delta t}I_{N_y+1} - \theta_y\bar{\mathcal{B}}\right)F(t_i, x_k, \mathbf{y}) = \frac{1}{\Delta t}U(x_k, \mathbf{y}) - \theta_y\bar{\mathcal{B}}F(t_{i+1}, x_k, \mathbf{y}) \tag{A.37}$$

Here we think of $U(\mathbf{x}, y_j) = (U(x_0, y_j), \ldots, U(x_{N_x}, y_j))^T$ and $U(x_k, \mathbf{y}) = (U(x_k, y_0), \ldots, U(x_k, y_{N_y}))^T$. Note that we still perform two steps at each time step as before, however it is not a half time step. But once again we calculate (A.36) for each $y_0, \ldots, y_{N_y}$ and then (A.37) for each $x_0, \ldots, x_{N_x}$.

With $\theta_x = \theta_y = \frac{1}{2}$ we obtain a scheme which is $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta y^2)$ accurate and unconditional stable, cf. page 14 in [Andreasen (2010)].

Bibliography

[Duffie (2001)] *Dynamic Asset Pricing Theory*, book by Darrell Duffie, Princeton University Press, 3rd edition, 2001

[Østerby (2012)] *Numerical Solutions of Parabolic Equations*, notes by Ole Østerby, Aarhus University, 2012

[Seydel (2009)] *Tools for Computational Finance*, book by Rüdiger U. Seydel, Köln University, 4th edition, 2009

[Andreasen (2011)] *Minimal Mortgage*, slides by Jesper Andreasen, Danske Bank, 2011

[Sjögren (2011)] *Mortgage Prepayment Modelling*, slides by Karolina Sjögren, Danske Bank, 2011

[Lando and Poulsen (2006)] *Lecture Notes for Finance 1 (and More)*, notes by David Lando and Rolf Poulsen, Copenhagen University, 2006

[Björk (2003)] *Arbitrage Theory in Continuous Time*, book by Tomas Björk, Oxford University Press, 2nd edition 2003

[Heath, Jarrow and Morton (1992)] *Bond Pricing and the Term Structure of Interest Rates: A New Methodology*, article by David Heath, Robert Jarrow and Andrew Morton, Econometrica Journal, Vol. 60, No. 1 1992

[Beyna (2013)] *Interest Rate Derivatives*, book by Ingo Beyna, Springer, 2013

[Karawia (2005)] *A computational algorithm for solving periodic penta-diagonal linear systems*, paper by A.A. Karawia, Mansoura University, 2005

[Karatzas and Shreve (1991)] *Brownian Motion and Stochastic Calculus*, book by Ioannis Karatzas and Steven E. Shreve, Springer-Verlag New York, 2nd edition 1991

[Lando (1998)] *On Cox Processes and Credit Risky Securities*, paper by David Lando, University of Copenhagen, 1998

[Stanton (1995)] *Rational Prepayment and the Valuation of Mortgage-Backed Securities*, paper by Richard Stanton, Haas School of Business, 1995

[Schwartz and Torous (1989)] *Prepayment and the Valuation of Mortgage-Backed securities*, paper by Eduardo S. Schwartz and Walter N. Torous, The Journal of Finance, 1989

[Kalbfleisch and Prentice (2002)] *The Statistical Analysis of Failure Time Data*, book by John D. Kalbfleisch and Ross L. Prentice, Wiley-Interscience Second Edition 2002

[Hansen (1982)] *The Statistical Analysis of Failure Time Data*, paper by Lars Peter Hansen, Econometrica 1982

[Hansen (1996)] *Finite-Sample Properties of Some Alternative GMM Estimators*, paper by Lars Peter Hansen, John Heaton and Amir Yaron, Journal of Business & Economic Statistics, 1996

[Tang and Chen (2009)] *Parameter estimation and bias correction for diffusion processes*, paper by Cheng Yong Tang and Song Xi Chen, Journal of Econometrics, 2009

[Andreasen (2010)] *Finite Difference Methods for Financial Problems, Part 4: ADI Schemes*, slides by Jesper Andreasen, Kwant Skool, 2010