- step 4 : $\langle H, \cdot, e \rangle$ is a group.

Take $x \in H$. $n \geq 2$ s.t. ~~$e \leq_{\mathcal{H}} x$~~ x^n idempotent in H, so it is necessarily $x^n = e$; $x \cdot x^{n-1} = e = x^{n-1} x$

□

<u>Corollary</u>: An $\mathcal{H}$-class of a finite monoid is a group iff it contains an idempotent element.

<u>Corollary</u>: The $\mathcal{J}$-class of the identity $(= 1)$ of a finite ~~gr~~ monoid is a group.

<u>proof</u>: TS: $\mathcal{H}$-class of 1 = $\mathcal{J}$-class of 1.

Take $x \in J$, $\mathcal{J}$-class of 1.

$1 \leq_r x$, hence $1 \mathcal{R} x$ by the EL

$1 \leq_\ell x$, hence $1 \mathcal{L} x$ by the EL.

Hence, $1 \mathcal{H} x$: $x \in H$, $H = J$, which then is a group.

IV] <u>Schützenberger's theorem</u>

For $\mathcal{M}$ a finite ~~semi~~ monoid, we write $x^\#$ for the idempotent power of x.

A finite monoid $\mathcal{M}$ is called <u>aperiodic</u> if for every $x \in \mathcal{M}$, $x^\# x = x^\#$.

<u>Schützenberger's theorem</u> [65] $L \subseteq \Sigma^*$, t.f.a.e.:

- L is FO;
- L is reco by some finite ~~syntac~~ aperiodic monoid;
- $\mathcal{M}_L$ is finite and aperiodic.

ii) iff iii) equivalent clearly (more or less).

2) From FO to aperiodicity

Reminder from a few months ago : $u \equiv_k v$ if for every formula $\varphi \in FO[<, \Sigma]$ of q.d. $\leq k$, $u \models \varphi$ iff $v \models \varphi$.

Proposition : • $\equiv_k$ has finitely many equivalence classes

• if $u_1 \equiv_k v_1$, $u_2 \equiv_k v_2$, then $u_1 u_2 \equiv_k v_1 \cdot v_2$

• $u^{2^k} \equiv_k u^{2^k+1}$

Proof : × first point by induction over k

× other two by EF games

Corollary : $\mathcal{M}_k = \{ [u]_{\equiv_k} \mid k \in \mathbb{N} \}$ is a finite aperiodic monoid.

(with the law $[u_1] \cdot [u_2] = [u_1 \cdot u_2]_{\equiv_k}$.

Corollary : If L is definable by an FO formula, then it is recognised by a finite aperiodic monoid.

proof : ~~Consider~~ Take φ FO defining L, and ~~take~~ consider k its quantifier depth.

$\mathcal{M}_k$ recognises L via $h : u \longmapsto [u]_{\equiv_k}$.

Indeed, let us show $h^{-1}(h(L)) \subseteq L$.

Consider $u \in h^{-1}(h(L))$: $h(u) \in h(L)$, so there exists $v \in L$ such that $[u]_{\equiv_k} = [v]_{\equiv_k}$. Since $v \models \varphi$, $u \models \varphi$, and $u \in L$. □

) From aperiodicity to FO / star free

Lemma 1: Let $\mathcal{M}$ be a finite aperiodic monoid. Then every $\mathcal{H}$-class of it is a singleton.

Proof: Consider $x \mathcal{H} y$ in $\mathcal{M}$, and take $\alpha, \beta \in \mathcal{M}$ such that $x = \alpha y$, $y = \beta x$: we get $x = \alpha^n x \beta^n$ for every $n \in \mathbb{N}$. Therefore, $x = \alpha^{\#} x \beta^{\#} = \alpha^{\#} x \beta^{\#} \beta = x\beta = y$. $\square$

~~Lemma 2~~ Now, we define, for $h : \Sigma^* \to \mathcal{M}$ any homomorphism, ~~R_x~~ and $x \in \mathcal{M}$,

R_x as the language $\{u \cdot v \mid u, v \in \Sigma^*, h(u) \mathcal{R} x\}$

L_x as the language $\{u \cdot v \mid u, v \in \Sigma^*, h(v) \mathcal{L} x\}$.

