Paper exercise:

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1.

To show:

equation *

$$\int \mathrm{Norm}_{\omega_t}\left[\mu_p + \Psi\omega_{t-1}, \Sigma_p\right] \mathrm{Norm}_{\omega_{t-1}}\left[\mu_{t-1}, \Sigma_{t-1}\right] d\omega_{t-1}$$

$$= \int \mathrm{Norm}_{\omega_t}\left[\mu_p + \Psi(\Psi^{-1}(\omega_t - \mu_p)), \Sigma_p\right] \mathrm{Norm}_{\omega_t}\left[\Psi\mu_{t-1} + \mu_p, \Psi\Sigma_{t-1}\Psi^t\right] d\omega_t$$

Proof:

equation 1:

$$\mathrm{Norm}_{\omega_t}\left[\mu_p + \Psi\omega_{t-1}, \Sigma_p\right] = \mathrm{Norm}\left[\mu_p + \Psi(\Psi^{-1}(\omega_t - \mu_p)), \Sigma_p\right]$$

because $(\omega_t = \Psi\omega_{t-1} + \mu_p) \Leftrightarrow \Psi^{-1}(\omega_t - \mu_p) = \omega_{t-1}$

equation 2:

$$\mathrm{Norm}_{\omega_{t-1}}\left[\mu_{t-1}, \Sigma_{t-1}\right] = \frac{1}{(2\pi)^{k/2}|\Sigma_{t-1}|^{1/2}} \exp\left(-0{,}5\,(\cancel{\omega}_{t-1} - \mu_{t-1})^t \Sigma_{t-1}^{-1}(\omega_{t-1} - \mu_{t-1}\right)$$

$$= \frac{1}{(2\pi)^{k/2}|\Sigma_{t-1}|^{1/2}} \exp\left(-0{,}5\cdot(\Psi(\omega_{t-1} - \mu_{t-1}))^t \Psi^{t^{-1}} \Sigma_{t-1}^{-1} \Psi^{-1}(\Psi(\omega_{t-1} - \mu_{t-1}))\right)$$

$$= \frac{1}{(2\pi)^{k/2}|\Sigma_{t-1}|^{1/2}} \exp\left(-0{,}5\cdot\underbrace{(\omega_t - (\Psi\mu_{t-1} + \mu_p))}_{\Psi\omega_{t-1} = \omega_t - \mu_p}{}^t (\Psi\Sigma_{t-1}\Psi^t)^{-1}(\omega_t - (\Psi\mu_{t-1} + \mu_p))\right)$$

$$= \frac{1}{(2\pi)^{k/2}\underbrace{|\Psi\Sigma_{t-1}\Psi^t|^{1/2}}} \exp\left(-0{,}5\cdot(\omega_t - (\Psi\mu_{t-1} + \mu_p))^t (\Psi\Sigma_{t-1}\Psi^t)^{-1}(\omega_t - (\Psi\mu_{t-1} + \mu_p))\right)$$

$$1 = \det(\Psi) = \sqrt{\det(\Psi\cdot\Psi^t)} = |\Psi\Psi^t|^{1/2}$$

$$= \mathrm{Norm}_{\omega_t}\left[\Psi\mu_{t-1} + \mu_p, \Psi\Sigma_{t-1}\Psi^t\right]$$

With equation 1 and 2 we can ~~proo~~ show equation *.

2.

To show:

$$\int \mathrm{Norm}_x[a,A]\ \mathrm{Norm}_x[b,B]\ dx$$

$$= \cancel{\int} \mathrm{Norm}_a[b, A+B] \int \mathrm{Norm}_x[\Sigma_*(A^{-1}a+B^{-1}b), \Sigma_*]\ dx$$

where $\Sigma_* = (A^{-1}+B^{-1})^{-1} = A+B$

equation *

Proof:

$$\mathrm{Norm}_x[a,A]\cdot \mathrm{Norm}_x[b,B]$$

$$= \frac{1}{(2\pi)^{k/2}\,|A|^{1/2}\,(2\pi)^{k/2}\,|B|^{1/2}} \exp\left(-0{,}5\,(x-a)^t A^{-1}(x-a) - 0{,}5\,(x-b)^t B^{-1}(x-b)\right)$$

$$= \frac{1}{(2\pi)^{k}\,|A|^{1/2}\,|B|^{1/2}} \exp\left(-0{,}5\left(a^t A^{-1}a + b^t B^{-1} b\right)\right) \exp\left(-0{,}5\left(x^t(A^{-1}+B^{-1})x - 2\underset{\text{symmetric}}{(A^{-1}a+B^{-1}b)}\,x\right)\right)$$

$$= \frac{1}{(2\pi)^{k}\,|A|^{1/2}\,|B|^{1/2}} \exp\left(-0{,}5\left(a^t A^{-1}a + b^t B^{-1} b\right)\right) \exp\left(0{,}5\cdot(A^{-1}a+B^{-1}b)(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)\right)$$

$$\cdot \underbrace{\exp\left(-0{,}5\left(x^t(A^{-1}+B^{-1})x - 2(A^{-1}a+B^{-1}b)\cancel{x}(A^{-1}a+B^{-1}b)(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)\right)\right)}_{\exp\left(-0{,}5\cdot\left(\left(x-(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)\right)^t (A^{-1}+B^{-1})\left(x-(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)\right)\right)\right)}$$

~~product of eigenvalues~~

$\cancel{|A|^{1/2}\cdot|B|^{1/2} = |A+B|^{1/2}}$

~~because~~ $\cancel{\det(A)^{1/2} = \prod_{j=1}^{k} \lambda_j^{1/2}}$

$$= \mathrm{Norm}_x\left[(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b), A+B\right] \cdot \frac{1}{(2\pi)^{k/2}\,|A|^{1/2}\,|B|^{1/2}\,\underbrace{|A^{-1}+B^{-1}|^{1/2}}}$$

because for the Norm_x we had to multiply $|(A^{-1}+B^{-1})^{-1}|^{1/2}$

$$\frac{1}{\det(A)} = \det(A^{-1})$$

$$\cdot \underbrace{\exp\left(-0{,}5\left(a^t A^{-1}a + b^t B^{-1}b\right) + 0{,}5\,(A^{-1}a+B^{-1}b)(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)\right)}$$

$$= -0{,}5\left((a-b)^t (A+B)^{-1}(a-b)\right.$$

$$= \mathrm{Norm}_x\left[(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b),\, A+B\right]$$

$$\cdot \underbrace{\frac{1}{(2\pi)^{k/2}\,|A+B|^{1/2}} \exp\left(-0{,}5\left((a-b)^t (A+B)^{-1}(a-b)\right)\right)}_{\mathrm{Norm}_a[b,\, A+B]}$$

$\square$

$$\frac{1}{(2\pi)^{k/2}|A|^{1/2}|B|^{1/2}\cdot|A^{-1}+B^{-1}|^{1/2}} = \frac{1}{(2\pi)^{k/2}|A|^{1/2}|B|^{1/2}\cdot(|A^{-1}|^{1/2}+|B^{-1}|^{1/2})}$$

$$= \frac{1}{(2\pi)^{k/2}(|B|^{1/2}+|A|^{1/2})} = \frac{1}{(2\pi)^{k/2}(|A+B|^{1/2})}$$

because $\det(A+B) = \det(A)+\det(B)$