

1

The Maxwell Equations

1.1

Complex amplitudes

1.1.1

Harmonic (monochromatic) oscillations

High-frequency electrodynamics deals with the electromagnetic field consisting of two vector fields: electric and magnetic. These fields depend on spatial coordinates (i. e., on the vector $\vec{r}(x, y, z)$) and time t . Almost throughout the book only those fields dependent on t by the so-called harmonic law are considered. Each field component $a(\vec{r}, t)$ consists of two summands proportional to $\cos \omega t$ and $\sin \omega t$, respectively:

$$a(\vec{r}, t) = A^c(\vec{r}) \cos \omega t + A^s(\vec{r}) \sin \omega t. \quad (1.1)$$

Fields that have all of their components in the above form are called *harmonic* or *monochromatic* fields.

The parameter ω is called the *circular frequency*. It has the dimensionality s^{-1} and differs from the frequency f , usually used in radioengineering, by the multiplier 2π : $\omega = 2\pi f$, where $f = 1/T$ (T is an oscillation period). In the further formulas a proportional value $k = \omega/c$ is used instead of ω , where c is the light velocity in vacuum, $c = 3 \times 10^8 \text{ m/s}$. The value k has the dimensionality cm^{-1} and is called the *wave number*. However, we call k itself the frequency, omitting the multiplier $2\pi/c$, by which k differs from f : $k = 2\pi/c \cdot f$. The *wavelength* $\lambda = 2\pi/k$ often participates in formulas. The frequency f is inversely proportional to λ : $f = c/\lambda$.

Monochromatic fields are of interest because the majority of radiating devices create fields with the time dependence close to (1.1). It is explained by resonant nature of radiating devices used in radioengineering. Harmonic oscillations are also of interest because almost every process is described by the time functions which can be expressed as a sum or integral of functions of the type (1.1), that is, as superposition of harmonic oscillations.

1.1.2

Complex amplitudes

There is a mathematical technique which permits us to exclude explicit time occurrence while describing harmonic oscillations. This technique implies that a complex function (*complex amplitude*) introduced as

$$A(\vec{r}) = A^c(\vec{r}) - iA^s(\vec{r}), \quad i = \sqrt{-1}, \quad (1.2)$$

is considered instead of the function $a(\vec{r}, t)$ (1.1), containing two functions of the coordinates $A^c(\vec{r})$ and $A^s(\vec{r})$. Then the function $a(\vec{r}, t)$ is expressed as

$$a(\vec{r}, t) = \operatorname{Re} [A(\vec{r}) e^{i\omega t}]. \quad (1.3)$$

This equality can be easily verified using the *Euler formula* $\exp(i\omega t) = \cos \omega t + i \sin \omega t$. The complex function $A(\vec{r})$ can be presented in the form

$$A(\vec{r}) = |A(\vec{r})| \exp [-i\alpha(\vec{r})], \quad (1.4)$$

where

$$\begin{aligned} |A(\vec{r})| &= \sqrt{[A^c(\vec{r})]^2 + [A^s(\vec{r})]^2}, \\ \cos \alpha(\vec{r}) &= \frac{A^c(\vec{r})}{|A(\vec{r})|}, \quad \sin \alpha(\vec{r}) = \frac{A^s(\vec{r})}{|A(\vec{r})|}. \end{aligned} \quad (1.5)$$

The function $a(\vec{r}, t)$ can now be expressed as

$$a(\vec{r}, t) = |A(\vec{r})| \cos [\omega t - \alpha(\vec{r})]. \quad (1.6)$$

The physical meaning of the modulus $|A(\vec{r})|$ of the complex amplitude $A(\vec{r})$ is that it equals the maximal value of $a(\vec{r}, t)$ as a function of time. In fact, the phase $\alpha(\vec{r})$ is not interesting; only the difference of the phases of two harmonic processes is essential. If one of the processes has the complex amplitude A and the other has the amplitude $B = |B| \exp(-i\beta)$, then they are synchronous if $\alpha - \beta = 0$; the first process is ahead of the second one in time if $\alpha - \beta > 0$ and lags behind it if $\alpha - \beta < 0$.

All linear operations on the fields can be performed on the complex amplitudes $A(\vec{r})$ without considering the physical amplitudes $a(\vec{r}, t)$, simply by replacing $a(\vec{r}, t)$ by $A(\vec{r}, t)$. This is possible because of the operation of extracting the real part of a complex number, which connects $a(\vec{r}, t)$ with $A(\vec{r}, t)$ by (1.3), is additive and, therefore, interchangeable with any other additive operation. For instance,

$$\operatorname{Re} [(A + B) e^{i\omega t}] = \operatorname{Re} [A e^{i\omega t}] + \operatorname{Re} [B e^{i\omega t}]. \quad (1.7)$$

Hence, the complex amplitude corresponding to the sum of two functions $a(\vec{r}, t)$ and $b(\vec{r}, t)$ equals the sum of the complex amplitudes corresponding to each of the functions. Similarly, the equality

$$\operatorname{Re} \left[\frac{\partial A(\vec{r})}{\partial x} \cdot e^{i\omega t} \right] = \frac{\partial}{\partial x} \operatorname{Re} [A(\vec{r})e^{i\omega t}] \quad (1.8)$$

together with (1.1) means that the complex amplitude corresponding to the function $\partial a(\vec{r}, t)/\partial x$ equals the derivative of the complex amplitude of the function $a(\vec{r}, t)$ with respect to x .

It is easy to verify that the complex amplitude of the function $\partial a(\vec{r}, t)/\partial t$ is $i\omega A(\vec{r})$, that is, differentiation of the function $a(\vec{r}, t)$ with respect to t is equivalent to multiplying the complex amplitude with $i\omega$, as follows:

$$\frac{\partial}{\partial t} \operatorname{Re} [Ae^{i\omega t}] = \operatorname{Re} [i\omega Ae^{i\omega t}]. \quad (1.9)$$

Introducing the complex amplitudes instead of the harmonic functions simplifies all the calculations. Any operation over the functions of the type (1.1) requires equating the terms at $\cos \omega t$ and $\sin \omega t$, which is more complicated than performing operations on complex functions. There is another advantage in using the complex amplitudes. In many diffraction problems, analytical properties of functions of a complex variable are used. Applying this technique to the functions which are not complex themselves would require very cumbersome derivations. This difficulty does not arise when operating with the complex amplitudes.

Introducing the complex amplitudes by (1.3) is usually described by the expression "Time dependence is taken in the form of $\exp(i\omega t)$." In nearly 50 per cent of monographs and papers the time dependence is taken as $\exp(-i\omega t)$. While reading a paper or a book, you must find out to which half it belongs. Since the imaginary unit is introduced by choosing the time dependence, the sign of i would be opposite throughout your book if the time dependence is taken as $\exp(-i\omega t)$. Note that the usual definition of the imaginary unit as a square root of -1 is ambiguous; it does not specify which of the two roots i is.

1.1.3

The period-average product of two harmonic functions

Nonlinear operations cannot be performed on the complex amplitudes. The product of two harmonic functions $a(\vec{r}, t)$ and $b(\vec{r}, t)$ equals

$$a(\vec{r}, t) \cdot b(\vec{r}, t) = \frac{1}{2} |A| |B| \{ \cos(\alpha - \beta) + \cos(2\omega t - \alpha - \beta) \}, \quad (1.10)$$

that is, it depends on t differently than in (1.1). However, the period-average product, introduced as

$$\overline{a(\vec{r}, t) \cdot b(\vec{r}, t)} = \frac{1}{T} \int_0^T a(\vec{r}, t) \cdot b(\vec{r}, t) dt, \quad (1.11)$$

equals the first summand in (1.10), which contains moduli and phases of the complex amplitudes, so that

$$\overline{a \cdot b} = \frac{1}{2} \operatorname{Re} [A \cdot B^*], \quad (1.12)$$

where asterisk means the complex conjugation.

Therefore, in almost all calculations related to the monochromatic fields, we can operate with the complex amplitudes $A(\vec{r})$ instead of passing to the physical values $a(\vec{r}, t)$.

The complex amplitudes of electric and magnetic fields are denoted by $\vec{E}(\vec{r})$ and $\vec{H}(\vec{r})$, respectively. These two complex vectors are called *electric* and *magnetic fields*.

1.2

The Maxwell equations

1.2.1

The conduction current and the extrinsic current

In this section we give the Maxwell equations describing the laws of the field $\vec{E}(\vec{r})$ and $\vec{H}(\vec{r})$ behavior in space. The equations contain two more fields, namely, the *electric* and *magnetic inductions*. The complex amplitudes of these fields are called the inductions, denoted as $\vec{D}(\vec{r})$ and $\vec{B}(\vec{r})$, respectively. Further, we will consider a connection between the inductions and fields.

Besides, two other vectors, namely, the volume *densities* of the *conduction current* and the *extrinsic current* take part in the Maxwell equations. Their complex amplitudes are denoted by $\vec{j}(\vec{r})$ and $\vec{j}^{\text{ext}}(\vec{r})$, respectively.

The conduction current density is proportional to the electric field \vec{E} ,

$$\vec{j} = \sigma \vec{E}. \quad (1.13)$$

The proportionality coefficient σ is called the *conductivity*; it is a property of the medium where the field \vec{E} and current density \vec{j} are considered. The conductivity has the dimensionality s^{-1} ; $\sigma = 0$ in vacuum and nonconducting media.

The extrinsic current density $\vec{j}^{\text{ext}}(\vec{r})$ has, in general, a different meaning than \vec{j} . In applied electrodynamics problems, the extrinsic current appears as a prescribed value, similar to the outside force in dynamics. It can be specified explicitly; then $\vec{j}^{\text{ext}}(\vec{r})$ is the known vector function. The current may also be given in an implicit form as a wave of specified magnitude and structure, coming from infinity. In this case there is no term \vec{j}^{ext} in the equation; instead, the solution must contain the oncoming wave.

Separation of the current into the conduction and extrinsic ones depends on the problem formulation. Let, for instance, an antenna in the form of a metallic cylinder be investigated. The current flowing on its surface can be treated as an extrinsic (given) current, and then the field created by it in the surrounding space is studied. Otherwise, the field created by a line feeding the radiating device may be considered as a given field. Then the current flowing on the line and creating the field is found from the requirement that this field together with the prescribed one satisfies specified conditions. The problem formulated in this way is closer to the actual one and the result obtained is more informative. Needless to say that solving the problem in this formulation is more difficult. The current on the cylinder should be considered as an induced one, that is, the current of conductivity created by the field, but not as an exterior one. The density $\vec{j}(\vec{r})$ is found from the solution of the problem.

1.2.2

The Maxwell equations

The *Maxwell equations* are a system of two vector (that is, of six scalar) partial differential equations of the first order. We write them for the complex amplitudes of the fields, inductions, and current densities. The time derivation is replaced by a multiplication by $i\omega$, and k stands for the *w/c* multiplier. The equations become

$$\text{rot } \vec{H} - ik\vec{D} = \frac{4\pi}{c}\vec{j} + \frac{4\pi}{c}\vec{j}^{\text{ext}}, \quad (1.14a)$$

$$\text{rot } \vec{E} + ik\vec{B} = 0. \quad (1.14b)$$

We give some elementary consequences of these equations. Taking the divergence of (1.14b) and keeping in mind the vector identity

$$\text{div rot } \vec{A} = 0, \quad (1.15)$$

we arrive at the scalar equation

$$\text{div } \vec{B} = 0. \quad (1.16)$$

Introducing the so-called *charge density* by the formula

$$\rho = \frac{1}{4\pi} \text{div } \vec{D} \quad (1.17)$$

and replacing (only in this formula) multiplication by $i\omega$ with the time differentiation $\partial/\partial t$, we obtain from (1.14a)

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\vec{j} + \vec{j}^{\text{ext}}). \quad (1.18)$$

This is one of the main equations of alternating current theory, the so-called *continuity equation*. There is usually no need for introducing the charge density (1.17) and using equation (1.18) when studying the harmonic fields, that is, in high-frequency electrodynamics. Similar to (1.17), equation (1.16) can be treated as an assertion that the absence of magnetic charges follows from (1.14).

