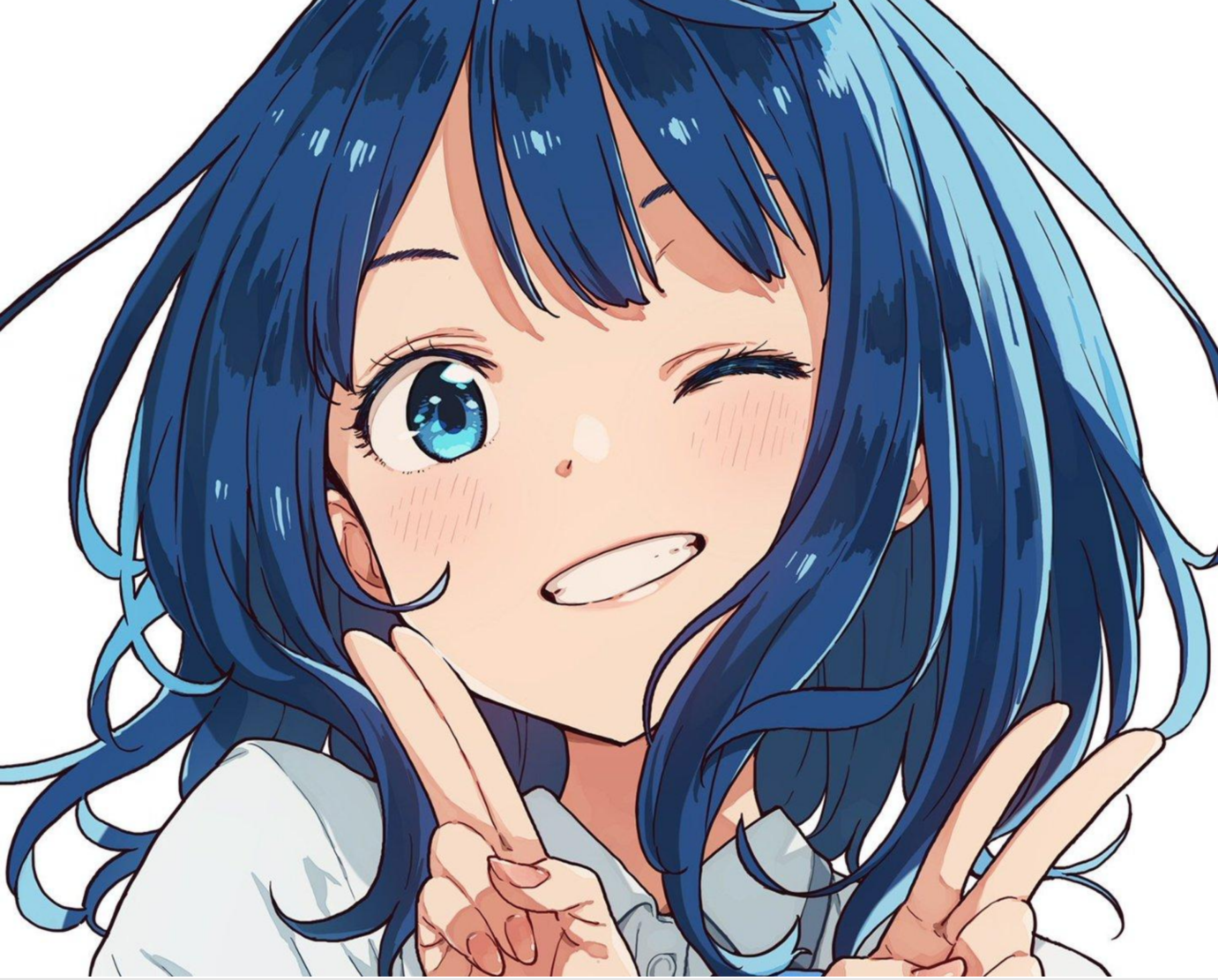

Basic Representation Theory

Learning and Thinking

Author: Yang Yihan

Institute: Department of Mathematics, Nankai University

Date: September 7, 2024

Email: matyyh@163.com

Beauty is the first test: there is no permanent place in the world for ugly mathematics

—— G.H. Hardy

Contents

Introduction

This note is made from the Basic Representation Theory in Nankai University in Fall, 2024.

