Chapter 1: LINEAR PROGRAMMING PROBLEM

Programming problems deal with determining optimal allocations of limited resources to meet given objectives. The resources may be in the form of men, machines and materials etc. and the objective may yield one or more products. There are certain restrictions on the total amount of each resource available and on the quantity or quality of each product made. Out of all allocations of resources, one has to find the one which optimize (maximize or minimize) the total profit or cost.

Linear programming deals with the class of programming problems for which all relations among the variables are linear. The relations must be linear both in the constraints and in the function to be optimized.

Formulation of Linear Programming Problem:

The linear function which is to be optimized is called the objective function and the conditions for the problem expressed as simultaneous linear equations (or inequalities) are referred as constraints.

LPP model is as follow

Max . or Min. $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq,\ =,\ \geq)\ b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq,\ =,\ \geq)\ b_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq,\ =,\ \geq)\ b_m$$

where $x_1, x_2, \ldots, x_n \geq 0$ are called decision variables

b_i= available amount of resources

a_{ij} = amount of resources (per unit)

c_j = worth per unit of activity

b_i, a_{ij}, c_j are constants.

For example:

A firm manufactures two type of products A and B and sells them at a profit of Rs.2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and two minutes on H; type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hour 40 minutes while machine H is available for 10 hours during any working day. Formulate the problem as a linear programming problem.

Formulation: let x_1 be the number of products of type A and x_2 be the number of products of type B.

According to question

Machine	Time of products		Available time
	Type A (x_1 units)	Type B (x_2 units)	
G H	1 2	1 1	400 600
Profit per unit	Rs.2	Rs.3	

Since the profit on type A is Rs. 2 per product, $2x_1$ will be the profit on selling x_1 units of type A. Similarly, $3x_2$ will be the profit on selling x_2 units of type B. therefore, total profit on selling x_1 units of type A and x_2 units of type B is given by

$$P = 2x_1 + 3x_2 \quad \text{(objective function)}$$

Machine G takes one minute time on type A and one minute time on type B, the total number of minutes required on machine G is given by $x_1 + x_2$

Also, machine G is not available for more than 6 hour 40 minutes (=400 minutes)

$$\therefore \qquad x_1 + x_2 \leq 400$$

Similarly, Machine H takes two minute time on type A and one minute time on type B, the total number of minutes required on machine H is given by $2x_1 + x_2$

Also, machine H is not available for more than 10 hours (=600 minutes)

$$\therefore \qquad 2x_1 + x_2 \leq 600$$

Now it is not possible to produce negative quantities, $x_1 \; and \; x_2 \geq 0$

Thus the LPP is Max. $P = 2x_1 + 3x_2$

Subject to constraints

$$x_1 + x_2 \leq 400$$

$$2x_1 + x_2 \leq 600$$

$$x_1 , x_2 \geq 0$$

Benefits of LPP:

1. Linear programming technique helps us in making the optimum use of productive resources. It also indicates how a decision maker can employ his productive factors most effectively by choosing and allocating these resources.
2. The quality of decisions may also be improved by linear programming techniques. The user of this technique becomes more objective and less subjective.
3. In production processes, high lighting bottlenecks is the most significant advantage of this technique. For example when a bottleneck occurs, some machines cannot meet the demand while others remain idle for some time.

Limitations of Linear Programming :

1. In some problems objective functions and constraints are not linear .Generally, in real life situations concerning business and industrial problems constraints are not linearly related to variables.
2. There is no guarantee of getting integer valued solutions, for example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems.
3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.
4. Sometime large scale problems cannot be solved with linear programming tech. even when the computer facility is available such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.

Graphically solution of LPP:

LPP involving two decision variables can be easily solved by graphical method

The outlines of graphical procedure are as follows:

Step 1. Consider each inequality-constraint as equation.

Step 2. Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality- constraint corresponding to the line is '≤ ', then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with '≥' sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step 4. Choose the convenient value of Z and plot the objective function line.

Step 5. Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6. Read the coordinates of the extreme points selected in step 5, and find the maximum or minimum value of Z.

