Chapter 5

Integer Linear Programming Problem

In linear programming, each of the decision variable as well as slack and/or surplus variables is allowed to take any real or fractional value. However, there are certain real life problems in which the fractional value of the decision variables has no significance. For example, it does not make sense saying 1.5 men working on a project or 1.6 machines in a workshop. This is the main reason why integer programming is so important. The integer solution to a problem can be obtained by rounding off the optimal value of the variables to the nearest integer value.

An Integer LPP has important applications in capital budgeting, construction scheduling, plant, location and size, routing and shipping schedule etc.

Types of IPP

1. **Pure (All) integer programming problem**: In a LPP if all variables are required to take integer values then it is called pure (all) IPP. i.e. if all $x_j \in X$ are integers.
2. **Mixed IPP:** If only some of the variables in the optimal solution of LPP are restricted to non-negative integer values while the remaining variables are free to take any non-negative values then is called a mixed integer programming problem i.e. if not all $x_j \in X$ are integers.

Method of solving IPP: There are two methods used to solve IPP namely

1. Gomory's Cutting Plane Method
2. Branch and Bound Method

Gomory's Cutting –Plane Algorithm

Step1. If the IPP is in the minimization form convert it to maximization form.

Step2. Then convert the inequalities into equations by introduction slack and/or surplus variables (if necessary) and obtain the optimum solution of the LPP (after ignoring the integer condition) by usual simplex method.

Step3. Now, test the integrality of the optimum solution which is obtained in step 2.

a) If the optimum solution contains all integer values, then an optimum integer basic feasible solution has been achieved.

b) If not, go to next step.

Step4. Examine the constraint equations corresponding to the current optimal solution. Let these constraints be expressed by

$$x_{Bi} = x_i + \sum_{j=m+1}^{n} x_{ij}\, x_j \qquad \text{(i=1, 2, 3, . . ., m)}$$

Select the largest fraction of x_{Bi} 's, i.e. find $\max_i [f_{Bi}]$. Let it be f_{Bk} $for\ i = k$.

Step5. Express the negative fraction, if any in the kth row of the optimum simplex table, as the sum of a negative integer and a non-negative fraction.

Step6. At this stage, construct the Gomorian Constraint:

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j \leq 0$$

As described in the preceding section, and then introduce the Gomorian equation

$$-f_{Bi} = -\sum_{j=m+1}^{n} f_{ij}\, x_j + g_i$$

To the current set of equality constraints

Step7. Starting with this new set of constraint equations, obtain the new optimum solution by using dual simplex method in order to clear infeasibility. The slack variable g_i will be the initial leaving basic variable.

Step8. Now two possibilities may arise:

a) If this new optimum solution for the Modified LPP is an all integer solution, it is also feasible and optimum for the given LPP

b) Otherwise, we return to step 4 and repeat the entire process until an optimum feasible integer solution is obtained.

Q1. Solve the following integer programming problem using Gomory's cutting plane method

$Max. Z = x_1 + x_2$

Subject to the constraint

$3x_1 + 2x_2 \leq 5$

$x_2 \leq 2$

$x_1 , x_2 \geq 0$ and are integers

Solution: $Max. Z = x_1 + x_2 + 0s_1 + 0s_2$

Subject to the constraints

$3x_1 + 2x_2 + s_1 = 5$

$x_2 + s_2 = 2$

$x_1 , x_2, s_1, s_2 \geq 0$

		c_j	1	1	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio ${x_B}/{x_1}$
s_1	0	5	$\boxed{3}$	2	1	0	←
s_2	0	2	0	1	0	1	×
	$z_j = C_B x_j$		0	0	0	0	
	$\Delta_j = c_j - z_j$		1	1	0	0	

	c_j		1	1	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio x_B/x_2
x_1	1	5/3	1	2/3	1/3	0	5/2
s_2	0	2	0	$\boxed{1}$	0	1	2 ←
	$z_j = C_B x_j$		1	2/3	1/3	0	
	$\Delta_j = c_j - z_j$		0	1/3	-1/3	0	

	c_j		1	1	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio
x_1	1	1/3	1	0	1/3	-2/3	
x_2	1	2	0	1	0	1	
	$z_j = C_B x_j$		1	1	1/3	1/3	
	$\Delta_j = c_j - z_j$		0	0	-1/3	-1/3	

$x_1 = 1/3, \quad x_2 = 2$ Max Z=7/2

In the current solution, all basic variables in the basis are not integers.

