Weighted automata

I] Semirings and series

- semiring $\langle R, +, \times, 0, 1 \rangle$
  - $\langle R, +, 0 \rangle$ commutative monoid
    $\|$ $(x+y)+z = x+(y+z)$
    $x + 0 = x = 0 + x$
    $x + y = y + x$
  - $\langle R, \times, 1 \rangle$ monoid (commutative)
    $\|$ $(x \times y) \times z = x \times (y \times z)$
    $x \times 1 = x = 1 \times x$
  - $\times$ distributive over $+$
    $\|$ $x \times (y+z) = x \times y + x \times z$
    $(x+y) \times z = x \times z + y \times z$
  - 0 is absorbent for $\times$
    $\|$ $0 \times x = 0 = x \times 0$

ex: $\langle \mathbb{N}, +, \times, 0, 1 \rangle$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$,
$\langle \mathbb{B}, \vee, \wedge, 0, 1 \rangle$, $\langle \mathbb{B}, \wedge, \vee, 1, 0 \rangle$
$\mathbb{B} = \{0, 1\}$

- series over Σ and R: $\Sigma^* \longrightarrow R$

  ex: $u \longmapsto |u|_a - |u|_b$

  $u \longmapsto |u|$

- $R\langle\langle \Sigma \rangle\rangle$ the set of series over Σ and R

II] Weighted Monadic Second Order Logic

Σ, SR α) Syntax

- $\varphi, \psi ::= r \mid a(x) \mid x<y \mid x=y \mid x \in X \mid$
  $\underset{R}{\overset{\cap}{}}$ $\neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid$
  $\exists x.\,\varphi \mid \forall x.\,\varphi \mid \exists X.\,\varphi \mid \forall X.\,\varphi$

ex: $\mathbb{Z}$, $\exists x.\, a(x) \vee [\underset{-1}{\vee} \wedge \exists x.\, \tfrac{3}{2}\, b(x)]$

- free variables, valuations

β) Semantics

$w = a_0 \cdots a_{|w|-1}$ $[|\varphi|]^w_\nu \in R$ $\nu \xrightarrow{val} w$

evaluation $[|\varphi|]^w_\nu \in R$ of φ over w via ν defined inductively

- $[|r|]^w_\nu = r$
- $[|a(x)|]^w_\nu = \begin{cases} 1 \text{ if } a_{\nu(x)} = 1 \\ 0 \text{ if not} \end{cases}$
- $[|x<y|]^w_\nu = \begin{cases} 1 \text{ if } \nu(x) < \nu(y) \\ 0 \text{ if not} \end{cases}$ other atoms ......
- $[|\neg\varphi|]^w_\nu = \begin{cases} 1 \text{ if } [|\varphi|]^w_\nu = 0 \\ 0 \text{ if not} \end{cases}$
- $[|\varphi \wedge \psi|]^w_\nu = [|\varphi|]^w_\nu \cdot [|\psi|]^w_\nu$
- $[|\varphi \vee \psi|]^w_\nu = [|\varphi|]^w_\nu + [|\psi|]^w_\nu$
- $[|\exists x.\,\varphi|]^w_\nu = \sum_{i \in dom(w)} [|\varphi|]^w_{\nu[i/x]}$
- $[|\forall x.\,\varphi|]^w_\nu = \prod_{i \in dom(w)} [|\varphi|]^w_{\nu[i/x]}$
- $[|\exists X.\,\varphi|]^w_\nu = \sum_{I \subseteq dom(w)} [|\varphi|]^w_{\nu[I/X]}$
- $[|\forall X.\,\varphi|]^w_\nu = \prod_{I \subseteq dom(w)} [|\varphi|]^w_{\nu[I/X]}$

consider order of $dom(w)$

consider lexorder of $\mathcal{P}(dom(w))$

ex: $[\![\exists x.\, a(x) \vee [(-1) \times \exists x.\, b(x)]]\!]^w_\emptyset$

$= [\![\exists x.\, a(x)]\!]^w_\mu + \left([\![-1]\!]^w_\emptyset \times [\![\exists x.\, b(x)]\!]^w_\emptyset \right)$

$= \sum_{i \in dom(w)} [\![a(x)]\!]^w_{x \mapsto i} + (-1) \times \sum_{i \in dom(w)} [\![b(x)]\!]^w_{x \mapsto i}$

$= |w|_a - |w|_b$

φ sentence defines a series

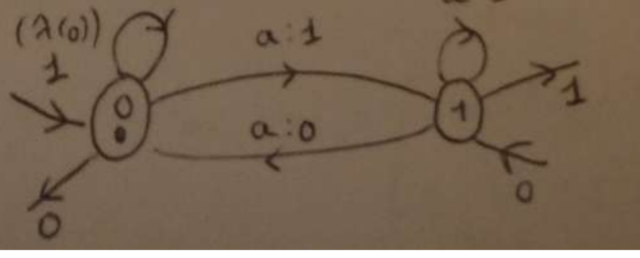
$$\mathcal{S}(\varphi) := \left(\begin{array}{l} \Sigma^* \longrightarrow R \\ w \longmapsto [\![\varphi]\!]^w_\emptyset \end{array} \right)$$

$s \in R\langle\langle \Sigma \rangle\rangle$ definable in wMSO if $s = \mathcal{S}(\varphi)$ for some φ.

