Foundations of Audio Signal Processing
Exercise sheet 6

Group members
Christopher Schmidt
Gerhard Mund
Robert Logiewa
Maren Pielka

1. Dezember 2018

Exercise 6.1

(a)

Need to show: $\widehat{f'}(\omega) = 2\pi i\omega \hat{f}(\omega)$.
Given: $f \in L^2(\mathbb{R})$ differentiable with $f' \in L^2(\mathbb{R})$.

$$
\begin{aligned}
\widehat{f'}(\omega) &= \int_{\mathbb{R}} f'(t)e^{-2\pi i\omega t}dt \\
&= f(t)e^{-2\pi i\omega t} \mid_{-\infty}^{\infty} - \int_{\mathbb{R}} f(t)(-2\pi i\omega)e^{-2\pi i\omega t}dt && \text{(Integration by parts)} \\
&= 0 - \int_{\mathbb{R}} f(t)(-2\pi i\omega)e^{-2\pi i\omega t}dt && (1) \\
&= 2\pi i\omega \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}dt && \text{(Constant factor)} \\
&= 2\pi i\omega \hat{f}(\omega). && \text{(Def. of the Fourier Transform)}
\end{aligned}
$$

In step (1), the first term evaluates to 0, because we assume that $f(t)$ has finite energy, meaning that $\lim_{t\to\pm\infty} f(t) = 0$.

(b)

Need to show: $\hat{f}'(\omega) = -2\pi i\hat{g}(\omega)$.

Given: $f \in L^2(\mathbb{R}), \hat{f}(\omega)$ differentiable and $g(t) = tf(t)$.

$$\begin{aligned}
\hat{f}'(\omega) &= \frac{d}{d\omega}\left(\int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}dt\right) \\
&= \int_{\mathbb{R}} \frac{d}{d\omega}(f(t)e^{-2\pi i\omega t})dt && \text{(Leibniz Integral Rule)} \\
&= \int_{\mathbb{R}} f(t)(-2\pi it)e^{-2\pi i\omega t}dt \\
&= -2\pi i \int_{\mathbb{R}} tf(t)e^{-2\pi i\omega t}dt && \text{(Constant factor)} \\
&= -2\pi i \int_{\mathbb{R}} g(t)e^{-2\pi i\omega t}dt && \text{(Def. of } g(t)\text{)} \\
&= -2\pi i\hat{g}(\omega). && \text{(Def. of the Fourier Transform)}
\end{aligned}$$

(c)

$$\begin{aligned}
\hat{f}(\omega) &= \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}dt \\
&= \int_{\mathbb{R}} f(t)\left(\cos(2\pi\omega t) - i \cdot \sin(2\pi\omega t)\right)dt \\
&= \int_{\mathbb{R}} Re(f)\cos(2\pi\omega t) + \cancel{Im(f)\sin(2\pi\omega t)} && \text{(because } f \text{ is real } \Rightarrow Im(f) = 0\text{)} \\
&\quad + i\left(Re(f)\sin(2\pi\omega t) + \cancel{Im(f)\cos(2\pi\omega t)}\right)dt \\
&= \int_{\mathbb{R}} f(t)\cos(2\pi\omega t) + i \cdot f(t)\sin(2\pi\omega t)dt
\end{aligned}$$

$$\begin{aligned}
\Rightarrow Re(\hat{f}) &= \int_{\mathbb{R}} f(t)\cos(2\pi\omega t)dt \\
&= \int_{\mathbb{R}} f^+(t)\cos(2\pi\omega t) + f^-(t)\cos(2\pi\omega t)dt && \text{(} f \text{ split in even and odd)} \\
&= \int_{\mathbb{R}} \underbrace{f^+(t)\cos(2\pi\omega t)}_{\text{even}}dt + \cancel{\int_{\mathbb{R}} \underbrace{\overbrace{f^-(t)}^{\text{odd}}\overbrace{\cos(2\pi\omega t)}^{\text{even}}}_{\text{odd}}dt} && \text{(because odd integrals are 0)} \\
\Rightarrow Im(\hat{f}) &= \int_{\mathbb{R}} f(t)\sin(2\pi\omega t)dt \\
&= \cancel{\int_{\mathbb{R}} \underbrace{\overbrace{f^+(t)}^{\text{even}}\overbrace{\sin(2\pi\omega t)}^{\text{odd}}}_{\text{odd}}dt} + \int_{\mathbb{R}} \underbrace{\overbrace{f^-(t)}^{\text{odd}}\overbrace{\sin(2\pi\omega t)}^{\text{odd}}}_{\text{even}}dt
\end{aligned}$$

$\Rightarrow Re(\hat{f})$ is even and $Im(\hat{f})$ is odd. $\square$

(d) Show that $\hat{f}$ is real, if f is real:

$$\begin{aligned}
\hat{f}(\omega) &= \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}dt \\
&= \int_{\mathbb{R}} f(t)\cos(2\pi\omega t) + i\cdot f(t)\sin(2\pi\omega t)dt && \text{(see previous part)} \\
&= \int_{\mathbb{R}} \underbrace{f(t)\cos(2\pi\omega t)}_{\text{even}} dt + \cancel{i\int_{\mathbb{R}} \underbrace{f(t)\sin(2\pi\omega t)}_{\text{odd}} dt} \\
&\Rightarrow \hat{f} \text{ is real.}
\end{aligned}$$

Show that $\hat{f}$ is even:

$$\begin{aligned}
\hat{f}(\omega) &= \int_{t=-\infty}^{t=\infty} f(t)e^{-2\pi i\omega t}dt \\
&= \int_{t=-\infty}^{t=\infty} f(-t)e^{-2\pi i\omega t}dt && \text{(because } f \text{ is even)} \\
&= \int_{u=\infty}^{u=-\infty} -f(u)e^{-2\pi i\omega(-u)}du && \text{(substitute } t \text{ with } -u) \\
&= \int_{u=-\infty}^{u=\infty} f(u)e^{-2\pi i(-\omega)u}du && \text{(flip the borders)} \\
&= \hat{f}(-\omega) \Rightarrow \hat{f} \text{ is even.}
\end{aligned}$$

$\Rightarrow \hat{f}$ is real and even.