Analysis H Notes

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1 Introduction

These are my notes for Gunn High School Analysis Honors Course.

2 Geometric Approach to Matrices (GAtM)

2.1 Groups

Group: a set of elements with a binary operation (two inputs, one output)

1. **Identity**: there is an identity element $I \in G$, $X \cdot I = I \cdot X = X$
2. **Inverse**: each element X has an inverse X^{-1} such that $X \cdot X^{-1} = X^{-1} \cdot X = X$
3. **Associativity**: $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$
4. **Closure**: If $X \in G$ and $Y \in G$, then $X \cdot Y \in G$

Order: the number of elements in a finite group

- **Snap Group** S_n (n-post snap group) has an order of $n!$
- **Dihedral Group** D_n (rotation and reflections of a regular n-gon) has an order of $2n$
- **Cyclic Group** C_n (rotation of a regular n-gon) has a order of n

Period of an element X: the least possible integer n such that $X^n = I$
Isomorphic (Isomorphism of a group): two groups with the same order, inverse, periods, and table
Generators of a set are elements that can express all elements of a group (also known as the smallest generating set).
Subgroups: elements in a group that are closed among themselves..

2.2 Infinities and Infinite Groups

Countable Infinity: numbers that can be put in one-to-one correspondence with the set of natural numbers $\mathbb{N}$
Uncountable Infinity: numbers that cannot be put in one-to-one correspondence with the set of natural numbers $\mathbb{N}$
Cardinality[1]: Two infinite sets have the same cardinality if their elements can be put into a one-to-one correspondence with each other.
Number Sets Since not all infinities are equally big, we can prove that infinite sets have the

Type of Number	Definitions	Examples
Natural Numbers $\mathbb{N}$	all positive integers from 1 to infinity	1,2,3,...
Integers $\mathbb{Z}$	a whole number that can be positive, negative, or zero	...,-1,0,1,...
Rational Numbers $\mathbb{Q}$	numbers that can be expressed as a fraction	$\frac{2}{3}, 0.5$
Irrational Numbers $\mathbb{I}$	numbers that cannot be expressed as a fraction	$\sqrt{2}, \pi$
Real Numbers $\mathbb{R}$	the union of both rational and irrational numbers	N/A
Imaginary Numbers	a square root of a negative number	i
Complex Numbers $\mathbb{C}$	the union of both real and imaginary numbers	$1 + i$

Table 1: Types of Numbers

[1]Cardinality refers to the number of elements in a set, while isomorphism refers to the structure. Groups that are isomorphic have the same cardinality, but not all groups with the same cardinality are isomorphic.

same cardinality by:

1. making a one-to-one correspondence between the sets, or
2. if you can prove that $|A| \geq |B|$ and $|A| \leq |B|$, then $|A| = |B|$.[2]

Comparing the set of standards for numbers, we get

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}|$$

For example, we can prove that Natural Numbers and Integers have the same cardinality ($|\mathbb{N}| = |\mathbb{Z}|$) by creating a one-to-one correspondence

$$\begin{array}{c} \mathbb{N} \quad \mathbb{Z} \\ \\ 1 \Leftrightarrow 0 \\ 2 \Leftrightarrow 1 \\ 3 \Leftrightarrow -1 \\ 4 \Leftrightarrow 2 \\ 5 \Leftrightarrow -2 \\ \ldots \end{array}$$

thus showing that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

We can also find a one-to-one correspondence between natural numbers and rational numbers greater than zero.

