Analysis H Notes

Hayden L

2024-2025

1 Introduction

These are my notes for Gunn High School Analysis Honors Course.

2 Geometric Approach to Matrices (GAtM)

2.1 Groups

Group: a set of elements with a binary operation (two inputs, one output)

1. **Identity**: there is an identity element $I \in G$, $X \cdot I = I \cdot X = X$
2. **Inverse**: each element X has an inverse X^{-1} such that $X \cdot X^{-1} = X^{-1} \cdot X = X$
3. **Associativity**: $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$
4. **Closure**: If $X \in G$ and $Y \in G$, then $X \cdot Y \in G$

Order: the number of elements in a finite group

- **Snap Group** S_n (n-post snap group) has an order of $n!$
- **Dihedral Group** D_n (rotation and reflections of a regular n-gon) has an order of $2n$
- **Cyclic Group** C_n (rotation of a regular n-gon) has a order of n

Period of an element X: the least possible integer n such that $X^n = I$
Isomorphic (Isomorphism of a group): two groups with the same order, inverse, periods, and table
Generators of a set are elements that can express <u>all elements of a group</u> (also known as the smallest generating set).
Subgroups: elements in a group that are closed among themselves..

2.2 Infinities and Infinite Groups

Countable Infinity: numbers that can be put in one-to-one correspondence with the set of natural numbers $\mathbb{N}$
Uncountable Infinity: numbers that cannot be put in one-to-one correspondence with the set of natural numbers $\mathbb{N}$
Cardinality[1]: Two infinite sets have the same cardinality if their elements can be put into a one-to-one correspondence with each other.
Number Sets Since not all infinities are equally big, we can prove that infinite sets have the

Type of Number	**Definitions**	**Examples**
Natural Numbers $\mathbb{N}$	all positive integers from 1 to infinity	1,2,3,...
Integers $\mathbb{Z}$	a whole number that can be positive, negative, or zero	...,-1,0,1,...
Rational Numbers $\mathbb{Q}$	numbers that can be expressed as a fraction	$\frac{2}{3}, 0.5$
Irrational Numbers $\mathbb{I}$	numbers that cannot be expressed as a fraction	$\sqrt{2}, \pi$
Real Numbers $\mathbb{R}$	the union of both rational and irrational numbers	N/A
Imaginary Numbers	a square root of a negative number	i
Complex Numbers $\mathbb{C}$	the union of both real and imaginary numbers	$1 + i$

Table 1: Types of Numbers

[1]Cardinality refers to the number of elements in a set, while isomorphism refers to the structure. Groups that are isomorphic have the same cardinality, but not all groups with the same cardinality are isomorphic.

same cardinality by:

1. making a one-to-one correspondence between the sets, or
2. if you can prove that $|A| \geq |B|$ and $|A| \leq |B|$, then $|A| = |B|$.[2]

Comparing the set of standards for numbers, we get

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}|$$

For example, we can prove that Natural Numbers and Integers have the same cardinality ($|\mathbb{N}| = |\mathbb{Z}|$) by creating a one-to-one correspondence

$$\begin{aligned} &\mathbb{N} \quad \mathbb{Z} \\ &1 \Leftrightarrow 0 \\ &2 \Leftrightarrow 1 \\ &3 \Leftrightarrow -1 \\ &4 \Leftrightarrow 2 \\ &5 \Leftrightarrow -2 \\ &\ldots \end{aligned}$$

thus showing that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

We can also find a one-to-one correspondence between natural numbers and rational numbers greater than zero.

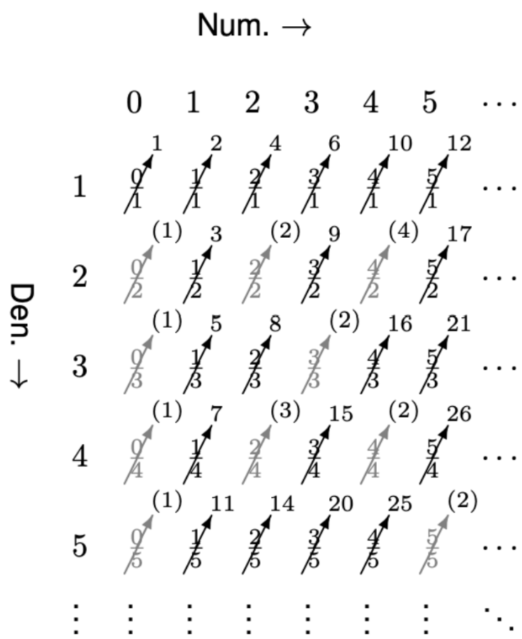

Figure 1: Image 1: $\mathbb{N}$ vs $\mathbb{Q}_{\geq 0}$

[2]This means if you find two injective functions, $f : A \to B$ and $g : B \to A$, then there exists a bijective function $h : A \to B$, according to the Cantor-Schroeder-Bernstein Theorem.

