Vasicek: Abstract to Concrete

Consider an economy where as usual zero-coupon bond (ZCB) prices are denoted by $P(t;T)$, and suppose that these are traded for any conceivable T. Assume that ZCB price dynamics are given by

$$dP(t,T) = \mu^{\mathbb{P}}(t;T)P(t,T)dt + \sigma_P(t;T)P(t,T)dW^{\mathbb{P}}(t),$$

where $W^{\mathbb{P}}$ is a 1-dimensional (largely to keep the notation simple) Brownian motion under some (real-world, actual, physical, objective, statistical, ...) probability measure $\mathbb{P}$. Let r denote the short rate, ie. $r(t) = f(t,t) = (-\partial_T \ln P(t,T))|_{T=t}$, where ∂_T of course denotes differentiation, and the "$|_{T=t}$"-notation, *evaluated at*, is used to point out that we "differentiate first, then put $T = t$'.

From where in Bjørk do we conclude that the short rate dynamics are

$$dr(t) = \text{something } dt - (\partial_T \sigma_P(t;T))|_{T=t} dW^{\mathbb{P}}(t)?$$

From where in Bjørk do we conclude that for this model to be arbitrage-free, there must exist a risk-premium process λ such that

$$\frac{\mu^{\mathbb{P}}(t;T) - r(t)}{\sigma_P(t;T)} = \lambda(t) \quad \text{for all } t \text{ and } T? \tag{1}$$

Suppose now that we consider the Vasicek-model, which we take to involve the following *two* assumptions

- The short rate dynamics (under $\mathbb{P}$) are given by the affine stochastic differential equation (SDE)
$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW^{\mathbb{P}}(t),$$
where κ, θ and σ are positive constants. (Alternatively, r may be referred to as a mean-reverting Gaussian process or an Ornstein/Uhlenbeck process.)

- The risk premium (as defined in (1)) is a deterministic constant, λ.

What sign (+/-) would we reasonably expect λ to have and why?

Define a probability measure $\mathbb{Q}$ by the Radon/Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^U \lambda dW^{\mathbb{P}}(t) - \frac{1}{2}\int_0^U \lambda^2 dt\right) \text{ on } \mathcal{F}_U,$$

where U is some terminal date (that you needn't worry about).

What theorem(s, if you're really picky) tells us that ZCB prices suitably discounted (by what?) are $\mathbb{Q}$-martingales, that $dW^{\mathbb{Q}} = dW^{\mathbb{P}} + \lambda dt$ defines a $\mathbb{Q}$-Brownian motion, and that

$$dr(t) = \kappa\left(\left(\theta - \frac{\sigma\lambda}{\kappa}\right) - r(t)\right)dt + \sigma dW^{\mathbb{Q}}(t)?$$

The important message here is that r is also governed by an affine SDE under the martingale measure $\mathbb{Q}$; the change of measure can simply been subsumed in the new parameter $\widetilde{\theta} = \theta - \sigma\lambda/\kappa$. (This is what Bjørk does in chapter 17.)

In the literature you can statements like the $\mathbb{Q}$-long term mean (or the $\mathbb{Q}$-typical short rate) is $\widetilde{\theta} = \theta + \widetilde{\lambda}$ where $\widetilde{\lambda}$ is the risk-premium". The first part shows that this is in conflict (sign- and scale-wise) with how we typically define risk-premia. But $\widetilde{\lambda}$ has a natural interpretation/nice identification: Up to small terms, $\widetilde{\lambda}$ is the typical/average observed difference between long and short maturity zero coupon yields. Use the ZCB price-formula in Bjørk and your knowledge of affine SDEs to prove this last statement.

For the sake of argument, let's now say that $\theta = 0.04$, $\widetilde{\theta} = 0.06$, $\kappa = 0.5$, and $\sigma = 0.01$. Suppose $r_0 = \widetilde{\theta}$. Then what does the term-structure/yield curve look like?
This we can interpret as the $\mathbb{Q}$-typical-shape of the yield curve, or as the typical shape in a risk-neutral world.
Suppose $r_0 = \theta$. Now what is the shape of the yield curve? With similar reasoning, this can be called the $\mathbb{P}$-typical"-shape of the yield curve.
Comment on the results.

But you shouldn't take our word for the parameter values being "reasonable". You should estimate them. In fact, you must.
First, use your knowledge of affine SDEs to conclude that given the information at time t, $r_{t+\Delta t}$ is Gaussian, specifically

$$r_{t+\Delta t}|r_t \sim N\left(r_t e^{-\kappa\Delta t} + \theta(1 - e^{-\kappa\Delta t}), \frac{\sigma^2(1 - e^{-2\kappa\Delta t})}{2\kappa}\right),$$

where the Markov-property ensures that it does not matter whether we condition on all information or just the time-t interest rate. This means that with Δt-equidistant observations ($n+1$ of them denoted $r_i = r_{i\Delta t}$) the likelihood function is

$$L(\text{data}; \theta, \kappa, \sigma) = \prod_{i=1}^{n} \phi\left(r_i; r_{i-1}e^{-\kappa\Delta t} + \theta(1 - e^{-\kappa\Delta t}), \frac{\sigma^2(1 - e^{-2\kappa\Delta t})}{2\kappa}\right),$$

where

$$\phi(x; \mu, s^2) = \frac{1}{\sqrt{2\pi}s} e^{-\frac{(x-\mu)^2}{2s^2}}$$

is the normal density function.

Estimating parameters by maximizing the likelihood function is a good idea. The best there is, in fact. Trust us. Thus endeth today's lesson in theoretical statistics. Now let's get some numbers out.

On Greg Duffee's homepage

`http://faculty.haas.berkeley.edu/duffee/affine.htm`

or more specifically

`http://faculty.haas.berkeley.edu/duffee/splicedmccul.dat`

you will find US data on ZC yields for various maturities from 1952 to 1998. (We're pretty sure they are quoted on yearly compounded basis; we're absolutely sure they're in %).

For simplicity, suppose the shortest maturity yield (actually 3M) is the instantaneous short rate, r.

Estimate the ($\mathbb{P}$-)parameters in the Vasicek model. Excel should have no problem maximizing the (logarithm of) the likelihood function. And then any piece of software with a numerical optimization routine will work. With some convenient reparametrizations and pen and paper, you can also find closed-form expressions for the estimators. Either way is fine.

How would you estimate the $\mathbb{Q}$-long term mean, $\widetilde{\theta}$, or in other words the λ- (or $\widetilde{\lambda}$-)parameter, or the risk premium? Does plotting the the average (over calender-time, of course) term structure give you any ideas?

Stochastic Volatility

A stochastic volatility model for a stock price, S, could look like this under the measure P:

$$\begin{aligned} dS(t) &= \beta S(t) + \sqrt{V(t)}S(t)dW_1(t), \\ dV(t) &= \xi V(t)dt + \gamma V(t)dW_2(t), \end{aligned}$$

where W_1 and W_2 are independent Brownian motions, and we think of ξ and γ as constants. Assume that we have a riskless money market account with interest rate $r > 0$, and that the stock is traded. However, the volatility is not the price of a traded asset.

1. Assume that a measure Q changes the drift of W_1 and W_2 by the processes λ_1 and λ_2, respectively. Is any choice of λ_1 consistent with Q being an equivalent martingale measure? How about λ_2?

2. Does the choice of λ_2 affect the price of a European call option on the stock?

3. Assume that λ_2 is constant. Show that the $Q-$evolution of the volatility is (also) a Geometric Brownian Motion.

4. Without thinking about closed form solutions, describe carefully how you would price the option based on Monte Carlo simulation