Lemma 2: Let $\mathcal{M}$ be a finite monoid, and let $x \in \mathcal{M}$ not $\mathcal{J}$-equivalent to 1.

Then $$R_x = \bigcup_{\langle y, a\rangle \in S} h^{-1}(y) \cdot a \cdot \Sigma^*,$$

with $$S = \left\{ \langle y, a\rangle \in \mathcal{M} \times \Sigma \;\middle|\; \begin{array}{l} x \mathcal{R} y h(a) \\ x <_{\mathcal{J}} y \end{array} \right\}$$

(+) similar equation for L_x.

proof: $\supseteq$ clear: if $w = u a v$, with $h(u) a \mathcal{R} x$, then $w \in R_x$ by definition.

$\subseteq$: let $w \in R_x$, and consider u' ~~small~~ shortest prefix of w such that $h(u') \mathcal{J} x$. Since $x \not\mathcal{J} 1$, $u' \neq \varepsilon$: $u' = u \cdot a$ with $x <_{\mathcal{J}} h(u)$, $x \mathcal{J} h(ua)$.

But since $x \leq_{\mathcal{R}} h(ua)$ (using the prefix of w $\mathcal{R}$-equiv to x), $x \mathcal{R} h(ua)$ by the eggbox lemma. [illegible]

Lemma 3: Let $\mathcal{M}$ be a finite aperiodic monoid, and let $h: \Sigma^* \to \mathcal{M}$ be a homomorphism.

For every $x \in \mathcal{M}$, we get

$$h^{-1}(x) = L_x \cap R_x \setminus \left(\bigcup_{\langle y, a \rangle \in T} L_y \cdot a \cdot \Sigma^* \right),$$

with $T = \{ \langle y, a \rangle \in \mathcal{M} \times \Sigma \mid y \mathcal{J} x \text{ but } y h(a) <_{\mathcal{J}} x \}$

proof:

$\subseteq$

Consider $w \in h^{-1}(x)$

- $w \in L_x \cap R_x$ is clear
- now, suppose that $w = u \cdot v \cdot a \cdot v'$, with, for some $y \in \mathcal{M}$, $h(v) \mathcal{L} y \mathcal{J} x$

$y h(a) <_{\mathcal{J}} x$.

We get $x = h(w) = h(u) h(v) h(a) h(v')$

$= h(u) z y h(a) h(v')$, with z s.t. $h(v) = zy$

Hence $x \leq_{\mathcal{J}} y h(a)$ and we have a contradiction.

therefore $w \notin \bigcup_{\langle y, a \rangle \in T} L_y \cdot a \cdot \Sigma^*$

$\supseteq$

Consider $w \in L_x \cap R_x \setminus \bigcup_{\langle y, a \rangle \in T} L_y \cdot a \cdot \Sigma^*$

Take u longest prefix of w with $h(u) R x$.

- step 1: ~~$h(w) \leq_{\mathcal{J}} x$~~ $x \mathcal{J} h(w)$

. $h(w) \leq_{\mathcal{J}} x$ is clear by belonging to R_x.

[Su]ppose now that $x \leq_j^{h(w)}$ does not hold, in order to [co]me to a contradiction.

Let v' be the shortest prefix of w such that $x \leq_j h(v')$ does not hold. Since $x \leq_j 1 = h(\varepsilon)$, v' is nonempty:
$v' = v\,a$ with $x \leq_j h(v)$, NOT $x \leq_j h(v)h(a)$

Now, we compare u and v

- If v strictly shorter than u,

$$x \mathrel{R} h(u) \leq_r h(v \cdot a) = h(v)h(a), \text{ contradiction.}$$

- Hence, u is shorter than v

$$x \leq_j h(v) \leq_r h(u) \mathrel{R} x. \text{ By the egglox lemma, } x \mathrel{R} h(v).$$

This means that $u = v$.