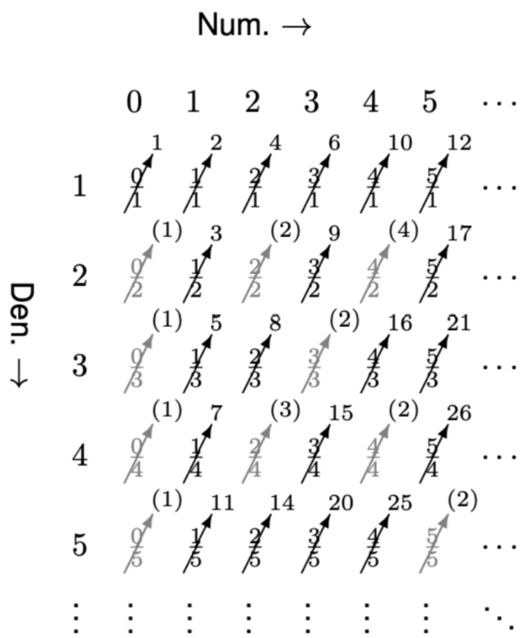

Figure 1: Image 1: $\mathbb{N}$ vs $\mathbb{Q}_{\geq 0}$

[2] This means if you find two injective functions, $f : A \to B$ and $g : B \to A$, then there exists a bijective function $h : A \to B$, according to the Cantor-Schroeder-Bernstein Theorem.

Now we can compare the infinite sets as listed on GAtM 5.9

	Infinite Sets	**Group?**	**Reason**
a	natural numbers, addition	No	0 not in group
b	integers, addition	Yes	
c	even integers, addition	Yes	
d	odd integers, addition	No	odd + odd = even
e	rational numbers, addition	Yes	
f	real numbers, addition	Yes	
g	complex numbers, addition	Yes	
h	integers, multiplication	No	0 has no identity
i	integer powers of 2, multiplication	No	
j	rational numbers, multiplication	No	0 has no identity
k	rational numbers excluding 0, multiplication	Yes	
l	real numbers excluding 0, multiplication	Yes	
m	complex numbers, multiplication	No	0 has no identity
n	rotation by a rational number of degrees	Yes	
o	rotation by a rational number of radians	Yes	
p	rotation by an integer number of radians	Yes	

Table 2: Groups of Infinite Sets

We can then see that groups b, c, i, and p are isomorphic to each other, groups e and o are isomorphic to each other, and groups f and g are isomorphic to each other.

2.3 Geometry of Complex Numbers

As a refresher of complex number terminology and notation:

Complex number in rectangular form: $z = a + bi$
Re(z): real part of z, $Re(z) = a$
Im(z): imaginary part of z, $Im(z) = b$
Arg(z): angle of z from the positive x-axis
|**z**|: the length of z, $|z| = \sqrt{a^2 + b^2}$
Conjugate of z: $\overline{z} = a - bi$
cis form: $z = |z| \operatorname{cis} \theta$ or $z = r \operatorname{cis} \theta$, where $\operatorname{cis} \theta = \cos \theta + i \sin \theta$

Now, we can begin with De Moivre's Thereom, where

$$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis} (n\theta).$$

We can prove De Moivre's Theorem by showing $r_1(\cos A + i \sin A) \cdot r_2(\cos B + i \sin B)$.[3][4]

[3]see slides
[4]Note that z and zi are always perpendicular.

n-th roots of a complex number: when finding the n-th roots of a complex number, you must make sure to find all of the solutions. For instance, when finding $\sqrt[n]{z}$,

$$\sqrt[n]{z}$$

$$= (r \operatorname{cis} \theta)^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}} \operatorname{cis} (\frac{\theta}{n} + \frac{2\pi k}{n})$$

$$\text{for } k = 0, 1, 2, ..., n-1$$

These n solutions are a result of the Fundamental Theorem of Algebra, which states that any polynomial of degree n has n-roots.

2.4 Mapping the Plane with Matrices

When mapping a matrices, consider a matrix representing a unit square $M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and a transformation matrix. The following are a list of transformation matrices.[5]

Identity: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection over x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflection over y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection over line y=x: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Rotation by θ[6]: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Stretch by k in x-direction: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

Stretch by k in y-direction: $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Shear[7] **by k in x-direction**: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Shear by k in y-direction: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

[5] The identity, dilation, and some rotations and reflections have a corresponding complex number.

[6] Identical to multiplying by $r \operatorname{cis} \theta$.

[7] Shear is a type of linear transformation that distorts the shape of an object such that its points shift parallel to a given axis. Line that were originally parallel remain parallel and area remains the same, but angles between lines and lengths of line segments may change.

