Chapter III
Transportation Problem

The transportation problem is one of the subclasses of LPPs in which the objective is to transport various quantities of a single homogeneous commodity (product) that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum.

For example: A tyre manufacturing concern has m factories located in m different cities. The total supply potential of manufactured product is absorbed by n retail dealers in n different cities of the country. Then, transportation problem is to determine the transportation schedule that minimizes the total cost of transporting tyres from various factory locations to various retail dealers.

To achieve this objective we must know the amount and location of available supplies and the quantities demanded. In addition we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

Mathematical Model of Transportation Problem:

Let $a_i =$ quantity of commodity available at origin 'i'

$b_j =$ quantity of commodity needed at destination 'j'

$c_{ij} =$ cost of transporting one unit of commodity from origin 'i' to destination 'j'

and $x_{ij} =$ quantity transported from origin 'i' to destination 'j'

then the problem is to determine the transportation schedule so as to minimize the total transportation cost satisfying supply and demand constraints.

The problem may be stated as a linear programming problem as follows:

$$Min. Z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij} \quad \text{(objective function)}$$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i\,, \qquad i = 1,2,\cdots,m \quad \text{(supply constraints)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j\,, \qquad j = 1,2,\cdots,n \quad \text{(demand constraints)}$$

And $x_{ij} \geq 0 \ \ \forall\, i \text{ and } j$

Existence of Feasible solution: A necessary and sufficient condition for the existence of a feasible solution to the transportation problem is

Total supply = Total demand

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \text{(also called Rim condition)}$$

Tabular Representation: Suppose there are m supply (factories) and n demand (warehouse) transportation problem is represented in a tabular form

To / From	D_1	D_2	…	D_n	Supply a_i
S_1	c_{11} x_{11}	c_{12} x_{12}	…	c_{1n} x_{1n}	a_1
S_2	c_{21} x_{21}	c_{22} x_{22}	…	c_{2n} x_{2n}	a_2
⋮	⋮	⋮	⋮	⋮	⋮
S_m	c_{m1} x_{m1}	c_{m2} x_{m2}	…	c_{mn} x_{mn}	a_m
Demand b_j	b_1	b_2	…	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Note: When the total supply equals total demand the problem is called a balanced transportation problem, otherwise an unbalanced TP.

Feasible Solution: Any set of non-negative allocations ($x_{ij} > 0$) which satisfies the row and column sum (Rim requirement) is called a feasible solution.

Basic Feasible Solution: A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to m+n-1 where m is the number of rows, n the number of columns in a transportation table.

The solution of a transportation problem can be obtained in two stages, namely initial solution and optimum solution. Initial solution can be obtained by using any one of the three methods

1. North West Corner Rule (NWCR)
2. Least cost method or Matrix minima method
3. Vogel's Approximation method (VAM)

1. North West Corner Rule (NWCR):

Step 1. The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is, $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1,1) of the transportation table.

Step 2. (i) If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).

(ii) If $b_1 < a_1$, move horizontally right-side to the second row and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).

(iii) If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude $x_{12} = \min(a_1 - a_1, b_2) = 0$ in the cell (1, 2) or $x_{21} = \min(a_1, b_1 - b_1) = 0$ in the cell (2, 1).

Step 3. Start from the new north-west corner of the transportation table and repeat step 1 and 2 until all the requirements are satisfied.

Q1. Obtain the initial basic feasible solution of a transportation problem by North East corner method

	D_1	D_2	D_3	supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
demand	7	9	18	34

Solution: Since $\sum a_i = 34 = \sum b_j$ there exist a feasible solution to the transportation problem we obtain initial feasible solution as follow

Start with the cell at the upper left corner of the transportation matrix and allocate as much as possible equal to the minimum of the rim values for the first row and first column i.e. min.(a_1, b_1)

	D_1	D_2	D_3	supply
O_1	2 (5)	7	4	$\cancel{5}$ 0
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
demand	$\cancel{7}$ 2	9	18	34

	D_1	D_2	D_3	supply
O_1	2 (5)	7	4	$\cancel{5}$ 0
O_2	3 (2)	3	1	$\cancel{8}$ 6
O_3	5	4	7	7
O_4	1	6	2	14
demand	$\cancel{7}$ $\cancel{2}$ 0	9	18	34

	D_1	D_2	D_3	supply
O_1	2 (5)	7	4	$\cancel{5}$ 0
O_2	3 (2)	3 (6)	1	$\cancel{8}$ $\cancel{6}$ 0
O_3	5	4	7	7
O_4	1	6	2	14
demand	$\cancel{7}$ $\cancel{2}$ 0	$\cancel{9}$ 3	18	34

	D_1	D_2	D_3	supply
O_1	2 (5)	7	4	$\cancel{5}$ 0
O_2	3 (2)	3 (6)	1	$\cancel{8}$ $\cancel{6}$ 0
O_3	5	4 (3)	7	$\cancel{7}$ 4
O_4	1	6	2	14
demand	$\cancel{7}$ $\cancel{2}$ 0	$\cancel{9}$ $\cancel{3}$ 0	18	34