Since x_1 is the basic variable whose value is a non-negative fraction. Thus x_1 is the source row

$$\frac{1}{3} = x_1 + 0x_2 + \frac{1}{3}s_1 - \frac{2}{3}s_2$$

Notice that each of the non-integer coefficient is factored into integer and fractional parts in such a manner that the fractional parts is strictly positive.

i.e.

$$0 + \frac{1}{3} = (1+0)x_1 + (0 + \frac{1}{3})s_1 + \left(-1 + \frac{1}{3}\right)s_2, \qquad \because 0 \le f_{ij} < 1$$

Rearrange the equation so that all of the integer coefficient appear on the left-hands

$$\frac{1}{3} + s_2 - x_1 = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

Since x_1 and s_2 are integers left hand side must satisfy

$$\frac{1}{3} \le \frac{1}{3}s_1 + \frac{1}{3}s_2$$

$$\frac{1}{3} + g_1 = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

$$\frac{1}{3} = \frac{1}{3}s_1 + \frac{1}{3}s_2 - g_1$$

$$-\frac{1}{3} = g_1 - \frac{1}{3}s_1 - \frac{1}{3}s_2$$

Where g_1 is Gomory's slack. By adding this equation (also called Gomory cut) at the bottom of above table

		c_j	1	1	0	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	g_1
x_1	1	1/3	1	0	1/3	-2/3	0
x_2	1	2	0	1	0	1	0
g_1	0	-1/3	0	0	-1/3	-1/3	1
	$z_j = C_B x_j$		1	1	1/3	1/3	0
	$\Delta_j = c_j - z_j$		0	0	-1/3	-1/3	0

Now we use dual simplex method to find optimal solution

Since $g_1 = -1/3,\ g_1$ leaves the basis

$$\min\left\{\frac{c_j - z_j}{a_{ik}},\ a_{ik} < 0\right\} = \min\left\{\frac{-1/3}{-1/3}, \frac{-1/3}{-1/3}\right\} = 1$$

		c_j	1	1	0	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	g_1
x_1	1	0	1	0	0	-1	1
x_2	1	2	0	1	0	1	0
s_1	0	1	0	0	1	1	-3
	$z_j = C_B x_j$		1	1	0	0	1
	$\Delta_j = c_j - z_j$		0	0	0	0	-1

since all $c_j - z_j \leq 0$ the optimal integer solution is $x_1 = 0,\quad x_2 = 2$ Max Z=2

Q2. $Max. Z = 4x_1 + 3x_2$

Subject to the constraint

$x_1 + 2x_2 \le 4$

$2x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$, and are integers

Solution: $Max. Z = 4x_1 + 3x_2 + 0s_1 + 0s_2$

Subject to the constraints

$x_1 + 2x_2 + s_1 = 4$

$2x_1 + x_2 + s_2 = 6$

$x_1, x_2, s_1, s_2 \ge 0$

		c_j	4	3	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio x_B/x_1
s_1	0	4	1	2	1	0	4
s_2	0	6	$\boxed{2}$	1	0	1	3←
	$z_j = C_B x_j$		0	0	0	0	
	$\Delta_j = c_j - z_j$		4	3	0	0	

		c_j	4	3	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio x_B/x_2
s_1	0	1	0	$\boxed{3/2}$	1	-1/2	2/3←
x_1	4	3	1	1/2	0	1/2	5
	$z_j = C_B x_j$		4	2	0	0	
	$\Delta_j = c_j - z_j$		0	1	0	0	

		c_j	4	3	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2
x_2	3	2/3	0	1	2/3	-1/3
x_1	4	8/3	1	0	-1/3	2/3
	$z_j = C_B x_j$		4	3	0	0
	$\Delta_j = c_j - z_j$		0	0	0	0

$x_1 = 8/3, \quad x_2 = 2/3$

In the current solution, all basic variables in the basis are not integers.

$$x_1 = 2 + \frac{2}{3}, \quad x_2 = \frac{2}{3}$$

To obtain the integer-valued solution with the help of x_1 row

$$2 + \frac{2}{3} = x_1 + \left(-\frac{1}{3}\right) s_1 + \frac{2}{3} s_2$$

$$2 + \frac{2}{3} = (1+0)x_1 + \left(-1 + \frac{2}{3}\right) s_1 + \frac{2}{3} s_2$$