Q1. A company makes two kinds of leather belts. Belt A is a high quality belt and belt B is of low quality. The respective profits are Rs. 4.00 and Rs. 3.00 per belt. Each belt of type A requires twice as much time as required by a belt of belt B, and the company would have time to make a maximum 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckle a day available for belt B. Determine the optimal product mix.

Sol. Let x_1 be the number of belts of type A and x_2 be the number of belts of type B.

$$\max Z = 4x_1 + 3x_2$$

Subject to constraints

$2x_1 + x_2 \leq 1000$ (time constraint)

$x_1 + x_2 \leq 800$ (availability of leather)

$x_1 \leq 400, \quad x_2 \leq 700$

$x_1, x_2 \geq 0$

Replace all the inequalities of the constraints into equations

$$2x_1 + x_2 = 1000$$

$$x_1 + x_2 = 800$$

$$x_1 = 400, \quad x_2 = 700$$

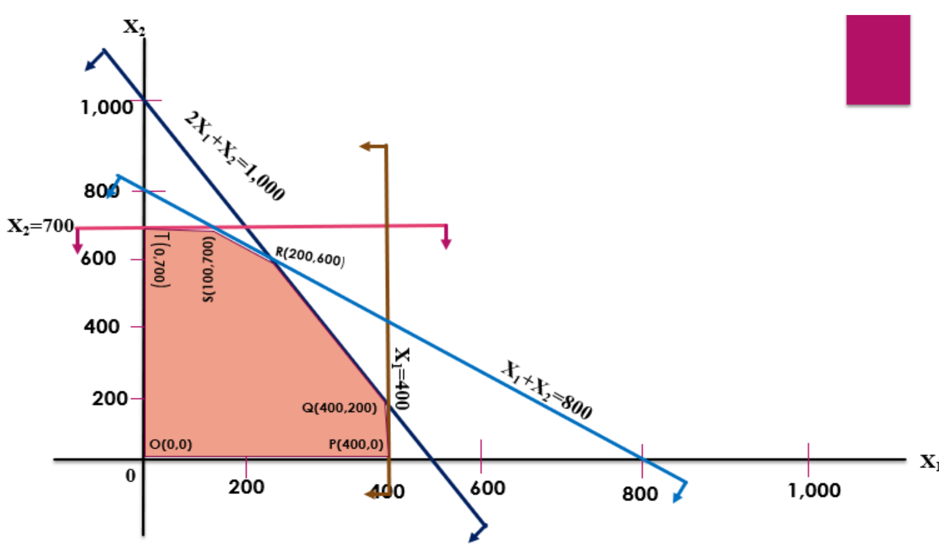
To plot the line $2x_1 + x_2 = 1000$, put $x_1 = 0 \Rightarrow x_2 = 1000$

Again put $x_2 = 0 \Rightarrow x_1 = 500$

Therefore, $2x_1 + x_2 = 1000$ passes through (0, 1000) and (500, 0)

Similarly, $x_1 + x_2 = 800$ passes through (0,800) and (800, 0)

Also, $x_1 = 400$, $x_2 = 700$



Corner points	$Z = 4x_1 + 3x_2$
O(0,0)	0
P(400,0)	1600
Q(400,200)	2200
R(200,600)	2600
S(100,700)	2500
T(0,700)	2100

The maximum value of Z occurs at R(200,600) the optimal solution is $x_1 = 200$, and $x_2 = 600$

Hence to maximize profit, the company should produce 200 belts of type A and 600 belts of type B.

Q2. A firm makes two products X and Y, and has a total production capacity of 9 tonnes per day, X and Y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tonnes of X and at least 3 tonnes of Y per day to another company. Each tonne of X requires 20 machine hours of production time and each tonne of Y requires 50 machine hours of production time. The daily maximum possible number of machine hours is 360. All the firm output can be sold and the profit made is Rs. 80 per tonne of X and Rs. 120 per tonne of Y. it is required to determine the production schedule for maximum profit and to calculate this profit.