Dilation with factor $\boldsymbol{k}$: $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Translation by $\boldsymbol{<\alpha, \beta>}$: $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+\alpha \\ y+\beta \\ 1 \end{bmatrix}$

Composite of two or more transformations: When doing two or more transformations, the matrix written first is done second. Consider a rotation by 90° and a reflection over the y-axis. If we were to do the reflection followed by the rotation, we would notate it

$$R_{90^\circ}\, r_y,$$

while if we were to the rotation first, we would notate it

$$r_y\, R_{90^\circ}.$$

The former order of transformations is identical to a reflection over the line $y = -x$, where the latter is identical to a reflection over the line $y = x$. You can also do a transformation with matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and translation by vector $<\alpha, \beta>$ by using 3D matrices. Consider doing the transformation first to get $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{bmatrix}$.

Mapping points onto a line: given a line $ax + by = 0$, we can use a matrix: $\begin{bmatrix} b & b \\ -a & -a \end{bmatrix}$ to map the points onto the line. To map the points onto a line $y = \frac{a}{b}x + c$, we can use a 3D matrix.

Reflection over the line $\boldsymbol{\theta = n^\circ}$ (i.e. line with equation $y = x\tan\theta$): $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

This can be derived from the fact that a composite of two reflections is a rotation, noting that the angle of the rotation is double the angle between the lines of reflection:

$$r_{xtan\theta} \cdot r_x = R_{2\theta}$$

$$r_{xtan\theta} \cdot r_x \cdot r_x^{-1} = R_{2\theta} \cdot r_x^{-1}$$

$$r_{xtan\theta} = R_{2\theta} \cdot r_x^{-1}$$

$$r_{xtan\theta} = R_{2\theta} \cdot r_x$$

3 Limits

3.1 Convergence

Definition of Convergence:[8] A sequence $\{a_n\}$ is said to converge to a limit A if for any neighborhood of A, there can be found a natural number M such that whenever $n \geq M$, $\{a\}$ is in the neighborhood of A.

In other words, for any given neighborhood, there is a finite number of terms outside the neighborhood.

Proof for a general ϵ:[9]

$$\text{Let } \epsilon > 0$$

$$\text{Given } \lim_{x \to \infty} f(x) = A^{10}$$

$$A - \epsilon < f(x) < A + \epsilon$$

Then solve for x in terms of ϵ to prove that: For all natural $n \geq$ n in terms of ϵ, $f(x)$ is within ϵ of A. Therefore, $\lim_{x\to\infty} f(x) = A$

3.2 Tests for Convergence and Divergence of Sequences

Comparison Principle: The Comparison Principle for sequences can be used to compare a sequence to the terms of a simpler sequence to prove divergence.

We can use sequences such as {n} and {-n} as they are both simple and obviously divergent.

Big Theorem: If a sequence can be proved to be everywhere increasing and bounded above, or everywhere decreasing and bounded below, then it converges. The lowest upper bound, or the highest lower bound, is its limit.

P-Series Test: For sequences n^p and p^n, the following cases diverge[11]

$$\{n^p\} \text{ where } p > 0$$

$$\{p^n\} \text{ where } p > 1 \text{ or } p \leq -1^{12}$$

Subsequence Principle: If b_n is everywhere increasing, a_n increases without bound, and a_n is a subsequence of b_n, then b_n increases without bound (is divergent).

Domination Principle: If two sequences $\{a_n\}$ and $\{c_n\}$ both converge to the same limit L and $\{a_n\} \leq \{b_n\} \leq \{c_n\}$ for all n greater than or equal to some natural number M, then $\{b_n\}$ converges to the same limit L.

[8] If a sequence does NOT converge, then it diverges.
[9] This is less complicated than the Delta-Epsilon Proof
[10] Assume A is a known finite number
[11] Other values of p will lead to convergence and is flipped for reciprocals
[12] When $p \leq -1$, p^n becomes an alternating sequence that is not practical for the comparison principle

3.3 Series

Series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$$

Sequence of Partial Sums:

$$\{S_1, S_2, S_3, ...\} \text{ where } S_1 = a_1, S_2 = a_1 + a_2, ...$$

A few important sequences to take note of are $\{\frac{1}{n^2}\}$ and $\{\frac{1}{n}\}$. [13]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... + \frac{1}{n^2} = \frac{\pi^2}{6}\text{; Converges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} = \infty \text{ or DNE; Diverges}$$

3.4 Tests for Convergence and Divergence of Series

Geometric Series Test: Given the geometric series $S = a + ar + ar^2 + ar + 3 + ...$, the geometric series converges to $\frac{a}{1-r}$ if $|r| < 1$, otherwise it diverges.

$$\text{If } |r| < 1 \implies \sum_{n=1}^{\infty} a(r)^{n-1} = \frac{a}{1-r}$$

nth Term Test: If the sequence $\{a_n\}$ does not converge to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges. Be careful in using this test as this test can NOT prove convergence of a series.