In the domain where neither material objects nor extrinsic currents exist, that is, where $\vec{j} \equiv 0$ and $\vec{j}^{\text{ext}} \equiv 0$, there are also no charges, that is, $\vec{D} \equiv \vec{E}$, and, similar to (1.16),

$$\operatorname{div} \vec{E} = 0. \quad (1.19)$$

Sometimes the set of the Maxwell equations consists not only of equation (1.14) but also of equations (1.16) (1.19), and (1.17). In these cases (1.17) is treated not as a definition of ρ , but as an equation for \vec{D} . In the text below, the system of the Maxwell equations is understood as system (1.14) or any of its other notations.

1.2.3

Dielectric permittivity and magnetic permeability of a medium

The inductions are linearly related to the fields. Coefficients of these relations constitute a medium in which the fields \vec{E} , \vec{H} and inductions \vec{D} , \vec{B} are considered.

In most media, \vec{D} depends only on \vec{E} , as well as \vec{B} on \vec{H} , so that

$$\vec{D} = \tilde{\epsilon} \vec{E}, \quad (1.20a)$$

$$\vec{B} = \mu \vec{H}. \quad (1.20b)$$

The difference in notation of $\tilde{\epsilon}$ and μ will be explained below. The coefficient $\tilde{\epsilon}$ is called the *dielectric permittivity* (further, permittivity) and μ the *magnetic permeability* (further, permeability) of the medium. We accept that $\tilde{\epsilon}$ and μ are dimensionless; $\tilde{\epsilon} = 1$, $\mu = 1$ in vacuum.

In anisotropic media the vector \vec{D} is not parallel to \vec{E} , and, generally speaking, \vec{B} and \vec{H} are not parallel, either. In such media, $\tilde{\epsilon}$ (sometimes also μ) is a tensor, not a scalar, and it is described by all its components $\tilde{\epsilon}^{\tau\sigma}$, where τ and σ take the values x, y, z . The expression $\tilde{\epsilon} \vec{E}$ is the product of a tensor and a vector; it is a vector with the components

$$D_x = \sum_{\sigma} \tilde{\epsilon}^{x\sigma} E_{\sigma}, \quad D_y = \sum_{\sigma} \tilde{\epsilon}^{y\sigma} E_{\sigma}, \quad D_z = \sum_{\sigma} \tilde{\epsilon}^{z\sigma} E_{\sigma}, \quad (1.21)$$

where the summation is made over the three values $\sigma = \{x, y, z\}$. The above equalities are basic for crystal optics and interdisciplinary sciences. Further, almost everywhere in the text $\tilde{\epsilon}$ and μ are scalar.

In nonhomogeneous media, $\tilde{\epsilon}$ and μ are functions of the coordinates. In general, they are complex, that is, for instance, \vec{D} and \vec{E} are not in-phase. In linear electrodynamics, which is only considered in the book, $\tilde{\epsilon}$ and μ do not depend on \vec{E} and \vec{H} . Such dependence takes place only for large fields.

1.2.4

The polarization current

In order to explain the difference between the induction \vec{D} and field \vec{E} , we rewrite (1.14) in the form

$$\text{rot } \vec{H} = \frac{4\pi}{c} \left[\vec{j}^{\text{ext}} + \vec{j} + \frac{i\omega}{4\pi} \vec{E} + \frac{i\omega}{4\pi} (\tilde{\epsilon} - 1) \vec{E} \right]. \quad (1.22)$$

The magnetic field is created by the currents \vec{j}^{ext} and \vec{j} , the *displacement current* $i\omega/4\pi\vec{E}$, and the current with density $i\omega/4\pi(\vec{D} - \vec{E})$. The assertion that the magnetic field is created by the time-varying electrical field in vacuum, that is, by the displacement current, is the basis of the whole Maxwell theory. The last term in (1.22) is called the *polarization current*. It exists only in material objects.

We explain the process of appearance of the polarization current on a model of a simplest medium. Imagine a medium containing a plenty of small metallic particles. The distance between any two particles is large in comparison with their sizes and small with the distance, where the field \vec{E} varies significantly, for instance, in comparison with the length of the wave propagating in the medium.

A particle is being polarized under the influence of the electric field \vec{E} , that is, positive and negative charges are gathering on opposite sides of the particle. If the field \vec{E} varies with time, then the charges displace; for instance, if \vec{E} changes its sign, then the charges interchange. The displacement of the charges is the polarization current. It is proportional to the time derivative of the field, that is, to the vector $i\omega\vec{E}$, and, together with three other summands in (1.22), creates the magnetic field, and, therefore, appears in (1.14).

We give two formulas connecting $\tilde{\epsilon}$ with medium parameters. Each particle is characterized by a parameter p (*polarization coefficient*) which is determined from the condition that the field \vec{E} causes the polarization current $i\omega p\vec{E}$. In the general case, p is a tensor and the current is not parallel to \vec{E} ; however, we assume it to be a scalar in our simplest model. Tensorial nature of p is not essential here because the value \vec{D} is the induction averaged over many particles, and, therefore, the particle orientation disappears. The value of p is obtained from the solution of the electrostatic problem. It has the order of the

particle volume (if the particle is not very prolate or flattened). For instance, in the case of a sphere, p is equal to the cube of its radius. A number N of the particles per unit volume of the medium is large; however, the total volume of all the particles in the unit volume (*particle concentration*) is assumed to be small. Under this condition, $\tilde{\epsilon}$ depends only on the dimensionless quantity Np . All the above considerations as well as the below formulas (1.23) and (1.24) for $\tilde{\epsilon}$ are valid only at not very high frequencies, more precisely, when the field varies slightly at the distance of the particle size.

The simplest formula for $\tilde{\epsilon}$ is obtained if Np is so small that the mutual influence of the particles can be neglected. Then the polarization current density is equal to the sum of all the polarization currents caused only by the field \vec{E} influence on the particles of unit volume. In this case

$$\tilde{\epsilon} = 1 + \alpha, \quad \alpha = 4\pi Np. \quad (1.23)$$

The above formula is known as the *formula of molecular optics*.

We derive a more precise formula by taking into consideration that each particle is influenced not only by the applied field \vec{E} but also by the fields of other polarized particles. Then

$$\tilde{\epsilon} = \frac{1 + 2\alpha/3}{1 - \alpha/3} \quad (1.24)$$

(the *Lorentz–Lorence formula*). Formula (1.23) coincides with the first two terms on the right-hand side of expansion (1.24) in the series in parameter α . Formula (1.24) is also valid only at $\alpha \ll 1$; however, this restriction is not so strong as for (1.23).

The mutual influence of the particles should be taken into account more properly when considering a larger particle concentration. In this case $\tilde{\epsilon}$ depends not only on the polarization factor but also on other electrodynamic parameters of the particle. The theory, resulting in formulas (1.23), (1.24) and other more complicated and accurate formulas, was developed for natural materials, in which molecules or atoms play the role of particles in our model. Needless to say that the polarization process in such materials is more complicated than in metallic particles; it cannot be treated as charge separation only. However, the process is also characterized by the quantity (tensor) p , and it causes the appearance of the polarization current $i\omega p\vec{E}$. In crystals the particles are not chaotically positioned and, in principle, the polarization current is not parallel to the field; $\tilde{\epsilon}$ is a tensor.

The theory based on the model described is also applicable to artificial media, so-called composites. These media contain small particles of macroscopic sizes, incorporated into a material with $\tilde{\epsilon} \approx 1$, $\mu \approx 1$. If the particle sizes are small, then the composite object behaves like the same object from a natural material having the effective $\tilde{\epsilon}$ and μ . The above formulas (as well as the

similar ones for μ) express the effective parameters of the composite in terms of the particle characteristics. The interest to the composites is caused by the possibility of creating the media with desired electrodynamic parameters by choosing the material, shape, size, and number of particles.

Another model for which the explicit expression for $\tilde{\epsilon}$ can be found is a set of free electrons (plasma). Being influenced by an electric field, they are fast oscillating with the field frequency. The amplitude of the oscillations is inversely proportional to the inertia of electrons, that is, to their mass m , and also to the force acting on them, that is, to their charge e . The amplitude decreases when the frequency increases. The field of the unit volume is proportional to the amplitude multiplied by the charge and the number N of electrons per unit volume. The permittivity of such medium is

$$\tilde{\epsilon}(\omega) = 1 - 4\pi \frac{e^2}{m\omega^2} N. \quad (1.25)$$

If the collisions and effects like “friction” are taken into account, then $\tilde{\epsilon}$ also acquires the imaginary part. According to (1.25), $\tilde{\epsilon} = 0$ at $\omega = \omega_0$, where

$$\omega_0^2 = 4\pi \frac{e^2}{m} N. \quad (1.26)$$

If $\omega < \omega_0$, then $\tilde{\epsilon} < 0$. In the medium of free electrons, the electric field and induction are oppositely directed at low frequencies. If $\omega > \omega_0$, then $0 < \tilde{\epsilon} < 1$. At $\omega \rightarrow \infty$, $\tilde{\epsilon} \rightarrow 1$, the sign of the field acting on the electrons changes fast and they have no time to displace; the polarization current is small at the high frequencies. This remark does not relate to this model only. The permittivity tends to unity as $\omega \rightarrow \infty$ in any media.

The permeability μ behaves in a similar way; $\mu \rightarrow 1$ as $\omega \rightarrow \infty$. The frequency value at which $|\tilde{\epsilon} - 1|$ and $|\mu - 1|$ become small depends on the physical mechanism of appearance of the polarization. For plasma, such frequencies are noticeably larger than ω_0 .

The mechanism causing the distinction between the induction \vec{B} and field \vec{H} can be explained similarly to that causing the distinction between \vec{D} and \vec{E} , which has been described for our simple model. Rewrite equation (1.14b) in the form

$$\text{rot } \vec{E} = -ik\vec{H} - ik(\vec{B} - \vec{H}). \quad (1.27)$$

The electric field \vec{E} is created not only by the time-varying magnetic field in vacuum (the first term in (1.27)) but also by the time-varying field $(\vec{B} - \vec{H})$, which exists only in the material medium. A set of permanent magnets can serve as a model of the medium for describing the appearance of the additional magnetic field. If there is no external magnetic field, then the magnet

axes are oriented chaotically and magnetic fields created by the magnets mutually cancel. Under the influence of the external magnetic field, the axes become mostly oriented along it and the magnetic fields created by the magnets are not fully compensated. An additional magnetic field is created. When the applied magnetic field changes, then the extra magnetic field also changes and, therefore, it creates the electric field. This effect is described by the second term in (1.27). The additional field $\vec{B} - \vec{H}$ is proportional to \vec{H} , so that if $\vec{B} - \vec{H} \neq 0$, then the factor μ in (1.20b) differs from unity. It is clear that the model describes only one (probably, the most demonstrative) process leading to $\vec{B} \neq \vec{H}$.

1.2.5

The frequency dispersion and the spatial dispersion (chirality)

Formula (1.20a) states that at a given point and time moment, the electric induction is determined only by the value of the electric field at the same point and time moment. This statement needs two specifications.

The first specification implies that the induction depends also on the time derivative of the field. For the fields varying by the harmonic law, the specification does not require more complicated formula (1.20a), but leads to $\tilde{\epsilon}$ becoming the function of frequency, $\tilde{\epsilon} = \tilde{\epsilon}(\omega)$. This effect is called the *frequency dispersion*.

The second specification is connected with the phenomenon of *spatial dispersion*. The induction \vec{D} depends not only on the field $\vec{E}(\vec{r})$ but also on its spatial derivatives. At the arbitrary dependence $\vec{E}(\vec{r})$, formula (1.20a) is not valid in media where the above effect is essential. It remains valid only if the field varies in space just like in the plane wave. However, in this case $\tilde{\epsilon}$ should be treated as a function of the normal direction to the wave front; in the same manner as for the frequency dispersion it is a function of the frequency. If we do not restrict ourselves to the fields of plane waves (as we did to the harmonic fields above), then it is not possible to take into account the existence of the spatial dispersion by assuming $\tilde{\epsilon}$ to be dependent on the normal.