	D_1	D_2	D_3	supply
O_1	2 (5)	7	4	$\cancel{5}$ 0
O_2	3 (2)	3 (6)	1	$\cancel{8}$ $\cancel{6}$ 0
O_3	5	4 (3)	7 (4)	$\cancel{7}$ $\cancel{4}$ 0
O_4	1	6	2	14
demand	$\cancel{7}$ $\cancel{2}$ 0	$\cancel{9}$ $\cancel{3}$ 0	$\cancel{18}$ 14	34

	D_1	D_2	D_3	supply
O_1	2 (5)	7	4	$\cancel{5}$ 0
O_2	3 (2)	3 (6)	1	$\cancel{8}$ $\cancel{6}$ 0
O_3	5	4 (3)	7 (4)	$\cancel{7}$ $\cancel{4}$ 0
O_4	1	6	2 (14)	$\cancel{14}$ 0
demand	$\cancel{7}$ $\cancel{2}$ 0	$\cancel{9}$ $\cancel{3}$ 0	$\cancel{18}$ $\cancel{14}$ 0	34

Number of non-negative allocation is 6=m+n-1.

Hence the initial basic feasible solution is

$$\text{Total cost} = 2 \times 5 + 3 \times 2 + 3 \times 6 + 4 \times 3 + 7 \times 4 + 2 \times 14 = 102$$

Q2. Obtain the initial basic feasible solution of a transportation problem

	D_1	D_2	D_3	D_4	supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
demand	6	10	15	4	

Answer : 128

Q3. Obtain the initial basic feasible solution of a transportation problem

	D_1	D_2	D_3	D_4	supply
O_1	21	16	15	3	11
O_2	17	18	14	23	13
O_3	32	27	18	41	19
demand	6	10	12	15	

2. Least cost method or Matrix minima method

Step 1: Determine the smallest cost matrix of the transportation table. Let it be c_{ij} . Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i,j) .

Step 2: (i) If $x_{ij} = a_i$, cross out the ith row of the transportation table and decrease b_j by a_i. Go to Step3.

(ii) If $x_{ij} = b_j$, cross out the jth column of the transportation table and decrease a_i by b_j. Go to Step3.

(iii) If $x_{ij} = a_i = b_j$, cross out the ith row and jth column.

Step 3: Repeat steps 1 &2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Q1. Obtain the initial basic feasible solution of a transportation problem by Least cost method

	D_1	D_2	D_3	D_4	supply
O_1	21	16	15	3	11
O_2	17	18	14	23	13
O_3	32	27	18	41	19
demand	6	10	12	15	43

Solution: Since $\sum a_i = 43 = \sum b_j$ there exist a feasible solution to the transportation problem we obtain initial feasible solution as follow

Start with the lowest cost entry (3) in the cell (1,4) and allocate as much as possible, i.e. $x_{14} = 11$.

	D_1	D_2	D_3	D_4	supply
O_1	21	16	15	3 (11)	~~11~~ 0
O_2	17	18	14	23	13
O_3	32	27	18	41	19
demand	6	10	12	~~15~~ 4	43

	D_1	D_2	D_3	D_4	supply
O_1	~~21~~	~~16~~	~~15~~	~~3 (11)~~	~~11~~ 0
O_2	17	18	14	23	13
O_3	32	27	18	41	19
demand	6	10	12	~~15~~ 4	43

The next lowest cost (14) lies in the cell (2,3) , so allocate $x_{23} = 12$

	D_1	D_2	D_3	D_4	supply
O_1	~~21~~	~~16~~	~~15~~	~~3 (11)~~	~~11~~ 0
O_2	17	18	~~14 (12)~~	23	~~13~~ 1
O_3	32	27	~~18~~	41	19
demand	6	10	~~12~~ 0	~~15~~ 4	43

The next lowest cost (17) lies in the cell (2,1) , so allocate $x_{21} = 1$

	D_1	D_2	D_3	D_4	supply
O_1	~~21~~	~~16~~	~~15~~	~~3 (11)~~	~~11~~ 0
O_2	~~17 (1)~~	~~18~~	~~14 (12)~~	~~23~~	~~13~~ ~~1~~ 0
O_3	32	27	~~18~~	41	19
demand	~~6~~ 5	10	~~12~~ 0	~~15~~ 4	43

	D_1	D_2	D_3	D_4	supply
O_1	~~21~~	~~16~~	~~15~~	~~3 (11)~~	~~11~~ 0
O_2	~~17 (1)~~	~~18~~	~~14 (12)~~	~~23~~	~~13~~ ~~1~~ 0
O_3	32 (5)	27 (10)	~~18~~	41(4)	~~19~~ ~~9~~ ~~4~~ 0
demand	~~6~~ ~~5~~ 0	~~10~~ 0	~~12~~ 0	~~15~~ ~~4~~ 0	43

Number of non-negative allocation is 6=m+n-1.