Rearrange the equation so that all of the integer coefficient appear on the left-hands

$$2 + \frac{2}{3} - x_1 + s_1 = \frac{2}{3} s_1 + \frac{2}{3} s_2$$

$$\frac{2}{3} \le \frac{2}{3} s_1 + \frac{2}{3} s_2$$

$$\frac{2}{3} + g_1 = \frac{2}{3} s_1 + \frac{2}{3} s_2$$

$$-\frac{2}{3} = g_1 - \frac{2}{3} s_1 - \frac{2}{3} s_2$$

		c_j	4	3	0	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	g_1
x_2	3	2/3	0	1	2/3	-1/3	0
x_1	4	8/3	1	0	-1/3	2/3	0
g_1	0	-2/3	0	0	-2/3	-2/3	1
	$z_j = C_B x_j$		4	3	2/3	5/3	0
	$\Delta_j = c_j - z_j$		0	0	-2/3	-5/3	0

Now we use dual simplex method to find optimal solution

Since $g_1 = -2/3$, g_1 leaves the basis

$$\min\left\{\frac{c_j - z_j}{a_{ik}}, \ a_{ik} < 0\right\} = \min\left\{\frac{-2/3}{-2/3}, \frac{-5/3}{-2/3}\right\} = 1$$

		c_j	4	3	0	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	g_1
x_2	3	0	0	1	0	-1	1
x_1	4	3	1	0	0	1	-1/2
s_1	0	1	0	0	1	1	-3/2
	$z_j = C_B x_j$		4	3	0	1	1
	$\Delta_j = c_j - z_j$		0	0	0	-1	-1

since all $c_j - z_j \leq 0$ the optimal integer solution is $x_1 = 3, \quad x_2 = 0$

Max Z=12

Q3. $Max. Z = x_1 + 2x_2$

Subject to the constraint

$2x_2 \leq 7$

$x_1 + x_2 \leq 7$

$2x_1 \leq 11$

$x_1, x_2 \geq 0$, and are integers

Answer: $x_1 = 4, \quad x_2 = 3$ Max Z=10

Q4. $Max. Z = 3x_1 + 12x_2$

Subject to the constraint

$2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$, and are integers

Answer: $x_1 = 1, \quad x_2 = 1$ Max Z=15

Q5. Use Branch and Bound technique to solve the Integer Linear Programming.

$Min. Z = 3x_1 + 2.5x_2$

Subject to the constraint

$x_1 + 2x_2 \geq 20$

$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$, and are integers

Solution: Using graphical method,

Replace all the inequalities of the constraints into equations

$$x_1 + 2x_2 = 20$$

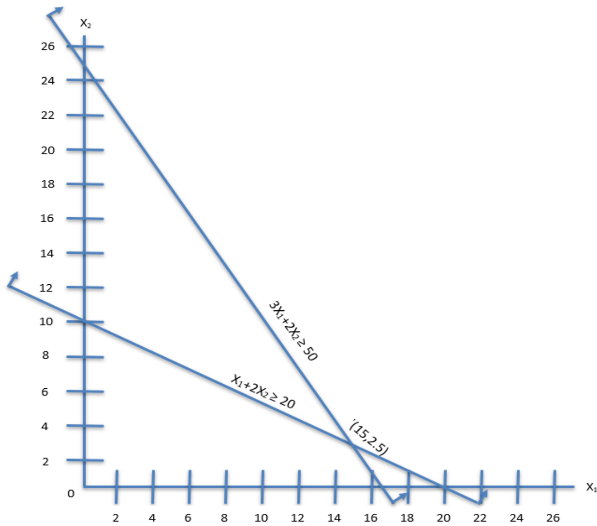
$$3x_1 + 2x_2 = 50$$

To plot the line $x_1 + 2x_2 = 20$, put $x_1 = 0 \Rightarrow x_2 = 10$

Again put $x_2 = 0 \Rightarrow x_1 = 20$

Therefore, $x_1 + 2x_2 = 20$ passes through (0, 10) and (20, 0)

Similarly, $3x_1 + 2x_2 = 50$ passes through (0,25) and (16.67, 0)



