Chapter 2: Duality in Linear Programming

Every linear programming problem is associated by another linear programming problem which is called a dual of the given problem. The original problem is called primal while the other is called its dual. The optimum solution of primal gives information about the optimum solution of dual and optimum solution of dual gives information about the optimum solution of primal.

Formulation of dual LPP: Suppose the primal LPP is given by

Maximum $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

where $x_1, x_2, \dots, x_n \ge 0$

Let $y_1, y_2, \dots, y_m$ are the dual variables then the corresponding dual LPP is

Minimum $Z_y = b_1y_1 + b_2y_2 + \cdots + b_my_m$

Subject to the constraints

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n$$

where $y_1, y_2, \dots, y_m \ge 0$

Primal-Dual relationship:

	Primal	**Dual**
i.	Objective is to maximize	Objective is to minimize
ii.	Variable $'x_j'$	Constraints ‘ j’
iii.	Constraints $'i'$	Variable $'y_i'$
iv.	Variable $'x_j'$ unrestricted	Constraints ‘ j’ is ‘=’ type
v.	Constraints $'i'$ is ‘=’ type	Variable $'y_i'$ unrestricted in sign
vi.	≤ type constraints	≥ type constraints

Note: If any variable x_i is unrestricted then replace it by $x_i' - x_i''$.

If a constraints has $' = '$ sign than it is replace by two constraint with $' \leq' \; and \; ' \geq '$ sign.

Q1. Write the dual of the following

$$Max. Z = x_1 + 2x_2 + x_3$$

Subject to the constraints

$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 10$$
$$x_1 , x_2, x_3 \geq 0$$

Solution: The given primal is

$$Max. Z_x = x_1 + 2x_2 + x_3$$

Subject to the constraints

$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 10$$
$$x_1 , x_2, x_3 \geq 0$$

Let y_1 , y_2 are the dual variables, then the dual of given primal is

$$Min. Z_y = 2y_1 + 10y_2$$

Subject to the constraints

$$2y_1 + 2y_2 \geq 1$$
$$y_1 - y_2 \geq 2$$
$$-y_1 + 5y_2 \geq 1$$
$$y_1 , y_2 \geq 0$$

Q2. $Min. Z = 7x_1 + 3x_2 + 8x_3$

Subject to the constraints

$$8x_1 + 2x_2 + x_3 \geq 3$$
$$3x_1 + 6x_2 + 4x_3 \geq 4$$
$$4x_1 + x_2 + 5x_3 \geq 1$$
$$x_1 + 5x_2 + 2x_3 \geq 7$$
$$x_1 , x_2, x_3 \geq 0$$

Q3. $Min. Z = 3x_1 - 2x_2 + 4x_3$

Subject to the constraints

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 + 4x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$
$$x_1 , x_2, x_3 \geq 0$$

Solution: The given primal is

$$Min. Z_x = 3x_1 - 2x_2 + 4x_3$$

Subject to the constraints

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 + 4x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

$$Min. Z_x = 3x_1 - 2x_2 + 4x_3$$

Subject to the constraints

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$-7x_1 + 2x_2 - 4x_3 \geq -10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

Let y_1, y_2, y_3, y_4, y_5 are the dual variables, then the dual of given primal is

$$Max. Z_y = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$$

Subject to the constraints

$$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$$
$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$$
$$4y_1 + 3y_2 - 4y_3 + 5y_4 - 2y_5 \leq 4$$
$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

Q4. $Min. Z = 2x_1 + 3x_2 + 4x_3$

Subject to the constraints

$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 = 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution: The given primal is

$$Min. Z_x = 2x_1 + 3x_2 + 4x_3$$

Subject to the constraints

$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 = 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

$$Min.Z_x = 2x_1 + 3x_2 + 4x_3$$

Subject to the constraints

$2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \geq 3$

$3x_1 + x_2 + 7x_3 \leq 3$ or $-3x_1 - x_2 - 7x_3 \geq -3$

$-x_1 - 4x_2 - 6x_3 \geq -5$

$x_1, x_2, x_3 \geq 0$

Let y_1, y_2', y_2'', y_3 are the dual variables, then the dual of given primal is