Hence the initial basic feasible solution is

$$\text{Total cost} = 3 \times 11 + 17 \times 1 + 14 \times 12 + 32 \times 5 + 27 \times 10 + 41 \times 4 = 812$$

Q2. Obtain the initial basic feasible solution of a transportation problem by Least cost method

	D_1	D_2	D_3	D_4	supply
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
demand	20	40	30	10	

3. Vogel's Approximation method (VAM)

Step 1: For each row of the transportation table identify the smallest and next-to-smallest cost. Determine the difference between them for each row. These are called 'penalties'. Put them alongside the transportation table by enclosing them in the parentheses against the respective rows. Similarly, compute these penalties for each column.

Step 2: Identify the row or column with the largest penalty among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the largest penalty correspond to i^{th} row and let c_{ij} be the smallest cost in the i^{th} row. Allocate the largest possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross-out the i^{th} row or the j^{th} column in the usual manner.

Step 3: Again compute the column and row penalties for the reduced transportation table and then go to step 2. Repeat the procedure until all the requirements are satisfied.

Q1. Obtain the initial basic feasible solution of a transportation problem by Vogel's approximation method

	D_1	D_2	D_3	D_4	D_5	supply
O_1	2	11	10	3	7	4
O_2	1	4	7	2	1	8
O_3	3	9	4	8	12	9
demand	3	3	4	5	6	21

Solution: Since $\sum a_i = 21 = \sum b_j$ there exist a feasible solution to the transportation problem we obtain initial feasible solution as follow

Step 1: Take the difference between the lowest and second lowest cost entries in each column beneath the corresponding column and put the difference between the lowest and second lowest cost entries of each row to the right of that row. Such individual differences can be thought of a penalty for making allocations in second lowest cost entries instead of lowest cost entries in each row or column.

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	2	11	10	3	7	4	1
O_2	1	4	7	2	1	8	1
O_3	3	9	4	8	12	9	1
demand	3	3	4	5	6	21	
Penalties	1	5	3	1	6		

Step 2: select the row or column for which the penalty is the largest i.e. 6 and allocate the maximum possible amount to the cell (2,5) with the lowest cost (1) in the particular column making $x_{25} = 6$. If there are more than one largest penalty rows (columns), select one of them arbitrarily.

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	2	11	10	3	7	4	1
O_2	1	4	7	2	1 (6)	~~8~~ 2	1
O_3	3	9	4	8	12	9	1
demand	3	3	4	5	~~6~~ 0	21	
Penalties	1	5	3	1	6 ↑		

Step 3: Cross-out that column in which the requirement has been satisfied. In this fifth column has been crossed-out. Then find the corresponding penalties from the reduce transportation table.

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	2	11	10	3	~~7~~	4	1
O_2	1	4	7	2	~~1~~ (6)	~~8~~ 2	1
O_3	3	9	4	8	~~12~~	9	1
demand	3	3	4	5	~~6~~ 0	21	
Penalties	1	5	3	1	×		

Step 4: Repeat steps 2&3 till all allocations have been made.

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	2	11	10	3	~~7~~	4	1
O_2	1	4 (2)	7	2	~~1~~ (6)	~~8~~ ~~2~~ 0	1
O_3	3	9	4	8	~~12~~	9	1
demand	3	~~3~~ 1	4	5	~~6~~ 0	21	
Penalties	1	5	3	1	×		

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	2	11	10	3	~~7~~	4	1
O_2	~~1~~	~~4~~ (2)	~~7~~	~~2~~	~~1~~ (6)	~~8~~ ~~2~~ 0	×
O_3	3	9	4	8	~~12~~	9	1
demand	3	~~3~~ 1	4	5	~~6~~ 0	21	
Penalties	1	2	6	5	×		

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	2	11	10	3	~~7~~	4	1
O_2	~~1~~	~~4~~ (2)	~~7~~	~~2~~	~~1~~ (6)	~~8~~ ~~2~~ 0	×
O_3	3	9	4 (4)	8	~~12~~	~~9~~ 5	1
demand	3	~~3~~ 1	~~4~~ 0	5	~~6~~ 0	21	
Penalties	1	2	6	5	×		

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	2	11	$\cancel{10}$	3	$\cancel{7}$	4	1
O_2	$\cancel{1}$	$\cancel{4\,(2)}$	$\cancel{7}$	$\cancel{2}$	$\cancel{1\,(6)}$	$\cancel{8}\ \cancel{2}\ 0$	×
O_3	3 (3)	9	$\cancel{4\,(4)}$	8	$\cancel{12}$	$\cancel{9}\ \cancel{5}\ 2$	5 ←
demand	$\cancel{3}\ 0$	$\cancel{3}\ 1$	$\cancel{4}\ 0$	5	$\cancel{6}\ 0$	21	
Penalties	1	2	×	5	×		

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	$\cancel{2}$	11	$\cancel{10}$	3	$\cancel{7}$	4	8
O_2	$\cancel{1}$	$\cancel{4\,(2)}$	$\cancel{7}$	$\cancel{2}$	$\cancel{1\,(6)}$	$\cancel{8}\ \cancel{2}\ 0$	×
O_3	$\cancel{3\,(3)}$	9	$\cancel{4\,(4)}$	8	$\cancel{12}$	$\cancel{9}\ \cancel{5}\ 2$	1
demand	$\cancel{3}\ 0$	$\cancel{3}\ 1$	$\cancel{4}\ 0$	5	$\cancel{6}\ 0$	21	
Penalties	×	2	×	5	×		