Now we can compare the infinite sets as listed on GAtM 5.9

	Infinite Sets	Group?	Reason
a	natural numbers, addition	No	0 not in group
b	integers, addition	Yes	
c	even integers, addition	Yes	
d	odd integers, addition	No	odd + odd = even
e	rational numbers, addition	Yes	
f	real numbers, addition	Yes	
g	complex numbers, addition	Yes	
h	integers, multiplication	No	0 has no identity
i	integer powers of 2, multiplication	No	
j	rational numbers, multiplication	No	0 has no identity
k	rational numbers excluding 0, multiplication	Yes	
l	real numbers excluding 0, multiplication	Yes	
m	complex numbers, multiplication	No	0 has no identity
n	rotation by a rational number of degrees	Yes	
o	rotation by a rational number of radians	Yes	
p	rotation by an integer number of radians	Yes	

Table 2: Groups of Infinite Sets

We can then see that groups b, c, i, and p are isomorphic to each other, groups e and o are isomorphic to each other, and groups f and g are isomorphic to each other.

2.3 Geometry of Complex Numbers

As a refresher of complex number terminology and notation:

Complex number in rectangular form: $z = a + bi$
Re(z): real part of z, $Re(z) = a$
Im(z): imaginary part of z, $Im(z) = b$
Arg(z): angle of z from the positive x-axis
|**z**|: the length of z, $|z| = \sqrt{a^2 + b^2}$
Conjugate of z: $\overline{z} = a - bi$
cis form: $z = |z| \operatorname{cis} \theta$ or $z = r \operatorname{cis} \theta,$ where $\operatorname{cis} \theta = \cos \theta + i \sin \theta$

Now, we can begin with De Moivre's Thereom, where

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis} (n\theta).$$

We can prove De Moivre's Theorem by showing $r_1(\cos A + i \sin A) \cdot r_2(\cos B + i \sin B)$.[3][4]

[3]see slides
[4]Note that z and zi are always perpendicular.

***n*-th roots of a complex number**: when finding the n-th roots of a complex number, you must make sure to find all of the solutions. For instance, when finding $\sqrt[n]{z}$,

$$\sqrt[n]{z}$$

$$= (r \operatorname{cis} \theta)^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}} \operatorname{cis} (\frac{\theta}{n} + \frac{2\pi k}{n})$$

$$\text{for } k = 0, 1, 2, ..., n - 1$$

These n solutions are a result of the Fundamental Theorem of Algebra, which states that any polynomial of degree n has n-roots.

2.4 Mapping the Plane with Matrices

When mapping a matrices, consider a matrix representing a unit square $M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and a transformation matrix. The following are a list of transformation matrices.[5]

Identity: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection over x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflection over y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection over line *y*=*x*: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Rotation by $\boldsymbol{\theta}$[6]: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Stretch by *k* in x-direction: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

Stretch by *k* in y-direction: $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Shear[7] **by *k* in x-direction**: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Shear by *k* in y-direction: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

[5] The identity, dilation, and some rotations and reflections have a corresponding complex number.

[6] Identical to multiplying by $r \operatorname{cis} \theta$.

[7] Shear is a type of linear transformation that distorts the shape of an object such that its points shift parallel to a given axis. Line that were originally parallel remain parallel and area remains the same, but angles between lines and lengths of line segments may change.

Dilation with factor k: $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Translation by $<\alpha, \beta>$: $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+\alpha \\ y+\beta \\ 1 \end{bmatrix}$

Composite of two or more transformations: When doing two or more transformations, the matrix written first is done second. Consider a rotation by 90° and a reflection over the y-axis. If we were to do the reflection followed by the rotation, we would notate it

$$R_{90^\circ}\, r_y,$$

while if we were to the rotation first, we would notate it

$$r_y\, R_{90^\circ}.$$

The former order of transformations is identical to a reflection over the line $y = -x$, where the latter is identical to a reflection over the line $y = x$. You can also do a transformation with matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and translation by vector $<\alpha, \beta>$ by using 3D matrices. Consider doing the transformation first to get $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{bmatrix}$.

Mapping points onto a line: given a line $ax + by = 0$, we can use a matrix: $\begin{bmatrix} b & b \\ -a & -a \end{bmatrix}$ to map the points onto the line. To map the points onto a line $y = \frac{a}{b}x + c$, we can use a 3D matrix.

Reflection over the line $\theta = n^\circ$ (i.e. line with equation $y = x\tan\theta$): $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

This can be derived from the fact that a composite of two reflections is a rotation, noting that the angle of the rotation is double the angle between the lines of reflection:

$$r_{xtan\theta} \cdot r_x = R_{2\theta}$$

$$r_{xtan\theta} \cdot r_x \cdot r_x^{-1} = R_{2\theta} \cdot r_x^{-1}$$

$$r_{xtan\theta} = R_{2\theta} \cdot r_x^{-1}$$

$$r_{xtan\theta} = R_{2\theta} \cdot r_x$$