$$Max.Z_y = 2y_1 + 3y_2' - 3y_2'' - 5y_3$$

Subject to the constraints

$2y_1 + 3y_2' - 3y_2'' - y_3 \leq 2$

$3y_1 + y_2' - y_2'' - 4y_3 \leq 3$

$5y_1 + 7y_2' - 7y_2'' - 6y_3 \leq 4$

$y_1, y_2', y_2'', y_3 \geq 0$

$$Max.Z_y = 2y_1 + 3(y_2' - y_2'') - 5y_3$$

Subject to the constraints

$2y_1 + 3(y_2' - y_2'') - y_3 \leq 2$

$3y_1 + (y_2' - y_2'') - 4y_3 \leq 3$

$5y_1 + 7(y_2' - y_2'') - 6y_3 \leq 4$

$y_1, y_2', y_2'', y_3 \geq 0$

Let $y_2' - y_2'' = y_2$

$$Max.Z_y = 2y_1 + 3y_2 - 5y_3$$

Subject to the constraints

$2y_1 + 3y_2 - y_3 \leq 2$

$3y_1 + y_2 - 4y_3 \leq 3$

$5y_1 + 7y_2 - 6y_3 \leq 4$

$y_1, y_3 \geq 0,$ y_2 is unrestricted.

Q5. $Max.Z = 3x_1 + x_2 + x_3 - x_4$

Subject to the constraints

$x_1 + 5x_2 + 3x_3 + 4x_4 \leq 5$

$3x_1 + x_2 = -1$

$x_3 - x_4 \geq -5$

$x_1, x_2, x_3, x_4 \geq 0$

Q6. $Min.Z = x_1 + x_2 + x_3$

Subject to the constraints

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0,\quad x_3$ is unrestricted.

Solution: Transform the given LPP into the standard primal form by substituting $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$.

$$Min.Z_x = x_1 + x_2 + (x_3' - x_3'')$$

Subject to the constraints

$x_1 - 3x_2 + 4(x_3' - x_3'') \geq 5$

$x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$ or $-x_1 + 3x_2 - 4(x_3' - x_3'') \geq -5$

$-x_1 + 2x_2 \geq -3$

$2x_2 - (x_3' - x_3'') \geq 4$

$x_1, x_2, x_3', x_3'' \geq 0$

Let y_1', y_1'', y_2, y_3 are the dual variables then the dual problem is

$$MaxZ_y = 5y_1' - 5y_1'' - 3y_2 + 4y_3$$

Subject to the constraints

$y_1' - y_1'' - y_2 \leq 1$

$-3y_1' + 3y_1'' + 2y_2 + 2y_3 \leq 1$

$4y_1' - 4y_1'' - y_3 \leq 1$

$-4y_1' + 4y_1'' + y_3 \leq -1$ or $4y_1' - 4y_1'' - y_3 \geq 1$

$y_1', y_1'', y_2, y_3 \geq 0$

Converting into standard form

$$MaxZ_y = 5y_1 - 3y_2 + 4y_3$$

Subject to the constraints

$y_1 - y_2 \leq 1$

$-3y_1 + 2y_2 + 2y_3 \leq 1$

$4y_1 - y_3 = 1$

$y_2, y_3 \geq 0$ and y_1 is unrestricted

Q7. $Min.Z = 2x_1 + 3x_2 + 4x_3$

Subject to the constraints

$2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 = 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2 \geq 0$ and x_3 is unrestricted.

Properties of duality:

1. The dual of the dual is primal.
2. If one is a maximization problem then the other is a minimization problem.
3. The necessary and sufficient condition for any LPP and its dual to have an optimal solution is that both must have feasible solution.
4. Fundamental duality theorem states if either the primal or dual problem has a finite optimal solution, then the dual problem also has a finite optimal solution and also the optimal values of the objective function in both the problems are the same i.e. $Max.Z = Min.Z_D$. The solution of the other problem can be read from the $c_j - z_j$ row below the column of slack, surplus variables.
5. Existence Theorem state that, if either problem has an unbounded solution then the dual problem has no feasible solution.