The optimal non-integer solution of the given integer LPP obtained by graphical method is

$$x_1 = 15, \quad x_2 = 2.5 \qquad Min.Z = 51.25$$

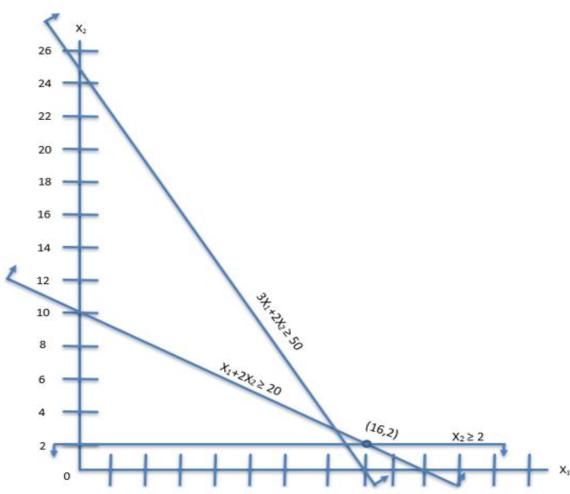
The variable $x_2 = 2.5$ is the only non-integer solution value and therefore is selected for dividing the given problem into two sub problems B and C. In order to eliminate the fractional part of 2.5, two new constraints $x_2 \leq 2$ *and* $x_2 \geq 3$ are created

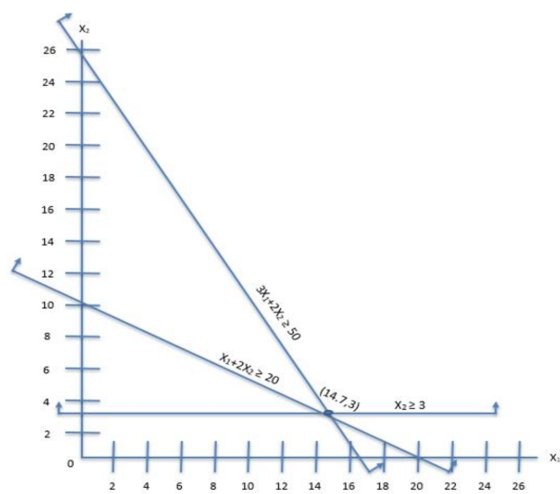
LP Sub problem B

$Min.Z = 3x_1 + 2.5x_2$
Subject to the constraints
$x_1 + 2x_2 \geq 20$
$3x_1 + 2x_2 \geq 50$
$x_2 \leq 2$
$x_1 , x_2 \geq 0$

LP Sub problem C

$Min.Z = 3x_1 + 2.5x_2$
Subject to the constraints
$x_1 + 2x_2 \geq 20$
$3x_1 + 2x_2 \geq 50$
$x_2 \geq 3$
$x_1 , x_2 \geq 0$





Sub-problem B and C are solved graphically. The solutions are

Sub-problem B: $x_1 = 16,\quad x_2 = 2 \quad Min.Z_B = 53$

Sub-problem C: $x_1 = 14.7,\quad x_2 = 3 \quad Min.Z_C = 51.5$

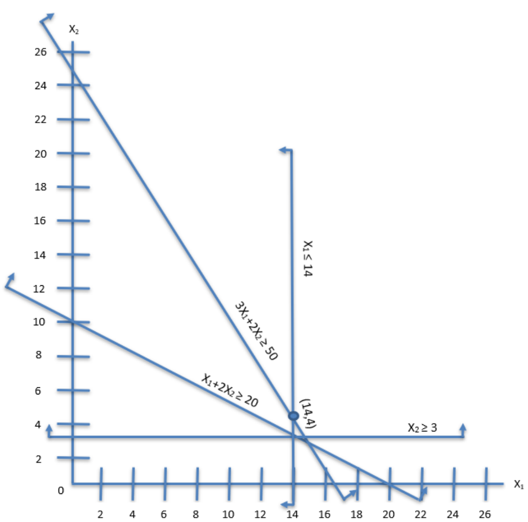
Since the solution of sub-problem B is all integer, we stop the search of this sub-problem. We divide sub-problem C into two new sub-problems D and E which are obtained by adding constraints $x_1 \leq 14 \ and\ x_1 \geq 15$