But then: $u \in L_{h(u)}$, $w \in L_{h(u)} \cdot a \cdot \Sigma^*$

with $\langle h(u), a\rangle \in T$:

$h(u) \mathrel{J} x$ (since $h(u) \mathrel{R} x$)

NOT $h(u)\,h(a) \mathrel{J} x$.

contradiction again

We conclude that $x \mathrel{J} h(w)$.

- step 2 : $h(w) = x$.

We have $h(w) \mathrel{J} x$, $h(w) \leq_r x$, therefore $h(w) \mathrel{R} x$ by the egglox lemma.

The same way, $h(w) \mathrel{L} x$.

We have $h(w) \mathrel{H} x$ and therefore $x = h(w)$ by Lemma 1. $\square$

Proposition : Let $L \subseteq \Sigma^*$, we suppose [illegible] by a finite aperiodic monoid. Then L is star free

Proof : ~~the~~ Let $\mathcal{M}$ finite ap recognizing L via $h: \Sigma^* \to \mathcal{M}$.
We show that for every $x \in \mathcal{M}$, $h^{-1}(x) \subseteq \Sigma^*$ is star free
(and then $L = \bigcup_{x \in h(L)} h^{-1}(x)$ will be as well).

We consider, for J a $\mathcal{J}$-class of $\mathcal{M}$, $\mathcal{P}(J)$ the property "for every $x \in J$, $h^{-1}(x)$ is s f".
We show by induction that $\mathcal{P}(J)$ holds for every J.

- base case : $\mathcal{P}(J_1)$ holds, J_1 being the $\mathcal{J}$-class of 1.
- induction step : if for every $J' >_{\mathcal{J}} J$, $\mathcal{P}(J')$ holds, then $\mathcal{P}(J)$ as well.

* base case : $J_1 = H_1$ (last week)
$= \{1\}$ by lemma 1.

But $h^{-1}(1) = \{ a_0 a_1 \cdots a_{n-1} \in \Sigma^* \mid h(a_i) = 1$ for every $i <$
(if $1 = h(a_0) h(a_1) \cdots h(a_{n-1})$, then
$1 \leq_{\mathcal{J}} h(a_i) \leq_{\mathcal{J}} 1$).

Hence $h^{-1}(1) = \left(\cancel{\emptyset^c} \bigcup_{a \in \Gamma} \emptyset^c \cdot a \cdot \emptyset^c \right)^c$, $\Gamma = \Sigma \setminus h^{-1}(1)$

* induction step : suppose $\mathcal{P}(J')$ holding for every $J' >_{\mathcal{J}} J$
~~Let $x \in$~~ J. – step 1 : for every $x \in J$, R_x s.f
~~By lemma 2,~~ ~~R_x is~~ s.f. Take E_y s.f.e for
~~By lemma 2,~~
every $y >_{\mathcal{J}} x$. By lemma 2,
$R_x = \bigcup_{(y,a) \in S} E_y \cdot a \cdot \Sigma^*$ is star free.

– step 2 : same for Lx, any $x \in J$.

- step 2: For every $x \in J$, $f^{-1}(x)$ s.f.

Take, for every $y \in J$, E_y^{ℓ} (resp. $E_y^{\hat{}}$) a s.f.e. for $\mathcal{L}_y$ (resp $\mathcal{R}_y$).

Then by lemma 2,

$$f^{-1}(x) = E_x^{\ell} \cap E_x^{\hat{}} \cap \left(\bigcup_{\langle y, a\rangle \in T} E_y^{\ell} \cdot a \cdot \emptyset^c \right).$$

is star free. □