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum_{n=1}^{\infty} a_n = \infty \text{ or DNE}$$

P-series Test: Since we know from sequences that $\{n^p\}$ diverges for all $p > 0$ and that $\{\frac{1}{n^p}\}$ converges for all $p \geq 0$, the series of $\frac{1}{n^p}$ will converge if and only if $p > 1$. [14]

$$p \leq 1 \implies \sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$$

Comparison Test[15]: If $\{a_n\}$ and $\{b_n\}$ are two sequences such that $0 < a_n \leq b_n$ for all natural numbers n, then:

[13] Also known as the harmonic series

[14] Note that $\sum_{n=1}^{\infty} \frac{1}{p^n}$ is just a geometric series and thus can be found through the Geometric Series Test.

[15] Not to be confused with the sequence test, comparison principle.

1. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

2. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Take note that you cannot prove that $\sum_{n=1}^{\infty} a_n$ diverges or that $\sum_{n=1}^{\infty} b_n$ converges.

Ratio Test: For a sequence $\{a_n\}$, consider b in:

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n} = b$$

1. If $0 \leq b < 1$, then the series converges.

2. If $b > 1$, then the series diverges.

3. If $b = 1$, then this test is inconclusive and another test must be used to prove its convergence or divergence.

Alternating Series Test: A series $\sum_{n=1}^{\infty} a_n$ will converge if and only if:

1. The sequence is strictly alternating in signs,

2. Everywhere decreasing in absolute value (approaches 0)

3. And the sequence converges to 0.

4 Calculus

4.1 Integral: Area Under Curve

Sum of Important Sums:[16]

$$1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + 4^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$$

Definite Integral: The definite integral is the area under the curve. One way to find the definite integral is by considering rectangles with a width of $\frac{1}{n}$. Consider for example $f(x) = 2x$. To find the definite integral over the interval $[0, 2]$

$$\frac{1}{n}\sum_{k=1}^{2n}[2(\frac{k}{n})] = A$$

Here, $\frac{1}{n}$ is the width of the rectangles, $2n$ is the number of rectangles, and $\frac{k}{n}$ is plugged into x. After simplifying using sum equations, find the limit as n approaches ∞ to find the indefinite integral

$$\lim_{n\to\infty}[A]$$

[16]These will be given on the test and do not need to be memorized.

When finding the definite integral over an interval $[1,2]$, add the starting x value to $\frac{k}{n}$:

$$\frac{1}{n}\sum_{k=1}^{n}[2(\frac{k}{n}+1)]$$

and solve for the limit as n approaches ∞.

Trapezoid Rule: To approximate the definite integral, you can use the trapezoid rule, where given $y = f(x)$,

$$A = \frac{h}{2}(f(a) + 2f(a+h) + 2f(a+2h) + ... + 2f(a+(n-1)h) + f(a+nh))$$

4.2 AROC vs IROC

Average Rate of Change: Calculated as if there is a straight line between points $(a, f(a))$ and $(b, f(b))$. Over the interval $[a, b]$, the average rate of change would be

$$\frac{f(b)-f(a)}{b-a}$$

Instantaneous Rate of Change: Also known as the derivative. This can be *approximated* by using the AROC of two points that are very close together.

$$f'(x) = \frac{dy}{dx}$$

4.3 Limits and Discontinuities

Limits: A limit exists at $x \to c$ if and only if

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$$

and the limit is a finite, non-infinite number.