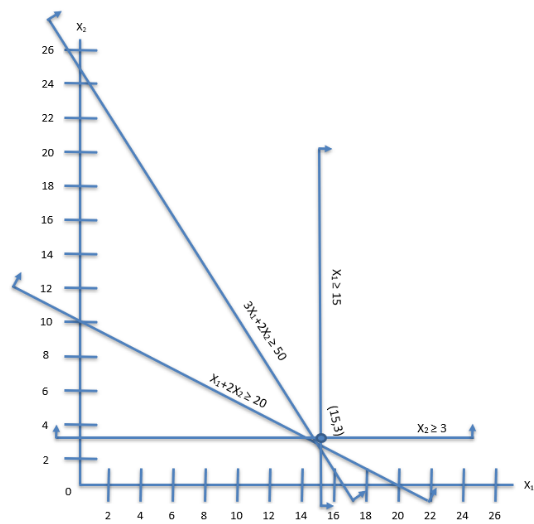
LP Sub problem D

$Min.Z = 3x_1 + 2.5x_2$
Subject to the constraints
$x_1 + 2x_2 \geq 20$
$3x_1 + 2x_2 \geq 50$
$x_2 \geq 3$
$x_1 \leq 14$
$x_1, x_2 \geq 0$

LP Sub problem E

$Min.Z = 3x_1 + 2.5x_2$
Subject to the constraints
$x_1 + 2x_2 \geq 20$
$3x_1 + 2x_2 \geq 50$
$x_2 \geq 3$
$x_1 \geq 15$
$x_1, x_2 \geq 0$