Textbook: Lectures on Representation Theory, J-S Huang.

Recommended Video: BiliBili: Representation theory, R.Borchard

Topic:

- Rep of finite groups
- Rep of complex semisimple Lie algebra
- Rep of compact Lie groups

Grading:

- 50% Quiz(open-book) in class
- 50% Take home final assignments:Exercises,Report/Eassy
- Bonus: 20% presentation on Undergraduate Forum

Abbreviation:

- rep : representation
- irre : irreducible

Chapter 1 Representations of Finite Groups

1.1 Group representations

Definition 1.1 (representation)

*Let G be a finite group. A **representation** (π, V) of G on a finite-dimensional vector space V over $\mathbb{C}$ is a group homomorphism*

$$\pi : G \to \mathrm{GL}(V) \simeq \mathrm{GL}(n, \mathbb{C})$$

♣

Note *The dimension of V is sometimes called the **degree** of π.*

Exercise 1.1 Is $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{H})$ a group?

Remark If π is injective(faithful rep)($G \hookrightarrow \mathrm{GL}(n,\mathbb{C})$), then π identify G as a subgroup(called a linear matrix group) of $\mathrm{GL}(n,\mathbb{C})$.

Remark π induces a linear group action on V: $gv := \pi(g)v$.

Example :

- Trivial reps: G a group, $V = \mathbb{C}$, $\pi : G \to \mathrm{GL}(1,\mathbb{C})$,$g \mapsto 1 \iff \pi(g)v = v$.
- $G = S_3$, $V = \mathbb{C}^3 = \mathrm{Span}\{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\}$, then $\pi : G \to V$

  $$e \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (1,2) \mapsto \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad (2,3) \mapsto \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, \quad (1,3) \mapsto \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

  $$(1,2,3) \mapsto \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad (1,3,2) \mapsto \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

  Then π is a rep of S_3, s.t. $\pi(\sigma)\mathbf{e}_i = \mathbf{e}_{\sigma(i)}$, $\sigma \in S_3$.
- $V' = \mathbb{C}$, $\pi' : S_3 \to \mathbb{C}$, $\sigma \mapsto \det(\pi(\sigma)) = \mathrm{sgn}\,\sigma = \begin{cases} 1, & \text{even} \\ -1, & \text{odd} \end{cases}$ is a (1-dim) rep.
- $V = \mathbb{C}^3 = V_1 \oplus V_2$, $V_1 = \mathrm{Span}\{(1,1,1)^T\}$, $V_2 = \mathrm{Span}\{(a,b,c)^T : \ a+b+c = 0\}$.

  Note that V_1 and V_2 are invariant subspace under G

Definition 1.2

*Two representations (π, V) and (π', V') of G are said to be **equivalent** if there is an isomorphism of vector spaces $T : V \to V'$ which is compatible with the operation of G:*

$$\pi'(g)T(v) = T(\pi(g)v), \quad \textit{for all } g \in G \textit{ and } v \in V$$

$$\begin{tikzcd} V \arrow[r, "\pi(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ V' \arrow[r, "\pi'(g)"] & V' \end{tikzcd}$$

♣

Definition 1.3

*Let (π, V) a representation of G. A subspace W of V is called G-**invariant**, if $\pi(g)W \subset W$ for all $g \in G$. And we call (π, W) a **subrepresentation**.* ♣

Definition 1.4

*The representation (π, V) is said to be **irreducible** if there is no non-trivial G-invariant subspace,* i.e., *the only G-invariant subspaces are $\{0\}$ and V.* ♣

✍ **Exercise 1.2** Prove (π, V_2) is irreducible.

Goal: Classify finite abelian group; Methods: Schur's Lemma $\&$ Character

Theorem 1.1 (Schur's Lemma)

Let (π, V) and (π', V') be two irreducible representations of G. Suppose there is a linear transformation $T : V \to V'$ such that

$$\pi'(g)T(v) = T(\pi(g)v), \quad \textit{for all } g \in G \textit{ and } v \in V$$

Then T is either zero or an isomorphism. ♡

✍ **Exercise 1.3** (Also Motivation)Find matrices commuting with all $n \times n$ matrices.