It can be shown that the first derivatives of $\vec{E}(\vec{r})$ appear in $\vec{D}(\vec{r})$ only as a combination of $\text{rot } \vec{E}$ if the fields depend on \vec{r} arbitrarily. Since at the points where no extrinsic currents exist, the fields $\vec{E}(\vec{r})$ and $\vec{H}(\vec{r})$ satisfy equations (1.14) with right-hand side zero. The relation between induction and fields can be written in the symmetric form, without derivatives, as follows:

$$\vec{D} = \tilde{\epsilon} \vec{E} - i\gamma \vec{H}, \quad \vec{B} = \mu \vec{H} + i\gamma \vec{E}. \quad (1.28)$$

It is a generalization of formula (1.20), valid only for the media where the spatial dispersion may be neglected. The “cross” terms $-i\gamma \vec{H}$ and $i\gamma \vec{E}$ are introduced into (1.28) in a special form: with the same factor γ (*chirality factor*),

opposite signs in front of them, and separated multiplier i . The expediency of this form of equations connecting inductions with fields will be shown later.

The appearance of the cross terms in the constitutive equations (1.27) for chiral media can be explained by physical reasons without referring to a formally nonlocal dependence between \vec{D} and \vec{E} (and \vec{B} and \vec{H} , respectively). Return to the model of dielectric as a set of metallic particles. The presence of the term proportional to \vec{E} in the expression for \vec{B} means that the current induced by the field \vec{E} creates not only the electric field (described by the difference $\tilde{\epsilon} - 1$) but also the magnetic field described by the term $-i\chi\vec{E}$ in the expression for \vec{B} .

This magnetic field is created only by the particles not having the plane of symmetry. For instance, if the particles are ball- or ellipsoid-shaped, then the field \vec{E} creates a very low magnetic field proportional to $(ka)^2$, where a is the particle size. The induction-to-field connection in such “nonchiral” dielectric, natural or artificial (that is, composite), is given by (1.20). The simplest example of the element possessing the pronounced chiral property is a planar open ring with free ends connected to the two oppositely directed linear wires lying in the plane perpendicular to the ring plane. The electric field parallel to the wires excites the current in them, flowing also along the ring. The current in the wires creates the electric field (resulting in $\tilde{\epsilon} \neq 1$), and the current in the ring creates the magnetic field, and, therefore, $\chi \neq 0$. In a similar way, one can explain the appearance of the term $-i\chi\vec{H}$ in the expression for \vec{D} when the particles have no plane symmetry.

In the chiral media, the electrodynamic processes are going on in a different way than in the nonchiral ones. The most known chirality phenomenon is the rotation of the polarization plane. It will be considered in Chapter 2.

1.2.6

Complex permittivity

Equation (1.14) is simplified after substituting the conduction current density and induction with use of (1.13) and (1.20), respectively. Then (1.14) becomes

$$\begin{aligned} \text{rot } \vec{H} - ik\epsilon\vec{E} &= \frac{4\pi}{c}\vec{j}^{\text{ext}}, \\ \text{rot } \vec{E} + ik\mu\vec{H} &= 0. \end{aligned} \tag{1.29}$$

In such a form the Maxwell equations are applicable only to the nonchiral media.

The factor ϵ in (1.29) equals

$$\epsilon = \tilde{\epsilon} - i\frac{4\pi\sigma}{\omega}. \tag{1.30}$$

We call it the *complex permittivity*, sometimes, however, omitting the word "complex." The Maxwell equations are understood as system (1.29). Rewrite it in the extended form in the Cartesian coordinates x, y, z :

$$\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - ik\epsilon E_x &= \frac{4\pi}{c} j_x^{\text{ext}}, & \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + ik\mu H_x &= 0, \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - ik\epsilon E_y &= \frac{4\pi}{c} j_y^{\text{ext}}, & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + ik\mu H_y &= 0, \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - ik\epsilon E_z &= \frac{4\pi}{c} j_z^{\text{ext}}, & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + ik\mu H_z &= 0, \end{aligned} \quad (1.31)$$

and in the cylindrical coordinates r, ϑ, φ :

$$\begin{aligned} \frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} - ik\epsilon E_r &= \frac{4\pi}{c} j_r^{\text{ext}}, & \frac{1}{r} \frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} + ik\mu H_r &= 0, \\ \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} - ik\epsilon E_\varphi &= \frac{4\pi}{c} j_\varphi^{\text{ext}}, & \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} + ik\mu H_\varphi &= 0, \\ \frac{1}{r} \frac{\partial(rH_\varphi)}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \varphi} - ik\epsilon E_z &= \frac{4\pi}{c} j_z^{\text{ext}}, & \frac{1}{r} \frac{\partial(rE_\varphi)}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \varphi} + ik\mu H_z &= 0. \end{aligned} \quad (1.32)$$

1.2.7

The radiation condition

System (1.29) should be complemented by the condition that the fields are created only by the current \vec{j}^{ext} , that is, there are no waves incoming from infinity. This condition can be written in the form of an asymptotic equality, which implies that the field has the structure of an outgoing wave in any domain outside the sphere containing all the objects and currents. This condition is called the *radiation* (or *Sommerfeld*) condition. In the spherical coordinates R, ϑ, φ it is written for the meridional E_ϑ, H_ϑ and azimuthal E_φ, H_φ components of the vectors \vec{E} and \vec{H} , and has the form

$$\begin{aligned} E_\vartheta &\equiv -H_\varphi = F_1(\vartheta, \varphi) \exp(-ikR) / kR \cdot \left[1 + O\left(\frac{1}{kR}\right) \right], \\ E_\varphi &\equiv H_\vartheta = F_2(\vartheta, \varphi) \exp(-ikR) / kR \cdot \left[1 + O\left(\frac{1}{kR}\right) \right]. \end{aligned} \quad (1.33)$$

This dependence of the complex amplitudes on R means that the physical fields are proportional to the functions

$$\frac{\cos(\omega t - kR)}{kR}, \quad \frac{\sin(\omega t - kR)}{kR} \quad (1.34)$$

in the higher order with respect to the parameter $1/kR$. The spheres $R = ct + \text{const}$ are (asymptotically) equiphase ones. The surfaces expand with

the velocity c as t increases, that is, (1.33) represents a divergent spherical wave. The complex amplitudes of the convergent wave incoming from the infinity would be proportional to $\exp(ikR)/kR$. Condition (1.33) means that such waves are not present in the field. The more simpler requirement that the field decreases not slower than $1/kR$ as $kR \rightarrow \infty$ would not be enough to exclude the incoming waves. For doing this it is also necessary to specify how the wave phase depends on R .

The functions $F_1(\vartheta, \varphi)$ and $F_2(\vartheta, \varphi)$ are not given in conditions (1.33). They can be found only after solving the problem, that is, after finding the fields $\vec{E}(\vec{r})$ and $\vec{H}(\vec{r})$ at the given \vec{r}^{ext} .

There is another way to exclude the waves incoming from infinity. It can be required that the field tends to zero as $R \rightarrow \infty$, if the real k is replaced by $k = k' + ik'', k'' < 0$. Obviously, at such k the field decreases only for waves which depend on R as $\exp(-ikR)/kR$, that is, for the outgoing waves and infinitely increases for the incoming waves as $R \rightarrow \infty$. One can prove that at $k'' \rightarrow 0$ the solution of the problem with the above requirement passes to the solution of the problem with $k'' = 0$ satisfying the radiation condition (1.33). The last statement is called the *limiting absorption principle*. This way of excluding the incoming waves is also applicable to the problems with object expanding to infinity, for which the radiation condition should be modified to exclude not only the incoming spherical waves but also the plane waves.

There are no waves with amplitudes of all components decreasing at $kR \rightarrow \infty$ faster than in (1.33), for instance, as $1/(kR)^2$. It follows from (1.29) that the fields with all components decreasing at infinity faster than $1/kR$ are identical zeros. Further, we will specify this assertion because under certain idealized conditions the fields are possible which exist only in the finite domain (closed resonators).

From (1.33), (1.16), (1.19) it follows that the asymptotic dependence of the radial field components on R also has a universal form. In spherical coordinates the equation $\text{div } \vec{E} = 0$ is written as

$$\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 E_R) + \frac{1}{R \sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta E_\vartheta) + \frac{1}{R \sin \vartheta} \frac{\partial E_\varphi}{\partial \varphi} = 0. \quad (1.35)$$

Substituting the asymptotic expressions (1.33) for E_ϑ and E_φ , we obtain that $\partial(R^2 E_R)/\partial R$ is asymptotically proportional to $\exp(-ikR)$; hence,

$$E_R = \Phi(\vartheta, \varphi) \exp(-ikR)/(kR)^2 \left[1 + O\left(\frac{1}{kR}\right) \right]. \quad (1.36)$$

A similar dependence of H_R on R follows from the equality $\text{div } \vec{H} = 0$, except that the angular dependence is given by the different function than for E_R . These angular functions are expressed by the angular functions from (1.33) and they can be found only after the problem is solved.

1.2.8

The wave equations

Equations (1.29) contain two vector functions, that is, six coordinate functions. Sometimes it is expedient to eliminate one of the fields from (1.29) and pass to the second-order equations in only either \vec{E} or \vec{H} . The equations of such type are called the *wave (Helmholtz) equations*.

Let no extrinsic currents exist in a domain and the medium be homogeneous and nonchiral, that is, $\epsilon = \text{const}$, $\mu = \text{const}$, $\kappa = 0$. Acting on both equations (1.29) by the operation “rot” and replacing the obtained terms $\text{rot} \vec{E}$ and $\text{rot} \vec{H}$ by the corresponding expressions from the second equation, we obtain the two independent equations

$$\begin{aligned}\text{rot rot } \vec{H} - k^2 \epsilon \mu \vec{H} &= 0, \\ \text{rot rot } \vec{E} - k^2 \epsilon \mu \vec{E} &= 0,\end{aligned}\tag{1.37}$$

each containing only one field.

Transform the first terms in these equations. For any vector \vec{A} , we introduce a vector $\Delta \vec{A}$ defined as

$$\Delta \vec{A} = \text{grad div } \vec{A} - \text{rot rot } \vec{A}.\tag{1.38}$$

One can verify that its Cartesian components equal ΔA_x , ΔA_y , ΔA_z , that is, they are the result of the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\tag{1.39}$$

acting on the Cartesian components of the vector \vec{A} . This fact is not valid for the non-Cartesian components; for instance, $(\Delta \vec{A})_R \neq \Delta A_R$. In domains where ϵ and μ are constant, (1.16) and (1.19) follow from (1.29), that is, $\text{rot rot } \vec{E} = -\Delta \vec{E}$, $\text{rot rot } \vec{H} = -\Delta \vec{H}$, and equations (1.37) become

$$\begin{aligned}\Delta \vec{E} + k^2 \epsilon \mu \vec{E} &= 0, \\ \Delta \vec{H} + k^2 \epsilon \mu \vec{H} &= 0.\end{aligned}\tag{1.40}$$

The first equation is separated into three independent scalar equations in the Cartesian components E_x , E_y , E_z , as follows:

$$\begin{aligned}\Delta E_x + k^2 \epsilon \mu E_x &= 0, \\ \Delta E_y + k^2 \epsilon \mu E_y &= 0, \\ \Delta E_z + k^2 \epsilon \mu E_z &= 0.\end{aligned}\tag{1.41}$$

The second equation (1.40) can be separated in a similar way.

Equations (1.41) follow from system (1.29), that is, any solution to (1.29) with right-hand side zero satisfies (1.41) and similar equations in H_x, H_y, H_z . However, the inverse statement is not valid; not all solutions to (1.40) satisfy system (1.29) because (1.29) is not a consequence of (1.40). System (1.40) does not contain the connection between the Cartesian components of the fields at all. For instance, $\vec{H} \equiv 0, \vec{E} \not\equiv 0$ may be its solution, but the system of Maxwell equations cannot have such a solution. After finding some field components from (1.40) we have to substitute them into system (1.29). The solutions for the found components are valid if the system for the remaining components is consistent. The solutions for the remaining components can be obtained from this consistent system.

In the domains where $\vec{j}^{\text{ext}} \neq 0$, the wave equations would involve the term $\text{rot } \vec{j}^{\text{ext}}$. In such domains it is simpler to introduce the auxiliary functions, so-called *potentials*. The fields are expressed in terms of the derivatives of these functions. Potentials may be introduced in a way that they satisfy the wave equations with extrinsic currents standing on the right-hand side themselves, instead of their derivatives.