	D_1	D_2	D_3	D_4	D_5	supply	Penalties
O_1	$\cancel{2}$	11	$\cancel{10}$	3 (4)	$\cancel{7}$	$\cancel{4}\ 0$	8 ←
O_2	$\cancel{1}$	$\cancel{4\,(2)}$	$\cancel{7}$	$\cancel{2}$	$\cancel{1\,(6)}$	$\cancel{8}\ \cancel{2}\ 0$	×
O_3	$\cancel{3\,(3)}$	9 (1)	$\cancel{4\,(4)}$	8 (1)	$\cancel{12}$	$\cancel{9}\ \cancel{5}\ \cancel{2}\ \cancel{1}\ 0$	1
demand	$\cancel{3}\ 0$	$\cancel{3}\ \cancel{1}\ 0$	$\cancel{4}\ 0$	$\cancel{5}\ \cancel{1}\ 0$	$\cancel{6}\ 0$	21	
Penalties	×	2	×	5	×		

Number of non-negative allocation is 7=m+n-1. Hence the initial basic feasible solution is

$$\text{Total cost} = 3\times 4 + 4\times 2 + 1\times 6 + 3\times 3 + 9\times 1 + 4\times 4 + 8\times 1 = 68$$

Q2. Obtain the initial basic feasible solution of a transportation problem by Vogel's approximation method

	D_1	D_2	D_3	D_4	supply
O_1	21	16	25	13	11
O_2	17	18	14	23	13
O_3	32	17	18	41	19
demand	6	10	12	15	43

Solution: Since $\sum a_i = 43 = \sum b_j$ there exist a feasible solution to the transportation problem we obtain initial feasible solution as follow

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	21	16	25	13	11	3
O_2	17	18	14	23	13	3
O_3	32	17	18	41	19	1
demand	6	10	12	15	43	
Penalties	4	1	4	10		

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	21	16	25	13 (11)	$\cancel{11}$ 0	3
O_2	17	18	14	23	13	3
O_3	32	17	18	41	19	1
demand	6	10	12	$\cancel{15}$ 4	43	
Penalties	4	1	4	10 ↑		

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	$\cancel{21}$	$\cancel{16}$	$\cancel{25}$	$\cancel{13 (11)}$	$\cancel{11}$ 0	×
O_2	17	18	14	23	13	3
O_3	32	17	18	41	19	1
demand	6	10	12	$\cancel{15}$ 4	43	
Penalties	4	1	4	10		

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	$\cancel{21}$	$\cancel{16}$	$\cancel{25}$	$\cancel{13 (11)}$	$\cancel{11}$ 0	×
O_2	17	18	14	23(4)	$\cancel{13}$ 9	3
O_3	32	17	18	41	19	1
demand	6	10	12	$\cancel{15}$ $\cancel{4}$ 0	43	
Penalties	15	1	4	18 ↑		

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	$\cancel{21}$	$\cancel{16}$	$\cancel{25}$	$\cancel{13\,(11)}$	$\cancel{11}$ 0	×
O_2	17	18	14	$\cancel{23(4)}$	$\cancel{13}$ 9	3
O_3	32	17	18	$\cancel{41}$	19	1
demand	6	10	12	$\cancel{15}$ $\cancel{4}$ 0	43	
Penalties	15	1	4	×		

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	$\cancel{21}$	$\cancel{16}$	$\cancel{25}$	$\cancel{13\,(11)}$	$\cancel{11}$ 0	×
O_2	17(6)	18	14	$\cancel{23(4)}$	$\cancel{13}$ $\cancel{9}$ 3	3
O_3	32	17	18	$\cancel{41}$	19	1
demand	$\cancel{6}$ 0	10	12	$\cancel{15}$ $\cancel{4}$ 0	43	
Penalties	15↑	1	4	×		

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	$\cancel{21}$	$\cancel{16}$	$\cancel{25}$	$\cancel{13\,(11)}$	$\cancel{11}$ 0	×
O_2	$\cancel{17\,(6)}$	18	14	$\cancel{23(4)}$	$\cancel{13}$ $\cancel{9}$ 3	3
O_3	$\cancel{32}$	17	18	$\cancel{41}$	19	1
demand	$\cancel{6}$ 0	10	12	$\cancel{15}$ $\cancel{4}$ 0	43	
Penalties	15↑	1	4	×		

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	$\cancel{21}$	$\cancel{16}$	$\cancel{25}$	$\cancel{13\,(11)}$	$\cancel{11}$ 0	×
O_2	$\cancel{17\,(6)}$	18	14 (3)	$\cancel{23(4)}$	$\cancel{13}$ $\cancel{9}$ $\cancel{3}$ 0	4
O_3	$\cancel{32}$	17	18	$\cancel{41}$	19	1
demand	$\cancel{6}$ 0	10	$\cancel{12}$ 9	$\cancel{15}$ $\cancel{4}$ 0	43	
Penalties	×	1	4↑	×		