Proof For $\forall v \in \text{Ker } T$, we have

$$T(\pi(g)v) = \pi'(g)T(v) = 0$$

So we know $\text{Ker } T$ is G-invariant, and given $\text{Ker } T$ is irreducible, we know $\text{Ker } T = 0$ or V.

(1) $\text{Ker } T = 0$, we have $V \simeq \text{Im } T \subset V'$, and we have $\text{Im } T \neq \{0\}$, so due to V' is irreducible, and for $v = T(u)$, $\pi'(g)v = T(\pi(g)u) \in \text{Im } T$, so $\text{Im } T$ is G-invariant, so we have $V = \text{Im } T = V'$. Therefore T is an isomorphism.

(2) $\text{Ker } T = V$, we can get $T = 0$. □

Definition 1.5 (G-equivariant)

$$\begin{tikzcd} V \arrow[r, "\pi(g)"] \arrow[d, "T"'] \arrow[dr, phantom, "\textit{commutes}"] & V \arrow[d, "T"] \\ V' \arrow[r, "\pi'(g)"'] & V' \end{tikzcd} \iff G\textit{-equivariant}$$

♣

Corollary 1.1

(π, V) is irre, $T : V \to V$ is G-equivariant linear transformation, then $T = \lambda I$, $\lambda \in \mathbb{C}$.

Proof As $\mathbb{C}$ is algebraic closed, T has an eigenvalue $\lambda \in \mathbb{C}$. The there exists an eigenvector v, s.t. $(T - \lambda I)(v) = 0$, i.e., $\mathrm{Ker}\,(T - \lambda I) \neq 0$.

Moreover, for any $v \in V$, we have

$$(T - \lambda I)(\pi(g)v) = T(\pi(g)v) - \lambda I(\pi(g)v) = \pi(g)T(v) - \pi(g)(\lambda I(v)) = \pi(g)(T - \lambda I)(v)$$

$T - \lambda I$ is G-equivariant, by Schur's Lemma, $\mathrm{Ker}\,(T - \lambda I) = V$, that is $T = \lambda I$. □

Corollary 1.2

(π_1, V_1), (π_2, V_2) are irreducible reps, $L_1, L_2 : V \to V'$ are invertible G-equivariant ((π_1, V_1), (π_2, V_2) are equivalent), then $L_2 = \lambda L_1$.

Theorem 1.2

G(finite) is abelian group, (π, V) is irre rep of G, then V is 1-dim.

Proof As G is abelian, we have

$$\pi(x)\pi(g) = \pi(xg) = \pi(gx) = \pi(g)\pi(x), \quad \forall x, g \in G$$

For any fixed $x \in G$, $\pi(x)$ is G-equivariant, by Schur's Lemma(Cor 1.1), $\pi(x) = \lambda I$.

So every subspace of V is G-invariant, so V is 1-dim. □

Definition 1.6

A Hermitian form on a vector space V over the complex field $\mathbb{C}$ is a function $f : V \times V \to \mathbb{C}$ such that for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$,

(1) $f\,(a\,u + b\,v,\ w) = a\ f\,(u,\ w) + b\ f\,(v,\ w)$.

(2)$f\,(u, v) = \overline{f\,(v,\ u)}$.

Here, the bar indicates tfe complex conjugate. It follows that

$$f\,(u, a\,v + b\,w) = \overline{a}\ f\,(u,\ v) + \overline{b}\ f\,(u,\ w)$$

which can be expressed by saying that f is antilinear on the second coordinate. Moreover, for all $v \in V$, $f\,(v, v) = \overline{f\,(v, v)}$ which means that $f\,(v, v) \in \mathbb{R}$

Note *An example is the dot product of $\mathbb{C}^n$, defined as*

$$(u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) = \sum_{i=1}^{n} u_i\,\overline{v_j}$$

Every Hermitian form on $\mathbb{C}^n$ is associated with an $n \times n$ Hermitian matrix A such that

$$f\,(\mathbf{X}, \mathbf{Y}) = \mathbf{X}\mathbf{A}\mathbf{Y}^{\mathrm{T}}$$

for all row vectors $\mathbf{X}$ and $\mathbf{Y}$ of $\mathbb{C}^n$. The matrix associated with the dot product is the $n \times n$ identity matrix.