Continuity: A function is continuous at point $x = c$ if and only if

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = f(c)$$

1. $\lim_{x\to c} f(x)$ exists
2. $f(c)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

Discontinuities:

1. Removable discontinuity (hole): The limit exists but is not continuous.
2. Jump discontinuity: $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist but $\lim_{x\to c} f(x)$ does not exist.
3. Infinite discontinuity (vertical asymptote): both one-sided limits approach infinity, and thus do not exist (DNE). Limit algebraically is $\frac{k}{0}$.

Definition of a Limit: A function $f(x)$ is said to approach a limit L as $x \to c$ if and only if for any positive ϵ there exists a positive number δ such that whenever x is within δ of c, but $x \neq c$, $f(x)$ is within ϵ of L.

$$\lim_{x\to c} f(x) = L \text{ iff } \forall\epsilon > 0, \exists\delta > 0 : 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

Delta-Epsilon Proof: Start with epsilon proof:

$$L - \epsilon < f(x) < L + \epsilon$$

and express x in terms of ϵ. Find δ_L and δ_R and set δ to be the smaller of the two. Finally, write a conclusion statement and plug in values as necessary:

$$\text{Let } \epsilon > 0, \text{for any } x \text{ within } \delta = \_ \text{ of } c, f(x) \text{ is within } \epsilon \text{ of } L$$

Limit Properties:

1. The limit of a produce equals the product of the limits.

$$\lim_{x\to c}(f(x) \times g(x)) = \lim_{x\to c}(f) \times \lim_{x\to c} g(x)$$

2. The limit of a sum equals the sum of the limits.

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c}(f) + \lim_{x\to c} g(x)$$

3. The limit of a quotient equals the quotient of the limits.

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c}(f)}{\lim_{x\to c} g(x)}$$

4. The limit of a constant times a function equals the constant time the limit.

$$\lim_{x\to c} Cf(x) = C \lim_{x\to c}(f)$$

5. The limit of the constant is that constant.

$$\lim_{x\to c} C = C$$

6. The limit of the identity function is "c".

$$\lim_{x\to c} x = c$$

7. The limit of a composite is the limit of the inside function with respect to the outside function.

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$$

Intermediate Value Theorem (IVT): Given that $f(x)$ is continuous for all x in the interval $[a, b]$, if y is a value between $f(a)$ and $f(b)$, then there is a number $x = c$ in $[a, b]$ for which $f(c) = y$.

4.4 Derivatives

Formal Definition of the Derivative at a Point (FDoDaP):

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Formal Definition of the Derivative (FDoD):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Power Rule:

$$f(x) = x^n \implies f'(x) = nx^{n-1}$$

Chain Rule:

$$h(x) = f(g(x)) \implies h'(x) = f'(g(x))g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Trigonometric Functions:

$$f(x) = \sin(x) \implies f'(x) = \cos(x)$$

$$f(x) = \cos(x) \implies f'(x) = -\sin(x)$$

$$f(x) = \tan(x) \implies f'(x) = \sec^2(x)$$

$$f(x) = \cot(x) \implies f'(x) = -\csc^2(x)$$

$$f(x) = \sec(x) \implies f'(x) = \sec(x)\tan(x)$$

$$f(x) = \csc(x) \implies f'(x) = -\csc(x)\cot(x)$$

4.5 Particle Motion

Relation of Derivatives: Given that $p(t)$ is the position of the particle, $p'(t) = v'(t)$ is the velocity and $s''(t) = v'(t) = a(t)$ is acceleration. Note that the speed of a particle is $|v(t)|$ and the particle is speeding up if $v(t)$ and $a(t)$ have the same sign.

4.6 Sinusoids

Sinusoids: Amplitude is A, Period is $\frac{2\pi}{B}$, Phase shift is C (positive to left)[17], and vertical shift is D (midline is $y = D$)

$$y = A\sin(B(x + C)) + D$$

4.7 Antiderivatives

Indefinite Integral: $g(x)$ is the antiderivative of f(x) if and only if $g'(x) = f(x)$.

$$\int f(x)dx = g(x) + c^{18}$$

4.8 Fundamental Theorem of Calculus

$$\int_a^b f'(t)dt = f(b) - f(a)$$

[17] also remember that $\sin(0) = 0$ and $\cos(0) = 1$

[18] Make sure to not forget the +c