	D_1	D_2	D_3	D_4	supply	Penalties
O_1	~~21~~	~~16~~	~~25~~	~~13~~ (11)	~~11~~ 0	×
O_2	~~17~~ (6)	~~18~~	~~14~~ (3)	~~23~~(4)	~~13~~ ~~9~~ ~~3~~ 0	4
O_3	~~32~~	17(10)	18 (9)	~~41~~	~~19~~ ~~9~~ 0	1
demand	~~6~~ 0	~~10~~ 0	~~12~~ ~~9~~ 0	~~15~~ ~~4~~ 0	43	
Penalties	×	1	4 ↑	×		

	D_1	D_2	D_3	D_4	supply
O_1	21	16	25	13 (11)	11
O_2	17 (6)	18	14 (3)	23 (4)	13
O_3	32	17 (10)	18 (9)	41	19
demand	6	10	12	15	43

Number of non-negative allocation is 6=m+n-1. Hence the initial basic feasible solution is

$$\text{Total cost} = 13 \times 11 + 17 \times 6 + 14 \times 3 + 23 \times 4 + 17 \times 10 + 18 \times 9 = 711$$

Q3. Obtain the initial basic feasible solution of a transportation problem by Vogel's approximation method

	D_1	D_2	D_3	D_4	supply
O_1	5	3	6	4	30
O_2	3	4	7	8	15
O_3	9	6	5	8	15
demand	10	25	18	7	

Answer: $x_{12} = 23,\ x_{14} = 7,\ x_{21} = 10,\ x_{22} = 2,\ x_{23} = 3,\ x_{33} = 15,$ min. cost= 231, having alternative solution.

Optimality test: An optimal solution is one where there is no other set of transportation routes (allocations) that will further reduce the total transportation cost. To perform optimality test, we use Modified Distribution Method (MODI) or u-v method.

1. Start with initial basic feasible solution consisting of m+n-1 allocations.
2. Determine a set of m+n numbers u_i $(i = 1,2,3, \dots, m)$ and v_j $(j = 1,2,3, \dots, n)$ such that for each occupied cell (r, s) $c_{rs} = u_r + v_s$.
3. Calculate cell evaluations (unit cost differences) d_{ij} for each empty cell (i, j) by using formula $d_{ij} = c_{ij} - (u_i + v_j)$

4. Finally, examine the matrix of cell evaluations d_{ij} for negative entries and conclude that
   i) Solution under test is optimal, if none is negative
   ii) Alternative optimal solution exist, if none is negative but any is zero
   iii) Solution under test is not optimal, if any is negative, then further improvement is required by repeating the above process.

Q1. Determine the optimum basic feasible solution to the following transportation problem.

	D_1	D_2	D_3	D_4	supply
O_1	5	3	6	4	30
O_2	3	4	7	8	15
O_3	9	6	5	8	15
demand	10	25	18	7	

Solution: Using Vogel's approximation method, the initial basic feasible solution is

	D_1	D_2	D_3	D_4	supply
O_1	5	3 (23)	6	4 (7)	30
O_2	3 (10)	4 (2)	7 (3)	8	15
O_3	9	6	5 (15)	8	15
demand	10	25	18	7	

Number of allocation =6=m+n-1. Hence the solution is non-degenerate basic feasible.

Initial total cost= 231

To find optimal solution: Applying u-v method, we determine a set of numbers u_i and v_j for each row and column, such that $c_{ij} = u_i + v_j$ for each occupied cell

Since the second row has maximum number of allocations put u_2=0

	D_1	D_2	D_3	D_4	
O_1	5	3 23	6	4 7	u_1=
O_2	3 10	4 2	7 3	8	u_2=0
O_3	9	6	5 15	8	u_3=
	v_1=	v_2=	v_3=	v_4=	

u_2=0,

$c_{21} = u_2 + v_1 \Rightarrow 3 = 0 + v_1,$

$v_1 = 3$,

$c_{12} = u_1 + v_2 \Rightarrow 3 = u_1 + 4,$

$u_1 = 3 - 7 = -1,$

$c_{22} = u_2 + v_2 \Rightarrow 4 = 0 + v_2,$

$v_2 = 4$,

$c_{14} = u_1 + v_4 \Rightarrow 4 = -1 + v_4,$

$v_4 = 4 + 1 = 5$

$c_{23} = u_2 + v_3 \Rightarrow 7 = 0 + v_3,$

$v_3 = 7$

$c_{33} = u_3 + v_3 \Rightarrow 5 = u_3 + 7,$

$u_3 = 5 - 7 = -2$

	D_1	D_2	D_3	D_4	
O_1	$\boxed{5}$	3 23	$\boxed{6}$	4 7	$u_1 = -1$
O_2	3 10	4 2	7 3	8	$u_2 = 0$
O_3	9	6	5 15	8	$u_3 = -2$
	$v_1 = 3$	$v_2 = 4$	$v_3 = 7$	$v_4 = 5$	