Definition 1.7 (unitary)

A representation (π, V) is called ***unitary*** *if there is a positive definite hermitian form $\langle , \rangle$ which is G-invariant,* i.e., *it satisfies*

$$\langle \pi(g)u, \pi(g)v \rangle = \langle u, v \rangle, \quad \text{for all } g \in G \text{ and } u, v \in V$$

♣

Theorem 1.3

Let (π, V) be a rep of a finite group G. Then there exists a G-invariant, positive definite hermitian form $\langle , \rangle$ on V. We say that (π, V) is unitarizable.

♡

Proof Let $(,)$ be an hermitian inner product on V, define an "averaging" inner product $\langle , \rangle$ on V by

$$\langle u, v \rangle = \frac{1}{|G|} \sum_{g \in G} (\pi(g) u, \pi(g) v)$$

We can have

$$\begin{aligned} \langle \pi(g_0)u, \pi(g_0)v \rangle &= \frac{1}{|G|} \sum_{g \in G} (\pi(gg_0)u, \pi(gg_0)v) \\ &= \frac{1}{|G|} \sum_{g' \in G} (\pi(g')u, \pi(g')v) \\ &= \langle u, v \rangle \end{aligned}$$

□

Definition 1.8

Let (π, V) and (π', V') be two reps of G. We can make the direct sum $V \oplus V'$ a rep of G by

$$(\pi \oplus \pi')(g)(v + v') = \pi(g)v + \pi'(g)v'$$

♣

Proposition 1.1

Let (π, V) be a unitary representation of G and let W be a G-invariant subspace. Then the orthogonal complement $W^\perp$ is also G-invariant. So $V = W \oplus W^\perp$ as subrep.

♠

Proof Easy to know $W^\perp$ is G-invariant, and we know $V = W \oplus W^\perp$ by linear algebra. □

Motivation

- All reps of finite abelian groups are 1-dim.
- Any reps of finite group is direct sum of irre reps.
  (1) Any reps of finite groups is unitary.
  (2) Unitary $\Longrightarrow$ completely reducible.
- Classify all irre reps of finite abelian group.

Proposition 1.2

(π, V) is unitary rep of a finite group G, $W \subset V$ is a subrep of (π, V). Then the orthogonal complement $W^\perp$ is also G-invariant, i.e. *$V = W \oplus W^\perp$ is a direct sum of $\pi|_W \oplus \pi|_{W^\perp}$.*

♠

Proof For any $v \in W^\perp$, we have $\langle v, W \rangle = 0$, so we know that $\langle \pi(g)v, W \rangle = \langle \pi(g)v, \pi(g)W \rangle =$

$\langle v, W\rangle = 0$.

So we know $W^{\perp}$ is G-invariant, so $V = W \oplus W^{\perp}$ is a direct sum of $\pi|_W \oplus \pi|_{W^{\perp}}$. □

Theorem 1.4

Every unitary representation (π, V) of a finite group G is a direct sum of irreducible reps.

Proof (1) If V is irre rep, the case is done.

(2) If V is not irre rep, there exists subrep $W \subsetneq V$. As (π, V) is unitary, we know $(\pi, v) = (\pi, W) \oplus (\pi, W^{\perp})$ from proposition 1.2.

As $\dim V$ is finite, both W and $W^{\perp}$ can be decomposed until irre reps by induction. □

Exercise 1.4 $G = R$ as an additive group, $V = \mathbb{C}$, $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, π is not completely irre and its only 1-dim subrep is $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Proposition 1.3

G is a finite abelian group, (π, v) is a n-dim rep. There exists a basis $\mathscr{B}$ of V,

s.t. $\pi(g)|_{\mathscr{B}} = \begin{pmatrix} * & & & \\ & * & & \\ & & \ddots & \\ & & & * \end{pmatrix}$ *for* $\forall g \in G$, *with* $|*| = 1$.

