Automata and logic over infinite words

In this whole chapter, Σ is an alphabet

I] ω-words

An ω-word (or infinite word) over Σ is any infinite sequence of letters of Σ $w = a_0 a_1 a_2 a_3 \cdots$

ex: $w = ababab ab \cdots$

$w = abab b ab b b ab b b b a \cdots$

By Σ^ω we denote the set of ω-words over Σ.

A language (or ω-language) is any set $L \subseteq \Sigma^\omega$.

Some new operations:

- $Inf(w)$ = set of letters occurring ∞ly often in w
- $u \cdot v$, for $u \in \Sigma^*$, and $v \in \Sigma^\omega$
- u^ω, for $u \in \Sigma^+ \ (:= \Sigma^* \setminus \{\varepsilon\})$
- $w \in \Sigma^\omega$ is ultimately periodic if of the shape $u \cdot v^\omega$, for $u \in \Sigma^*$, $v \in \Sigma^+$.

II) Automata over infinite words

α) General definition

An ω-automaton over Σ is a tuple

$$\langle Q, I, \delta, Acc \rangle, \text{ where:}$$

- Q finite set of states
- $I \subseteq Q$ set of initial states
- $\delta \subseteq Q \times \Sigma \times Q$ is the set of transitions
- Acc is the accepting condition (actually defines the type of the auto)

Notion of run ρ via w: "$\rho \in Q^\omega$", accepting if $\rho \models Acc$

(one can see Acc as a subset of Q^ω)

$\mathcal{L}(\mathcal{A}) = \{ w \in \Sigma^\omega \mid \text{exists } \rho \text{ via } w \text{ accepting over } \mathcal{A} \}$

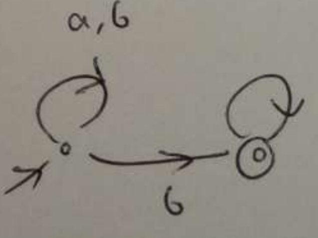
$\mathcal{A}$ deterministic if $\delta : Q \times \Sigma \to Q$, and $I = \{i\}$

Fact: if deterministic, every ω-word admits a unique run.

β) Büchi automata (NBA / DBA)

Acc consists of a subset of final states / accepting states $F \subseteq Q$ meant to be visited ∞ly often: ρ accepting if $Inf(\rho) \cap F \neq \emptyset$

ex: $\Sigma = \{a, b\}$

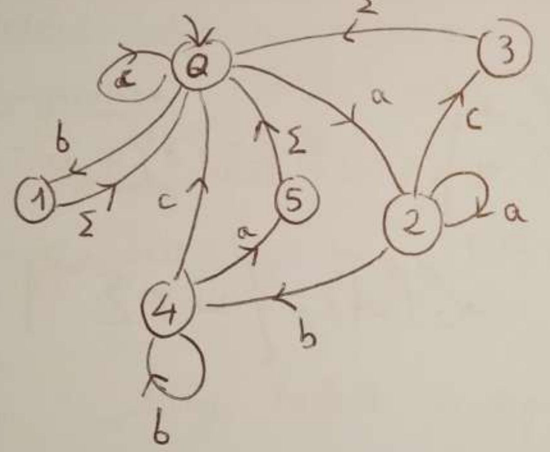


$\mathcal{L}(\mathcal{A}) = \{ w \in \{a,b\}^\omega \mid a \notin Inf(w) \}$

γ) Parity automata

$Q = \{0, \dots, N-1\}$ and ρ is accepting if $\max \mathrm{Inf}(\rho)$ is even.

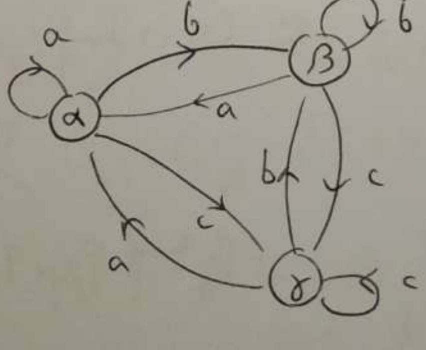
ex: $\Sigma = \{a, b, c\}$



δ) Müller automata

Acc consists of a family $\mathcal{F} \subseteq \mathcal{P}(Q)$ (every element of $\mathcal{F}$ is a subset of Q)

ρ accepting if $\mathrm{Inf}(\rho) \in \mathcal{F}$



$$\mathcal{F} = \{ \{\alpha, \beta\}, \{\beta, \gamma\}, \{\gamma, \alpha\} \}$$

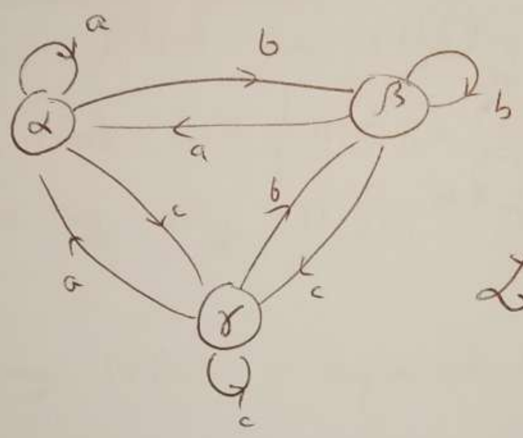
$$\mathcal{L}(\mathcal{A}) = \{ w \in \Sigma^\omega \mid |\mathrm{Inf}(w)| = 2 \}$$

ε) Rabin automata

Acc consists of a pair (B, G) of sets of states:

