Monadic Second-Order Logic

A) Syntax and semantics

basically the same as for finite words

$\varphi ::= a(x) \mid x<y \mid x \in X \mid \neg \varphi \mid \exists x.\varphi \mid \ldots$

notion of valuation $\rightsquigarrow V \xrightarrow{\text{val}} w \in \Sigma^\omega$

B) Correspondence with ω-regular expressions

Prop: $L \subseteq \Sigma^\omega$ is regular iff L is definable in MSO

B₁) From automata to MSO

Considering $\mathcal{A} = \langle \underset{\{q_0,\ldots,q_{n-1}\}}{Q}, I, \delta, F \rangle$ NBA, the formula

$$\varphi_{\mathcal{A}} ::= \exists X_0, \ldots, X_{n-1}.\ \text{Partition}(X_0, \ldots, X_{n-1})$$
$$\wedge\ \forall x.\ \text{first}(x) \rightarrow \bigvee_{q_i \in I} x \in X_i$$
$$\wedge\ \forall x,y.\ \text{succ}(x,y) \rightarrow$$
$$\bigwedge_{\substack{a \in \Sigma \\ q_i \in Q}} (x \in X_i \wedge a(x)) \rightarrow \bigvee_{\substack{q_j \in Q \\ q_i \xrightarrow{\delta} q_j}} y \in X_j$$
$$\wedge \bigvee_{q_f \in F} \forall x.\ \exists y > x.\ y \in X_f$$

defines the language $\mathcal{L}(\mathcal{A})$.

β.) From MSO to automata

similar as for finite words (have valuation on second coordinate)

VI) Temporal Logics

~~[illegible]~~ Every definition / statement in this section also holds in the realm of finite words

α) Syntax and semantics of Linear Temporal Logic

Notation: if $w \in \Sigma^\omega = a_0 a_1 a_2 a_3 \cdots$, then $w\lceil_i$ denotes the ω-word $a_i a_{i+1} a_{i+2} \cdots$ $(w\lceil_0 = w)$

$((w\lceil_i)\lceil_j = w\lceil_{i+j})$

Linear Temporal Logic:

$$\lambda, \mu ::= a \mid \lambda \underset{\text{until}}{U} \mu \mid \overset{\text{next}}{X} \lambda \mid \neg\lambda \mid \lambda \vee \mu \mid \lambda \wedge \mu$$

- $w = a_0 a_1 a_2 \cdots \models a$ if $a_0 = a$
- $w \models X\lambda$ if $w\lceil_1 \models \lambda$
- $w \models \lambda U \mu$ if there exists $j \in \mathbb{N}$ such that
  - $w\lceil_j \models \mu$
  - $w\lceil_i \models \lambda$ for every $i < j$

(if $w \models \mu$, then $w \models \lambda U \mu$ $(j = 0)$)

ex: $\Sigma = \{a, b, c\}$:

$(a \vee b \vee c) U (a \wedge X b)$ tells that the word admits an occurrence of a directly followed by an occurrence of b.

β) Correspondence with First-Order Logic

Theorem [Kamp '68]: $L \subseteq \Sigma^\omega$ is first order iff it is defined by an LTL formula

β_1) From LTL to FO

For every λ LTL, define $\varphi_\lambda(x)$ FO such that $w \models \varphi_\lambda(i)$ iff $w\restriction_i \models \lambda$ for every $w \in \Sigma^\omega$, every $i \in \mathbb{N}$.

$$a \rightsquigarrow a(x)$$

$$\lambda \cup \mu \rightsquigarrow \exists z > x.\ \varphi_\mu(z) \wedge \forall x \leq y < z.\ \varphi_\lambda(y)$$

$$X\lambda \rightsquigarrow \exists y.\ \varphi_\lambda(y) \wedge \forall z.\ z \leq x \vee y \leq z$$

NB: only three variables are needed..

β_2) From FO to LTL

A formula $\varphi(x_0, \dots, x_{k-1})$ is in linearly-existential-universal form if it is of the shape $\overrightarrow{\exists\forall}$

$$\exists y_0 < y_1 < y_2 \cdots < y_{n-1}.$$

$$\bigwedge_{i<k} x_i = y_{k_i}$$

$$\wedge \bigwedge_{j<n} \alpha_j(y_j)$$

$$\wedge\ \forall y < y_0.\ \beta_0(y)$$

$$\wedge\ \forall y. \bigwedge_{1\leq j<k} \left(y_{j-1} < y < y_j \rightarrow \beta_j(y)\right)$$

$$\wedge\ \forall y > y_{n-1}.\ \beta_n(y)$$

quantifier free + only one variable

$\lceil \cdot \rceil$

~~translation~~

Prop: Any $\vec{\exists}\forall$ formula φ can be described by an LTL formula using U, X, and G (for "globally")

$w \models G\psi$ if ~~exists $i \in \mathbb{N}$ such that $w[i \models \psi$ and $w[j \models \psi$ for every $j > i$~~ for every $i \in \mathbb{N}$, $w[i \models \psi$

(G expressed in terms of U and X in exercise)

Hence, the goal is the following lemma

$\vec{\exists}\forall$ Lemma: Any FO φ can be translated to an $\vec{\exists}\forall$ formula.

Lemma 1: A conjunction of an $\vec{\exists}\forall$ formula is equivalent to an $\vec{\exists}\forall$ formula $\square$

Lemma 2: Any $\vec{\exists}\forall$ formula is equiv to a conjunction of $\vec{\exists}\forall$ with at most two free variables

Lemma 3: The negation of a $\vec{\exists}\forall$ formula with at most two free variables is equivalent to a disjunction of $\vec{\exists}\forall$ formulae

Then the $\vec{\exists}\forall$-lemma can be proven by structural induction.

Corollary: $FO[<] = FO^3[<]$, the fragment of FO where only three variables x, y, z, are allowed.