Foundations of Audio Signal Processing
Exercise sheet 3

Group members
Christopher Schmidt
Gerhard Mund
Robert Logiewa
Maren Pielka

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Exercise 3.1

(a) $z = 4 + i4\sqrt{3}$

$r = \sqrt{4^2 + (4\sqrt{3})^2} = \sqrt{16 + 48} = \sqrt{64} = 8$
$4 = 8\cos(\phi) \Leftrightarrow \frac{1}{2} = \cos(\phi)$
$4\sqrt{3} = 8\sin(\phi) \Leftrightarrow \frac{\sqrt{3}}{2} = \sin(\phi)$
$\Rightarrow \phi = \frac{\pi}{3}$
$\Rightarrow z = 8e^{i\frac{\pi}{3}}$

(b) $z = (-1 + i\sqrt{3})^4$

$(-1 + i\sqrt{3})^4 = ((-1 + i\sqrt{3})^2)^2 = (-2 - i2\sqrt{3})^2 = -8 + i8\sqrt{3}$
$r = \sqrt{(-8)^2 + (8\sqrt{3})^2} = \sqrt{64 + 192} = \sqrt{256} = 16$
$-8 = 16\cos(\phi) \Leftrightarrow -\frac{1}{2} = \cos(\phi)$
$8\sqrt{3} = 16\sin(\phi) \Leftrightarrow \frac{\sqrt{3}}{2} = \sin(\phi)$
$\Rightarrow \phi = \frac{2\pi}{3}$
$\Rightarrow z = 16e^{i\frac{2\pi}{3}}$

(c) $z = \frac{(-1+i\sqrt{3})^4}{4+i4\sqrt{3}}$
$\frac{(-1+i\sqrt{3})^4}{4+i4\sqrt{3}} = \frac{-8+i8\sqrt{3}}{4+i4\sqrt{3}} = \frac{(-8+i8\sqrt{3})(4-i4\sqrt{3})}{(4+i4\sqrt{3})(4-i4\sqrt{3})} = \frac{64+i64\sqrt{3}}{64} = 1 + i\sqrt{3}$

$r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$
$1 = 2\cos(\phi) \Leftrightarrow \frac{1}{2} = \cos(\phi)$
$\sqrt{3} = 2\sin(\phi) \Leftrightarrow \frac{\sqrt{3}}{2} = \sin(\phi)$
$\Rightarrow \phi = \frac{\pi}{3}$

$\Rightarrow z = 2e^{i\frac{\pi}{3}}$

(d) $z = 2e^{i\pi/2}(1+i)$

First, transfer the left hand expression to Cartesian coordinates:
$a = 2\cos(\frac{\pi}{2}) = 2 * 0 = 0$
$b = 2\sin(\frac{\pi}{2}) = 2 * 1 = 2$
$\Rightarrow z = 2i(1+i) = 2i - 2$

Then, transfer this number back to Polar coordinates:
$r = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$
$-2 = \sqrt{8}\cos(\phi) \Leftrightarrow -\frac{2}{\sqrt{8}} = \cos(\phi)$
$2 = \sqrt{8}\sin(\phi) \Leftrightarrow \frac{2}{\sqrt{8}} = \sin(\phi)$
$\Rightarrow \phi = \frac{\pi}{4}$
$\Rightarrow z = \sqrt{8}e^{i\frac{\pi}{4}}$

Exercise 3.2

(a)

