List 6

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6.1 Operations on structures

Exercise 6.1

Given two τ-structures $\mathfrak{A}$, $\mathfrak{B}$, their Cartesian product $\mathfrak{A} \times \mathfrak{B}$ is a τ-structure whose universe is $\mathfrak{A} \times \mathfrak{B}$, each constant symbol $c \in \tau$ is inerpreted as $(c^{\mathfrak{A}}, c^{\mathfrak{B}})$, and every k-ary predicate R is interpreted as $\{((a_1, b_1), \ldots, (a_k, b_k)) | (a_1, \ldots, a_k) \in R^{\mathfrak{A}}, (b_1, \ldots, b_k) \in R^{\mathfrak{B}}\}$

Assuming that $\mathfrak{A}_1 \equiv_m \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_m \mathfrak{B}_2$ hold, show that $\mathfrak{A}_1 \times \mathfrak{B}_1 \equiv_m \mathfrak{A}_2 \times \mathfrak{B}_2$.

solution

We know that $\mathfrak{A}_1 \equiv_m \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_m \mathfrak{B}_2$ hold, so there is winning strategy for duplicator in m-round Ehrenfeucht-Fraiisse game on structures $\mathfrak{A}_1, \mathfrak{B}_1$ and $\mathfrak{A}_2, \mathfrak{B}_2$.

To show that $\mathfrak{A}_1 \times \mathfrak{B}_1 \equiv_m \mathfrak{A}_2 \times \mathfrak{B}_2$ we will play m-round game as duplicator. Our strategy is following:

If spoiler picks elements $(a_1, b_1) \in \mathfrak{A}_1 \times \mathfrak{B}_1$ then we can with tuple (a_2, b_2) where is a_2 is duplicator answer for spoiler play a_1 in game on $\mathfrak{A}_1, \mathfrak{B}_1$, and b_2 is decided in the same way

The same happens when spoiler picks $(a_2, b_2) \in \mathfrak{A}_2 \times \mathfrak{B}_2$

Exercise 6.2

Fix a finite purely-relational signature τ, a τ-strucutre $\mathfrak{A}$, and a non empty set of indices I. We define the structure $\mathfrak{A} \times I$ as follows: the domain of $\mathfrak{A} \times I$ is $A \times I$, and for each k-ary relational symbol R we put $((a_1, i_1), \ldots, (a_k, i_k)) \in R^{\mathfrak{A} \times I}$ if and only if $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$ holds. Show by structural induction that for all eqaulity-free $FO[\tau]$-formulae $\psi(x_1, \ldots, x_k)$, all tuples $\bar{a} \in A^k$ and $\bar{i} \in I^k$ we have $\mathfrak{A} \times I \vdash \psi[(a_1, i_1), \ldots, (a_k, i_k)]$ if and only if $\mathfrak{A} \vdash \psi[a_1, \ldots a_k]$

Exercise 6.3

Let $\mathfrak{A}$, $\mathfrak{B}$be countably-infinite structures that are ω-elementary equivalent(i.e. duplicator can surwive ω rounds in any E-F game on $\mathfrak{A}$ and $\mathfrak{B}$). Show that $\mathfrak{A} \cong \mathfrak{B}$.

During the game all elements of structure a and b can be selected, and duplicator surived so there is a isomorphism beetwen all elements of $\mathfrak{A}$and all elements of $\mathfrak{B}$. Which means that these structures are isomorphic.