The *Hertz electric vector* $\vec{\Pi}(\vec{r})$ is such a potential. It is proportional to the vector potential, usually introduced in electrodynamics. The Hertz vector satisfies the nonhomogeneous wave equation

$$\Delta \vec{\Pi} + k^2 \varepsilon \mu \vec{\Pi} = \frac{4\pi i}{\omega} \vec{j}^{\text{ext}}. \quad (1.42)$$

The fields are expressed in terms of the Hertz vector by the formulas

$$\vec{E} = k^2 \varepsilon \mu \vec{\Pi} + \text{grad div } \vec{\Pi}, \quad \vec{H} = ik\varepsilon \text{rot } \vec{\Pi}. \quad (1.43)$$

It is easy to check that the fields (1.43) satisfy system (1.29) if $\vec{\Pi}$ satisfies equation (1.42); we suppose that $\varepsilon = \text{const}, \mu = \text{const}$.

In Cartesian coordinates, equation (1.42) is equivalent to three scalar equations in Π_x, Π_y, Π_z having the form

$$\Delta u + k^2 \varepsilon \mu u = f. \quad (1.44)$$

This equation follows from the Maxwell system when the medium is homogeneous. However, (1.44) is also of interest in investigation of the field in nonhomogeneous media; in particular, it is valid for certain two-dimensional nonhomogeneous media. The equation itself and the properties of the function $u(x, y, z)$ satisfying it are interesting mainly because they describe the qualitative peculiarities of propagation and diffraction of the electromagnetic wave, which are not determined by the vector nature of the wave. If the wave polarization (that is, directions of \vec{E} and \vec{H}) and the mutual connection between different components of the fields are not essential, then many qualitative

properties of the field, as well as many peculiarities of various methods of their finding, may be investigated on the instance of the scalar wave equation (1.44).

1.2.9

The reciprocity conditions

Consider the connection between the fields created by the two different extrinsic currents $\vec{j}^{\text{ext}(1)}$ and $\vec{j}^{\text{ext}(2)}$ in some medium. The fields created by these two currents are denoted as $\vec{E}^{(1)}, \vec{H}^{(1)}$ and $\vec{E}^{(2)}, \vec{H}^{(2)}$, respectively. The functions $\epsilon(\vec{r}), \mu(\vec{r})$ (and $\varkappa(\vec{r})$, if the medium is chiral) in the constitutive equations are the same for both cases, that is, the currents $\vec{j}^{\text{ext}(1)}$ and $\vec{j}^{\text{ext}(2)}$ are placed into the same medium.

Consider the expression $\text{div}(\vec{E}^{(1)} \times \vec{H}^{(2)} - \vec{E}^{(2)} \times \vec{H}^{(1)})$. According to the vector identity

$$\text{div}(\vec{A} \times \vec{B}) = \vec{B} \cdot \text{rot} \vec{A} - \vec{A} \cdot \text{rot} \vec{B}, \quad (1.45)$$

the above expression equals

$$\vec{H}^{(2)} \cdot \text{rot} \vec{E}^{(1)} - \vec{E}^{(1)} \cdot \text{rot} \vec{H}^{(2)} - \vec{H}^{(1)} \cdot \text{rot} \vec{E}^{(2)} + \vec{E}^{(2)} \cdot \text{rot} \vec{H}^{(1)}. \quad (1.46)$$

Substituting the value of the rotor of the fields from the Maxwell equations (1.29) into (1.46), we obtain

$$\begin{aligned} & \frac{4\pi}{c} \left(\vec{E}^{(2)} \vec{j}^{\text{ext}(1)} - \vec{E}^{(1)} \vec{j}^{\text{ext}(2)} \right) \\ & + ik \left(\vec{H}^{(1)} \vec{B}^{(2)} - \vec{H}^{(2)} \vec{B}^{(1)} + \vec{E}^{(2)} \vec{D}^{(1)} - \vec{E}^{(1)} \vec{D}^{(2)} \right). \end{aligned} \quad (1.47)$$

We find the conditions under which the second bracket in (1.47) equals zero. If the constitutive equations are $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$, then the bracket equals

$$\left(\vec{H}^{(1)} \cdot \mu \vec{H}^{(2)} - \vec{H}^{(2)} \cdot \mu \vec{H}^{(1)} \right) + \left(\vec{E}^{(2)} \cdot \epsilon \vec{E}^{(1)} - \vec{E}^{(1)} \cdot \epsilon \vec{E}^{(2)} \right). \quad (1.48)$$

If ϵ and μ are scalar, then the above two brackets equal zero. However, if ϵ or μ is a tensor, then the brackets are zero only if the tensor is symmetric, that is, if $\epsilon^{xy} = \epsilon^{yx}$, and so on. The media with ϵ or μ being a nonsymmetrical tensor do not possess the property of “reciprocity” which other media do.

If the medium is chiral, that is, the constitutive equations have a form (1.28), then the second bracket in (1.47) remains zero. If there were different factors \varkappa in the coefficients at the cross terms of (1.28), then this bracket would not be zero, and the chiral medium also would not possess the reciprocity property. Such situation is possible if, for instance, the particles forming a composite consist of the material for which ϵ or μ is a nonsymmetrical tensor.

Hence, if ε and μ are scalars or symmetrical tensors, then

$$\operatorname{div} \left(\vec{E}^{(1)} \times \vec{H}^{(2)} - \vec{E}^{(2)} \times \vec{H}^{(1)} \right) = \frac{4\pi}{c} \left(\vec{E}^{(2)} \vec{j}^{\text{ext}(1)} - \vec{E}^{(1)} \vec{j}^{\text{ext}(2)} \right) \quad (1.49)$$

for any $\vec{j}^{\text{ext}(1)}$ and $\vec{j}^{\text{ext}(2)}$.

Integrate (1.49) over a sphere of radius a so large that there are no currents $\vec{j}^{\text{ext}(1)}$, $\vec{j}^{\text{ext}(2)}$ and material objects outside the sphere. The asymptotic radiation conditions (1.33) hold on the surface of such a sphere. On the left-hand side of the obtained integral form of equation (1.49) we have the flux of the vector standing under the “div” operation in (1.49), through the sphere surface. The vector flux is equal to the integral of the radial component of this vector, that is, of the quantity

$$E_\theta^{(1)} H_\varphi^{(2)} - E_\varphi^{(1)} H_\theta^{(2)} - E_\theta^{(2)} H_\varphi^{(1)} + E_\varphi^{(2)} H_\theta^{(1)}, \quad (1.50)$$

taken over the angles. The terms of the order $1/a^2$ in this sum mutually cancel whatever the functions $F_1(\vartheta, \varphi)$, $F_2(\vartheta, \varphi)$ are in (1.33), so that the above quantity becomes of the order not lower than $1/a^3$. Since the surface element is proportional to a^2 , the integral over the sphere surface decreases at $a \rightarrow \infty$ not slower than $1/a$. It equals the volume integral of the right-hand side of equality (1.49). However, this value differs from zero only inside the sphere of finite radius; it does not depend on the radius a if a is sufficiently large. Hence, the integral of (1.50) is zero at large a , and it follows from (1.49) that if the integrals are taken over the volume V containing all the extrinsic currents, then

$$\int_V \vec{E}^{(1)} \vec{j}^{\text{ext}(2)} dV = \int_V \vec{E}^{(2)} \vec{j}^{\text{ext}(1)} dV. \quad (1.51)$$

This formula is called the *reciprocity condition*.

The physical sense of (1.51) is most clearly seen if both the currents are the point sources, that is, they are described by the δ -functions. Then (1.51) becomes

$$\vec{E}^{(1)}(\vec{r}^{(2)}) \cdot \vec{a}_2 = \vec{E}^{(2)}(\vec{r}^{(1)}) \cdot \vec{a}_1. \quad (1.52)$$

Here $\vec{r}^{(1)}$ and $\vec{r}^{(2)}$ are the points where the currents $\vec{j}^{\text{ext}(1)}$ and $\vec{j}^{\text{ext}(2)}$ are located, respectively; \vec{a}_1 and \vec{a}_2 are the unit vectors along which the sources (elementary dipoles, see Subsection 6.1.1) are oriented; $\vec{E}^{(1)}(\vec{r})$ and $\vec{E}^{(2)}(\vec{r})$ are the fields created by the currents. The field at the point $\vec{r}^{(2)}$, created by the elementary dipole located at the point $\vec{r}^{(1)}$, equals the value of the field at the point $\vec{r}^{(1)}$, created by the elementary dipole located at $\vec{r}^{(2)}$. In this statement the specification about the field directions at the points $\vec{r}^{(1)}$ and $\vec{r}^{(2)}$ is omitted for brevity. This is the simplest formulation of the reciprocity condition.

Condition (1.51) holds for any set of bodies if the medium possesses the properties mentioned above. If all the parts of the medium have these properties, then the whole medium has them, either. A set of reciprocal elements is reciprocal itself. The nonreciprocity of an element is a property caused by the existence of a particular direction in its material, for instance, an internal magnetostatic field.

It follows from the reciprocity condition that for transmitting energy in a channel only in one direction it is necessary to insert a nonreciprocal element into the channel.

1.2.10

Average energy losses

We derive the expression for average energy losses, that is, the quantity of the electromagnetic field energy transforming into other types of energy in the unit volume per unit time. We begin with the expression for *thermal (joule) losses*. In the unit volume, they equal the conduction current density multiplied by the electric field. Denote the period-average losses by P_E . According to (1.13) and (1.11), if expressed by the amplitudes of the current \vec{j} and the field \vec{E} , the losses are $P_E = \sigma |\vec{E}|^2 / 2$. Following (1.30), the conductivity factor σ can be expressed by the imaginary part of permittivity as $\sigma = -\epsilon'' \omega / 4\pi$, so that

$$P_E = -\frac{\omega}{8\pi} \epsilon'' |\vec{E}|^2. \quad (1.53)$$

We show that the above definition of P_E is, in general, valid for any physical sense of ϵ'' ; it is not necessarily related to the joule losses only. The energy flux through a closed surface is the integral of the Poynting vector, equal (with accuracy to the multiplier $c/4\pi$) to the vector product of the electric and magnetic fields. The average value of the flux expressed by the complex amplitudes \vec{E} and \vec{H} is

$$\vec{S} = \frac{c}{8\pi} \operatorname{Re} (\vec{E} \times \vec{H}^*), \quad (1.54)$$

where the average mark over \vec{S} is omitted. Integral of $\operatorname{div} \vec{S}$ over any volume is the energy decrease in this volume. For infinitely small volume the decrease equals the density of the average losses, that is

$$\operatorname{div} \vec{S} = -(P_E + P_H), \quad (1.55)$$

where P_H is the average losses describing the decrease of the magnetic field.

We express $\operatorname{div} \vec{S}$ in terms of \vec{E} and \vec{H} with use of equations (1.29). Applying identity (1.45), substituting $\operatorname{rot} \vec{E}$ and $\operatorname{rot} \vec{H}$ into (1.45) and taking the real part, we obtain

$$\operatorname{div} \vec{S} = \frac{\omega}{8\pi} \left(\epsilon'' |\vec{E}|^2 + \mu'' |\vec{H}|^2 \right) - \frac{1}{2} \operatorname{Re} (\vec{j}^{\text{ext}*} \cdot \vec{E}). \quad (1.56)$$

According to (1.55), it follows from this equality that formula (1.53) and the similar expression

$$P_H = -\frac{\omega}{8\pi}\mu''|\vec{H}|^2 \quad (1.57)$$

are valid for any losses proportional to the squared electric and magnetic fields.

Media exist for which $\epsilon'' > 0$, that is, "losses" are negative. In such media not the absorption, but generation of the electromagnetic energy occurs, which is proportional to the squared electromagnetic field. Formula (1.53) describes not the process of the field energy decreasing, but the inverse process of its increasing. The physical sense of the above phenomenon is that the population of quantum levels is inverse in such media: the higher levels are more populated than the lower ones. Sometimes such a situation is formally called as "negative absolute temperature." It is clear that its creation requires energy consumption, so-called pumping. The processes occurring in coherent sources of the electromagnetic oscillations are based on the energy generation caused by the field influence.