For unoccupied cells we find the sum of u_i and v_j

	D_1	D_2	D_3	D_4	
O_1	$\boxed{5}$ $\boxed{2}$	3 23	$\boxed{6}$ $\boxed{6}$	4 7	$u_1 = -1$
O_2	3 10	4 2	7 3	$\boxed{8}$ $\boxed{5}$	$u_2 = 0$
O_3	$\boxed{9}$ $\boxed{1}$	$\boxed{6}$ $\boxed{2}$	5 15	$\boxed{8}$ $\boxed{3}$	$u_3 = -2$
	$v_1 = 3$	$v_2 = 4$	$v_3 = 7$	$v_4 = 5$	

Next we find $d_{ij} = c_{ij} - (u_i + v_j)$ for each unoccupied cell

	D_1	D_2	D_3	D_4	
O_1	$\boxed{5}$ $\boxed{2}$ $\boxed{3}$	3 (23)	$\boxed{6}$ $\boxed{6}$ $\boxed{0}$	4 (7)	$u_1 = -1$
O_2	3 (10)	4 (2)	7 (3)	$\boxed{8}$ $\boxed{5}$ $\boxed{3}$	$u_2 = 0$
O_3	$\boxed{9}$ $\boxed{1}$ $\boxed{8}$	$\boxed{6}$ $\boxed{2}$ $\boxed{4}$	5 (15)	$\boxed{8}$ $\boxed{3}$ $\boxed{5}$	$u_3 = -2$
	$v_1 = 3$	$v_2 = 4$	$v_3 = 7$	$v_4 = 5$	

Since all $d_{ij} \geq 0$, the solution is optimal having alternative solution. Therefore the total cost is 231

Note: If at least one $d_{ij} < 0$ (negative) select the most negative d_{rs} and construct a loop that starts and ends at the cell (r, s) and connects some of the occupied (basic) cells. Make $\pm$ adjustment in the cells at the corners of the loop in such a manner that the availabilities and requirements remain satisfied. Then select the smallest quantity amongst the cell marked with minus sign on the corners of closed loop. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs and subtract it from the occupied cells marked with minus sign.

Q2. Determine the optimum basic feasible solution to the following transportation problem.

	D_1	D_2	D_3	D_4	supply
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
demand	5	8	7	14	

Solution: Using Vogel's approximation method, the initial basic feasible solution is

	D_1	D_2	D_3	D_4	supply
O_1	19 (5)	30	50	10 (2)	7
O_2	70	30	40 (7)	60 (2)	9
O_3	40	8 (8)	70	20 (10)	18
demand	5	8	7	14	

Number of allocation =6=m+n-1. Hence the solution is non-degenerate basic feasible.

Initial total cost= 779

To find optimal solution: Applying u-v method, we determine a set of numbers u_i and v_j for each row and column, such that $c_{ij} = u_i + v_j$ for each occupied cell

Since the fourth column has maximum number of allocations put v_4=0

	D_1	D_2	D_3	D_4	
O_1	19 5	30	50	10 2	$u_1 =$
O_2	70	30	40 7	60 2	$u_2 =$
O_3	40	8 8	70	20 10	$u_3 =$
	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	

	D_1	D_2	D_3	D_4	
O_1	$\boxed{19}$ 5	$\boxed{30}$	$\boxed{50}$	$\boxed{10}$ 2	u_1 =10
O_2	$\boxed{70}$	$\boxed{30}$	$\boxed{40}$ 7	$\boxed{60}$ 2	u_2 =60
O_3	$\boxed{40}$	$\boxed{8}$ 8	$\boxed{70}$	$\boxed{20}$ 10	$u_3 = 20$
	v_1 =9	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$	

For unoccupied cells we find the sum of u_i and v_j

	D_1	D_2	D_3	D_4	
O_1	19 5	30 -2	50 -10	10 2	u_1 =10
O_2	70 69	30 48	40 7	60 2	u_2 =60
O_3	40 29	8 8	70 0	20 10	$u_3 = 20$
	v_1 =9	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$	

Next we find $d_{ij} = c_{ij} - (u_i + v_j)$ for each unoccupied cell

	D_1	D_2	D_3	D_4	
O_1	19 5	30 -2 32	50 -10 60	10 2	u_1 =10
O_2	70 69 1	30 48 -18	40 7	60 2	u_2 =60
O_3	40 29 11	8 8	70 0 70	20 10	$u_3 = 20$
	v_1 =9	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$	

It is observed that cell $(2,2)$, $d_{22} = -18$ is negative therefore the solution is not optimal.

