Exercise 2 Sigmoid Function

a)

$$\sigma(z) = \frac{1}{1+e^{-z}} = (1+e^{-z})^{-1}$$

① $$\frac{d}{dz}\left(\frac{1}{1+e^{-z}}\right) = \frac{d}{dz}(1+e^{-z})^{-1} = g'(z)\cdot h'(g)$$

chain rule ② $g(z) = 1+e^{-z}$ $\quad h(g) = g^{-1}$

$g'(z) = -e^{-z}$ $\quad h'(g) = -g^{-2}$

③ $$\Rightarrow \underline{\underline{e^{-z}\,\frac{1}{(1+e^{-z})^2} = \sigma'(z)}}$$

b)

$$\underline{\underline{\frac{e^{-z}}{(1+e^{-z})^2}}} = \sigma'(z) = \sigma(z)\cdot(1-\sigma(z))$$

$$= \frac{1}{1+e^{-z}}\cdot\left(1 - \frac{1}{1+e^{-z}}\right)$$

$$= \frac{1}{1+e^{-z}} - \frac{1}{(1+e^{-z})^2}$$

$$= \frac{(1+e^{-z})-1}{(1+e^{-z})^2} = \underline{\underline{\frac{e^{-z}}{(1+e^{-z})^2}}}$$

c) $$\varsigma(z) = -\log(\sigma(-z))$$

1) ① $$\frac{d}{dz}(-\log(\sigma(-z))) \overset{③}{=} g'(z)\cdot h'(g) = -\frac{e^{z}}{(1+e^{z})^2}\cdot\frac{-1}{\sigma(-z)}$$

chain rule ② $g(z) = \sigma(-z)$ $\quad h(g) = -\log(g)$

$g'(z) = -\sigma'(-z)$ $\quad h'(g) = -\frac{1}{g}$

④ $$\Rightarrow -\frac{e^{z}}{(1+e^{z})^2}\cdot -(1+e^{z}) = \underline{\underline{\frac{e^{z}}{1+e^{z}} = \varsigma'(z)}}$$

2) ① $$\frac{d}{dz}\left(\frac{e^{z}}{1+e^{z}}\right) = \frac{d}{dz}\left(e^{z}(1+e^{z})^{-1}\right)$$

product rule ② $f(z) = g(z)\cdot h(z) \rightarrow f'(z) = g'(z)\cdot h(z) + g(z)\cdot h'(z)$

$g(z) = e^{z}$ $\quad h(z) = (1+e^{z})^{-1}$

$g'(z) = e^{z}$ $\quad h'(z) = -e^{z}(1+e^{z})^{-2}$

③ $$\Rightarrow e^{z}\cdot\frac{1}{1+e^{z}} + e^{z}\cdot -e^{z}\cdot\frac{1}{(1+e^{z})^2} = \frac{e^{z}}{1+e^{z}} - \frac{e^{2z}}{(1+e^{z})^2}$$

$$= \frac{e^{z}(1+e^{z}) - e^{2z}}{(1+e^{z})^2}$$

$$= \frac{e^{z}}{(1+e^{z})^2} = \varsigma''(z)$$

Asymptotes for $z \to \pm\infty$:

- $\varsigma(z) = -\log\left(\frac{1}{1+e^z}\right)$

$$\lim_{z\to\infty} \varsigma(z) = \lim_{z\to\infty} \underbrace{-\log(1)}_{0} + \underbrace{\log(1+e^z)}_{\infty} = \infty$$

$$\lim_{z\to-\infty} \varsigma(z) = \lim_{z\to-\infty} \underbrace{-\log(1)}_{0} + \underbrace{\log(1+e^z)}_{0} = 0$$

- $\varsigma'(z) = \frac{e^z}{1+e^z}$

$$\lim_{z\to\infty} \varsigma'(z) = \lim_{z\to\infty} \frac{e^z}{1+e^z}$$ divide the numerator and denominator by e^z

$$= \lim_{z\to\infty} \frac{1}{\underbrace{e^{-z}}_{0}+1} = 1$$

$$\lim_{z\to-\infty} \varsigma'(z) = \lim_{z\to-\infty} \frac{e^z}{1+e^z} = \lim_{z\to-\infty} e^z \underbrace{(1+e^z)^{-1}}_{1} = 0$$