In chiral media, the Maxwell equations must be written not in the form (1.29), but in another one with \vec{D} and \vec{B} expressed in terms of \vec{E} and \vec{H} by (1.28). However, it is easy to prove that formula (1.56) remains valid, that is, the losses are expressed in terms of fields by the same formulas (1.53), (1.57) as for nonchiral media. For proving this statement the fact is used that the coefficients of the cross terms in (1.28) coincide (with accuracy to the sign). As was mentioned, this follows from the fact that the chiral media are, in general, reciprocal. In order that the expressions for P_E and P_H were the same in the chiral media as those in the nonchiral ones, it is sufficient that the real parts of the coefficients coincide.

1.2.11

The dispersion relations

The real and imaginary parts of the permittivity $\epsilon = \epsilon' + i\epsilon''$ are functions of the frequency ω . At any fixed frequency they can be independent of each other. However, in the whole frequency interval $0 < \omega < \infty$ the functions $\epsilon'(\omega)$ and $\epsilon''(\omega)$ are dependent. Setting one function, we thereby set the other.

We show in an example that the inconsistent setting of both the functions may lead to a paradox. It is clear that the time-dependent impulse, containing the oscillations of the wide frequency band, should be considered instead of the harmonic oscillations.

Assume that a material exists for which $\epsilon'(\omega) = \text{const}$ at any frequency, and $\epsilon''(\omega)$ is zero at almost all frequencies except for a narrow band near a frequency ω_0 . An impulse (the field equal to zero at $t < 0$) falls onto a thin plate

of such material at $t = 0$. This field can be expanded in the Fourier integral, that is, written as an integral of the harmonic fields or, more precisely, an integral of the factor $\exp(i\omega t)$ multiplied by a function of ω , and taken over ω . The form of the function is not essential here. The fact that the integral equals zero at $t < 0$ means that its components corresponding to different frequencies compensate each other at $t < 0$. After passing the plate, the components with frequencies not close to ω_0 obtain the same phase factor because they have the same values of ϵ' and their amplitudes do not change because $\epsilon'' = 0$ for them. The amplitudes of the other components (with frequencies close to ω_0) for which $\epsilon'' \neq 0$ decrease. The conditions of the mutual compensation of all harmonics at $t < 0$ are violated. This fact means the same that, besides the initial impulse equal to zero at $t < 0$, a narrow group of harmonics appears after passing the plate, equal (with opposite sign) to the values by which the harmonics of these frequencies decreased. These harmonics exist at any t , and they do not compensate each other at $t < 0$. If the material with the above-mentioned properties of the functions $\epsilon'(\omega)$ and $\epsilon''(\omega)$ existed, then the field outside the plate would appear before the impulse incidence onto the plate. The consequence would precede the cause; the *causality principle* would be violated. The paradox also takes place if assuming that $\epsilon'(\omega) = \text{const}$, $\epsilon''(\omega) > 0$ in a narrow frequency band and $\epsilon''(\omega) = 0$ outside.

We outline the mathematical technique allowing us to deduce the connection between the two functions $\epsilon'(\omega)$, $\epsilon''(\omega)$ of the frequency using the causality principle. This connection is called the *dispersion relation*. It holds for any material medium and it should be considered while creating composites. The requirements imposed on $\epsilon'(\omega)$ and $\epsilon''(\omega)$ should be subjected to the relation.

The induction $\vec{D}(t)$ is the medium response to the field $\vec{E}(t)$. It may depend on the field value at the given time moment and before it, but it cannot depend on the field at the succeeding moments. This requirement can be expressed in the form

$$\vec{D}(t) = \vec{E}(t) + \int_0^{\infty} \vec{E}(t - \tau) f(\tau) d\tau. \quad (1.58)$$

Separation of the first term on the right-hand side is not important. It only simplifies the formulas given below. The essential is that only positive values of the integration variable τ participate in the integral, so that only the field values at the time moments less than t are considered. The function $f(\tau)$ may be arbitrary but such that the integral converges.

For nonharmonic oscillations $\vec{D}(t)$ can be expressed as

$$\vec{D}(t) = \int_{-\infty}^{\infty} \vec{D}(\omega) e^{i\omega t} d\omega \quad (1.59)$$

(for simplicity, we denote the function and its Fourier spectrum by the same symbol). The field $\vec{E}(t)$ can be written in a similar form. Inverting (1.59) gives

$$\vec{D}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{D}(t) e^{-i\omega t} dt. \quad (1.60)$$

A similar formula exists also for $\vec{E}(\omega)$. Multiply both sides of equality (1.59) by $\exp(-i\tilde{\omega}t)$ and integrate from $t = -\infty$ to $t = +\infty$. Here $\tilde{\omega}$ is any frequency. Interchanging the integration order over τ and t on the right-hand side, introducing a new integration variable $t' = t - \tau$, $dt' = dt$, instead of t in the inner integral, and using (1.60) and similar expression for $\vec{E}(\tilde{\omega})$, we obtain

$$\vec{D}(\tilde{\omega}) = \vec{E}(\tilde{\omega}) + \vec{E}(\tilde{\omega}) \int_{\tau=0}^{\infty} f(\tau) e^{-i\tilde{\omega}\tau} d\tau. \quad (1.61)$$

From the function $\varepsilon(\omega)$ definition it follows that

$$\vec{D}(\tilde{\omega}) = \varepsilon(\tilde{\omega}) \vec{E}(\tilde{\omega}). \quad (1.62)$$

Hence, there exists the integral representation of $\varepsilon(\omega)$:

$$\varepsilon(\omega) = 1 + \int_0^{\infty} f(\tau) e^{-i\omega\tau} d\tau. \quad (1.63)$$

Due to the causality principle, the integration in this formula is made over only $\tau \geq 0$. We determine the mathematical properties of the functions represented in the above form. If ω is replaced by the complex variable $\Omega = \omega' + i\omega''$ in (1.63), then the multiplier $\exp(i\omega''\tau)$ with $\tau \geq 0$ appears under the integral. At $\omega'' < 0$ this multiplier is smaller than unity and it tends to zero as $\omega'' \rightarrow -\infty$. Hence, for every value of the complex variable Ω lying in its lower half-plane, the integral in (1.63) converges and tends to zero on the infinitely remote half-circle. Thus, $\varepsilon(\omega) - 1$ is a function analytical in the lower half-plane ($\omega'' < 0$) of the complex frequency Ω including its real axis, and vanishing at $\omega'' \rightarrow -\infty$.

The value of the function having such properties at any point of the real axis (at $\omega'' = 0$) can be expressed in the form of the Cauchy integral (as the principal value) of its values on the whole real axis. Separating the real and imaginary parts in the obtained equality, we get the sought formulas, that is, the function $\varepsilon'(\omega) - 1$ as the integral of $\varepsilon''(\omega)$ and the function $\varepsilon''(\omega)$ as the integral of $\varepsilon'(\omega) - 1$. For instance, at any fixed frequency ω_0 ,

$$\varepsilon'(\hat{\omega}_0) - 1 = \frac{2}{\pi} \lim_{\alpha \rightarrow +0} \left\{ \int_0^{\omega_0 - \alpha} + \int_{\omega_0 + \alpha}^{\infty} \right\} \frac{\omega \varepsilon''(\omega)}{\omega^2 - \omega_0^2} d\omega. \quad (1.64)$$

In the above example leading to a paradox, it is assumed that $\epsilon''(\omega)$ equals zero not for all frequencies, and $\epsilon'(\omega)$ is the same for all frequencies, which contradicts (1.64).

Equation (1.64) and the similar expression for $\epsilon''(\omega)$ as an integral of $\epsilon'(\omega)^{-1}$ are known as the principal *Kramers–Kronig equations*. It must be kept in mind that in the theoretical physics books where the time dependence is taken in the form of $\exp(-i\omega t)$, the function $\epsilon(\Omega)$ is analytic and vanishes in the upper half-plane of the complex frequency Ω .

1.3

Idealized objects

In the high-frequency electrodynamics just like in any other physical theory, a number of idealizations is accepted, that is, some mathematical models are introduced having properties close to those of the real objects. It is easier to solve problems for such models than those for the real objects. Solutions of the idealized problems are assumed to be close to those of the real problems taking into account the details dropped at the idealization. Use of the idealized objects is possible if these details are not essential. However, this condition cannot always be formalized. As in any other physical theory, the introduction of idealized objects is highly based on the intuition.

1.3.1

Interface of two media

It follows from the Maxwell equations and the inductions finiteness that the fields \vec{E} and \vec{H} are continuous (even differentiable) at points where ϵ , μ , and χ are continuous. A surface on which ϵ and μ are discontinuous, that is, an interface of the two different media, can be considered as a limit image of the thin layer with very large gradients of ϵ and μ . On such a surface, some components of the fields \vec{E} and \vec{H} are discontinuous, that is, they have different values on the surface sides. Conditions for tangential and normal components are different. Conditions for tangential components follow from relations on both sides of the thin layer. As is known, these relations are obtained from the Maxwell equations written in the integral form. According to the *Stokes theorem*, the integral of the tangential component H_{t_1} taken over a closed contour passing through both the sides of the thin layer is equal to the integral of $(\text{rot } \vec{H})_{t_2}$ over the area inside the contour (shaded area in Fig. 1.1). Equation (1.14a) implies that it is equal to the integral of $ikD_{t_2} + (4\pi/c)j_{t_2}$. Here t_1 and t_2 are two tangential unit vectors perpendicular to each other. Since D_{t_2} and j_{t_2} are finite (we will return to this statement), the surface integral is proportional to the thickness of the layer and has the same order. Hence, the difference of

the components H_{t_1} has also the same order. If the thickness of the transition layer tends to zero, then this difference also does. This conclusion does not require for the gradients of ϵ and μ to be finite. It also holds for models with interface, in which the gradients are infinite and the result does not depend on the way of passing to the discontinuous functions ϵ and μ . The tangential components of the field \vec{H} (as well as of the field \vec{E}) are equal on both the sides of the interface

$$H_t^+ - H_t^- = 0, \quad (1.65a)$$

$$E_t^+ - E_t^- = 0. \quad (1.65b)$$

Here t is any tangential unit vector, that is, each equation contains two conditions. The signs + and – in the superscript refer to the different sides of the interface.

The normal components of the fields \vec{E} and \vec{H} may be discontinuous. The normal components of the inductions \vec{D} and \vec{B} are continuous

$$D_N^+ - D_N^- = 0, \quad (1.66a)$$

$$B_N^+ - B_N^- = 0. \quad (1.66b)$$

For instance, if μ is the same and only permittivity ϵ is different on the interface sides, then five of six components of the fields \vec{E} and \vec{H} are continuous and only the component E_N is discontinuous.

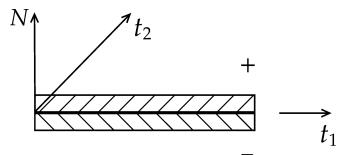


Fig. 1.1 Illustration to the Stokes theorem

Formulas (1.66) do not impose any additional conditions; they are a consequence of formulas (1.65): if conditions (1.65) are valid, then (1.66) are valid, too. For instance, D_N^+ can be expressed by the integral of $\text{rot } \vec{D}^+$ taken over a closed contour lying on the interface, that is, by the integral of H_t^+ . Since $H_t^+ = H_t^-$, then $D_N^+ = D_N^-$.

Note that in electrostatics, where no magnetic field exists, condition (1.66a) does not depend on (1.65b). It is seen after accurate transferring to the electrostatic equation. In electrostatics both conditions (1.65b) and (1.66a) must be fulfilled.

1.3.2

The impedance (one-side) boundary conditions

An often used idealization is the conditions connecting the tangential components of \vec{E} and \vec{H} on one side of the surface. The fields on the other side are not involved in such one-side conditions in contrast to the two-side conditions (1.65). The one-side conditions are the boundary conditions for the fields in a domain bounded by the surface.

In the simplest case, the *impedance conditions* have the form

$$E_{t_1} = -wH_{t_2}, \quad E_{t_2} = wH_{t_1}. \quad (1.67)$$

The factor w is called the *surface impedance*. It does not depend on the fields. It characterizes the medium lying on the other side on the surface. The unit vectors \vec{t}_1, \vec{t}_2 together with the normal \vec{N} directed outward the domain bounded by the surface on which (1.67) holds, make up a right-hand triple. In the local Cartesian coordinate system (x, y, z) , in which the body occupies the domain $z < 0$, we get $t_1 = x, t_2 = y, N = z$.