Q3. Solve by graphical method

$$\text{Min. Z} = 1.5x_1 + 2.5x_2$$

Subject to constraints

$$x_1 + 3x_2 \geq 3$$
$$x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

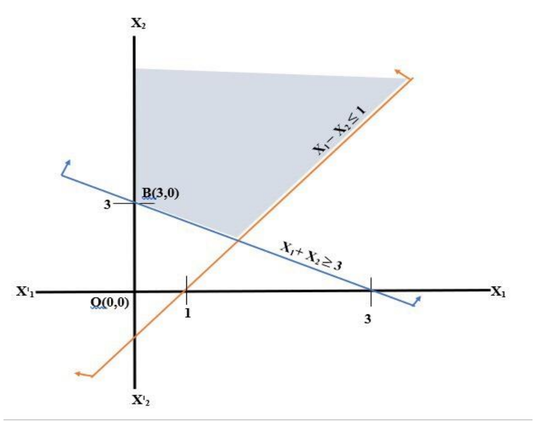
Answer : $x_1 = {}^3/_2,\ x_2 = {}^1/_2$ and min.Z=3.5

Q4. Problem having unbounded solution

$$\text{Max. Z} = 3x_1 + 2x_2$$

Subject to constraints

$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \geq 3,$$
$$x_1, x_2 \geq 0$$



Q5. Problem having infeasible solution

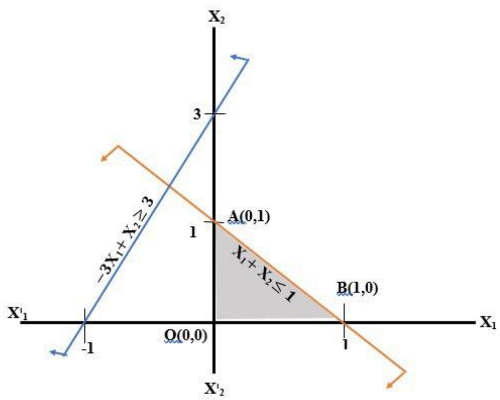
Max. Z= $x_1 + x_2$

Subject to constraints

$x_1 + x_2 \leq 1$

$-3x_1 + x_2 \geq 3$

$x_1 , x_2 \geq 0$



Q6. Problem having alternative solution

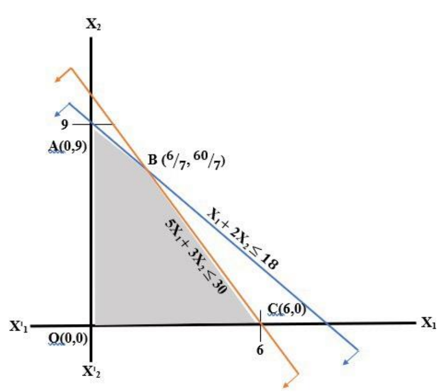
Max. Z= $10x_1 + 6x_2$

Subject to constraints

$5\,x_1 + 3x_2 \leq 30$

$x_1 + 2x_2 \leq 3$

$x_1 , x_2 \geq 0$



Some important definitions

Feasible Solution: Any set X = $\{x_1, x_2, \cdots x_{n+m}\}$ of variables is called a feasible solution of LPP, if it satisfies the set of constraints and non-negativity restrictions.

Basic solution: Given a system of m linear equations with n variables (m<n) any solution which is obtained by solving for m variables keeping the remaining n-m variables zero is called basic solution such m variables are called basic variables and the remaining variables are called non basic variables.

Example : $x_1 + 2x_2 + x_3 = 4$

$2x_1 + x_2 + 5x_3 = 5$

the basic solutions are

Let, $x_3 = 0 \Rightarrow x_1 = 2,\ x_2 = 1$

Let, $x_2 = 0 \Rightarrow x_1 = 5,\ x_3 = -1$

Let, $x_1 = 0 \Rightarrow x_2 = \frac{5}{3},\ x_3 = \frac{2}{3}$

Basic feasible solution: A basic feasible solution is a basic solution where all basic variables are non-negative.

