
Concept of Duality

In linear programming, **duality** refers to a relationship between two optimization problems:

- The **primal problem**: The original linear programming problem.
- The **dual problem**: A derived problem that provides bounds on the value of the primal problem's objective function.

Every linear programming problem (primal) has a corresponding **dual** problem. Solving either the primal or the dual gives information about the other. If one has an optimal solution, so does the other, and their optimal objective function values are equal under certain conditions (strong duality).

Fundamental Properties of Duality

1. **Weak Duality Theorem**

   For any feasible solution to the primal and any feasible solution to the dual:
   $Z_{\text{primal}} \leq W_{\text{dual}} \quad (\text{for maximization primal})$
   That is, the value of the primal is always less than or equal to the value of the dual.

2. **Strong Duality Theorem**

   If both the primal and dual have feasible solutions and are bounded, then:
   $Z_{\text{primal}} = W_{\text{dual}}$
   This means solving either problem gives the same optimal value.

3. **Complementary Slackness**

   If a primal constraint is **not tight** (i.e., slack exists), then the corresponding dual variable is zero, and vice versa. This provides a condition to check optimality:
   $y_i (a_i x - b_i) = 0 \quad \text{and} \quad x_j (a_j^T y - c_j) = 0$

4. **Dual of the Dual is the Primal**

When you take the dual of a dual problem, you get back the original primal problem.

5. **Infeasibility and Unboundedness**

    - If the primal is **unbounded**, then the dual is **infeasible**, and vice versa.
    - If one of the problems has no feasible solution, the other may be unbounded or also infeasible.

6. **Economic Interpretation**

    In many practical applications (e.g., resource allocation), the dual variables represent the **shadow prices**—the value of one additional unit of a resource.