In general, the impedance w is complex $w = w' + iw''$. According to (1.54) and (1.67), the period-average flux of the energy outgoing from the domain through the unit area of the boundary equals

$$S_N = -\frac{c}{8\pi}w' |\vec{H}_t|^2. \quad (1.68)$$

If $w' > 0$, then $S_N < 0$, that is, the energy goes into the domain.

If the surface is anisotropic, that is, its properties are different in different directions, then w is not a scalar, but a two-dimensional tensor. The impedance boundary conditions (1.67) keep the simple form only if \vec{t}_1 and \vec{t}_2 are directed along the principal axes of the tensor. In this case, the coefficients in both equalities of (1.67) are equal to the principal values of the tensor and, in general, they are not equal to each other.

1.3.3

Skin layer

Conditions (1.67) are fully valid only in the case when the fields in the given domain do not depend on the fields on the opposite side of the boundary. The more precisely this requirement holds, the larger any parameter describing the medium on the opposite side is. An example of such a medium is the homogeneous material with

$$|\epsilon\mu| \gg 1. \quad (1.69)$$

The stronger this inequality, the more accurately conditions (1.67) hold on the boundary of such a medium.

Introduce the Cartesian coordinate system originating on the surface, in which the axes x and y are directed along the unit vectors \vec{t}_1 and \vec{t}_2 , respectively, and z -axis is normal to the surface and directed toward the medium with property (1.69) (Fig. 1.2). From the existence of boundary conditions (1.65) it follows that in this coordinate system, the fields in the tangential direction, that is, along the axes x and y , vary just as in the main domain. The velocity of varying in the z -direction is significantly larger. This follows from the wave equations (1.41), since under condition (1.69), the action of the operator Δ (see 1.39) is equivalent to multiplication by $-k^2\epsilon\mu$ only if at least one of the derivatives is large. Hence, $\partial^2/\partial z^2 \approx -k^2\epsilon\mu$, or

$$\frac{\partial}{\partial z} = \pm ik\sqrt{\epsilon\mu}; \quad (1.70a)$$

$$\frac{\partial}{\partial z} \gg \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial z} \gg \frac{\partial}{\partial y}. \quad (1.70b)$$

In this approximation the wave equation becomes an ordinary differential equation

$$\frac{d^2u}{dz^2} + k^2\epsilon\mu u = 0, \quad (1.71)$$

where u is any Cartesian component of the field. Its solution is a linear combination of the two functions $\exp(\pm ik\sqrt{\epsilon\mu}z)$. Only the second function describes the wave going deep into the medium and decreasing due to $\epsilon'' < 0$ in this direction $z \rightarrow -\infty$. Therefore, all the components of \vec{E} and \vec{H} depend on z as

$$\exp(-ik\sqrt{\epsilon\mu}z). \quad (1.72)$$

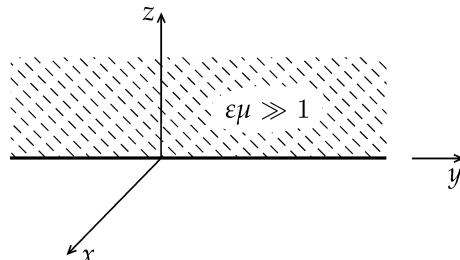


Fig. 1.2 Interface of two media

We derive the relations between different field components, which exist under conditions (1.72), (1.69), (1.70b). At $\vec{j}^{\text{ext}} = 0$, from (1.31) it follows that the ratio of E_z to E_x has the same order as that of $\partial/\partial x$ to $\partial/\partial z$, that is

$$\frac{E_z}{E_x} = O\left(\frac{1}{|\sqrt{\epsilon\mu}|}\right), \quad \frac{H_z}{H_x} = O\left(\frac{1}{|\sqrt{\epsilon\mu}|}\right), \quad (1.73)$$

where the second equality is written by analogy. Dropping in (1.31) the terms $\partial E_z / \partial x$, $\partial E_z / \partial y$, and so on, and considering (1.72), we obtain the following relation between the tangential components of the fields:

$$E_x = -\sqrt{\frac{\mu}{\epsilon}} H_y, \quad E_y = \sqrt{\frac{\mu}{\epsilon}} H_x. \quad (1.74)$$

The tangential components of the fields coincide on both the sides of the surface; therefore, the above relations are valid on the other side, as well. Thus, if (1.69) holds in some medium, then the impedance conditions (1.67) are valid on its boundary, and the impedance is equal to

$$w = \sqrt{\frac{\mu}{\epsilon}}. \quad (1.75)$$

As usual, $|\epsilon|$ is large because such is the conductivity σ , and $|\epsilon''| \gg \epsilon'$. It means that the conduction currents $\sigma \vec{E}$ are much larger than the displacement currents $i\omega \vec{E}$. For simplicity, we ignore ϵ' in comparison with ϵ'' in (1.30), that is, we assume that $\epsilon = -4\pi i\sigma/\omega$. Then the exponential multiplier (1.72) can be written as

$$\exp \left[-(1+i) \frac{z}{d} \right], \quad (1.76)$$

where the quantity d having the dimension of length equals

$$d = \sqrt{\frac{c}{2\pi k\mu\sigma}}. \quad (1.77)$$

The quantity d describes the velocity of the amplitude decreasing in the medium in the direction normal to the interface. It is called the *thickness of the skin layer* and is the main characteristics of the well-conducting material placed into the high-frequency field. The thickness d is zero for a conductor with infinite conductivity ($\sigma = \infty$). With increasing frequency, d decreases in inverse proportion to its square root ($d \sim \omega^{-1/2}$). The thickness of the skin layer is not large. For instance, $d = 0.4 \times 10^{-6}$ m for copper ($\sigma = 5 \times 10^{17}$ s $^{-1}$) at the frequency of 30 GHz ($\lambda = 0.01$ m). The skin layer is a thin layer on the good conductor surface, under which the high-frequency field almost does not penetrate. For a good conductor (with $\epsilon \approx -4\pi i\sigma/\omega$) the impedance is complex and proportional to the thickness of the skin layer:

$$w = \frac{1+i}{2} \mu k d. \quad (1.78)$$

1.3.4

Ideal conductor

The medium with $\sigma = \infty$ is called the *ideal conductor*. The impedance w is zero on its surface, and the boundary conditions (1.67) become

$$E_{t_1} = 0, \quad E_{t_2} = 0 \quad (1.79)$$

The field does not penetrate inside the ideal conductor, but from this it does not follow that the tangential magnetic field equals zero on its surface. At any depth and $\sigma = \infty, d = 0$, the field is detached from the exterior surface by the skin layer on which the tangential magnetic field has a jump. The current flowing in the skin layer remains finite if passing $\sigma \rightarrow \infty, d \rightarrow 0$. According to (1.74), the electric field tends to zero in the skin layer if $\sigma \rightarrow \infty$ as $\sigma^{-1/2}$. The conduction current density σE tends to infinity as $\sigma^{1/2}$. The thickness of the skin layer has the order $\sigma^{-1/2}$ (1.77), and the current flowing in the skin layer at $\sigma \rightarrow \infty$ does not depend on σ and remains finite.

The current with infinite volume density but with finite surface one flows on the ideal conductor surface. Denote the two-dimensional vector of the current volume density by \vec{I} , so that $I_t = \lim_{\sigma \rightarrow \infty} (j_t \cdot d)$. It can be expressed by the tangential magnetic field. From the integral form of the Maxwell equations it follows that

$$I_x = -\frac{c}{4\pi} H_y, \quad I_y = \frac{c}{4\pi} H_x. \quad (1.80)$$

The surface current density is equal (with accuracy to the multiplier $c/4\pi$) to the magnetic field, and perpendicular to it on the surface. The field \vec{H} can be found after the problem with boundary conditions (1.79) is solved in the domain adjacent to the ideal conductor. The surface density of the current (1.80) can be determined by H_t .

The normal component B_N of the magnetic induction, as well as the tangential component of the electric field, equals zero on the surface of the ideal conductor. The normal component D_N of the electric induction, together with the tangential component of the magnetic field, differs from zero. According to (1.14a) and (1.80), D_N is proportional to the two-dimensional divergence of the surface current:

$$D_N = \frac{4\pi i}{\omega} \left(\frac{\partial I_x}{\partial x} + \frac{\partial I_y}{\partial y} \right). \quad (1.81)$$

Since the quantity in the brackets equals $-i\omega\rho_{\text{surf}}$, where ρ_{surf} is the surface density of the charge (1.18), equality (1.81) means that the surface charge density appears on the surface of the ideal conductor.

The above results are also valid for material with large displacement currents instead of the conduction currents, that is, for material in which $\epsilon' \gg 1$.

For simplicity, we assume that $\epsilon'' = 0$, that is, the conductivity of the material is zero, and there are no other losses in it. In this case the field is not concentrated in the thin layer on the surface; it exists in the entire volume, but it changes the sign fast with the z -coordinate increasing. According to (1.72), the thickness of the layer, in which the field is of the same sign, has the order $1/(k\sqrt{\epsilon\mu})$, being small. The tangential components of the induction \vec{D} are large, $D_t = \epsilon E_t$, that is, they have the order $\sqrt{\epsilon\mu}H_t$. The product of D_t and the thickness is finite; it does not depend on ϵ if ϵ is large. Therefore, the variation of the magnetic field H_t along the thickness of the layer is finite. This fact is analogous to the finiteness of the magnetic field variation over the thickness of the skin layer at $\sigma \gg 1$. The assertion “the impedance conditions (1.67) taking the form (1.79) at the limit $|\epsilon| \rightarrow \infty$, are valid on the surface of the object with $|\epsilon| \gg 1$,” does not depend on the relation between ϵ' and ϵ'' . It is true either for the metallic objects or for the objects with large real permittivity.

1.3.5

Singularities of fields near the edge or vertex

One of the idealized objects is also an edge which is a fracture line of the two media interface (Fig. 1.3(a)). One of the surface curvatures is infinite on this line. As will be shown below, some of the field components become infinite at the edge, that is, they have a singularity there. Behavior of the fields when becoming infinite at some point is a local property, that is, it only depends on the medium structure near the point. When approaching the edge, the field increases with the velocity depending only on the angle between planes, material of the media separated by these planes, and polarization of the field near the edge. Therefore, the field structure at the edge can be investigated with use of a two-dimensional model, when considering the edge to be a straight line, and the fields to be independent of the coordinate parallel to this line.

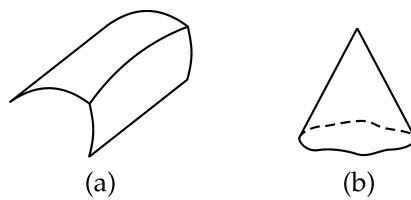


Fig. 1.3 Geometry of the edge and vertex

The fields become infinite not only on the edge but also on the linear source. The distinction between the fields structure near the edge and that in the source neighborhood consists in the fact that the density of the electromag-

netic field energy is integrable near the edge, that is

$$\int |\vec{E}|^2 dV < \infty, \quad (1.82a)$$

$$\int |\vec{H}|^2 dV < \infty, \quad (1.82b)$$

whereas in the source neighborhood these conditions are violated. The integrals are taken over an arbitrary volume containing the edge segment, or the source.

Conditions (1.82) are necessary and sufficient for the line on which some components of the fields become infinite, not to be a linear source, but an edge. Below we show that if these conditions hold, then the energy flux out of the line is zero. Note that in contrast to (1.82) the equality to zero of the energy flux is only necessary (but not sufficient) for the line to be an edge. In certain cases it is valid for a linear source, too. Although the linear source does not radiate in such cases, the field is so large near it that condition (1.82) violates.

Assuming that (1.82) holds, we find the field structure near the edge. We restrict ourselves to the case of a metallic wedge, that is, a wedge on the edges of which condition (1.79) is valid (Fig.1.4). Align the z -axis of the cylindrical coordinate system along the wedge edge. The angle φ is measured from a wedge face. Denote the angle between the wedge faces by α , and its complement to 2π by $\tau\pi$, that is, $\tau = (2\pi - \alpha)/\pi$. Then the equations of the wedge faces are $\varphi = 0$ and $\varphi = \tau\pi$, respectively. For the half-plane, $\alpha = 0$, $\tau = 2$; for the wedge, $\alpha > 0$, $\tau < 2$.

