Foundations of Audio Signal Processing
Exercise sheet 5

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Exercise 5.1

(a) Need to show: $f \in L^1(\mathbb{R})$ and $f \notin L^2(\mathbb{R})$.

1.

$$\begin{aligned}
\int_{\mathbb{R}} |f(t)|^1 dt &= \int_{\mathbb{R}} f(t) dt && \text{(Definition of 1-norm)} \\
&= \begin{cases} \int_0^1 t^{-\frac{1}{2}} dt & \text{if } t \in (0,1] \\ \int_{\mathbb{R}} 0 dt & \text{otherwise} \end{cases} \\
&= \begin{cases} \lim\limits_{\epsilon \to 0} (2 * 1^{\frac{1}{2}} - 2 * \epsilon^{\frac{1}{2}}) = 2 & \text{if } t \in (0,1] \\ 0 & \text{otherwise} \end{cases} \\
&< \infty \text{ in both cases.}
\end{aligned}$$

$\Rightarrow f \in L^1(\mathbb{R})$.

2.

$$\begin{aligned}
\int_{\mathbb{R}} |f(t)|^2 dt &= \int_{\mathbb{R}} \langle f(t), f(t) \rangle^2 dt && \text{(Definition of 2-norm)} \\
&= \begin{cases} (\int_0^1 t^{-1} dt)^{\frac{1}{2}} & \text{if } t \in (0,1] \\ \int_{\mathbb{R}} 0 dt & \text{otherwise} \end{cases} \\
&= \begin{cases} \lim\limits_{\epsilon \to 0} (\ln(1) - \ln(\epsilon)) = 0 - (-\infty) = \infty & \text{if } t \in (0,1] \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$\Rightarrow f \notin L^2(\mathbb{R})$.

$\Rightarrow f \in L^1(\mathbb{R}) \backslash L^2(\mathbb{R})$.

(b) Need to show: $g \notin L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$.

1.

$$\int_{\mathbb{R}} |g(t)|^1 dt = \int_{\mathbb{R}} g(t) dt \qquad \text{(Definition of 1-norm)}$$
$$= \begin{cases} \int_1^\infty t^{-1} dt & \text{if } t \in [1, \infty) \\ \int_{\mathbb{R}} 0 dt & \text{otherwise} \end{cases}$$
$$= \begin{cases} \lim\limits_{n\to\infty} (\ln(n) - \ln(1)) = \infty - 0 = \infty & \text{if } t \in (0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow g \notin L^1(\mathbb{R})$.

2.

$$\int_{\mathbb{R}} |g(t)|^2 dt = \int_{\mathbb{R}} \langle g(t), g(t) \rangle^2 dt \qquad \text{(Definition of 2-norm)}$$
$$= \begin{cases} (\int_1^\infty t^{-2} dt)^{\frac{1}{2}} & \text{if } t \in [1, \infty) \\ \int_{\mathbb{R}} 0 dt & \text{otherwise} \end{cases}$$
$$= \begin{cases} \lim\limits_{n\to\infty} (-n^{-1} - (-1))^{\frac{1}{2}} = (0+1)^{\frac{1}{2}} = 1 & \text{if } t \in [1, \infty) \\ 0 & \text{otherwise} \end{cases}$$
$$< \infty \text{ in both cases.}$$

$\Rightarrow g \in L^2(\mathbb{R})$.

$\Rightarrow g \in L^2(\mathbb{R}) \backslash L^1(\mathbb{R})$.

Exercise 5.2

Need to show: $\sum\limits_{n\in\mathbb{Z}} |x_n|^p \overset{!}{<} \infty$ with $p \in [1, \infty]$ for given x

(a)

$$\lim_{n\to\infty} x(n) = \lim_{n\to\infty} e^n = \infty$$

$\Rightarrow \sum\limits_{n\in\mathbb{Z}} |x(n)|^p = \infty$ and $x \notin \ell^p(\mathbb{Z})$ for every $p \in [1, \infty]$

(b)

$$|x_n| = |e^{2\pi i n}| = |\cos(2\pi n) + i \cdot \sin(2\pi n)| = 1$$

$\Rightarrow \sum\limits_{n\in\mathbb{Z}} |x(n)|^p = \infty$ and $x \notin \ell^p(\mathbb{Z})$ for every $p \in [1, \infty]$

(c)

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 & \\
&\Rightarrow \sum_{n\in\mathbb{Z}} |\frac{1}{\sqrt{n}}|^1 < \infty && \text{(because } x(n) \text{ goes to zero, the sum is limited)} \\
&\Rightarrow x \in \ell^1(\mathbb{Z}) \\
&\Rightarrow \forall p \in [1,\infty] : x \in \ell^p(\mathbb{Z}) && \text{(because } \ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \cdots \subset \ell^\infty(\mathbb{Z})\text{)}
\end{aligned}
$$