Multinomial and ordered logits

Silje Synnøve Lyder Hermansen

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Reminder: What is a GLM?

Regressions aim to describe (a linear) relationship between x and y with one number, β.

- Assumes a continuous and unbounded variable.
- When y is neither (e.g. binary), we relied on a latent continuous variable
- To approximate the latent variable, we calculated the logodds (i.e. we compare)

$\Rightarrow$ *Probability distribution maps unoberved variable to observed outcomes.*

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What is an ordered variable?

A ranked variable with unknown distance between categories.

- Often the result of binning: Close connection to latent formulation.
- We can choose how to treat it: As linear, categorical or **ordinal**.

$\Rightarrow$ *estimate a single set of regression parameters, but keep the information on the order without assuming a continuous variable.*

Two conceptions of ordered logisitc regression

There are two ways of uncerstanding the ordered logit:

- Latent variable: useful for interpretation.
- Parallel regressions: useful for understanding and checking estimation.

Latent variable approach: cutpoints

Cutpoints

We rely on cutpoints to slice up the latent variable and determine outcomes

- **Binomial logistic:** One cutpoint. $\rightarrow$ Rarely estimated.
- **Ordinal logistic:** Serveral cutpoints. $\rightarrow$ Explicit.

$\Rightarrow$ *Model estimates both regression parameters (β) and cutpoints (τ).*

A series of cutpoints

You are in the category m when the latent variable is between its two cutpoints: $\tau_{m-1} < y^{\star} < \tau_m$

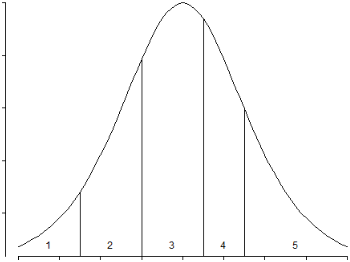


Figure 1: Slicing up a latent variable

The regression coefficients

The model calculates the odds of being lower than τ_m

- The first cutpoint (τ_0) is 0 ($-inf$): you cant be lower than the lowest.
- The last cutpoint is 1 ($+inf$): all observations are in some category.
- You end up with $m-1$ cutpoints.

The regression output

The regression output reports both β and τ

- **Regression coefficient** β is reported in relation to *upper* cutpoint of the category: $\tau_m - \beta x_i$
- **Cutpoints** serve also as intercepts.

The predicted value

The predicted probability of being in category *m*:

$$Pr(y_i = m) = \frac{exp(\tau_m - \beta x_i)}{1 + exp(\tau_m - \beta x_i)} - \frac{exp(\tau_{m-1} - \beta x_i)}{1 + exp(\tau_{m-1} - \beta x_i)} \tag{1}$$

An example: Attitudes towards redistribution

An example:

ESS respondents (that voted H or FrP) are asked to what extent they believe the state should engage in redistribution (1 = disagree; 5 = agree).

```
#Load in data
df <- read.table(
  "https://siljehermansen.github.io/teaching/stv4020b/kap10.txt")

#Check distribution
barplot(table(df$Utjevn))
```

An example:

ESS respondents (that voted H or FrP) are asked to what extent they believe the state should engage in redistribution ($1 = $ disagree; $5 = $ agree).

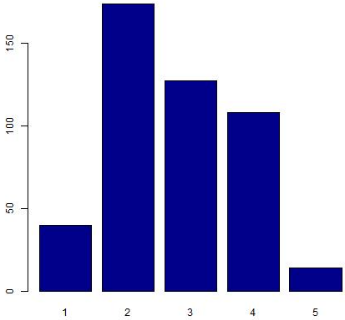


Figure 2: Attitudes towards redistribution is an ordered variable

Attitudes towards redistribution as a function of income

```r
#Library for ordinal regression
library(MASS)
#Recode into ordered factor
df$Utjevn.ord <- as.ordered(as.factor(df$Utjevn))
#Run regression
mod.ord <- polr(Utjevn.ord ~ Inntekt,
                df,
                method = "logistic",
                Hess = TRUE)
summary(mod.ord)
```

Attitudes towards redistribution as a function of income

```
## Call:
## polr(formula = Utjevn.ord ~ Inntekt, data = df, Hess = TRUE,
##     method = "logistic")
##
## Coefficients:
##           Value Std. Error t value
## Inntekt 0.08387    0.03128   2.681
##
## Intercepts:
##     Value   Std. Error t value
## 1|2 -2.0119  0.2422    -8.3052
## 2|3  0.3029  0.1994     1.5190
## 3|4  1.4724  0.2107     6.9883
## 4|5  3.9305  0.3317    11.8496
##
## Residual Deviance: 1218.94
## AIC: 1228.94
## (17 observations deleted due to missingness)
```

We learn two things from the regression output

Regression coefficient reports effect of x on probability to be placed one category higher

- Effect in logodds: `0.084`
- We can backtransform to one unit increase in x: $(exp(\beta) - 1) \times 100$ $= 9\%$ increase in likelihood of a higher category.