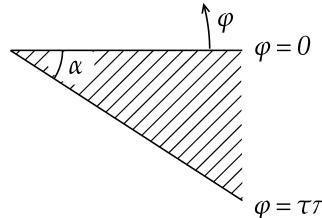


Fig. 1.4 Cross-section of the wedge

For the fields independent of z , the equation system (1.32) with $\vec{j}^{\text{ext}} = 0$ split into two independent systems. One of them contains the components E_z , H_r , H_φ , and the other – H_z , E_r , E_φ . If $\partial/\partial z \equiv 0$ and $H_z \equiv 0$ for the field in which the wedge is placed, then only the components E_z , H_r , H_φ appear (the *electric polarization*). The second component triple appears for the *magnetic polarization*.

For the electric polarization,

$$H_r = -\frac{1}{ik\mu} \frac{\partial E_z}{r \partial \varphi}, \quad (1.83a)$$

$$H_\varphi = \frac{1}{ik\mu} \frac{\partial E_z}{\partial r}, \quad (1.83b)$$

and for the magnetic one,

$$E_r = \frac{1}{ik\epsilon} \frac{\partial H_z}{r \partial \varphi}, \quad (1.84a)$$

$$E_\varphi = -\frac{1}{ik\epsilon} \frac{\partial H_z}{\partial r}. \quad (1.84b)$$

The components E_z and H_z satisfy the same wave equation (1.41), which at $\partial/\partial z \equiv 0$ obtains the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial z} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + k^2 \epsilon \mu u = 0, \quad (1.85)$$

where u is one of the components E_z, H_z . In the edge neighborhood (i.e., at $kr \ll 1$), the last summand in (1.85) can be neglected at $kr \ll 1$. Then the functions $\Phi(r) \cos(n\varphi), \Phi(r) \sin(n\varphi)$ satisfy equation (1.85) with the conditions $E_z = 0$ (electric polarization) and $E_r = 0$ (magnetic polarization) at $\varphi = 0, \varphi = \tau\pi$.

Begin from the case of the electric polarization. In general, E_z can be a superposition of the terms

$$r^n \sin(n\varphi), \quad r^{-n} \sin(n\varphi), \quad (1.86)$$

where $n = 1/\tau, 2/\tau, \dots$ However, condition (1.82b) implies that near the edge, E_z contains only the positive powers of r , with $r^{1/\tau}$ to be the lowest one. Indeed, if E_z contained terms proportional to r^{-n} , then, according to (1.83), H_r and H_φ would contain terms proportional to r^{-n-1} , and the integral (1.82b) would have a form $\int_0^\infty r^{-(2n+1)} dr$ and diverge. The magnetic field has a singularity $r^{1/\tau-1}$ at $\tau > 1$, that is, for acute angles ($\alpha < \pi$). The singularity order is higher for the half-plane ($\tau = 2$) than for the wedge with $\alpha > 0$. The magnetic field has a singularity $1/\sqrt{r}$ near the half-plane edge. The current I_z proportional to H_z has a singularity as well. On both planes near the edge, the current flows in the same direction parallel to the edge; its surface density becomes infinite when approaching the edge.

The energy flux out of the edge is zero at any α . Since $H_\varphi^* \sim \partial E_z^*/\partial r$, the energy flux (1.54) is proportional to the imaginary part of the integral $\int_0^{2\pi} E_z \cdot \partial E_z^*/\partial r \cdot r d\varphi$. The integrand involves only the positive powers of r ;

therefore, the integral tends to zero as $r \rightarrow 0$. The energy flux equals the integral of $\operatorname{Re}(\operatorname{div} \vec{S})$ over the volume inside the surface. According to (1.56), $\operatorname{div} \vec{S} = 0$ and therefore the integral cannot depend on r ; hence it equals zero.

Thus, if the field does not have a singularity in the domain or its singularity is so weak that condition (1.82) holds, then the energy flux out of the domain equals zero. We show that this assertion is also valid in the magnetic polarization case, that is, at $E_z = 0$. Since the component H_z consists of the terms $r^{\pm n} \cos n\varphi$, the boundary condition $E_z = 0$ at $\varphi = 0$ and $\varphi = \tau\pi$ holds for the same values of n ($n = 1/\tau, 2/\tau, \dots$) as for the electric polarization. For condition (1.82a) to hold, H_z must not contain the terms r^{-n} . When approaching the edge, a singularity appears in the electromagnetic field (1.84), which depends on r as $r^{1/\tau-1}$ at small r . For the half-plane $\tau = 2$, this dependence is the strongest when the field is proportional to $1/\sqrt{r}$. Then the integral in (1.82a) converges, and the energy flux outgoing from the edge is equal to zero.

The surface current flows perpendicularly to the edge, $I_r \neq 0$, for this polarization. The current has no singularities; the function I_r is finite. The surface density ρ_{surf} of the charge proportional to E_φ at $\varphi = 0$ and $\varphi = \tau\pi$ (i.e., to the normal component of the electric field) has a singularity. According to (1.81), ρ_{surf} is proportional to the two-dimensional divergence of the surface current density. Since I_r is proportional to \sqrt{r} for the half-plane, ρ_{surf} is proportional to $1/\sqrt{r}$.

The simple formulas for the field near the edge exist only in the case of the edges with the ideal-conducting faces. If the faces are impedance or they separate two dielectrics, then the explicit expressions for the field exist only in the electrostatic approximation, analogous to (1.86). Just as for the metallic wedge, from these formulas it follows that if the conditions (1.82) hold, then the field can have only such singularity, at which there is no energy flux out of the edge. In this case the singularity can be even a little stronger than for metallic faces. However, for the edge with arbitrary boundary conditions on its faces, conditions (1.82) are necessary and sufficient for the field singularities in the edge neighborhood to be weaker than those near the linear source.

Similar result is valid for another idealized model, namely, for a pyramid or cone vertex (Fig. 1.3(b)), that is for a point in which both the surface curvatures are infinite. At this point the fields are also singular; some of their components become infinite. At a singular point of another kind, namely, at the point source, the fields are infinite, as well. Similarly as for the linear source or edge, the same condition (1.82) holds for the vertex and does not hold for the point source.

We give another, more formal, proof of the fact that if in the neighborhood of a point or line the fields are finite or have singularities weak enough for satisfying condition (1.82), then the energy flux out of this point or line is zero. This proof is shorter, but it does not allow us to determine nature of the surface currents and charges.

It follows from the Maxwell equations and identity (1.45) that

$$\operatorname{div}(\vec{E} \times \vec{H}^*) = ik \left(|\vec{E}|^2 - |\vec{H}|^2 \right) \quad (1.87)$$

at $j^{\text{ext}} = 0, \varepsilon = 1, \mu = 1$. If the fields are finite everywhere, then this equality is valid at any point and it can be integrated over any volume. Transforming the integral on the left-hand side and taking the real part of the obtained equality, we get $\int S_N dS = 0$, where \vec{S} is the Poynting vector and the integral is taken over a closed surface having no singularities inside. The surface interior surrounds a singular point (vertex) or singular line (edge). Conditions (1.82) imply that the integral has a limit when the surface interior is contracting to a point or line. The integral over the exterior part of the surface remains zero, which means that the energy flux out of the vertex or edge equals zero.

1.3.6

The line current and the point current

Above we have considered the additional conditions which are required in the formulation of electromagnetic problems in the case when the idealized objects are introduced into the theory. Since the Maxwell equations are not satisfied on such objects, the fields can have singularities or discontinuities. On the surfaces, where the medium properties are discontinuous, the normal components can also be discontinuous, and only the tangential components remain to be continuous. For instance, on the impedance surfaces the fields are subjected to the one-side boundary conditions. At points and on lines, where the media interface has infinite curvature, the fields tend to infinity but not so fast for the energy density to be nonintegrable. The radiation condition (1.33) implies that if the field exists at the infinite point, then it must have a concrete structure in the infinity “neighborhood.”

In each of the above cases, the local conditions imposed on the fields near the idealized objects exclude the existence of the extrinsic currents on them. The continuity of H_t on the interface implies that there is no extrinsic surface current on it. Recall that the current flowing on the surface of the ideal conductor is an induced one. The radiation conditions mean that there is also no extrinsic current at the infinite point. The conditions near the edge or vertex imply that there are neither line nor point extrinsic currents there.

In this subsection we consider the field structure near the true extrinsic currents. If the volume density \vec{j}^{ext} does not become infinite, then the fields have no singularities. In the neighborhood of both the *line extrinsic current* and the *point extrinsic* one, the fields tend to infinity so fast that the conditions (1.82) do not hold. Nature of the infinity does not depend on the existence of the other objects; it is local for the edges and vertices.

In the case of the line extrinsic current, the fields can be found from the problem about a straight-line current of a constant value. Such a current is an idealized model. The volume density j_z^{ext} is infinite; it exists in an infinitely thin straight-line cylinder. The product of j_z^{ext} and the cross-section area of the cylinder is finite. It is called the *line current density* and has the only component I_z , independent of the cross-section shape.

The fields are infinite near the line current. They are found from the Maxwell equations in which \vec{j}_z is proportional to $\delta(r)$, so that the integral of \vec{j}_z over any part of the surface $z = \text{const}$ containing the point $r = 0$, is finite and equal to I_z^{ext} . The fields depend neither on z nor on the azimuthal coordinate φ , and have only the components H_φ and E_z . In the domain of interest, where $kr \ll 1$, the field E_z is much smaller than H_φ ; it has a singularity so weak that the electric energy density is integrable. The Maxwell equations are reduced to the equation for the direct currents $\text{rot } \vec{H} = (4\pi/c)\vec{j}^{\text{ext}}$. Integrating it over a circumference of a small radius a , and using the definition of I^{ext} , we get

$$H_\varphi = \frac{2}{c} \cdot \frac{1}{a} I_z^{\text{ext}}. \quad (1.88)$$

The magnetic energy density is proportional to a^{-2} , the integrand has the order a^{-1} , and the integral diverges at $a \rightarrow 0$.

The question about the energy flux from the line current is not as simple as in the case of the edge. In general, this flux is not zero. It cannot be determined by the near field and, therefore, it cannot be calculated using values of the fields at $kr \ll 1$. The flux depends on the fields created by other sources at the place where the line current is located. The integral of the vector $\vec{E} \times \vec{H}^*$ flux through the surface $z = a$ is proportional to $\ln ka$; it tends to infinity as $ka \rightarrow 0$. However, at $kr \ll 1$ the vectors \vec{E} and \vec{H} are in-quadrature, that is, the real part of their product equals zero. The energy flux is determined by the next terms of the expansion of these fields in powers of kr , that is, by the terms having no singularities.

If the line current is located in vacuum and there are no other objects in its neighborhood, then the energy flux out of it can be calculated by expressing $H_\varphi(r)$ and $E_z(r)$ by exact formulas valid for the direct straight-line current at all r . These formulas contain the Hankel functions. Omitting the derivations, we only point out that the flux per unit length equals $\pi k / (2c) \cdot I^2$.

If there are any other objects in the field of the extrinsic current, then the energy flux created by this current changes, so that the factor at I^2 differs from $\pi k / (2c)$. The fields of currents, induced on those objects, interfere with the field of the line current. For instance, if the current is located near a mirror, then the induced field is the field of the current reflected in this mirror. Since in the neighborhood of the line current the field of other currents is finite, the type of the field singularity does not change. However, in this case, at the

same value of J^{ext} the radiated energy can be larger or smaller than in the case when the other fields are absent, or even equal to zero.

