II] Green's relation

Let M a monoid.

The right ideal of $x \in M$ is the set $xM := \{xy \mid y \in M\}$

The left ideal of $y \in M$ is the set $My := \{xy \mid y \in M\}$

The two sided ideal of $y \in M$ is the set $MyM := \{xyz \mid x, z \in M\}$

- y prefix of x, written $x \leq_r y$, if there exists $z \in M$ such that $x = yz$
  or equivalently: - $x \in yM$
  - $xM \subseteq yM$
- z suffix of x, written $x \leq_\ell z$, if exists $y \in M$ s.t.
  - $x = yz$
  equivly: - $x \in Mz$
  - $Mx \subseteq My$
- y infix of w, written $w \leq_j y$ if exist $x, z \in M$ s.t.
  - $w = xyz$
  or equivalently: - $w \in MyM$
  - $MwM \subseteq MyM$

Facts: • $\leq_{r, \ell \text{ and } j}$ are pre-orders

not antisymmetric a priori ($\mathbb{Z}_5$: $1 \leq_r 4$)

$$\begin{array}{ccc} \leq_r & \subseteq & \leq_j \\ & & \rotatebox{90}{$\subseteq$} \\ & & \leq_\ell \end{array}$$

Define the corresponding equivalence classes

- $x \mathcal{R} y$ if $x \leq_r y$ and $y \leq_r x$

$$xM = yM$$

- $x \mathcal{L} y$ ″
- $x \mathcal{J} y$ ″

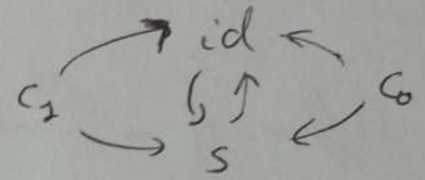
⇝ $\mathcal{R}$-class / $\mathcal{L}$-class / $\mathcal{J}$-class

Fact every $\mathcal{L}$-class is included in some $\mathcal{J}$-class.

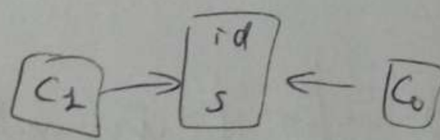
ex: $\Omega = 2^2$:

$\circ$	id	s	c_0	c_1
id	id	s	c_0	c_1
s	s	id	c_1	c_0
c_0	c_0	c_0	c_0	c_0
c_1	c_1	c_1	c_1	c_1

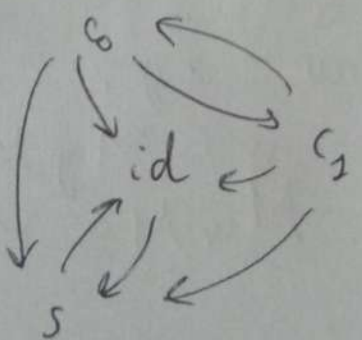
$\leq_r$-graph



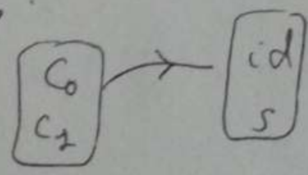
$\mathcal{R}$-classes :



$\leq_l$-graph :



$\mathcal{L}$-classes :



$$\leq_l = \leq_j .$$

Egglox lemma

Consider a finite monoid M, $x, y \in M$.
Suppose that $x \mathcal{J} y$ and $x \leq_R y$.
Then ~~y~~ $y \leq_R x$ (and therefore $x \mathcal{R} y$)

proof: Take $\alpha, \beta, \gamma \in M$ such that
- $x = y\beta$
- $y = \alpha x \gamma$,

to show: $y = x$ something

$$
\begin{aligned}
y &= \alpha x \gamma \\
&= \alpha y \beta \gamma \\
&= \alpha (\alpha x \beta \gamma) \beta \gamma \\
&= \alpha^2 x (\beta\gamma)^2 \\
&\vdots \\
&= \alpha^n x (\beta\gamma)^n \quad \text{for every } n \in \mathbb{N}
\end{aligned}
$$