$\Rightarrow$ *Hypothesis testing as in a binomial logit*

We learn two things from the regression output

We have one intercept per cutpoint

- e.g.: intercept of passing from 1 to 2 is `-2.012`
- e.g.: intercept is reported as significant (with standard errors)

$\Rightarrow$ *The model does a fair job in distinguishing between categories.*

Predicted scenarios

We interpret predicted probability by choosing one level of x and one category (two cutpoints) of y: What is the probability of m?

$$Pr(y_i = m) = \frac{exp(\tau_m - \beta x_i)}{1 + exp(\tau_m - \beta x_i)} - \frac{exp(\tau_{m-1} - \beta x_i)}{1 + exp(\tau_{m-1} - \beta x_i)} \tag{2}$$

Example

Let's choose low-income respondents (x = 1) and category 3 (diff between cutpoints 2 and 3)

```
z = mod.ord$zeta
x = 1

logodds1 <- z[3] - coefficients(mod.ord) * x
logodds2 <- z[3-1] - coefficients(mod.ord) * x
## Probabilities
p1 <- exp(logodds1)/(1 + exp(logodds1)) #3/4 or lower
p2 <- exp(logodds2)/(1 + exp(logodds2)) #2/3 or lower
## Difference between cutpoints
p1 - p2 #cat 3
```

An example

Predicted proportion in category

```
paste(round((p1-p2)*100),
"% of low-income respondents are predicted to answer x = 3 ('neutral')." )
```

[1] "25 % of low-income respondents are predicted to answer x = 3 ('neutral')."

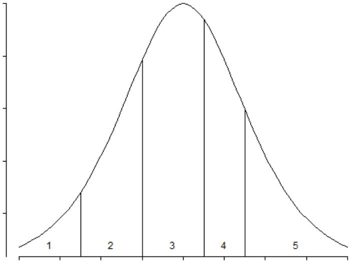
Cumulative probability

```
paste(round((p1)*100),
"% of low-income respondents are predicted to answer x = 3 ('neutral') or l
```

[1] "80 % of low-income respondents are predicted to answer x = 3 ('neutral') or lower to the question of whether they support redistribution."

Two ways of viewing the slicing

We can report the probability (e.g. `0.25`) of ending up between two cutpoints, or the *cumulative* probability (e.g. `0.8`) to be below each point.



Exercice:

Increase the τ (z) within each value of Income (x)

```
##Create empty plot
plot(y = 0,
     x = 0,
     axes = FALSE,
     xlim = c(1,4),
     ylim = c(0,1),
     ylab = "Probability of z or below",
     xlab = "Thresholds",
     main = "Cumulative probability \nof support for redistribution",
     type = "n")
axis(1, at = 1:length(p1),
     labels = names(p1))
axis(2)
```

Exercice:

Increase the τ (z) within each value of Income (x)

```
#Set values for prediction
x = 10 #Let this go from 1 to 10; check the shape of 10
z = mod.ord$zeta
#Logodds
logodds1 <- z - coefficients(mod.ord) * x
#Probabilities
p1 <- exp(logodds1)/(1 + exp(logodds1)) #3/4 or lower

#Plot probabilities
lines(y = p1,
      x = 1:length(p1),
      type = "b")
#Set legend (report x-value)
legend("topleft",
       bty = "n",
       cex = 0.8,
     paste("Income = ", x))

#Plot probabilities
text(y = p1+0.06,
```

Parallel regressions approach: for assessment

Parallel regressions approach

The parallel regression approach is useful to understand how the model is estimated

- The y is recoded into $m-1$ dummy variables indicating if $y \leq m$
- Run a series of regressions where all β are fixed (i.e.: the same).

$\Rightarrow$ *This is also useful when we assess the model*

How good is our model?