III] Weighted automata

- Σ alphabet, R semiring
- w aut over Σ and R: quadruple $\mathcal{A} = \langle N, \lambda, (\delta_a)_a, \mu \rangle$
  - N nat number ($\simeq$ set of states $Q = \{0, 1, \dots, N-1\}$)
  - λ and μ are functions from N to R: entry / exit functions
  - for each $a \in \Sigma$, $\delta_a : N \times N \to R$

ex: $N = 2$, $\begin{cases} \lambda(0) = 1 \\ \lambda(1) = 0 \end{cases}$, $\begin{cases} \mu(0) = 0 \\ \mu(1) = 1 \end{cases}$, $\begin{cases} \delta_a(0,0) = 1 \\ \delta_a(0,1) = 1 \\ \delta_a(1,0) = 0 \\ \delta_a(1,1) = 1 \end{cases}$



NB: λ can be seen as a ~~row~~ line

$$L^{in} = [\lambda(0) \quad \lambda(1) \quad \cdots \quad \lambda(N-1)] \in \mathcal{M}_{1\times N}[\mathbb{R}]$$

μ can be seen as a column

$$C^{OUT} = \begin{bmatrix} \mu(0) \\ \mu(1) \\ \vdots \\ \mu(N-1) \end{bmatrix} \in \mathcal{M}_{N\times 1}[\mathbb{R}]$$

δ_a can be seen as a matrix

$$M^{TR_a} = \begin{bmatrix} \delta_a(0,0) & \delta_a(0,1) & \cdots & \delta_a(0,N-1) \\ \delta_a(1,0) & \delta_a(1,1) & \cdots & \delta_a(1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_a(N-1,0) & \delta_a(N-1,1) & \cdots & \delta_a(N-1,N-1) \end{bmatrix} \in \mathcal{M}_{N\times N}[\mathbb{R}]$$

- $w \in \Sigma$

$\overset{\shortparallel}{a_0 a_1 \cdots a_{n-1}}$

~~We have the same number of notions of runs over~~

A run over $\mathcal{A}$ via w is any $(n+1)$-tuple

$$\rho = \langle \rho_0, \rho_1, \cdots, \rho_n \rangle, \text{ with } \rho_i \in N$$

The weight of ρ is

$$\mathrm{wght}(\rho) := \lambda(\rho_0) \times \delta_{a_0}(\rho_0, \rho_1) \times \delta_{a_1}(\rho_1, \rho_2) \cdots \times \delta_{a_{n-1}}(\rho_{n-1}, \rho_n) \times \mu(\rho_n)$$

Transition weight

Then, the ~~run~~ weight of w is the sum

$$\mathrm{wght}(w) = \sum_{\rho \text{ run via } w} \mathrm{wght}(\rho)$$

NB:

$$\mathrm{wght}(w) = L^{IN} \times M^{TR_{a_0}} \times M^{TR_{a_1}} \times \cdots \times M^{TR_{a_{n-1}}} \times C^{OUT}$$

Every automaton over Σ and R defines a series

$$\mathcal{S}(\mathcal{A}) := \begin{pmatrix} \Sigma^* \longrightarrow R \\ w \longmapsto \mathrm{wght}(w) \end{pmatrix}.$$

example above: $\mathcal{S}(\mathcal{A})(w) := |w|$

s ~~regular~~ rational if $\mathcal{S}(\mathcal{A}) = s$ for some $\mathcal{A}$.

$\cap$

$R\langle\langle\Sigma\rangle\rangle$

IV] Correspondence between wMSO and weighted automata

Prop: $s \in R\langle\langle\Sigma\rangle\rangle$ definable in wMSO if ~~regular~~ rational

$\mathrm{first}(x) := \forall y.\ x < y \vee x = y \rightsquigarrow 0$ or 1

$\mathrm{last}(y) := \forall x.\ x < y \vee x = y \rightsquigarrow 0$ or 1

$\mathrm{succ}(x,y) := x < y \wedge \forall z.\ z < x \vee z = x \vee y = z \vee y < z$

$$\begin{aligned}
\varphi_{\mathcal{A}} := \exists X_0, \dots, X_{n-1}.\ & \forall x.\ \bigvee_{p \in N} x \in X_p \wedge \bigwedge_{q \neq p} \neg\, x \in X_q \\
& \wedge \forall x.\ \neg\,\mathrm{first}(x) \vee \left[\begin{array}{l} \mathrm{first}(x) \\ \wedge \bigvee_{a \in \Sigma} a(x) \wedge \bigwedge_{p=0}^{N-1} \lambda(p) \wedge \bigvee_{q=0}^{N-1} (x \in X_q \wedge \delta_a(p,q)) \end{array}\right] \\
& \wedge \forall x, y.\ \neg\,\mathrm{succ}(x,y) \vee \left[\begin{array}{l} \mathrm{succ}(x,y) \\ \wedge \bigvee_{\substack{a \in \Sigma \\ p,q \in N}} a(y) \wedge x \in X_p \wedge y \in X_q \wedge \delta_a(p,q) \end{array}\right] \\
& \wedge \forall y.\ \neg\,\mathrm{last}(y) \vee \left[\begin{array}{l} \mathrm{last}(y) \\ \wedge \bigvee_{q \in N} x \in X_q \wedge \mu(q) \end{array}\right]
\end{aligned}$$