Duality and Simplex Method:

Q1. Use duality to solve $Min.Z = 3x_1 + x_2$

Subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Answer: $y_1 = 1, y_2 = 0, \quad \max Z_y = 1$

$x_1 = 0, x_2 = 1, \quad minZ_x = 1$

Q2. $Min.Z = x_1 - x_2$

Subject to the constraints

$$2x_1 + x_2 \geq 2$$
$$-x_1 - x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

Solution: The dual of given problem is

$$Max.Z_y = 2y_1 + y_2$$

Subject to the constraints

$$2y_1 - y_2 \leq 1$$
$$y_1 - y_2 \leq -1$$
$$y_1, y_2 \geq 0$$

Converting into standard form

$$Max.Z_y = 2y_1 + y_2 + 0s_1 + 0s_2 - MA_1$$

Subject to the constraints

$$2y_1 - y_2 + s_1 = 1$$
$$-y_1 + y_2 - s_2 + A_1 = 1$$
$$y_1, y_2, s_1, s_2, A_1 \geq 0$$

		c_j	2	1	0	0	-M	
Basic variable	C_B	Y_B	y_1	y_2	s_1	s_2	A_1	Min. Ratio
s_1	0	1	2	-1	1	0	0	×
A_1	-M	1	-1	1	0	-1	1	←
	$z_j = C_B y_j$		M	-M	0	M	-M	
	$\Delta_j = c_j - z_j$		-M+2	M+1	0	-M	0	

$R_1 \rightarrow R_1 + R_2$

		c_j	2	1	0	0	
Basic variable	C_B	Y_B	y_1	y_2	s_1	s_2	Min. Ratio
s_1	0	2	1	0	1	-1	←
y_2	1	1	-1	1	0	-1	×
	$z_j = C_B y_j$		-1	1	0	-1	
	$\Delta_j = c_j - z_j$		3	0	0	1	

$R_2 \rightarrow R_2 + R_1$

		c_j	2	1	0	0	
Basic variable	C_B	Y_B	y_1	y_2	s_1	s_2	Min. Ratio
y_1	2	2	1	0	1	-1	×
y_2	1	3	0	1	1	-2	×
	$z_j = C_B y_j$		2	1	3	-4	
	$\Delta_j = c_j - z_j$		0	0	-3	4	

Since the dual problem has unbounded solution hence the primal does not possess any optimum basic feasible solution.

Dual Simplex Method:

- The main advantage of dual simplex method over the usual simplex method is that we do not require any artificial variables in the dual simplex method.
- The dual simplex method works toward feasibility (all solution set $x_j \geq 0$) while simplex method works towards optimality $(c_j - z_j \leq 0)$

Q1. Solve LPP by Dual Simplex Method

$$Min. Z = 3x_1 + x_2$$

Subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Solution: Converting into standard form

$$Max. Z_x = -3x_1 - x_2$$

Subject to the constraints

$$-x_1 - x_2 \leq -1$$
$$-2x_1 - 3x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

$$Max. Z_x = -3x_1 - x_2 + 0s_1 + 0s_2$$

Subject to the constraints

$$-x_1 - x_2 + s_1 = -1$$
$$-2x_1 - 3x_2 + s_2 = -2$$
$$x_1, x_2, s_1, s_2 \geq 0$$

		c_j	-3	-1	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2
s_1	0	-1	-1	-1	1	0
s_2	0	-2	-2	-3	0	1
		$z_j = C_B x_j$	0	0	0	0
		$\Delta_j = c_j - z_j$	-3	-1	0	0

Since all $c_j - z_j \leq 0$ and all $x_{Bi} < 0$, the basic solution is optimal but infeasible

To find feasible solution:

Since $x_{B2} = -2$ is the most negative the corresponding basic variable s_2 leaves the basis (R_2 is the pivot row)

Also, $\min\left\{\frac{c_j - z_j}{a_{ik}}\ ,\ a_{ik} < 0\right\} = \min\left\{\frac{-3}{-2},\ \frac{-1}{-3}\right\} = \frac{1}{3}$

Therefore x_2 is the pivot column

$$R_2 \to \frac{R_2}{-3}\ , \qquad R_1 \to R_1 + R_2$$

The non-basic variable x_2 enters the basis

		c_j	-3	-1	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2
s_1	0	-1/3	-1/3	0	1	-1/3
x_2	-1	2/3	2/3	1	0	-1/3
		$z_j = C_B x_j$	-2/3	-1	0	1/3
		$\Delta_j = c_j - z_j$	-7/3	0	0	-1/3