~~n~~ take $n \geq 1$ idempotent power

$$
\begin{aligned}
y = \alpha^n x (\beta\gamma)^n (\beta\gamma)^n &= y(\beta\gamma)^n \\
&= y\beta(\gamma\beta)^{n-1}\gamma \\
&= x(\gamma\beta)^{n-1}\gamma
\end{aligned}
$$

$\square$

$\mathcal{H} = \mathcal{R} \cap \mathcal{J}$

ex (previous): $\mathcal{H}$-classes of $\langle 2^2, \circ, id \rangle$:

$$\boxed{c_0} \rightsquigarrow \boxed{\begin{matrix} id \\ s \end{matrix}} \quad \boxed{c_1}$$

$\mathcal{H}$-dichotomy lemma / Green's theorem [70]

Consider $\mathcal{M}$ finite monoid,
H a $\mathcal{H}$-class included in a $\mathcal{J}$-class J.

Then • for all $x, y \in H$, $xy \notin J$

XOR • H is a group

proof: suppose $x_0 y_0 \in J$ for some $x_0, y_0 \in H$

- step 1: for all $x, y \in H$, $xy \in J$.
  - $x \mathcal{R} x_0$, hence $x = \alpha x_0$
  - $y \mathcal{L} y_0$, hence $y = y_0 \beta$.

  therefore, $xy = \alpha x_0 y_0 \beta$: $xy \mathcal{J} x_0 y_0$, meaning $xy \in J$.
  and symmetrically $x_0 y_0 \leq_{\mathcal{J}} xy$

- step 2: for all $x, y \in H$, $xy \in H$.
  By def, $xy \leq_{\mathcal{R}} x$ and $xy \mathcal{J} x$ by step 1
  therefore, $xy \mathcal{R} x$ by the ESgbox lemma
  Similarly, $xy \mathcal{L} x$, and therefore $x \mathcal{H} xy$:
  $xy \in H$.

So, H is a semigroup. It admits an idempotent e ($e^2 = e$)

- step 3: $\langle H, \cdot, e \rangle$ monoid
  take $x \in H$, $x \leq_{\mathcal{L}} e$, so there exists $y \in \mathcal{M}$ such that
  $x = ye$, so $xe = ye^2 = ye = x$
  Similarly, $ex = x$, so e is the neutral element of H.

- step 4 : $\langle H, \cdot, e \rangle$ is a group.

Take $x \in H$. $n \geq 2$ s.t. x^n idempotent in H, so it is necessarily $x^n = e$; $x \cdot x^{n-1} = e = x^{n-1} x$

□

<u>Corollary</u> : An $\mathcal{H}$-class of a finite monoid is a group iff it contains an idempotent element.

<u>Corollary</u> : The $\mathcal{J}$-class of the identity $(= 1)$ of a finite ~~gr~~ monoid is a group.

<u>proof</u> : TS : $\mathcal{H}$-class of 1 = $\mathcal{J}$-class of 1.

Take $x \in J$, $\mathcal{J}$-class of 1.

$1 \leq_r x$, hence $1 \mathcal{R} x$ by the EL

$1 \leq_\ell x$, hence $1 \mathcal{L} x$ by the EL.

Hence, $1 \mathcal{H} x$: $x \in H$, $H = J$, which then is a group.

IV] <u>Schützenberger's theorem</u>

For $\mathcal{M}$ a finite ~~semi~~ monoid, we write $x^\#$ for the idempotent power of x.

A finite monoid $\mathcal{M}$ is called <u>aperiodic</u> if for every $x \in \mathcal{M}$, $x^\# x = x^\#$.

<u>Schützenberger's theorem</u> [65] $L \subseteq \Sigma^*$, t.f.a.e.:

- L is FO ;
- L is reco by some finite ~~syntac~~ aperiodic monoid ;
- $\mathcal{M}_L$ is finite and aperiodic.