Chapter 2 - Logic over finite words

I] Syntax and semantics

α) Syntax

- inf set of fo variables $x, y, z, x_0 \dots$
- inf set of mso variables $X, Y, Z, X_0, \dots$
- mso : $a(x) \mid x < y \mid x \in X \mid \varphi \vee \psi \mid \neg \varphi \mid \exists x. \varphi \mid \exists X. \varphi$

ex : $\exists x. \exists y. x < y \wedge a(x) \wedge b(y)$

β) Semantics

- dom (w), valuation of V to w
- w satisfying $\varphi(V)$ via ν, syntactic sugar
- free variables, sentence
- $\mathcal{L}(\varphi)$

II] Correspondence between regular languages and MSO

- statement : L regular iff definable in MSO

α) From automata to MSO

$\mathcal{A}$ with n states $q_0, \dots q_{n-1}$

$$\begin{array}{cc} a & b & \dots \\ X_2 & X_3 & \dots \end{array}$$

← "after reading ab, I am in state q_3"

- first (x)
- succ (x, y)
- last (x)
- partition $(X_0, \dots, X_{n-1})$
- $\exists X_0, \dots, X_{n-1}.$ Parti-$(X_0, \dots, X_{n-1})$

$$\wedge \forall x, y.\ succ(x,y) \rightarrow \bigwedge_{i=0}^{n-1} x \in X_i \rightarrow \bigvee_{q_i \xrightarrow{a} q_j} a(y) \wedge y \in X_j$$

$$\wedge \forall x.\ first(x) \rightarrow \bigvee_{\text{s.t. } \exists q_i \xrightarrow{a} q_j} a(x) \wedge x \in X_j$$

$$\wedge \forall x.\ last(x) \rightarrow \bigvee_{q_i \in F} x \in X_j$$

β) From MSO to automata

◦ homomorphism

Fact: determined by values on letter

Prop: regular languages are closed under homomorphic images

◦ apparent problem: no "subsentences"

idea: for every $\varphi(V)$, construct automaton recognising

$$\mathcal{L}(\varphi(V)) := \left\{ \langle w, \nu : V \xrightarrow{val} dom(w) \rangle \;\middle|\; w, \nu \models \varphi(V) \right\}$$

representing $\langle w, \nu \rangle$?

→ put variables below letters

$$\binom{a}{z, X}\binom{b}{x}\binom{c}{X}\binom{a}{\;}\binom{c}{y} \models b(x) \wedge c(y) \wedge z \in X \wedge y \notin Y \wedge z < x$$

◦ V finite set of variables

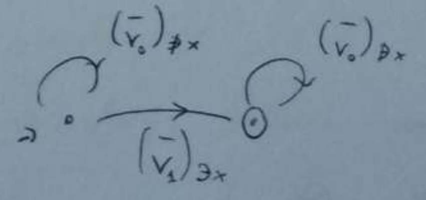
V-word over Σ: finite word over the product alphabet $\Sigma \times \mathcal{P}(V)$, written $\binom{\Sigma}{\mathcal{P}(V)}$

every V-word defines a word over Σ and a word over $\mathcal{P}(V)$, of the same domain, we write $\binom{w}{\nu} \in \binom{\Sigma}{\mathcal{P}(V)}^*$

◦ $\binom{w}{\nu}$ describes valuation if for every f.o. variable x, there exists a unique $i \in dom(w) = dom(\nu)$ s.t. $x \in V_i$, with $\nu = V_0 V_1 \cdots V_{|\nu|-1}$.

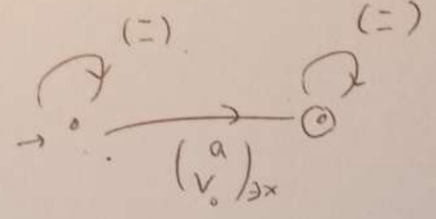
inductive construction

◦ $\mathcal{A}_{\text{unique } x}$

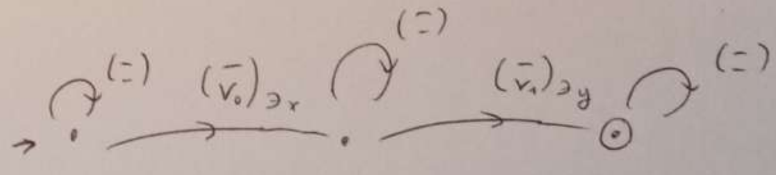


$$\mathcal{A}_{val} = \bigcap_{\text{f.o. } x \in V} \mathcal{A}_{\text{unique } x}$$

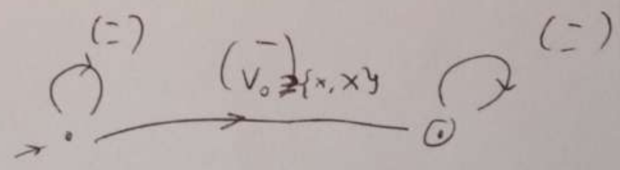
• $a(x)$:

$\mathcal{A}_{a(x)} := \mathcal{A}_{val} \cap$ 

• $x < y$

$\mathcal{A}_{x<y} := \mathcal{A}_{val} \cap$ 

• $x \in X$

$\mathcal{A}_{x \in X} := \mathcal{A}_{val} \cap$ 

• $\neg \psi$: suppose having $\mathcal{A}_\psi$ for $\mathcal{L}(\psi)$ (w/ free variables V)

$\mathcal{A}_{\neg\psi} := \mathcal{A}_{val} \cap \neg \mathcal{A}_\psi$

• $\psi \wedge \lambda$: standard conjunction of automata

• $\exists x.\ \varphi(V, x)$:
claim: $h_{\setminus\{x\}}\big(\mathcal{L}(\varphi(V,x))\big) = \mathcal{L}(\exists x.\ \varphi(V,x))$,
where $h_{\setminus\{x\}}$ homomorphism defined by $h_{\setminus\{x\}}\binom{a}{V_0} = \binom{a}{V_0 \setminus \{x\}}$

hence, if $\mathcal{L}(\varphi(V,x))$ is regular (reco by $\mathcal{A}_{\varphi(V,x)}$), then $\mathcal{L}(\exists x.\ \varphi(V,x))$ is regular as well.

• $\exists X.\ \varphi(V, X)$:
similar ✓

After all these steps, obtain an automaton over $\binom{\Sigma}{\emptyset}$ recognising our sentence.