Proof G is finite, so for any $g \in G$, we have $g^N = e$, where $N = |G|$.

Then we have $(\pi(g))^N = \pi(g^n) = \pi(e) = I$, which tells us that minimal polynomial $p(x)$ of $\pi(g)$ satisfies $f(x)|x^n - 1$, so we know all roots of $f(x)$ are distinct, i.e., $\pi(g)$ is diagonalizable.

As G is also abelian, we have $\pi(g_1)\pi(g_2) = \pi(g_2)\pi(g_1)$, which tells us that they are simultaneously diagonalizable.

And the rep is unitary, so $|*| = 1$. □

Motivation

- Build new rep from an original one.

Note *Direct sum $(\pi, V), (\pi', V')$, we have*

$$\pi \oplus \pi'(g)|_{(\mathscr{B}_1\ \mathscr{B}_2)} = \begin{pmatrix} \pi(g) & O \\ O & \pi'(g) \end{pmatrix}$$

Definition 1.9 (Dual rep)

Dual rep $V^* = \{f : V \to \mathbb{C} | f$ is linear function$\}$, then we define

$$\pi^*(g)f(v) \overset{\text{def}}{=} f(\pi(g^{-1})v)$$

Motivation of inverse

- Why inverse?
$$\pi^*(g_1g_2)\cdot f(v)=\pi^*(g_1)(\pi^*(g_2)\cdot f)(v)=\pi^*(g_2)\cdot f(\pi(g_1^{-1})v)=f(\pi(g_2^{-1}g_1^{-1})v)$$
- G-action on V, V^* cancel?
$$(\pi^*(g)f)(\pi(g)v)=f(\pi(g^{-1})\pi(g)v)=f(v)$$
- Suppose π is unitary, $\pi^*(g):\langle\cdot,w\rangle\mapsto\langle\cdot,\pi(g)w\rangle$.
Proof: As π is unitary, there exists unique $w\in V$ s.t. $f=\langle\cdot,w\rangle$. So we have
$$(\pi^*(g)f)v=f(\pi(g)^{-1}v)=\langle\pi(g^{-1})v,w\rangle=\langle v,\pi(g)w\rangle$$
- Linear algebra interpretation: $\pi(g)^*=\pi(g)^{-1}$.
Proof: Choose an orthogonal basis $\mathscr{B}$ of V and orthogonal basis of V^*, then
$$\langle v,w\rangle=\langle[v]_{\mathscr{B}},[w]_{\mathscr{B}}\rangle=[w]_{\mathscr{B}}^*[v]_{\mathscr{B}}$$
As $\langle\pi(v),\pi(w)\rangle=\langle v,w\rangle$, we know
$$w^*\pi(g)^*\pi(g)v=w^*v,\forall v,w\in V$$
So we know $\pi^*(g)=\pi(g^{-1})$.
This also tells us that π is unitary $\Longrightarrow\pi^*$ is unitary.
$$\langle\langle\cdot,v>,\langle\cdot,w\rangle\rangle_*\overset{\text{def}}{=}\overline{\langle v,w\rangle}=\langle w,v\rangle$$
- Complex conjugate rep $(\overline{\pi},\overline{V})$, where $\overline{V}:=V$ as an abelian group but $i\cdot\overline{v}=-i\cdot v$(new number multiplication with $1\cdot\overline{v}=1\cdot v$), and
$$\langle u,v\rangle_{\overline{V}}:=\langle v,u\rangle,\quad\overline{\pi}(g)v:=\pi(g)v$$

✍ **Exercise 1.5** $T:\overline{V}\to V^*, v\mapsto\langle\cdot,v\rangle$ is an equivalent of $(\overline{\pi},\overline{V})$ and (π^*,V^*).

1.2 Character of Representations

Definition 1.10 (Tensor product of vectors)

Tensor product of vectors: If $x,y\in V$ and have length M and N respectively, then the tensor product $x\otimes y=(x_iy_j)_{ij}=xy^T$, which is an $M\times N$ matrix. ♣

Similarly, we can generalize vector into matrix (which is vector in fact in matrix space).