Here R_1 is the pivot row, the most negative value $x_{B1} = -1/3$

$\min\left\{\frac{-7/3}{-1/3},\ \frac{-1/3}{-1/3}\right\} = 1,$ $\quad s_2$ is the pivot column

$$R_1 \to -3R_1\ , \qquad R_2 \to R_2 + \frac{1}{3}R_1$$

		c_j	-3	-1	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2
s_2	0	1	1	0	-3	1
x_2	-1	1	1	1	-1	0
		$z_j = C_B x_j$	-1	-1	1	0
		$\Delta_j = c_j - z_j$	-2	0	-1	0

Since all $c_j - z_j \leq 0$ and all $x_{Bi} \geq 0$ an optimum basic feasible solution has been obtained.

i.e. $x_1 = 0,\ \ x_2 = 1,\quad max.Z_x = -1\ ,\ min.Z = 1$

Q2. $\qquad Max.Z = -2x_1 - x_3$

Subject to the constraints

$x_1 + x_2 - x_3 \geq 5$

$x_1 - 2x_2 + 4x_3 \geq 8$

$x_1\,, x_2, x_3 \geq 0$

Solution: $Max. Z = -2x_1 - x_3$

Subject to the constraints

$-x_1 - x_2 + x_3 \leq -5$

$-x_1 + 2x_2 - 4x_3 \leq -8$

$x_1, x_2, x_3 \geq 0$

$Max. Z = -2x_1 - x_3 + 0\, s_1 + 0s_2$

Subject to the constraints

$-x_1 - x_2 + x_3 + s_1 = -5$

$-x_1 + 2x_2 - 4x_3 + s_2 = -8$

$x_1, x_2, x_3 \geq 0$

		c_j	-2	0	-1	0	0
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2
s_1	0	-5	-1	-1	1	1	0
s_2	0	-8	-1	2	-4	0	1
		$z_j = C_B x_j$	0	0	0	0	0
		$\Delta_j = c_j - z_j$	-2	0	-1	0	0

Since all $c_j - z_j \leq 0$ and all $x_{Bi} < 0$, the basic solution is optimal but infeasible .Since $x_{B2} = -8$ is the most negative the corresponding basic variable s_2 leaves the basis (R_2 is the pivot row)

Also, $\min\left\{\frac{c_j - z_j}{a_{ik}}, \ a_{ik} < 0\right\}$ =min$\left\{\frac{-2}{-1}, \ \frac{-1}{-4}\right\} = \frac{1}{4}$

Therefore x_3 is the pivot column

$R_2 \to \frac{R_2}{-4}$, $\quad R_1 \to R_1 - R_2$

		c_j	-2	0	-1	0	0
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2
s_1	0	-7	-5/4	-1/2	0	1	1/4
x_3	-1	2	1/4	-1/2	1	0	-1/4
		$z_j = C_B x_j$	-1/4	1/2	-1	0	1/4
		$\Delta_j = c_j - z_j$	-7/4	-1/2	0	0	1/4

Here R_1 is the pivot row, the most negative value $x_{B1} = -7$

$$\min\left\{\frac{-7/4}{-5/4}, \ \frac{-1/2}{-1/2}\right\} = 1, \qquad x_2 \text{ is the pivot column}$$

$$R_1 \to -2R_1 \ , \qquad R_2 \to R_2 + \frac{1}{2}R_1$$

		c_j	-2	0	-1	0	0
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2
x_2	0	14	5/2	1	0	-2	-1/2
x_3	-1	9	3/2	0	1	-1	-1/2
	$z_j = C_B x_j$		-3/2	0	-1	1	1/2
	$\Delta_j = c_j - z_j$		-1/2	0	0	-1	-1/2

Since all $c_j - z_j \leq 0$ and all $x_{Bi} \geq 0$ an optimum basic feasible solution has been obtained.

i.e. $x_1 = 0, \ x_2 = 14, \quad x_3 = 9 \ , \ max.Z = -9$

Q3. $Min. Z = x_1 + 2x_2$

Subject to the constraints

$$2x_1 + x_2 \geq 4$$
$$x_1 + 2x_2 \geq 7$$
$$x_1 \, , x_2 \geq 0$$

Answer: $x_1 = 0, \ x_2 = 2$

Q4. $Min. Z = 10x_1 + 6x_2 + 2x_3$

Subject to the constraints

$$-x_1 + x_2 + x_3 \geq 1$$
$$3x_1 + x_2 - x_3 \geq 2$$
$$x_1 \, , x_2 , x_3 \geq 0$$

Answer: $x_1 = 1/4, \ x_2 = 5/4$