ρ accepting if $B \cap Inf(\rho) = \emptyset$
and $G \cap Inf(\rho) \neq \emptyset$.

ex:



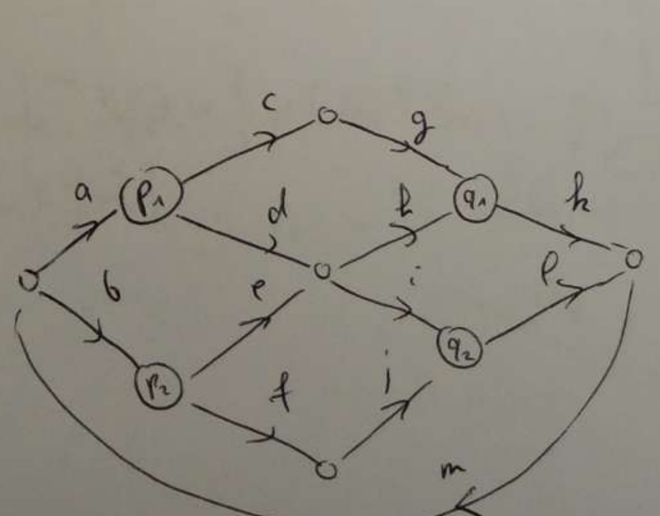
$B = \{\gamma\}, \ G = \{\alpha\}$

$$\mathcal{L}(\mathcal{A}) = \left\{ w \in \Sigma^\omega \;\middle|\; \begin{array}{l} a \in Inf(w) \\ c \notin Inf(w) \end{array} \right\}$$

η) Streett automata

Acc consists of a family of pairs (B_i, G_i) of sets of states

ρ accepting if for every i, $B_i \cap Inf(\rho) = \emptyset$
or $G_i \cap Inf(\rho) \neq \emptyset$



$B_1 = \{p_1, q_1\}, \ G_1 = \{p_2, q_2\}$

$B_2 = \{p_1, p_2\}, \ G_2 = \{q_1\}$

Tir.

Proposition: nondeterministically, all these classes of automata have the same expressive power.

$$NBA \equiv NPA \equiv NMA \equiv NRA \equiv NSA$$

Prop: $\mathcal{A}$ NBA: if $\mathcal{L}(\mathcal{A}) \neq \emptyset$, then $\mathcal{L}(\mathcal{A})$ admits an u.p. word.

III | Determinisation and Boolean closures of automata

α) Determinisation and negation

Non-proposition: DBA are strictly weaker than NBA.

proof: show that $L = \{ w \in \{a,b\} \mid a \notin Inf(w) \}$ cannot be recognised by any DBA.

Theorem: Any NBA is equivalent to a DMA (of exp size) [Safra '89]

Proposition: $NBA \equiv DPA \equiv DMA \equiv DRA \equiv DSA$

Corol: NBA can be ~~determ~~ negated: exists $\mathcal{B}$ such that
$$\mathcal{L}(\mathcal{A}) = \Sigma^\omega \setminus \mathcal{L}(\mathcal{B}).$$

β) ~~Composition of~~ Intersecting automata

Construction: $\mathcal{A}$, $\mathcal{B}$ NBA, the following NBA reco $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$:

$$\mathcal{C} = \langle Q_\mathcal{A} \times Q_\mathcal{B} \times \{1,2\}, I_\mathcal{A} \times I_\mathcal{B}, \delta, F_\mathcal{A} \times Q_\mathcal{B} \times \{1\} \rangle$$

where, for $p_1 \xrightarrow{\delta_\mathcal{A}} q_1$, $p_2 \xrightarrow{\delta_\mathcal{B}} q_2$,

$$\langle p_1, p_2, i \rangle \xrightarrow{\delta_\mathcal{C}} \langle q_1, q_2, i \rangle \text{ if } q_i \notin F_\mathcal{A} / F_\mathcal{B}$$

$$\langle p_1, p_2, i \rangle \xrightarrow{\delta_\mathcal{C}} \langle q_1, q_2, 3-i \rangle \text{ if } q_i \in F_\mathcal{A} / F_\mathcal{B}$$

IV) ω-regular expressions and languages

α) Def°

$$E_\infty ::= E \cdot E_\infty \mid E^\omega \mid E_\infty \cup F_\infty \mid E_\infty \cap F_\infty \mid E_\infty^c$$

E: reg expr (over finite words!)

E^ω: regexpr not containing ε

$w \models E^\omega$ if $w = u_0 \cdot u_1 \cdot u_2 \cdots$, each $u_i \models E$

$w \models E \cdot E_\infty$ if $w = u \cdot v$, $u \models E$, $v \models E_\infty$

$\mathcal{L}(E_\infty) := \{ w \in \Sigma^\omega \mid w \models E_\infty \}$

L (ω-)regular if $L = \mathcal{L}(E_\infty)$ for some E_∞.

β) Correspondence with automata

Proposition: $L \subseteq \Sigma^\omega$ is regular iff it is recognised by a NBA.

β_1) From ω-regexpr to NBA

rather clear once we have Boolean operations

β_2) From NBA to ω-regexpr

Consider a NBA $\mathcal{A} = \langle Q, I, \delta, F \rangle$

Define $\mathcal{A}_{q \to q'}$ as $\langle Q, \{q\}, \delta, \{q'\} \rangle$ over finite words

Get $E_{q \to q'}$ for it (it defines a language of finite words)

Then $\mathcal{L}(\mathcal{A}) = \bigcup_{q_0 \in I, q \in F} E_{q_0 \to q} \cdot (E_{q \to q} \setminus \{\varnothing\})^\omega$. $\square$