Foundations of Audio Signal Processing
Exercise sheet 4

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Exercise 4.1

$$
\begin{aligned}
\|\sum_{j=1}^{n} x_j\|^2 &= \langle \sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j \rangle && \text{(Def. of Inner Product Norm)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle x_i, x_j \rangle && \text{(Linearity of Inner Product)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \begin{cases} \langle x_i, x_j \rangle & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} && \text{(Because } x_i \text{ and } x_j \text{ are pairwise orthogonal)} \\
&= \sum_{j=1}^{n} \langle x_j, x_j \rangle \\
&= \sum_{j=1}^{n} = \|x_j\|^2 && \text{(Def. of Inner Product Norm)}
\end{aligned}
$$

$\square$

Exercise 4.2

(a) Mapping: $d(x,y) = |x-y| = |(a-c)+i\cdot(b-d)| = \sqrt{(a-c)^2+(b-d)^2}$

- Show non-negativity: $d(x,y) \geq 0$:
  Yes, because neither $(a-c)^2$ nor $(b-d)^2$ can be negative.
- Show identity of indiscernibles: $d(x,y)=0$ iff. $x=y$:
  - **case:** $x = y \Rightarrow a = c \wedge b = d \Rightarrow d(x,y) = \sqrt{0+0} = 0$
  - **case:** $x \neq y \Rightarrow a \neq c$ and/or $b \neq d \Rightarrow (a-c)^2 > 0$ and/or $(b-d)^2 > 0 \Rightarrow d(x,y) \geq 0$
- Show symmetry $d(x,y) = d(y,x)$:
  $d(x,y) = \sqrt{(a-c)^2+(b-d)^2} = \sqrt{(c-a)^2+(d-b)^2} = d(y,x)$
- Show triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$:

$$\begin{aligned}
d(x,z) &= |x-z| \\
&= |x-z+y-y| \\
&= |(x-y)+(y-z)| \\
&\leq |x-y|+|y-z| && \text{(because } |x| \geq 0\text{)} \\
&= d(x,y)+d(y,z)
\end{aligned}$$

$\Rightarrow$ The mapping is a metric $\square$

(b) Mapping: $d(x,y) = |x|\cdot|y|$

- Show identity of indiscernibles: $d(x,y)=0$ iff. $x=y$:
  Let $x = 0 + i\cdot 0$ and $y = 1 + i \cdot 1$.
  Then $d(x,y) = |0+i|\cdot|1+i\cdot 1| = 0 \cdot \sqrt{2} = 0$
  ↯Because $x \neq y$ it should be $d(x,y) \neq 0$
  $\Rightarrow$ not a metric.

(c) Mapping: $d(x,y) = \begin{cases} 1, \text{if } x \neq y \\ 0, \text{else} \end{cases}$

- Show non-negativity $d(x,y) \geq 0$ and identity of indiscernibles $d(x,y) = 0$ iff. $x = y$:
  Obviously by definition.
- Show symmetry $d(x,y) = d(y,x)$:
  Trivially, because $x = y \Leftrightarrow y = x$
- Show triangle inequality $d(x,z) \leq d(x,y)+d(y,z)$:
  Consider all possible cases:
  - **case:** $x = y = z$: $0 \leq 0 + 0$ ✓
  - **case:** $x = y$ and $z \neq x$: $1 \leq 0 + 1$ ✓
  - **case:** $x \neq y$ and $z = y$: $1 \leq 1 + 0$ ✓
  - **case:** $x \neq y$ and $z = x$: $0 \leq 0 + 1$ ✓
  - **case:** $x \neq y$ and $x \neq z \neq y$: $1 \leq 1 + 1$✓

$\Rightarrow$ The mapping is a metric $\square$