- $\varsigma''(z) = \frac{e^z}{(1+e^z)^2}$

$$\lim_{z\to\infty} \varsigma''(z) = \lim_{z\to\infty} \frac{e^z}{(e^z)^2} = \lim_{z\to\infty} e^{-z} = 0$$

$$\lim_{z\to-\infty} \varsigma''(z) = \lim_{z\to-\infty} e^z \underbrace{(e^z+1)^{-2}}_{0} = 0$$

f) $$C_1 = (\sigma(x)-1)^2 = \left(\frac{1}{1+e^{-x}} - 1\right)^2 = \left(\frac{1-(1+e^{-x})}{1+e^{-x}}\right)^2$$

$$= \left(\frac{-e^{-x}}{1+e^{-x}}\right)^2$$

1) $$= \frac{e^{-2x}}{(1+e^{-x})^2}$$

① $\underline{C_1'} = \frac{d}{dx} e^{-2x}(e^{-x}+1)^{-2} = -2e^{-2x}\cdot(e^{-x}+1)^{-2} + e^{-2x}\cdot 2e^{-x}(e^{-x}+1)^{-3}$

② product rule $\quad g(x) = e^{-2x} \quad h(x) = (e^{-x}+1)^{-2}$

$\quad g'(x) = -2e^{-2x} \quad h'(x) = 2e^{-x}(e^{-x}+1)^{-3}$

③ $$\frac{-2e^{-2x}(e^{-x}+1) + 2e^{-3x}}{(e^{-x}+1)^3} = \underline{\frac{-2e^{-2x}}{(e^{-x}+1)^3}}$$

2) $c'' = \frac{d}{dx} = \frac{-2e^{-2x}}{(e^{-x}+1)^3} = 4e^{-2x}(e^{-x}+1)^{-3} - 6e^{-3x}(e^{-x}+1)^{-4}$

① product rule $\quad g(x) = -2e^{-2x} \quad h(x) = (e^{-x}+1)^{-3}$

$g'(x) = 4e^{-2x} \quad h'(x) = 3e^{-x}(e^{-x}+1)^{-4}$

③ $\frac{4e^{-2x}(e^{-x}+1) - 6e^{-3x}}{(e^{-x}+1)^4} = \frac{4e^{-2x} - 2e^{-3x}}{(e^{-x}+1)^4} = \frac{2e^{-2x}(2-e^{-x})}{(e^{-x}+1)^4}$

④ $\Rightarrow \frac{2e^{-2x}(2-e^{-x})}{(e^{-x}+1)^4} = 0 \quad | : 2$

$\frac{e^{-2x}(2-e^{-x})}{(e^{-x}+1)^4} = 0$

Subst. $\quad y = e^{-x}$

$\frac{y^2(2-y)}{(y+1)^4} = 0 \quad | \cdot (y+1)^4$

$y^2(2-y) = 0$

$y_{1,2} = 0 \quad y_3 = 2$

$2 = e^{-x} \quad | \log$

$-\log(2) = x$

$\Rightarrow \quad x < 0 \quad \Rightarrow$ non-convex

g) $c_2(x) = -\left(y \cdot \log\left(\frac{1}{1+e^{-wx}}\right) + (1-y)\log\left(1 - \frac{1}{1+e^{-wx}}\right)\right)$

$= -\left(y \cdot (\log(1) - \log(1+e^{-wx})) + (1-y)\log\left(\frac{e^{-wx}}{1+e^{-wx}}\right)\right)$

$= -y\log(1) + y\log(1+e^{-wx}) - (1-y)\log\left(\frac{e^{-wx}}{1+e^{-wx}}\right)$

$\frac{d}{dx} c_2(x) = -\frac{w((y-1)e^{wx}+y)}{e^{wx}+1} = c_2'(x)$

$\frac{d}{dx} c_2'(x) = \frac{w^2 e^{wx}}{(e^{wx}+1)^2} = c_2''(x)$

$\Rightarrow \quad c_2''(x) = 0 \quad \Rightarrow$ no solutions