To determine the energy flux, given up by a given extrinsic current located in the field of other sources, it is sufficient to know the value of this field in the place where the current is located. This statement follows from formula (1.56), according to which the energy flux from the domain V through its surface S equals

$$\int_S S_N dS = -\frac{1}{2} \int_V \operatorname{Re} (\vec{j}^{\text{ext}*} \cdot \vec{E}) dV \quad (1.89)$$

if there are no losses in the medium. The immediate use of this formula for the line current is practically impossible, since \vec{E} has singularities at the points where \vec{j}^{ext} is infinite, but the volume of this domain is infinitesimal. Divide the field \vec{E} into the field \vec{E}_0 of the current \vec{j}^{ext} in vacuum and field \vec{e} of all other currents, $\vec{E} = \vec{E}_0 + \vec{e}$. Since the field \vec{E} participates in (1.85) linearly, the expression for the flux is divided into two terms. The first of them is the energy flux created by the field of the current J^{ext} itself in vacuum. The second one is easily calculated in any concrete problem, since the field \vec{e} has no singularities near the line current. The first term was given above for the direct straight-line current \vec{j}^{ext} , so that

$$\int_S S_N dS = \frac{\pi k}{2c} |\vec{I}^{\text{ext}}|^2 - \frac{1}{2} \operatorname{Re} \int_V \vec{j}^{\text{ext}*} \cdot \vec{e} dV. \quad (1.90)$$

In the second term the field \vec{e} of the induced currents can be taken out of the integral, so that

$$\int_S S_N dS = \frac{\pi k}{2c} |\vec{I}^{\text{ext}}|^2 - \frac{1}{2} \operatorname{Re} [\vec{j}^{\text{ext}*} \cdot e_z(0)]. \quad (1.91)$$

For a nonstraight-line current the first term in (1.91) is different, the second one is an integral over the line length, and the flux should be calculated per whole line, not per its unit length as in the last two formulas. However, the structure of the formula for the flux of the energy radiated by the line current in the presence of other fields is kept.

Finally, we consider the last idealized model introduced into the theory, namely, the *point current* (elementary electric dipole). Formally, we have already used this notion when considering the reciprocity conditions.

Let the volume density \vec{j}^{ext} differ from zero only in some domain, and \vec{j}^{ext} do not change its direction there. Introduce a notion of the *elementary dipole* as a source whose largest linear size tends to zero, and the amplitude of its volume current increases infinitely in such a way that the product of the source volume

and its density remains finite. This product is called the *dipole momentum* \vec{P} . The field of the elementary dipole does not depend on the intermediate shapes which the domain obtains while passing to limit. The same image can be obtained if considering an infinitely short segment l of the line current with infinite density I , such that the product $I \cdot l = P$ is finite.

The fields have strong singularities near the elementary dipole. Approaching the dipole, the electric field increases as an inverse cube of the distance to it, whereas the magnetic field as an inverse square. The densities of both electric and magnetic energies are nonintegrable, that is, the two conditions (1.82) are violated. This violation distinguishes singularities of the fields near the elementary dipole from those near the vertex.

Similarly as for the line current, the energy flux from the dipole cannot be calculated using its near field. If the dipole is not located in the field of other sources so that its field does not interfere with another field, then the flux equals $k^2 / (3c) |\vec{P}|^2$. This formula will be derived later in the chapter devoted to the spherical waves. In the presence of other objects and fields created by the currents induced on them, the factor at $|\vec{P}|^2$ can be smaller or larger than $k^2 / 3c$, or even zero. Similarly as for the line current, if the field $\vec{e}(0)$ existing (in the absence of the dipole) at the point of the dipole location is known, then the radiated energy can be found by the formula

$$\int_S S_N dS = \frac{k^2}{3c} |\vec{P}|^2 - \frac{1}{2} \operatorname{Re} [\vec{P}^* \cdot \vec{e}(0)] \quad (1.92)$$

following from (1.56).

1.4 Uniqueness and existence of solution

1.4.1 Uniqueness of solution

There are two types of problems in high-frequency electrodynamics – *direct* and *inverse* problems. In direct problems the fields created by the prescribed extrinsic currents in the given set of objects are found. In inverse problems the currents and objects should be found such that the arisen fields possess prescribed properties. For instance, some functionals of fields have to reach the maximal possible values under the given conditions.

Before solving problems of both types, we should clarify if the solution exists under formulated conditions, and if it is unique. The existence of a solution means that there are not too many conditions imposed and not too many

idealizations accepted, that is, they do not contradict each other. The uniqueness implies that there are enough conditions imposed in order that the only one solution exists, that is, there are not too few conditions. Below we show that the conditions formulated earlier and idealizations accepted provide the uniqueness, but still, only *almost* always. The case when this is not true, that is, when there exist more than one solution, is considered in the next subsection. Now we note that in this case it may turn out that there is no solution at all, that is, the accepted idealizations are not consistent.

It is necessary to prove that the Maxwell equations (1.29) supplemented by the conditions near a set of the idealized objects have the unique solution. These supplementary conditions are as follows: the radiation condition (1.33) describing the field near the infinite point, condition (1.65) of continuity of the tangential components on the interface, conditions (1.67) on the impedance surface and those (1.79) on the ideal conductor, conditions (1.82) near the edge or vertex as well as the conditions imposed on the constitutive constants ($\epsilon'' \leq 0, \mu'' \leq 0$) and the impedance ($w' \geq 0$). In fact, the assertion that at any t ($-\infty < t < \infty$) there exist harmonic fields dependent on t by (1.1) is also an idealization.

Since the Maxwell equations are linear, the assertion “the only one solution exists for the given extrinsic currents” is equivalent to “there exist only the solution $\vec{E} \equiv 0, \vec{H} \equiv 0$ if there are no extrinsic currents.” If two solutions exist for the same currents, then their difference is also a solution in the absence of currents. If there exists a solution for zero extrinsic currents, then its sum with a solution existing for certain current is also a solution for this current.

The assertion “there are no fields if the extrinsic currents are absent” is based, in fact, on the reason that if there are no sources increasing the field energy, and the energy leakage takes place at the same time, then the energy equals zero. Consequently, the fields are zero, either.

The proof does not use these energetic considerations explicitly. It is based on formula (1.56), which, at $\vec{j}^{\text{ext}} = 0$, becomes

$$\operatorname{div} \vec{S} = \frac{\omega}{8\pi} \left(\epsilon'' |\vec{E}|^2 + \mu'' |\vec{H}|^2 \right). \quad (1.93)$$

As we have noted earlier, this consequence from the Maxwell equations remains to be valid also for the chiral media if their constitutive equations have the form (1.28).

Integrate (1.93) over the volume V_a of the sphere with the radius a so large that there are no objects outside the sphere and fields have the form (1.33) on its surface. The interfaces of different media, as well as the edges and vertices, can lie inside the sphere. According to (1.82), the volume integrals

over domains where the fields are infinite converge. We use the *Gauss formula*

$$\int_{V_a} \operatorname{div} \vec{A} dV = \int_{S_a} A_N dS. \quad (1.94)$$

Since it requires that the vector \vec{A} is continuous inside the domain V_a , the interfaces must be separated out during the integration. When integrating over the two domains separated by the interface, the integral of A_N over this interface appears twice. The continuity of S_N follows from the continuity of E_t and H_t . However, the normals to the interface are oppositely directed in these two domains, and, therefore, the surface integrals of S_N cancel each other.

After integrating, formula (1.93) becomes

$$\int_{S_a} S_N dS + \int_{S_w} S_N dS = \frac{\omega}{8\pi} \int_{V_a} \left(\epsilon'' |\vec{E}|^2 + \mu'' |\vec{H}|^2 \right) dV. \quad (1.95)$$

Here S_w is an impedance surface if presented in the field.

Flux of the vector \vec{S} through the sphere S_a is proportional to the squared modulus of the functions $F_1(\vartheta, \varphi)$, $F_2(\vartheta, \varphi)$ in (1.33) describing the outgoing spherical wave. The flux equals

$$\frac{c}{8\pi k^2} \int_{S_a} \left(|F_1|^2 + |F_2|^2 \right) d\Omega, \quad d\Omega = \sin \vartheta d\vartheta d\varphi. \quad (1.96)$$

The flux of the vector \vec{S} into the impedance surface is calculated by (1.68). Hence, equality (1.95) (the *energy conservation law*) has the form

$$\int_{S_a} \left(|F_1|^2 + |F_2|^2 \right) d\Omega + k^2 \int_{S_w} |H_t|^2 w' dS - k^3 \int_{V_a} \left(\epsilon'' |\vec{E}|^2 + \mu'' |\vec{H}|^2 \right) dV = 0. \quad (1.97)$$

The left-hand side of (1.97) contains the sum of three nonnegative values: the radiation losses, the losses through the impedance surface and the volume losses. If at least one of the losses takes place in the system, then the above equality holds only if the field equals zero everywhere. If the system is open, that is, the field exists at infinity, then the radiation losses differ from zero. They would be zero if $F_1 \equiv 0$, $F_2 \equiv 0$. In this case the field would decrease faster than $1/R$. Then, as has already been marked in connection with formula (1.33), the field would be equal to zero in any domain (containing the infinite point), in which the field is analytical.

If the system contains the impedance surface, on which $w' > 0$, then there are no losses on it only if $H_t = 0$, and, consequently, $E_t = 0$ on the surface. Then, as can be shown, the field equals zero in any domain in which it is

analytical and which borders on this surface. If there are domains with $\epsilon'' < 0$ or $\mu'' < 0$, then the losses would be zero if $\vec{E} \equiv 0$ and $\vec{H} \equiv 0$ in these domains. Then the field would be zero also in any larger domain where it is analytical and which contains the domain with absorbent material.

Hence, if the fields \vec{E} and \vec{H} differ from zero, then the presence of any losses makes equality (1.97) impossible, that is, it proves the assertion about the solution uniqueness under the above assumptions.

1.4.2

Violation of the uniqueness theorem: eigenoscillations

The uniqueness can be violated if the field is nonanalytical. If there is an ideal conductor in the domain, then the analyticity on its surface is broken, due to the discontinuity of H_t . If there is a closed surface in the system, on which $E_t = 0$, that is, the closed surface of the ideal conductor, and there are no losses in the material inside this surface ($\epsilon'' = 0$, $\mu'' = 0$), then equality (1.97) may also hold for nonzero fields there. Then a solution to the Maxwell equations can exist in the domain without the extrinsic currents. Note that this takes place also if conditions (1.67) with $w' = 0$ hold instead of (1.79) on the surface. Further we do not mention about this possibility.

In such a domain (*resonator*) the homogeneous Maxwell equations can have nonzero solutions. In this case the *uniqueness theorem* is violated and, simultaneously, the assertion that the solution exists for any extrinsic currents, that is, the *existence theorem*, may be violated, too. In oscillation theory the absence of a solution is known as the “infinite resonance” in a system without losses. It occurs as a result of introducing certain idealizations turned inconsistent: harmonic time dependence, infinite conductivity, absence of losses, given extrinsic current.

Such a situation occurs in the resonator with fixed parameters only at certain frequencies, or, at a fixed frequency and certain values of some resonator parameter. At other frequencies or values of the parameters, the assertions about the existence and uniqueness of solution remain valid.

These specific combinations of the frequency and parameters are interesting just because they permit the existence of solutions to the homogeneous Maxwell equations, so-called *eigenoscillations*. The corresponding frequency values at the fixed resonator parameters are called the *eigenfrequencies*. A set of fields of eigenoscillations corresponding to different eigenfrequencies makes up a complete set of fields. Any field (with certain reservations) can be expanded in a convergent series by these fields. Introducing in an appropriate way the notion “product of two oscillations,” we can show that the product of two eigenoscillations equals zero (this assertion also needs to be specified), which is formulated as the *orthogonality* of eigenoscillations. Here the notion

of the orthogonality has more general meaning than the vectorial orthogonality.

At the frequency, being not an eigenfrequency, the solutions of direct problems can be sought in the form of expansions by the fields of eigenoscillations. This method for solving the problems, known as the *spectral method*, will be considered in Section 4.1.

A set of the eigenoscillations may correspond not only to the different values of the frequency k_n ($n = 1, 2, \dots$) at fixed parameters describing the domain where these oscillations exist. Let, for instance, a dielectric body with permittivity ϵ be located in a closed resonator with ideal-conducting walls. For any given frequency, the values of the permittivity ϵ_n ($n = 1, 2, \dots$) can be found, at which the eigenoscillations exist. The fields of all eigenoscillations corresponding to different values of ϵ_n at the fixed k also make up a complete set of fields. The field in a resonator with certain permittivity ϵ excited by an extrinsic current, can be sought in the form of a series by the these fields. In such a method the values ϵ_n are the eigenvalues and all the eigenoscillations correspond to the same frequency equal to the frequency of excitation current. Other parameters can also play a role of the eigenvalues, for instance, the wall impedance w_n ($n = 1, 2, \dots$). These generalizations of the spectral method are expedient to use in the theory of *open resonators*, when the field exists at infinity. We return to these questions in Chapter 4.

