Paper exercise :

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2. A function is submodular when it satisfies the equation: $P(\beta,\gamma)+P(\alpha,\delta)-P(\beta,\delta)-P(\alpha,\gamma)\geq 0$ for all $\alpha,\beta,\gamma,\delta$ such that $\beta>\alpha$ and $\delta>\gamma$. Show that

2.1. the Quadratic Function $P(\omega_m,\omega_n)=c(\omega_m-\omega_n)^2$ is submodular

2.2. the Potts model $P(\omega_m,\omega_n)=c(1-\delta(\omega_m-\omega_n))$ is not submodular, by providing a counter-example to the above criterion.

2.1. $\forall\,\alpha,\beta,\gamma,\delta:\ \beta>\alpha$ and $\delta>\gamma$

Show that: $P(\beta,\gamma)+P(\alpha,\delta)-P(\beta,\delta)-P(\alpha,\gamma)\geq 0$

$$P(\beta,\gamma)+P(\alpha,\delta)-P(\beta,\delta)-P(\alpha,\gamma)$$

$$=c\cdot(\beta-\gamma)^2+c\cdot(\alpha-\delta)^2-c\cdot(\beta-\delta)^2-c\cdot(\alpha\cdot\gamma)^2$$

$$=c\cdot\left((\cancel{\beta^2}-2\beta\gamma+\cancel{\gamma^2})+(\cancel{\alpha^2}-2\alpha\delta+\cancel{\delta^2})-(\cancel{\beta^2}-2\beta\delta+\cancel{\delta^2})-(\cancel{\alpha^2}-2\alpha\gamma+\cancel{\gamma^2})\right)$$

$$=c\cdot(-2\beta\gamma-2\alpha\delta+2\beta\delta+2\alpha\gamma)$$

$$=c\cdot 2\cdot(\beta\cdot(-\gamma+\delta)+\alpha\cdot(-\delta+\gamma))$$

$$=2c\cdot(\beta\cdot(-\gamma+\delta)-\alpha\cdot(-\gamma+\delta))$$

$$=2c\cdot(\underbrace{(\beta-\alpha)}_{>0}\cdot\underbrace{(-\gamma+\delta)}_{>0})\quad\geq 0\quad\text{for } c\geq 0$$

2.2. $$\delta(x)=\begin{cases}0 & \text{for } x\neq 0\\ 1 & \text{for } x=0\end{cases}$$

for $\delta>\beta=\gamma>\alpha$ is $\beta>\alpha$ and $\delta>\gamma$ and

$P(\beta,\gamma)+P(\alpha,\delta)-P(\beta,\delta)-P(\alpha,\gamma)=-c\leq 0$ for $c\geq 0$ $\Rightarrow$ not submodular for $c>0$

$$P(\omega_m,\omega_n)=\begin{cases}0 & \text{for } \omega_m=\omega_n\\ c & \text{for } \omega_m\neq\omega_n\end{cases}$$

Counter example: $c=1$ and $\alpha=1,\ \beta=\gamma=2,\ \delta=3$

$P(\beta,\gamma)+P(\alpha,\delta)-P(\beta,\delta)-P(\alpha,\gamma)=-1<0$ $\Rightarrow$ not submodular

3. Provide a graph using the alpha expansion method that encodes the inital state of 6 nodes (a, b, c, d, e, f) with the inital states $\beta \beta \gamma \alpha \alpha \gamma$ for the case where the lable α is expanded.