	D_1	D_2	D_3	D_4	
O_1	19 5	30 -2 32	50 -10 60	10 2	u_1 =10
O_2	70 69 139	30 48 (+) -18	40 7	60 (-) 2	u_2 =60
O_3	40 29 11	8 (-) 8	70 0 70	20 (+) 10	$u_3 = 20$
	v_1 =9	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$	

Here min(8,2)=2

Using uv-method

	D_1	D_2	D_3	D_4	
O_1	19 5	30	50	10 2	$u_1 = 0$
O_2	70	30 2	40 7	60	u_2 =32
O_3	40	8 6	70 0	20 12	u_3 =10
	v_1 =19	$v_2 = -2$	v_3 =8	v_4 =10	

	D_1	D_2	D_3	D_4	
O_1	$\boxed{19}$ 5	$\boxed{30}$ $\boxed{-2}$ $\boxed{32}$	$\boxed{50}$ $\boxed{8}$ $\boxed{42}$	$\boxed{10}$ 2	$u_1 = 0$
O_2	$\boxed{70}$ $\boxed{51}$ $\boxed{19}$	$\boxed{30}$ 2	$\boxed{40}$ 7	$\boxed{60}$ $\boxed{42}$ $\boxed{18}$	$u_2 = 32$
O_3	$\boxed{40}$ $\boxed{29}$ $\boxed{11}$	$\boxed{8}$ 6	$\boxed{70}$ $\boxed{18}$ $\boxed{52}$	$\boxed{20}$ 12	$u_3 = 10$
	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$	

Since all $d_{ij} > 0$, the solution is optimal. The total cost is

$$= 19 \times 5 + 10 \times 2 + 30 \times 2 + 40 \times 7 + 8 \times 6 + 20 \times 12 = 743$$

Q3. Determine the optimum basic feasible solution to the following transportation problem.

	D_1	D_2	D_3	supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
demand	7	9	18	34

Answer: 76

Degeneracy in Transportation Problem

A basic feasible solution for the general transportation problem must consist of exactly m+n-1 positive allocation in independent position in the transportation table. A solution will be called degenerate when the number of occupied cells is less than the required number m+n-1. In such cases the current solution cannot be improved because it is not possible to draw a closed path for every occupied cell. Also the values of u_i and v_j which are used to test a optimality cannot be computed. Thus we need to remove the degeneracy to improve the given solution. The degeneracy in the transportation problems may occur at two stages:

1. When obtaining an initial solution we may have less than m+n-1 allocations.
2. At any stage while moving towards optimal solution. This happens when two or more occupied cells with the same minimum allocation become unoccupied simultaneously.

Resolution of degeneracy

Case 1: Degeneracy in the initial solution

To resolve degeneracy at the initial solution, we proceed by allocating a very small quantity Δ close to zero to one or more unoccupied cell so as to get m+n-1 number of occupied cell. This quantity would not affect the total cost as well as supply and demand values. In a minimization transportation problem it is better to allocate Δ to unoccupied cells that have lowest transportation costs. Hence, the quantity Δ is used to evaluate unoccupied cell and to find an optimal solution.

Case 2: Degeneracy at Subsequent Iterations

To resolve degeneracy which occur during optimality test, the quantity may be allocated to one or more cell which have become unoccupied recently to have m+n-1 number of occupied cells in the new solution.

Q4. A company has three plants A, B and C, 3 warehouses X, Y, Z. the number of units available at the plants is 60, 70, 80 and the demand at X, Y, Z are 50, 80, 80 respectively. The unit cost of the transportation is given in the following table

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

Find the allocation so that the total transportation cost is minimum.

Solution: Using vogel's method

	X	Y	Z	supply
A	8	7	3 (60)	60
B	3 (50)	8	9 (20)	70
C	11	3 (80)	5	80
demand	50	80	80	210

The number of allocated cell is 4 < m+n-1. The problem is degenerate. To remove this degeneracy we assign Δ to unoccupied cell (C,Z) which has minimum cost among unoccupied cells

	X	Y	Z	supply
A	8	7	3 (60)	60
B	3 (50)	8	9 (20)	70
C	11	3 (80)	5 (Δ)	80
demand	50	80	80	210

Solution is non-degenerate.

Total cost $= 3 \times 60 + 3 \times 50 + 9 \times 20 + 3 \times 80 = 750$

Using uv-method

	X	Y	Z	
A	8	7	3 (60)	$u_1 =$
B	3 (50)	8	9 (20)	$u_2 =$
C	11	3 (80)	5 (Δ)	$u_3 =$
	$v_1 =$	$v_2 =$	$v_3 =$	

For occupied cell put $v_3 = 0$

	X	Y	Z	
A	8	7	3 (60)	$u_1 = 3$
B	3 (50)	8	9 (20)	$u_2 =9$
C	11	3 (80)	5 (Δ)	$u_3 =5$
	$v_1 = -6$	$v_2 = -2$	$v_3 = 0$	

For unoccupied cell $d_{ij} = c_{ij} - (u_i + v_j)$

	X	Y	Z	
A	8 -3 11	7 1 6	3 60	$u_1 = 3$
B	3 50	8 7 1	9 20	$u_2 =9$
C	11 -1 12	3 80	5 Δ	$u_3 =5$
	$v_1 = -6$	$v_2 = -2$	$v_3 = 0$	

Since all $d_{ij} > 0$, the solution is optimal. The total cost is 750

Unbalanced Transportation Problem

Q5. Determine the optimum basic feasible solution to the following transportation problem.