Slack variable: If the constraints of a general LPP be $\sum_{j=1}^{n} a_{ij}x_i \leq b_i, \quad (i = 1,2, \dots, m)$

then the non-negative variables s_i which are introduced to convert the inequalities '≤' to the equalities i.e. $\sum_{j=1}^{n} a_{ij}x_i + s_i = b_i, \quad (i = 1,2, \dots, m)$ are called slack variables.

Surplus variable: If the constraints of a general LPP be $\sum_{j=1}^{n} a_{ij}x_i \geq b_i, \quad (i = 1,2, \dots, m)$

then the non-negative variables s_i which are introduced to convert the inequalities '≥' to the equalities i.e. $\sum_{j=1}^{n} a_{ij}x_i - s_i = b_i, \quad (i = 1,2, \dots, m)$ are called slack variables.

Simplex Method:

Q1. Solve the following LPP using the simplex method

$$Max. Z = 3x_1 + 2x_2$$

Subject to the constraint

$$x_1 + x_2 \leq 4$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Solution:

Step 1: Formulation of the mathematical model:

(i) Formulate the mathematical model of given LPP.

(ii) If objective function is of minimization type then convert it into one of maximization by following relationship

Min. Z = – Max. Z*   Where Z* = -Z

(iii) Ensure all b_i values [all the right side constants of constraints] are positive. If not, it can be changed into positive value on multiplying both side of the constraints by-1.

(iv) Next convert the inequality constraints to equation by introducing the non-negative slack or surplus variable. The coefficients of slack or surplus variables are zero in the objective function.

$$Max. Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

Subject to the constraints

$$x_1 + x_2 + s_1 = 4$$
$$x_1 - x_2 + s_2 = 2$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Step 2: Set up the initial solution. Write down the coefficients of all the variables in given LPP in the tabular form, as shown in table below to get an initial basic feasible solution.

		c_j	3	2	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio x_B/x_j
s_1	0	4	1	1	1	0	
s_2	0	2	1	-1	0	1	
		$z_j = C_B x_j$	0	0	0	0	
		$\Delta_j = c_j - z_j$	3	2	0	0	

Step 3: Test for optimality:

(i) If all $\Delta_j \leq 0$, the solution under test will be optimal.

(ii) If at least one Δ_j is positive, the solution under test is not optimal, then proceed to improve the solution in step 4.

(iii) If corresponding to most positive Δ_j, all elements of the column X_j are negative or zero (≤ 0), then the solution under test will be unbounded

Applying this rule for testing the optimality of starting basic feasible solution, it is observed that Δ_1, and Δ_2 both are positive. Hence proceed to improve this solution in step 4.

Step 4:

In order to improve this basic feasible solution, the vector or entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules, such vectors are usually named as "incoming vector" and "outgoing vector" respectively.

(i) To select "incoming vector" we find such value of 'k' for which Δ_k= max Δ_j. Then the vector coming into the basis matrix will be Δ_k

(ii) To select "outgoing vector". The vector going out of the basis matrix will be β_r, if we determine the suffix 'r' by the minimum ratio rule $\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0\right]$, for predetermined value of k.

pivot element

		c_j	3	2	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio x_B/x_1
s_1	0	4	1	1	1	0	4/1=4
s_2	0	2	1	-1	0	1	2/1=2 ⬅ Pivot row
	$z_j = C_B x_j$		0	0	0	0	
	$\Delta_j = c_j - z_j$		3 ⬆	2	0	0	

Pivot column

Step 5: We now construct the next improvement table by using the simple matrix transformation rules.

		c_j	3	2	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio x_B/x_2
s_1	0	2	0	2	1	-1	
x_1	3	2	1	-1	0	1	×
	$z_j = C_B x_j$		3	-3	0	3	
	$\Delta_j = c_j - z_j$		0	5	0	-3	

$R_1(new) \rightarrow R_1(old) - R_2(\text{new})$

Step 6: Now return to step 3, 4 and 5 if necessary. This process is repeated till we reach the desired conclusion.