The basic assumption

The basic assumption is that all parallel regressions have (about) the same regression coefficient

- Check the mean of the predictor for each value of y. Does it trend?

```
tapply(df$Inntekt, df$Utjevn, mean, na.rm = T)
```

```
##        1        2        3        4        5
## 4.742857 5.547059 5.438017 6.205607 6.571429
```

- Run parallel regressions without contstraint on β. Are they similar?

An example of parallel regressions

Recode into dummies

The dummies flag cases below a cumulative threshold of *outcomes*

```
##
df$ut1 <- ifelse(df$Utjevn > 1, 1 , 0) #2 or above
df$ut2 <- ifelse(df$Utjevn > 2, 1 , 0) #3 or above
df$ut3 <- ifelse(df$Utjevn > 3, 1 , 0) #4 or above
df$ut4 <- ifelse(df$Utjevn > 4, 1 , 0) #5
```

$\Rightarrow$ *The model then runs 4 regressions where* β *reports an aggregated value from all 4 coefficients (think: weigted mean).*

Run four regressions

Let's examplify with the parallel regressions without fixed β:

```
##Parallel regressions:
mod1 <- glm(ut1 ~ Inntekt, df, family = "binomial")
mod2 <- glm(ut2 ~ Inntekt, df, family = "binomial")
mod3 <- glm(ut3 ~ Inntekt, df, family = "binomial")
mod4 <- glm(ut4 ~ Inntekt, df, family = "binomial")
```

Compare coefficients from four regressions

```
##
## =======================================================
##                          Dependent variable:
##                  --------------------------------------
##                    ut1       ut2       ut3       ut4
##                    (1)       (2)       (3)       (4)
## -------------------------------------------------------
## Inntekt          0.133**   0.057*  0.109***    0.127
##                  (0.067)  (0.035)   (0.039)   (0.101)
##
## Constant         1.772***  -0.155  -1.629*** -4.204***
##                  (0.367)  (0.216)   (0.259)   (0.711)
##
## -------------------------------------------------------
## Observations       447      447       447       447
## Log Likelihood   -120.685 -306.937 -257.053   -61.452
## Akaike Inf. Crit. 245.370  617.875   518.105   126.903
## =======================================================
## Note:                      *p<0.1; **p<0.05; ***p<0.01
```

Coefficient should be a weighted average from four regressions

These βs are weighted by the number of observations in each category:

```
table(df$Utjevn)
```

```
##
##   1   2   3   4   5
##  40 174 127 108  14
```

We can plot the βs for comparison:

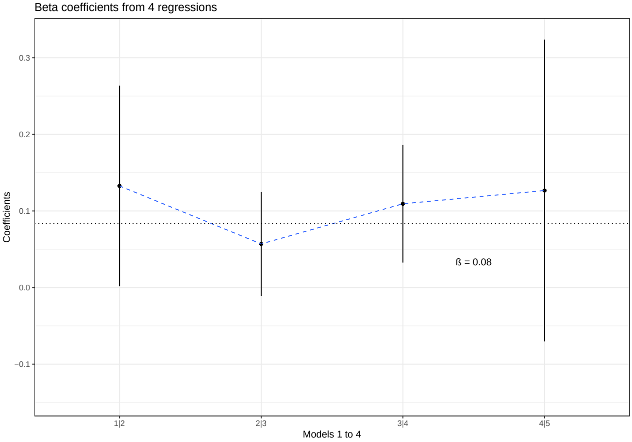
```
results <- rbind(summary(mod1)$coefficients[2, c(1,2)],
                 summary(mod2)$coefficients[2, c(1,2)],
                 summary(mod3)$coefficients[2, c(1,2)],
                 summary(mod4)$coefficients[2, c(1,2)])
thresholds <- c("1|2","2|3","3|4","4|5")
```

We can plot the βs for comparison:

```
ggplot() +
  geom_point(aes(y = results[, "Estimate"],
                 x = thresholds)) +
  geom_smooth(aes(y = results[, "Estimate"],
                  x = 1:4),
              lty = 2,
              lwd = 0.5) +
  geom_segment(aes(x = 1:4,
               xend = 1:4,
               y = results[, "Estimate"]-results[, "Std. Error"]*1.96,
               yend = results[, "Estimate"]+results[, "Std. Error"]*1.96)) +
  theme_bw() +
  ylim(c(results[, "Estimate"][2]-results[, "Std. Error"][4]*2,
         results[, "Estimate"][4]+results[, "Std. Error"][4]*2)) +
  geom_hline(yintercept = mod.ord$coefficients,
             lty = 3) +
  geom_text(aes(y = mod.ord$coefficients-0.05,
                x = 3.5,
                label = paste("\u03b2 =", round(mod.ord$coefficients,2))
                ),
            parse = F) +
  labs(title = "Beta coefficients from 4 regressions") +
  ylab("Coefficients") +
  xlab("Models 1 to 4")
```