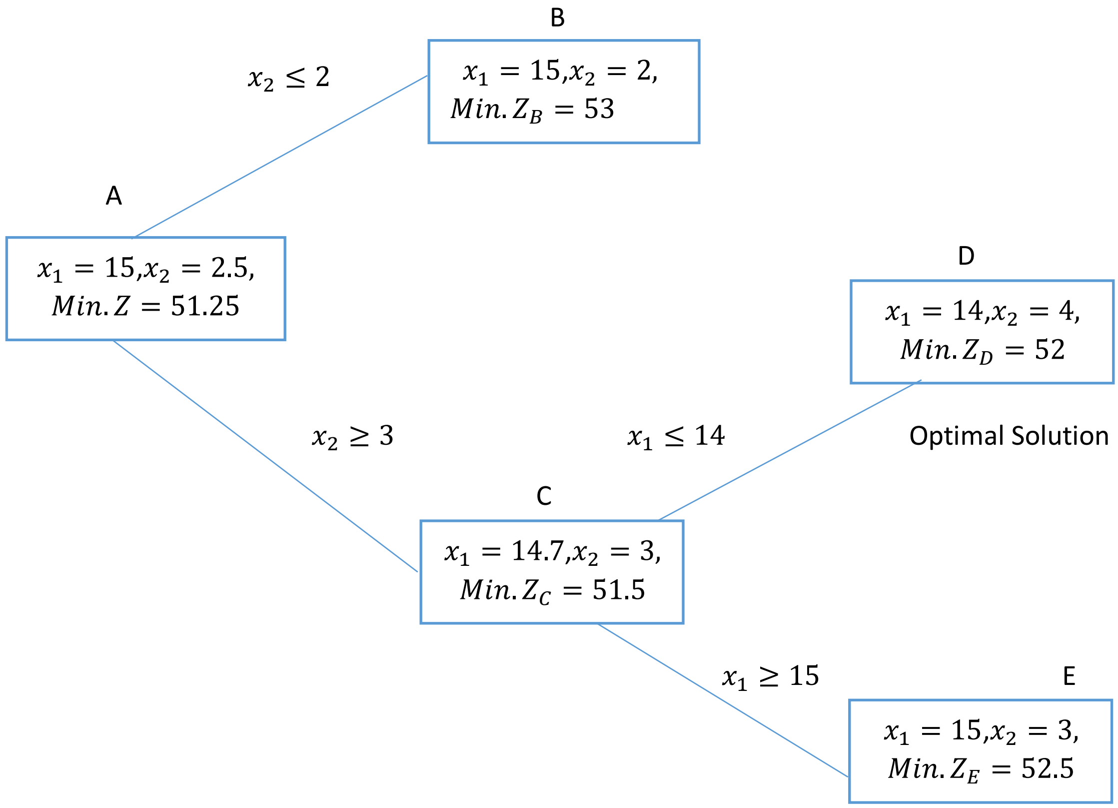


Sub-problem D and E are solved graphically. The solutions are

Sub-problem D: $x_1 = 14,\quad x_2 = 4 \quad Min.Z_D = 52$

Sub-problem E: $x_1 = 15,\quad x_2 = 3 \quad Min.Z_E = 52.5$

The solutions of sub-problem D and E are both all integer and therefore branch and bound algorithm is terminated. The optimal integer solution to the given LPP is at sub-problem D where the value of objective function is lowest among the values



Q6. Use Branch and Bound technique to solve the Integer Linear Programming.

$$Max. Z = 7x_1 + 6x_2$$

Subject to the constraint

$$2x_1 + 3x_2 \leq 12$$
$$6x_1 + 5x_2 \leq 30$$
$$x_1, x_2 \geq 0 \text{, and are integers}$$

Solution: Using graphical method,

Replace all the inequalities of the constraints into equations

$$2x_1 + 3x_2 = 12$$

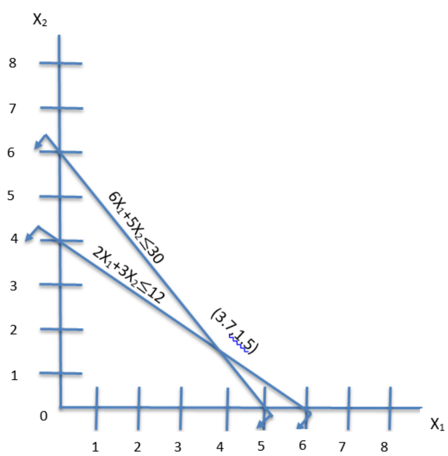
$$6x_1 + 5x_2 = 30$$

To plot the line $2x_1 + 3x_2 = 12$, put $x_1 = 0 \Rightarrow x_2 = 4$

Again put $x_2 = 0 \Rightarrow x_1 = 6$

Therefore, $2x_1 + 3x_2 = 12$ passes through (0, 4) and (6, 0)

Similarly, $6x_1 + 5x_2 = 30$ passes through (0,6) and (5, 0)



The optimal non-integer solution of the given integer LPP obtained by graphical method is

$$x_1 = 3.7, \quad x_2 = 1.5 \qquad Max. Z = 34.9$$

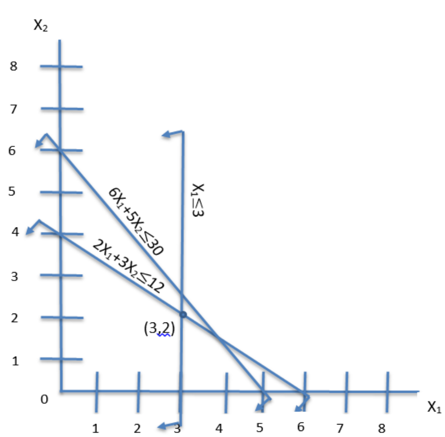
The variable $x_1 = 3.7$ is the maximum non-integer solution value and therefore is selected for dividing the given problem into two sub problems B and C. In order to eliminate the fractional part of 3.7, two new constraints $x_1 \leq 3$ and $x_1 \geq 4$ are created

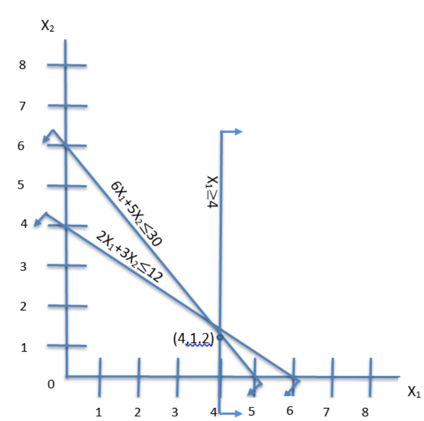
LP Sub problem B

$Max. Z = 7x_1 + 6x_2$
Subject to the constraints
$2x_1 + 3x_2 \leq 12$
$6x_1 + 5x_2 \leq 30$
$x_1 \leq 3$
$x_1, x_2 \geq 0$