		c_j	3	2	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio ${x_B}/{x_2}$
x_2	2	1	0	1	1/2	-1/2	
x_1	3	3	1	0	1/2	1/2	
	$z_j = C_B x_j$		3	2	5/2	1/2	
	$\Delta_j = c_j - z_j$		0	0	-5/2	-1/2	

$R_2(new) \to R_2(old) + R_1$(new), $\quad R_1(new) \to \frac{R_1(old)}{2}$

Since all $c_j - z_j \leq 0$ the optimal solution has been obtained

$x_1 = 3, \quad x_2 = 1, \quad s_1 = 0, \quad s_2 = 0$ Max. Z = 11

Q2. $Max. Z = 3x_1 + 2x_2 + 5x_3$

Subject to the constraint

$x_1 + 2x_2 + x_3 \leq 430$

$3x_1 + 2x_3 \leq 460$

$x_1 + 4x_2 \leq 420$

$x_1, x_2, x_3 \geq 0$

Solution: $Max. Z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to the constraint

$x_1 + 2x_2 + x_3 + s_1 = 430$

$3x_1 + 2x_3 + s_2 = 460$

$x_1 + 4x_2 + s_3 = 420$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

		c_j	3	2	5	0	0	0	
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	Min. ratio ${x_B}/{x_3}$
s_1	0	430	1	2	1	1	0	0	430/1=430
s_2	0	460	3	0	$\boxed{2}$	0	1	0	460/2=230
s_3	0	420	1	4	0	0	0	1	×
	$z_j = C_B x_j$		0	0	0	0	0	0	
	$\Delta_j = c_j - z_j$		3	2	5	0	0	0	

$R_2 \to \frac{R_2}{2}$, $R_1(new) \to R_1(old) - R_2(new)$

		c_j	3	2	5	0	0	0	
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	Min. ratio x_B/x_2
s_1	0	200	-1/2	$\boxed{2}$	0	1	-1/2	0	200/2=100
x_3	5	230	3/2	0	1	0	1/2	0	×
s_3	0	420	1	4	0	0	0	1	420/4=105
	$z_j = C_B x_j$		15/2	0	5	0	5/2	0	
	$\Delta_j = c_j - z_j$		-9/2	2	0	0	-5/2	0	

$R_1 \to \frac{R_1}{2}$, $R_3(new) \to R_3(old) - 4R_1(new)$

		c_j	3	2	5	0	0	0
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3
x_2	2	100	-1/4	1	0	1/2	-1/4	0
x_3	5	230	3/2	0	1	0	1/2	0
s_3	0	20	2	0	0	-2	1	1
	$z_j = C_B x_j$		7	2	5	1	2	0
	$\Delta_j = c_j - z_j$		-4	0	0	-1	-2	0

Since all $c_j - z_j \leq 0$ the optimal solution has been obtained

$x_1 = 0, \quad x_2 = 100, \quad x_3 = 230 \;\; s_1 = 0, \quad s_2 = 0, \;\; s_3 = 20$ Max. Z= 1350

Q3. $Max. Z = 5x_1 + 3x_2$

Subject to the constraint

$x_1 + x_2 \leq 2$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

$x_1 , x_2 \geq 0$

Answer: $x_1 = 2, \quad x_2 = 0, \;\; s_1 = 0, \quad s_2 = 0$

Max. Z= 10

Q4. $Min. Z = x_1 - 3x_2 + 3x_3$

Subject to the constraint

$3x_1 - x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Answer : $x_1 = 4, \quad x_2 = 5, \quad x_3 = 0 \;\; s_1 = 0, \quad s_2 = 0, \quad s_3 = 11$

