Foundations of Audio Signal Processing
Exercise sheet 2

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Exercise 2.1

(a) Simplify the complex number:

$$\begin{aligned} 2e^{\frac{\pi}{2}i}(1+i) &= 2\left(\cos\left(\frac{\pi}{2}\right) + i\cdot\sin\left(\frac{\pi}{2}\right)\right)(1+i) \\ &= (0+2i)(1+i) \\ &= -2+2i \end{aligned}$$

Get the magnitude: $r = \sqrt{(-2)^2+2^2} = \sqrt{8}$
Get the angle: $\varphi = \underbrace{\pi +}_{\text{because } a<0} \arctan(-\frac{b}{a}) = \pi + \arctan(-\frac{2}{2}) = \frac{3\pi}{4}$
$\Rightarrow$ The number in polar coordinates is: $z = \sqrt{8}\cdot e^{\frac{3\pi}{4}i}$

(b)

$$\begin{aligned} z\cdot\overline{z} &= re^{i\phi}\cdot re^{-i\phi} \\ &= r^2\cdot\cancel{e^{i\phi}}\cdot\cancel{e^{-i\phi}} \\ &= r^2 \\ &= |z|^2 \qquad \square \end{aligned}$$

(c)

$$\begin{aligned} \frac{1}{2i}\left(e^{i\alpha}-e^{-i\alpha}\right) &= \frac{1}{2i}\left(\cancel{\cos(\alpha)} + i\sin(\alpha) - \cancel{\cos(-\alpha)} - i\sin(-\alpha)\right) \\ &= \frac{1}{2i}\left(i\sin(\alpha) - i\sin(-\alpha)\right) \\ &= \frac{1}{2i}\left(2i\sin(\alpha)\right) && \text{(because } \sin(-\alpha) = -\sin(\alpha)\text{)} \\ &= \sin(\alpha) \quad \square \end{aligned}$$

We can see in figure 1 of the unit circle, that $\sin(-\alpha) = -\sin(\alpha)$ holds.

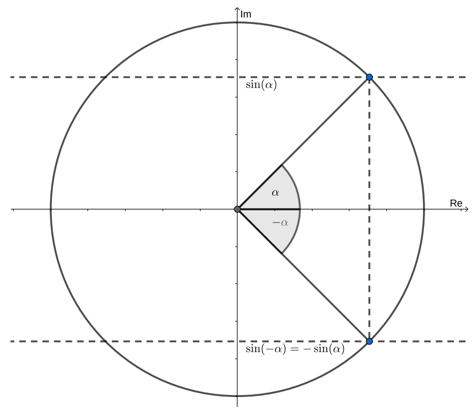

Figure 1: Unit circle

Exercise 2.2

(a)

Roots of unity for $n \in \mathbb{N}$: $C_n = \{e^{\frac{2k\pi i}{n}} = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})\}$

$n = 4:$

$\Omega_4^0 = \cos(0) + i\sin(0) = 1 + 0i = 1$
$\Omega_4^1 = \cos(\pi/2) + i\sin(\pi/2) = 0 + i = i$
$\Omega_4^2 = \cos(\pi) + i\sin(\pi) = -1 + 0i = -1$
$\Omega_4^3 = \cos(3\pi/2) + i\sin(3\pi/2) = 0 - i = i$

$\Omega_4^1 = i$ and $\Omega_4^3 = -i$ are primitive roots of unity for $n = 4$.

$n = 6:$

$\Omega_6^0 = \cos(0) + i\sin(0) = 1 + 0i = 1$
$\Omega_6^1 = \cos(\pi/3) + i\sin(\pi/3) = 1/2 + \sqrt{3}i/2$
$\Omega_6^2 = \cos(2\pi/3) + i\sin(2\pi/3) = -1/2 + \sqrt{3}i/2$
$\Omega_6^3 = \cos(\pi) + i\sin(\pi) = -1$
$\Omega_6^4 = \cos(4\pi/3) + i\sin(4\pi/3) = -1/2 - \sqrt{3}i/2$
$\Omega_6^5 = \cos(5\pi/2) + i\sin(5\pi/2) = 1/2 - \sqrt{3}i/2$

$\Omega_6^1 = 1/2 + \sqrt{3}i/2$ and $\Omega_6^5 = 1/2 - \sqrt{3}i/6$ are primitive roots of unity for $n = 6$.

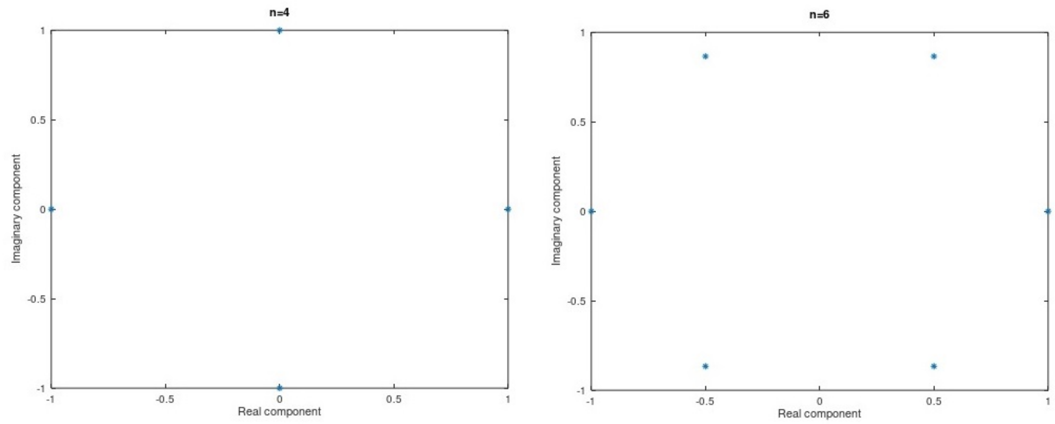

Figure 2: Roots of unity for n=4 and n=6, plotted

(b)

Need to show: For all $n > 1, \sum_{k=0}^{n-1} \Omega_n^k = 0$.

$$\begin{aligned}
\sum_{k=0}^{n-1} \Omega_n^k &= \sum_{k=0}^{n-1} e^{\frac{2k\pi i}{n}} \\
&= \sum_{k=0}^{n-1} (e^{\frac{2\pi i}{n}})^k \\
&= \frac{1-(e^{2\pi i/n})^n}{1-e^{2\pi i/n}} \quad (1) \\
&= \frac{1-1}{1-e^{2\pi i/n}} \quad (2) \\
&= 0.
\end{aligned} \tag{1}$$

(1) uses the geometric series formula: $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$.
(2) is possible because we know that $(\Omega_n^k)^n = 1$, by the definition of root of unity.