V] Fliess' theorem

α) Lex and shortlex orders, Hankel matrix

We define a linear order $\leqslant$ over the alphabet Σ,

typical examples: $\cdot\ \Sigma = \{a, b\}$, $a < b$

$\cdot\ \Sigma = \{a_0, \dots, a_{n-1}\}$, $a_0 < a_1 < \dots < a_{n-1}$.

We define the lexical order over Σ^* in the following way:
for $u, v \in \Sigma^*$, $u \leq_{lex} v$ if $\cdot$ u prefix of v ($u \cdot u' = v$ for some u')

or $\cdot$ for some u_0, u_1, v_1, $a < b$,

$$u = u_0 \cdot a \cdot u_1$$
$$v = u_0 \cdot b \cdot v_1$$

$\leq_{lex}$ is a linear order

We define the shortlex order in the following way:

$u \leq_{slex} v$ if $\cdot\ |u| < |v|$

or $\cdot\ |u| = |v|$ and $u \leq_{lex} v$

$$\varepsilon <_{slex} a <_{slex} b <_{slex} aa <_{slex} ab <_{slex} ba <_{slex} bb <_{slex} aaa <_{slex} \cdots$$

Let $s \in R\langle\langle\Sigma\rangle\rangle$. The (infinite) Hankel matrix of s is defined as follow: $H_s(u, v) = s(u \cdot v)$

Fact: $H_s(u \cdot v, w) = H_s(u, v \cdot w)$

ex: $\Sigma = \{a, b\}$

$H_{length} :=$

	ε	a	b	aa	ab	ba	bb	aaa	…
ε	0	1	1	2	2	2	2	3	…
a	1	2	2	3	3	3	3	4	…
b	1	2	2	3	3	3	3	4	…
aa	2	3	3	4	4	4	4	5	…
ab	2	3	3	4	4	4	4	5	…
ba	2	3	3	4	4	4	4	5	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮			

Statement and proof

Now, we suppose that R is actually a field $\mathbb{F}$ ($\times$ is commutative and $+, \times$ have inverses)

We have a notion of linearly independence (of columns, lines...)
(rows)
& a notion of ranks

Theorem [Fliess, 1974]

$s \in \mathbb{F}\langle\langle \Sigma \rangle\rangle$ is rational iff H_s has finite rank. Moreover, in that case, the finite rank n gives the minimal size of a weighted automaton recognising s.

proof

- Suppose s recognised by $\mathcal{A} = \langle N, \lambda, (\delta_a)_a, \mu \rangle$, for each $i \in N$, ~~and run~~ we define, for $w \in \Sigma^*$ and ρ a run via w ending on i (meaning $q_{|w|} = i$), we define the left weight of ρ as $\mathrm{Lwght}(\rho) = \lambda(q_0) \times \mathrm{runweight}(\rho)$ (meaning w/out $\mu(q_n)$)

and $L_i(u)$ is the sum $\displaystyle\sum_{\substack{\rho \text{ run via } w \\ \text{ending on } i}} \mathrm{Lwght}(\rho)$.
$$\begin{bmatrix} L_i(\varepsilon) \\ L_i(a) \\ \vdots \end{bmatrix}$$

L_i is the infinite ~~line~~ column ~~$[L_i(\varepsilon)\ L_i(a)\ \ldots\ L_i(ab)\ \ldots]$~~

Similarly, we have R_i as the infinite ~~column~~ line
where $R_i(v)$ is defined symmetrically (right weight)

~~$\begin{bmatrix} R_i(\varepsilon) \\ R_i(a) \\ \vdots \\ R_i(b) \end{bmatrix}$~~ $[R_i(\varepsilon)\ R_i(a) \ldots]$

Then, for ~~every~~ all $u, v \in \Sigma^*$, we have
$$H_s(u,v) = \sum_{i \in N} L_i(u) \times R_i(v)$$
$$H_s = \sum_{i \in N} L_i \times R_i, \text{ of rank } \leq N$$ (every $L_i \times R_i$ has rank at most 1)

• suppose now that H_s has finite rank N

$(L_{u_i})_{i<N}$ base of the matrix.

i) ~~for every~~ exist $\lambda(0), \lambda(1), \dots, \lambda(N-1)$ such that

$$L_\varepsilon = \sum_{i<N} \lambda(i)\cdot L_{u_i}$$

ii) for every $i<N$, $a \in \Sigma$, exists $\delta_a(i,0), \delta_a(i,1), \dots, \delta_a(i,N-1)$ such that

$$L_{u_i\cdot a} = \sum_{j<N} \delta_a(i,j)\, L_{u_j}$$

iii) finally, define $\mu(i) = L_{u_i}(\varepsilon)$

$\langle N, \lambda, (\delta_a)_a, \mu \rangle$ ~~define~~ recognises s !

proof by example:

$$
\begin{aligned}
s(a\cdot b) &= H_s(\varepsilon, a\cdot b) \\
&= L_\varepsilon(ab) \\
&= \sum_{i<N} \lambda(i)\, L_{u_i}(ab) && \longrightarrow H_s(u_i, ab) \\
& && \quad\;\; = H_s(u_i\cdot a, b) \\
&= \sum_{i<N} \lambda(i)\, L_{u_i\cdot a}(b) \\
&= \sum_{i<N} \lambda(i) \sum_{j<N} \delta_a(i,j)\, L_{u_j\cdot b}(\varepsilon) \\
&= \sum_{i<N} \lambda(i) \sum_{j<N} \delta_a(i,j) \sum_{k<N} \delta_b(j,k)\, L_{u_k}(\varepsilon) \\
&= \sum_{i<N} \lambda(i) \sum_{j<N} \delta_a(i,j) \sum_{k<N} \delta_b(j,k)\, \mu(k) && \text{iii} \\
&= \mathrm{wght}_{\mathcal{A}}(a\cdot b) && \text{// think of } \mathcal{A} \text{ in terms of matrices} \quad \square
\end{aligned}
$$