Min. Z= -11

Note: $Max. Z' = -x_1 + 3x_2 - 3x_3$

Q5. $Max. Z = -2x_1 + 3x_2$

Subject to the constraint

$x_1 \leq 5$

$2x_1 - 3x_2 \leq 6$

$x_1, x_2 \geq 0$

Solution: $Max. Z = -2x_1 + 3x_2 + 0s_1 + 0s_2$

Subject to the constraint

$x_1 + s_1 = 5$

$2x_1 - 3x_2 + s_2 = 6$

$x_1, x_2, s_1, s_2 \geq 0$

		c_j	-2	3	0	0	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	Min. ratio x_B/x_2
s_1	0	5	1	0	1	0	×
s_2	0	6	2	-3	0	1	×
	$z_j = C_B x_j$		0	0	0	0	
	$\Delta_j = c_j - z_j$		-2	3	0	0	

Since x_2 is pivot column but the elements of pivot column are negative and zero therefore the solution is unbounded.

Linear programming problems in which constraints may also have '≥' and '=' signs after ensuring that all $b_i \geq 0$, are considered in this section. In such problems basis matrix is not obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable called the artificial variable. These variables have no physical meaning. The artificial variable technique is used to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained.

There are two methods for eliminating these variables from the solution

(i) Two – Phase Method
(ii) Big-M Method or Method of Penalties

Two-Phase Method: The Two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows:

Phase I. In this phase, the simplex method is applied to a specially constructed auxiliary linear programming problem leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1. Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function.

Step 2. Construct the auxiliary linear programming problem in which the new objective function Z^* is to be maximized subject to the given set of constraints.

Step 3. Solve the auxiliary problem by simplex method until either of the following three possibilities do arise:

(i) Max. $Z^* < 0$ and at least one artificial variable is present in the basis with positive value than the original LPP does not possess any feasible solution.
(ii) Max. $Z^* = 0$ and at least one artificial variable is present in the basis at zero value than the original LPP possess the feasible solution. In this case proceed to Phase II.
(iii) Max. $Z^* = 0$ and no artificial variable present in the basis, than the basis feasible solution to the original LPP has been found. Go to Phase II.

Phase II: Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints.

Q1. Use Two-Phase simplex method to solve the problem

$$Min.Z = x_1 - 2x_2 - 3x_3$$

Subject to the constraints

$$-2x_1 + x_2 + 3x_3 = 2$$
$$2x_1 + 3x_2 + 4x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Solution: First convert the objective function into maximization form

$$Max.Z' = -x_1 + 2x_2 + 3x_3$$ where Z'=-Z

Subject to the constraints

$$-2x_1 + x_2 + 3x_3 + A_1 = 2$$
$$2x_1 + 3x_2 + 4x_3 + A_2 = 1$$

$$x_1, x_2, x_3, A_1, A_2 \geq 0$$

Phase I: $Max. Z'^{*} = 0x_1 + 0x_2 + 0x_3 - 1A_1 - 1A_2$

		c_j	0	0	0	-1	-1	
Basic variable	C_B	x_B	x_1	x_2	x_3	A_1	A_2	Min. ratio x_B/x_3
A_1	-1	2	-2	1	3	1	0	2/3=0.667
A_2	-1	1	2	3	4	0	1	1/4=0.25
		$z_j = C_B x_j$	0	-4	-7	-1	-1	
		$\Delta_j = c_j - z_j$	0	4	7	0	0	

$R_2 \rightarrow \frac{R_2}{4}$, $R_1(new) \rightarrow R_1(old) - 3R_2(new)$

		c_j	0	0	0	-1
Basic variable	C_B	x_B	x_1	x_2	x_3	A_1
A_1	-1	5/4	-7/2	-5/4	0	1
x_3	0	1/4	1/2	3/4	1	0
		$z_j = C_B x_j$	7/2	5/4	0	-1
		$\Delta_j = c_j - z_j$	-7/2	-5/4	0	0

Max. $Z'^{*} =$ -5/4

Since all $c_j - z_j \leq 0$. But at the same time $Max. Z'^{*} < 0$ and the artificial variable a_1 appears in the basic solution. Hence the original problem does not possess any feasible solution.