	D_1	D_2	D_3	D_4	supply
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	10	12	41	7	70
demand	85	35	50	45	

Solution: In this problem, the total demands (=215) is greater than the total supplies (=195). Therefore, the problem is of unbalanced type. So introduce a dummy row O_4 having all transportation costs equal to zero and having supply 20. The modified transportation problem is

	D_1	D_2	D_3	D_4	supply
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	10	12	41	7	70
O_4	0	0	0	0	20
demand	85	35	50	45	

Using 'VAM' the following initial BFS is obtained:

	D_1	D_2	D_3	D_4	supply
O_1	6 (65)	1 (5)	9	3	70
O_2	11	5(5)	2(50)	8	55
O_3	10	12(25)	41	7 (45)	70
O_4	0 (20)	0	0	0	20
demand	85	35	50	45	

Number of allocation =7=m+n-1. Hence the solution is non-degenerate basic feasible.

Initial total cost= 1135

To find optimal solution: Applying u-v method, we determine a set of numbers u_i and v_j for each row and column, such that $c_{ij} = u_i + v_j$ for each occupied cell

Since the second column has maximum number of allocations put v_2=0

	D_1	D_2	D_3	D_4	
O_1	6 65	1 5	9	3	$u_1 = 1$
O_2	11	5 5	2 50	8	$u_2 = 5$
O_3	10	12 25	41	7 45	$u_3 = 12$
O_4	0 20	0	0	0	$u_4 = -5$
	$v_1 = 5$	v_2 =0	$v_3 = -3$	$v_4 = -5$	

For unoccupied cells we find the sum of u_i and v_j

	D_1	D_2	D_3	D_4	
O_1	6 65	1 5	$\boxed{9}$ $\boxed{-2}$	$\boxed{3}$ $\boxed{-4}$	$u_1 = 1$
O_2	$\boxed{11}$ $\boxed{10}$	5 5	2 50	$\boxed{8}$ $\boxed{0}$	$u_2 = 5$
O_3	$\boxed{10}$ $\boxed{17}$	12 25	$\boxed{41}$ $\boxed{9}$	7 45	$u_3 = 12$
O_4	0 20	$\boxed{0}$ $\boxed{-5}$	$\boxed{0}$ $\boxed{-8}$	$\boxed{0}$ $\boxed{-10}$	$u_4 = -5$
	$v_1 = 5$	v_2 =0	$v_3 = -3$	$v_4 = -5$	

For unoccupied cell $d_{ij} = c_{ij} - (u_i + v_j)$

	D_1	D_2	D_3	D_4	
O_1	6 65	1 5	9 -2 11	3 -4 7	$u_1 = 1$
O_2	11 10 1	5 5	2 50	8 0 8	$u_2 = 5$
O_3	10 17 -7	12 25	41 9 32	7 45	$u_3 = 12$
O_4	0 20	0 -5 5	0 -8 8	0 -10 10	$u_4 = -5$
	$v_1 = 5$	$v_2 = 0$	$v_3 = -3$	$v_4 = -5$	

It is observed that cell $(3,1)$, $d_{31} = -7$ is negative therefore the solution is not optimal

	D_1	D_2	D_3	D_4	
O_1	6 (-) 65	1 (+) 5	9 -2 11	3 -4 7	$u_1 = 1$
O_2	11 10 1	5 5	2 50	8 0 8	$u_2 = 5$
O_3	10 17 (+) -7	12 (-) 25	41 9 32	7 45	$u_3 = 12$
O_4	0 20	0 -5 5	0 -8 8	0 -10 10	$u_4 = -5$
	$v_1 = 5$	$v_2 = 0$	$v_3 = -3$	$v_4 = -5$	

Now min.(25,65)=25

	D_1	D_2	D_3	D_4	
O_1	6 40	1 30	9	3	$u_1 =$
O_2	11	5 5	2 50	8	$u_2 =$
O_3	10 25	12	41	7 45	$u_3 =$
O_4	0 20	0	0	0	$u_4 =$
	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	

Using uv-method

	D_1	D_2	D_3	D_4	
O_1	6 40	1 30	9 -2 11	3 3 0	$u_1 = 6$
O_2	11 10 1	5 5	2 50	8 7 1	$u_2 = 10$
O_3	10 25	12 5 7	41 9 32	7 45	$u_3 = 10$
O_4	0 20	0 -5 5	0 -8 8	0 -3 3	$u_4 = 0$
	$v_1 = 0$	$v_2 = -5$	$v_3 = -8$	$v_4 = -3$	

Since all $d_{ij} > 0$, the solution is optimal. The total cost is

$= 6 \times 40 + 1 \times 30 + 5 \times 5 + 2 \times 50 + 10 \times 25 + 7 \times 45 + 0$ =960.

Q6. The Bombay Transport company has trucks available at four different sites in the following numbers:

Site	:	A	B	C	D
No. of Trucks	:	5	10	7	3

Customers W, X and Y require trucks as shown :

Customer	:	W	X	Y
No. of Trucks	:	5	8	10

Variable costs of getting trucks to the customers are :

From A to W → Rs 7, to X → Rs 3, to Y → Rs 6; From B → Rs 4, to X → Rs 6, to Y → Rs 8;

From C to W → Rs 5, to X → Rs 8, to Y → Rs 4; From D to W → Rs 8, to X → Rs 4, to Y → Rs 3

Solve the above transportation problem.

Answer: 90