III] First-Order Logic and Ehrenfeucht-Fraïssé games

α) FO and syntactic equivalence

- FO: $a(x) \mid x<y \mid \varphi \vee \psi \mid \neg \varphi \mid \exists x.\ \varphi$
- qd
- $u, v \in \Sigma^*$, $i_0, \dots, i_{p-1} \in dom(u)$, $j_0, \dots, j_{p-1} \in dom(v)$

$u[i_0, \dots, i_{p-1}] \equiv_m^p v[j_0, \dots, j_{p-1}]$

$u[i_0, \dots, i_{p-1}] \equiv^p v[j_0, \dots, j_{p-1}]$

$u \equiv v$, $u \equiv_m v$

<u>Facts</u>: $\equiv_m^p$ has finitely many eq classes
$L \in FO$ iff disj° of (finitely many) $\equiv_m$ classes for some m

β) EF games

- $u = a_0 a_1 \cdots a_{|u|-1}$, $v = b_0 b_1 \cdots b_{|v|-1}$
- E-F game ~~with~~ m turns ~~with~~ on $u[i_0, \dots, i_{p-1}]$ and $v[j_0, \dots, j_{p-1}]$:
  - 2 players: Spoiler and Duplicator
  - m turns: i_{p+n}, j_{p+n} → defines $i_0, \dots, i_{p+m-1}$; $j_0, \dots, j_{p+m-1}$
  - Spoiler wins if for ~~every~~ all k, ℓ:
    - $a_{i_k} = b_{j_k}$ • $i_k = i_\ell$ iff $j_k = j_\ell$
    - $i_k < i_\ell$ iff $j_k < j_\ell$

- $u[i_0, \dots, i_{p-1}]_m^p \cong_m^p v[j_0, \dots, j_{p-1}]$ $\qquad u[\dots] \cong^p v[\dots]$

<u>ex</u>:

$$\begin{array}{l} \qquad\qquad\quad\ \ i_0 \\ \qquad\qquad\quad\ \ \downarrow \\ u = bbbacacacacbccbac \\ v = bbacacacacabcccbac \\ \qquad\quad\ \ \uparrow \\ \qquad\quad\ \ j_0 \end{array}$$

<u>Fact</u>: $u[i_0, \dots, i_{p-1}] \cong_{m+1}^p v[j_0, \dots, j_{p-1}]$ iff for every $i_p \in dom(u)$, exists j_p such that $u[i_0, \dots, i_p] \cong_m^{p+1} v[j_0, \dots, j_p]$ ⊕ reverse

<u>Theorem</u> [Ehrenfeucht 61]: $\equiv_m^p = \cong_m^p$

proof by induc° over m

IV] Cuts

- 1-cut : any $T \subseteq \Sigma$
- $(m+1)$-cut : set of triples $\langle c_\ell, a, c_r \rangle$ with $a \in \Sigma$
  c_ℓ, c_r m-cuts
- $Cuts_m^\Sigma$
- $cuts_m(w)$: inductive definition

Prop : ($u \equiv_m v$ iff) $u \cong_m v$ iff $cuts_m(u) = cuts_m(v)$

V] Star-free languages

- star-free expressions
- ex : $\Sigma = \{a, b\}$, $\mathcal{L}(\{a,b\}^*) = \emptyset^c$, $\mathcal{L}(\forall x. a(x)) = (\emptyset^c \cdot b \cdot \emptyset^c)^c$ a^*

Theorem [McNaughton-Papert] : $L \subseteq \Sigma^*$ is FO iff it is star-free
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α) From star-free to FO

define, for every E, $\varphi_E^{seg}(x,y)$ s.t. $w \models \varphi_E^{seg}(i,j)$
by induction
iff $w|_{[i,\dots,j]} \models E$

then $\varphi_E := \exists x, y.\ first(x) \wedge last(y) \wedge \varphi_E^{seg}(x,y)$

β) From FO to star-free

If L ~~xxxxx~~ $\in$ FO, then union of m-cuts for some m

⇝ express each cut in star-free, and it is done.