Q2. $Min. Z = \frac{15}{2}x_1 - 3x_2$

Subject to the constraints

$3x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Solution: Convert the objective function into the maximization

$$Max. Z' = -\frac{15}{2}x_1 + 3x_2 \qquad \text{where Z'=-Z}$$

Introducing the surplus variables $s_1, s_2 \geq 0$ and artificial variables $A_1, A_2 \geq 0$ the constraints becomes

$$3x_1 - x_2 - x_3 - s_1 + A_1 = 3$$
$$x_1 - x_2 + x_3 - s_2 + A_2 = 2$$
$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

Phase I: Assigning a cost -1 to artificial variables A_1 and A_2 and cost 0 to all other variables, the objective function for auxiliary LPP

$$Max. Z'^{*} = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - A_1 - A_2$$

		c_j	0	0	0	0	0	-1	-1	
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2	A_1	A_2	Min. ratio x_B/x_1
A_1	-1	3	3	-1	-1	-1	0	1	0	3/3=1
A_2	-1	2	1	-1	1	0	-1	0	1	2/1=2
	$z_j = C_B x_j$		-4	2	0	1	1	-1	-1	
	$\Delta_j = c_j - z_j$		4	-2	0	-1	-1	0	0	

$R_1 \to \frac{R_1}{3}$, $R_2(new) \to R_2(old) - R_1(new)$

		c_j	0	0	0	0	0	-1	-1	
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2	A_1	A_2	Min. ratio x_B/x_3
x_1	0	1	1	-1/3	-1/3	-1/3	0	×	0	×
A_2	-1	1	0	-2/3	4/3	1/3	-1	×	1	3/4
	$z_j = C_B x_j$		0	2/3	-4/3	-1/3	1	×	-1	
	$\Delta_j = c_j - z_j$		0	-2/3	4/3	1/3	-1	×	0	

$R_2 \to \frac{3R_2}{4}$, $R_1(new) \to R_1(old) + \frac{1}{3} R_2(new)$

		c_j	0	0	0	0	0	-1	-1	
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2	A_1	A_2	Min. ratio x_B/x_3
x_1	0	5/4	1	-1/2	0	-1/4	-1/4	×	×	
x_3	0	3/4	0	-1/2	1	1/4	-3/4	×	×	
	$z_j = C_B x_j$		0	0	0	0	0	×	×	
	$\Delta_j = c_j - z_j$		0	0	0	0	0	×	×	

Since Max. Z'*=0 and no artificial variable present in the basis, an optimum solution to the auxiliary LPP has been obtained.

Phase II: $Max. Z' = -\frac{15}{2}x_1 + 3x_2 + 0x_3 + 0s_1 + 0s_2$

		c_j	-15/2	3	0	0	0	
Basic variable	C_B	x_B	x_1	x_2	x_3	s_1	s_2	Min. ratio x_B/x_3
x_1	-15/2	5/4	1	-1/2	0	-1/4	-1/4	
x_3	0	3/4	0	-1/2	1	1/4	-3/4	
	$z_j = C_B x_j$		-15/2	15/4	0	15/8	15/8	
	$\Delta_j = c_j - z_j$		0	-3/4	0	-15/8	-15/8	

Since all $c_j - z_j \leq 0$ the optimal solution has been obtained

$x_1 = \frac{5}{4}$, $x_2 = 0$, $x_3 = 3/4$, Min. Z= 75/8

Q3. $Min. Z = 5x_1 + 3x_2$

Subject to the constraints

$2x_1 + 4x_2 \leq 12$

$2x_1 + 2x_2 = 10$

$5x_1 + 2x_2 \geq 10$

$x_1, x_2 \geq 0$

Answer: $x_1 = 4$, $x_2 = 1$, Min. Z = 23

Q4. $Min. Z = x_1 + x_2$

Subject to the constraints

$$2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

Answer: $x_1 = \frac{21}{13}, \quad x_2 = \frac{10}{13},$ Min. Z = 31/13

Q5. $Max. Z = 5x_1 + 8x_2$

Subject to the constraints

$$3x_1 + 2x_2 \geq 3$$
$$x_1 + 4x_2 \geq 4$$
$$x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Answer: $x_1 = 0, \quad x_2 = 5,$ Max. Z = 40

Big –M Method (Method of Penalties): The Big-M method is another method of removing variables from the basis. In this method we assign a very high penalty say M>0 to the artificial variables in the objective function.