LP Sub problem C

$Max. Z = 7x_1 + 6x_2$
Subject to the constraints
$2x_1 + 3x_2 \leq 12$
$6x_1 + 5x_2 \leq 30$
$x_1 \geq 4$
$x_1, x_2 \geq 0$





Sub-problem B and C are solved graphically. The solutions are

Sub-problem B: $x_1 = 3, \quad x_2 = 2 \qquad Max. Z_B = 33$

Sub-problem C: $x_1 = 4,\quad x_2 = 1.2\qquad Max.Z_C = 35.2$

Since the solution of sub-problem B is all integer, we stop the search of this sub-problem. We divide sub-problem C into two new sub-problems D and E which are obtained by adding constraints

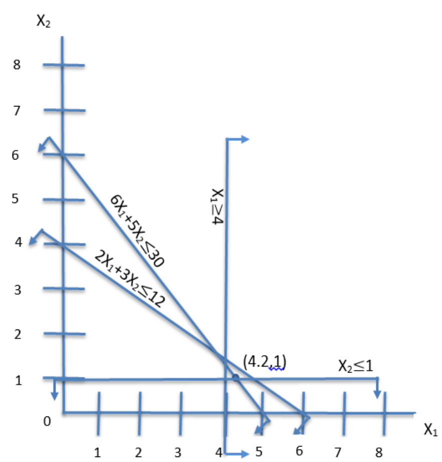
$x_2 \le 1\ \ and\ x_2 \ge 2$

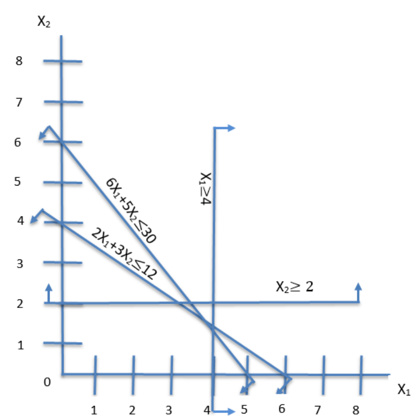
LP Sub problem D

$Max.Z = 7x_1 + 6x_2$
Subject to the constraints
$2x_1 + 3x_2 \le 12$
$6x_1 + 5x_2 \le 30$
$x_1 \ge 4$
$x_2 \le 1$
$x_1\,, x_2 \ge 0$

LP Sub problem E

$Max.Z = 7x_1 + 6x_2$
Subject to the constraints
$2x_1 + 3x_2 \le 12$
$6x_1 + 5x_2 \le 30$
$x_1 \ge 4$
$x_2 \ge 2$
$x_1\,, x_2 \ge 0$





Sub-problem D and E are solved graphically. The solutions are

Sub-problem D: $x_1 = 4.2,\qquad x_2 = 1\qquad Max.Z_D = 35.4$

Sub-problem E: infeasible solution

Since the solution of sub-problem D is non- integer, we divide sub-problem D into two new sub-problems F and G which are obtained by adding constraints

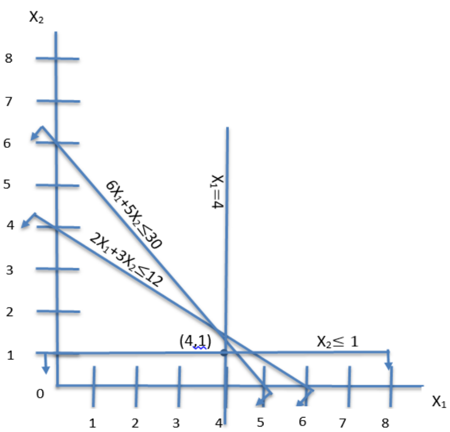
$x_1 \le 4\ and\ x_1 \ge 5$

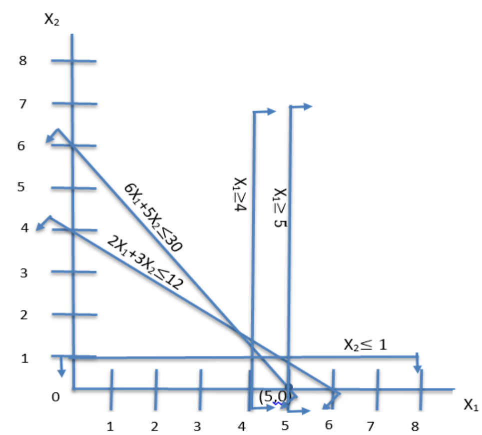
LP Sub problem F

$Max. Z = 7x_1 + 6x_2$
Subject to the constraints
$2x_1 + 3x_2 \leq 12$
$6x_1 + 5x_2 \leq 30$
$x_1 \geq 4$
$x_2 \leq 1$
$x_1 \leq 4$
$x_1 , x_2 \geq 0$