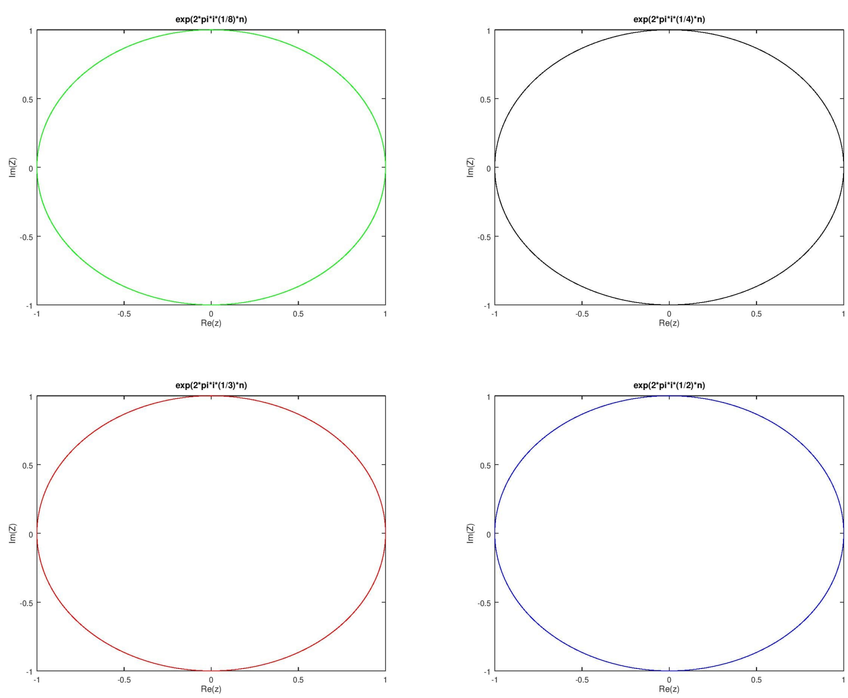

Figure 1: Plots of Exercise 3.2a

(b) For a periodic function there $\exists p \in \mathbb{N} : \forall n \in \mathbb{Z} : f(n+p) = f(n)$
Given: $f_\omega(n) := e^{2\pi i \omega n}$
To show: $\exists p \in \mathbb{N} : \forall n \in \mathbb{Z} : e^{2\pi i \omega n} = e^{2\pi i \omega (n+p)}$
Proof:

$$\begin{aligned} e^{2\pi i \omega n} &= e^{2\pi i \omega (n+p)} \\ &= e^{(2\pi i \omega n + 2\pi i \omega p)} \\ &= e^{2\pi i \omega n} \cdot e^{2\pi i \omega p} \\ \Rightarrow e^{2\pi i \omega p} &= 1 \end{aligned}$$

$$\begin{aligned} e^{2\pi i w p} &= \cos(2\pi \omega p) + i \sin(2\pi \omega p) = 1 \\ &\Rightarrow i \sin(2\pi \omega p) = 0 \text{ and } \cos(2\pi \omega p) = 1 \\ &\Rightarrow 2\pi \omega p = 2\pi n \text{ and } 2\pi \omega p = 4\pi n \qquad \forall n \in \mathbb{N} \end{aligned}$$

So we are looking for a p s.t. $\frac{1}{2} \cdot p \cdot \omega \in \mathbb{N}$.
Because $\omega \in \mathbb{Q}$, there exists a $\omega^{-1} \in \mathbb{Q}$ s.t. $\omega \cdot \omega^{-1} = 1$.
$\Rightarrow$ We choose $p = 2 \cdot \omega^{-1}$ and a $k \in \mathbb{N}$ s.t. $p' = p \cdot k \in \mathbb{N}$
$\Rightarrow p' \cdot \omega = 2k \in \mathbb{N}$
$\Rightarrow p'$ exists $\Rightarrow f_\omega(n)$ is periodic. $\square$

Exercise 3.3

$$\begin{aligned} \cos^3(x) &= \cos(x) \cdot \cos(x) \cdot \cos(x) \\ &= \left(\frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}\right)\left(\frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}\right)\left(\frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}\right) \\ &= \left(\frac{1}{4}e^{2ix} + \frac{1}{4}e^{0} + \frac{1}{4}e^{0} + \frac{1}{4}e^{-2ix}\right)\left(\frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}\right) \\ &= \left(\frac{1}{4}e^{2ix} + \frac{1}{4}e^{-2ix} + \frac{1}{2}\right)\left(\frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}\right) \\ &= \frac{1}{8}e^{3ix} + \frac{1}{8}e^{-ix} + \frac{1}{4}e^{ix} + \frac{1}{8}e^{ix} + \frac{1}{8}e^{-3ix} + \frac{1}{4}e^{-ix} \\ &= \frac{1}{4}\cos(3x) + \frac{3}{8}e^{-ix} + \frac{3}{8}e^{ix} \qquad // \text{ with } \cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \\ &= \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x) \qquad \square \end{aligned}$$