Definition 1.11 (Tensor product of matrices)

Tensor product of matrices: If X,Y are matrices, then $X\otimes Y=(x_{i,j}Y)_{ij}$,where each $x_{i,j}Y$ is a block matrix.
$$X\otimes Y=\begin{pmatrix}x_{11}Y & x_{12}Y & \cdots & x_{1n}Y\\ x_{21}Y & x_{22}Y & \cdots & x_{2n}Y\\ \vdots & \vdots & & \vdots\\ x_{n1}Y & x_{n2}Y & \cdots & x_{nn}Y\end{pmatrix}$$
♣

The definition of the tensor product of two”vector spaces” follow the above definitions.

Definition 1.12 (Tensor product of vector space)

Tensor product of vector space: Given V, W vector spaces. Then their tensor product is the set:

$$V \otimes W = \left\{ \sum_{i,j} c_{ij} v_i \otimes w_j : c_{i,j} \in \mathbb{C} \right\}$$

with some relation:
(1) $(c_1 v_1 + c_2 v_2) \otimes w = c_1(v_1 \otimes w) + c_2(v_2 \otimes w)$;
(2) $v \otimes (d_1 w_1 + d_2 w_2) = d_1(v \otimes w_1) + d_2(v \otimes w_2)$.
One can check that the tensor product of two vector space is still a vector space. and naturally we can expect that the basis of $V \otimes W$, $\mathcal{B}_{V\otimes W} = \{v_i \otimes w_j : v_i \in \mathcal{B}_V, w_j \in \mathcal{B}_W\}$. ♣

Definition 1.13 (Tensor product)

$V = \mathrm{Span}\{\mathbf{e}_i\}$, $V' = \mathrm{Span}\{\mathbf{f}_j\}$, *then we define*

$$V \otimes V' = \mathrm{Span}\{\mathbf{e}_i \otimes \mathbf{f}_j\}$$

and

$$(\pi_1 \otimes \pi_2)(g)(v \otimes v') = \pi(g)v \otimes \pi'(g)v'$$

♣

✍ **Exercise 1.6** π, π' is unitary $\Longrightarrow \pi \otimes \pi'$ is unitary, where

$$\langle v \otimes v', w \otimes w' \rangle_{\otimes} := \langle v, v' \rangle \cdot \langle w, w' \rangle$$

Example 1.1 V, V': 1-dim, basis $\{\mathbf{e}\}, \{\mathbf{f}\}$, then we have

$$(\pi \otimes \pi')(\mathbf{e} \otimes \mathbf{f}) = (\pi(g)\mathbf{e}) \otimes (\pi'(g)\mathbf{f}) = (\pi(g) \otimes \pi'(g))(\mathbf{e} \otimes \mathbf{f})$$

thanks to the equation :$(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$.
Implies that $(\pi \otimes \pi')(g) = \pi(g) \otimes \pi'(g)$ as matrix.

Definition 1.14 (Character)

*A function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{tr}(\pi(g))$ is called the **character** of (π, V).* ♣

Remark Fix a basis of V, let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of matrix $\pi(g)$, then $\chi(g) = \sum_{k=1}^{n} \lambda_k$.

Proposition 1.4

Let χ_V be the character of a representation (π, V) of a finite group G. Then
(1) $\chi_V(e)$ is the dimension of V.
(2) $\chi_V(g) = \chi_V(hgh^{-1})$ for all $g, h \in G$.
(3) $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g \in G$.
(4) If $\chi_{V'}$ be the character of another representation (π', V'), then the character of $(\pi \oplus \pi', V \oplus V')$ is $\chi_V + \chi_{V'}$.
(5) The character of $(\pi \otimes \pi', V \otimes V')$ is $\chi_V \cdot \chi_{V'}$. ♠

Proof Fix a basis of V and regard $\pi(g)$ and $\pi'(g)$ as matrices.

(1) $\chi(e) = \mathrm{tr}(I) = \dim V$.