LP Sub problem G

$Max. Z = 7x_1 + 6x_2$
Subject to the constraints
$2x_1 + 3x_2 \leq 12$
$6x_1 + 5x_2 \leq 30$
$x_1 \geq 4$
$x_2 \leq 1$
$x_1 \geq 5$
$x_1 , x_2 \geq 0$



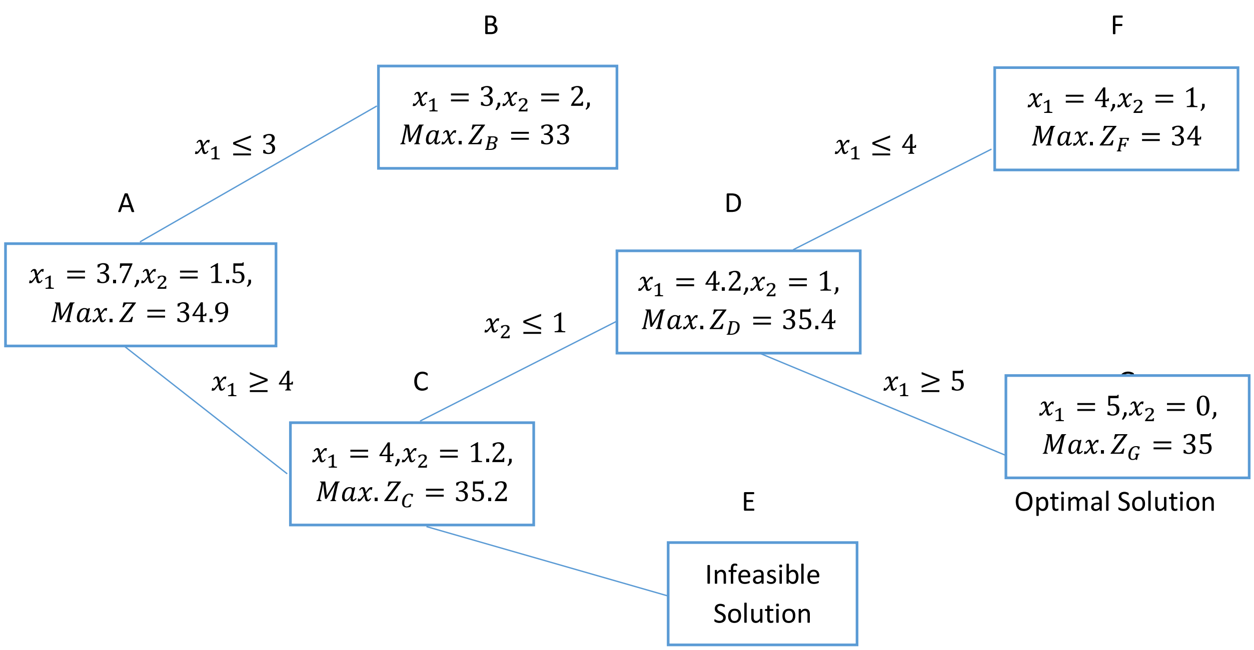


Sub-problem F and G are solved graphically. The solutions are

Sub-problem F: $x_1 = 4,$ $x_2 = 1$ $Max. Z_E = 34$

Sub-problem G: $x_1 = 5,$ $x_2 = 0$ $Max. Z_F = 35$

The solutions of sub-problem E and F are both all integer and therefore branch and bound algorithm is terminated. The optimal integer solution to the given LPP is at sub-problem G where the value of objective function is highest among the value



Q7. Use Branch and Bound technique to solve the Integer Linear Programming.

$$Max. Z = 8x_1 + 5x_2$$

Subject to the constraint

$9x_1 + 5x_2 \leq 45$

$x_1 + x_2 \leq 6$

$x_1 , x_2 \geq 0$, and are integers