Q1. $Min. Z = 5x_1 + 3x_2$

Subject to the constraints

$$2x_1 + 4x_2 \leq 12$$
$$2x_1 + 2x_2 = 10$$
$$5x_1 + 2x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

Solution: $Max. Z' = -5x_1 - 3x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

Subject to the constraints

$$2x_1 + 4x_2 + s_1 = 12$$
$$2x_1 + 2x_2 + A_1 = 10$$
$$5x_1 + 2x_2 - s_2 + A_2 = 10$$
$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

		c_j	-5	-3	0	0	-M	-M	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	A_1	A_2	Min. ratio x_B/x_1
s_1	0	12	2	4	1	0	0	0	12/2=6
A_1	-M	10	2	2	0	0	1	0	10/2=5
A_2	-M	10	5	2	0	-1	0	1	10/5=2
	$z_j = C_B x_j$		-7M	-4M	0	M	-M	-M	
	$\Delta_j = c_j - z_j$		7M-5	4M-3	0	-M	0	0	

$$R_3 \to \frac{R_3}{5}, \; R_2(new) \to R_2(old) - 2R_3(new), \quad R_1(new) \to R_1(old) - 2R_3(new)$$

		c_j	-5	-3	0	0	-M	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	A_1	Min. ratio $^{x_B}/_{x_2}$
s_1	0	8	0	16/5	1	2/5	0	40/16=2.5
A_1	-M	6	0	6/5	0	2/5	1	5
x_1	-5	2	1	2/5	0	-1/5	0	5
	$z_j = C_B x_j$		5	$\frac{-6M}{5} - 2$	0	$\frac{-2M}{5} + 1$	-M	
	$\Delta_j = c_j - z_j$		0	$\frac{6M}{5} - 1$	0	$\frac{2M}{5} - 1$	0	

$R_1 \to \frac{5R_1}{16}$, $R_2(new) \to R_2(old) - \frac{6}{5}R_1(new)$, $R_3(new) \to R_3(old) - \frac{2}{5}R_1(new)$

		c_j	-5	-3	0	0	-M	
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2	A_1	Min. ratio $^{x_B}/_{s_2}$
x_2	-3	5/2	0	1	5/16	1/8	0	20
A_1	-M	3	0	0	-3/8	1/4	1	12
x_1	-5	1	1	0	-1/8	-1/4	0	×
	$z_j = C_B x_j$		5	-3	$\frac{3M}{8} - \frac{5}{16}$	$\frac{-2M}{5} + 1$	-M	
	$\Delta_j = c_j - z_j$		0	0	$\frac{-3M}{8} + \frac{5}{16}$	$\frac{2M}{5} - 1$	0	

$R_2 \to 4\,R_2$, $R_1(new) \to R_1(old) - \frac{1}{8}R_2(new)$, $R_3(new) \to R_3(old) + \frac{1}{4}R_2(new)$

		c_j	-5	-3	0	0
Basic variable	C_B	x_B	x_1	x_2	s_1	s_2
x_2	-3	1	0	1	1/2	0
s_2	0	12	0	0	-3/2	1
x_1	-5	4	1	0	-1/2	0
	$z_j = C_B x_j$		-5	-3	1	0
	$\Delta_j = c_j - z_j$		0	0	-1	0

Since all $c_j - z_j \leq 0$ the optimal solution has been obtained

$x_1 = 4$, $x_2 = 1$, $s_1 = 0$, $s_2 = 12$ Max. Z' = -23, Min. Z=23

Q2. $Max. Z = 6x_1 + 4x_2$

Subject to the constraints

$2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

Answer: $x_1 = 8$, $x_2 = 0$, Max. Z= 48

Q3. $Max. Z = 3x_1 + 2x_2$

Subject to the constraints

$2x_1 + x_2 \leq 2$

$3\ x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$