(2) $\pi(hgh^{-1}) = \pi(h)\pi(g)\pi(h)^{-1}$, thus we have $\chi(hgh^{-1}) = \chi(g)$.

(3) We know $\pi^*(g) \sim \pi(g^{-1})$, so we know

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$$

(4) As $(\pi \oplus \pi')(g)|_{(\mathscr{B}_1\ \mathscr{B}_2)} = \begin{pmatrix} \pi(g) & O \\ O & \pi'(g) \end{pmatrix}$, we know $\chi_{V\oplus V'}(g) = \chi_V(g) + \chi_{V'}(g)$.

(5) Similarly, we know the eigenvalues of $(\pi \otimes \pi')$ are $\lambda_i\mu_j$, so we know

$$\chi_{V\otimes V'} = \sum_{i,j} \lambda_i\mu_j = (\sum_i \lambda_i)(\sum_j \mu_j) = \chi_V \cdot \chi_{V'}$$

□

Definition 1.15 (Class function)

A complex function $\phi : G \to \mathbb{C}$, which is constant on each conjugacy class is called a class function. ♣

Definition 1.16

We define $C(G)$ the space of all class functions of G, and

$$\langle \chi, \chi' \rangle := \frac{1}{|G|} \sum_{g\in G} \chi(g)\overline{\chi'(g)}, \quad \chi, \chi' \in C(G)$$

♣

Theorem 1.5

G is a finite group whith $|G| = N$ and $(\pi_1, V_1), \cdots, (\pi_r, V_r)$ are distinct isomorphism classes of irre reps of G. Let χ_i be the charcater of π_i and $d_i = \dim V_i$, then we have:

(1) $\langle \chi_i, \chi_j \rangle = \delta_{ij}$.

(2) The irre characters form an orthogonal basis of $(C(G), \langle, \rangle)$.

(3) $\#\{$isomorphism classes of irre reps$\} = \#\{$conjugacy classes in the group$\}$.

(4) $\sum_{k=1}^{r} d_k^2 = |G|$. ♡

Proof wait □

Application of theorem 1.5

- (3) $\Rightarrow$ Classify irre reps of finite abelian group.
- Classification of finite abelian group $G \simeq \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, $n_i | n_{i+1}$.
- Irre reps of cyclic group:
  $\mathbb{Z}_n = \{0, 1, \cdots, n-1\}$ is cyclic group and $\pi : \mathbb{Z}_n \to \mathrm{GL}(1, \mathbb{C}) \simeq C^* = C \backslash \{0\}$.
  $\pi(0) = 1$, $\pi(1) = C$(determines the rep) so $\pi(m) = C^m$, which leads to $1 = \pi(n) = C^n$, indicating that $C = e^{\frac{k}{n}2\pi i}$, $k = 0, 1, \cdots, n-1$.
  By (3), there are n distinct isomorphism classes of irre reps of $\mathbb{Z}_n$.
- Irre reps of non-cyclic
  Let π be an irreducible rep of $\mathbb{Z}_n \oplus \mathbb{Z}_m$, consider the subgroup $\mathbb{Z}_n \oplus \{0\}$, $\{0\} \oplus \mathbb{Z}_m$, we have

$$\pi(a, b) = \pi[(a, 0) + (0, b)] = \pi(a, 0) \cdot \pi(0, b) = e^{\frac{k_1}{n}2\pi i} e^{\frac{k_2}{m}2\pi i}$$

Where $k_1 \in \mathbb{Z}_n, k_2 \in \mathbb{Z}_m$, by induction we can get that irre rep π of $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$

$$\pi \underset{1-1}{\longleftrightarrow} (m_1, \cdots, m_k)\text{lattice on 'sphere'}, \quad m_i \in \mathbb{Z}_{n_i}$$

Remark All irreducible reps form a group $\simeq G$.

Definition 1.17 (Dual group)

$G^* := \{\chi : G \to \mathbb{C}^*\}$ where χ are homomorphisms, $(\chi_1\chi_2)(g) := \chi_1(g)\chi_2(g)$ is called the dual group of G. ♣

Theorem 1.6 (Pontryagin duality)

For a finite abelian group G, we know $G^* \simeq G$. ♡