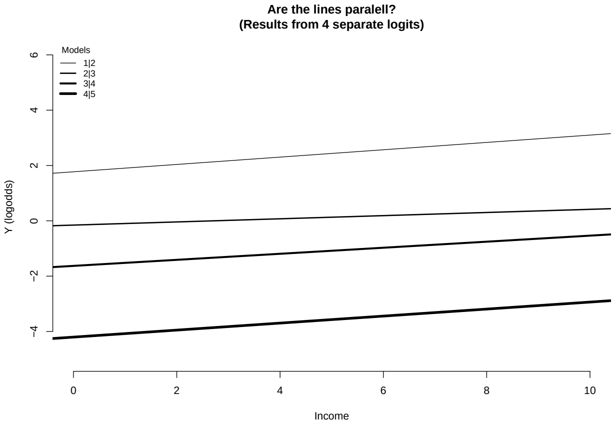
We can plot the βs for comparison:

The overall β is 0.08. If the ordered model describes the data well, then all the unconstrained βs should ressemble that description.



A visual inspection

A more visual way of checking the “parallel lines assumption” is to inspect if the regression lines are parallel.



When is it smart to run an ordered logit?

- You have few categories
- Fairly equal spread of observations between categories

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Dependent variable: nominal

The discrete choice models describe mutually exclusive choices.

- The choice variable is nominal: we cannot rank it
- Our *appreciation* of it is continuous. Two sets of models:
  - Multinomial: Models *chooser* characteristics
  - Conditional logit: Models *choice* characteristics

Multinomial logistic regression

Two conceptions of multinomial regression

- **A series of binomial logits** with the same reference category.
- **Latent variable approach:** Our utility of each choice.

Two conceptions of multinomial regression

A series of binomial logits with the same reference category.
* Data is subset to compare two groups $\rightarrow$ data/variation intensive model choice. * Categories/choice are mutually exclusive $\rightarrow$ Different β for each subset/choice

$\Rightarrow$ *All choices are given a probability and they sum up to one.*

Two conceptions of multinomial regression

Latent variable approach: Imagine k choices modeled as $y_m = \alpha_m \times \beta_m x$

- $\beta_m x_i$ reflects the utility of a choice k for the chooser i with x characteristic. $\rightarrow$ systematic term
- α_m reflects the baseline utility of that choice $\rightarrow$ stochastic term

$\Rightarrow$ *The preferred choice is the one with the highest utility because both or either are high*

Main assumption: IIA

Independence of irrelevant alternatives:

- there are no choices beyond what is modeled
- consistency: if we prefer A $>$ B and B $>$ C, then also A $>$ C

$\Rightarrow$ *The β does not depend on on other values of y (other alternatives).*

Testing the main assumption:

The Hausmann-McFadden test: Removes an alternative (supposed to be irrelevant) and check if β changes.

- Restricted model (a choice is removed) vs. unrestricted model (original)
- if IIA holds, then unrestricted model has smaller variance.

$\Rightarrow$ χ^2*-test with smaller value indicatee IIA holds.*

Prediction testing

- **Predict outcome**
  - predicted outcome/choice is the one with the highest probability/utility
  - confusion matrix (Proportion of correct predictions: $\frac{\textit{sum of diagonal}}{\textit{N observations}}$)
- **Probability of all outcomes separately**: ROC curve and separation plots

$\Rightarrow$ *as in binomial regression, where you have one category vs. the rest*

Interpretation

All the possibilities of the binomial logit are open:

- The regression table
- Predicted probabilities (and comparisons/scenarios) for each category
  - as with binomial logit, one line per category
  - *cumulative* predicted probabilitites $\rightarrow$ illustrates tradeoffs

$\Rightarrow$ *Remember reference cateogry is* $1-$ *the sum of all other probabilities*

Specific visual interpretations

If you have three categories (if $M = 3$) * The three dimensional simplex * The ternary plot: a sort of scatterplot for predicted probabilities

$\Rightarrow$ *Illustrates tradeoffs*

The conditional logit

From the chooser's perspectives

The conditional logit holds the chooser constant, and considers alternative choices

- x refers to characteristics of the choice (not chooser)
- Parallel regressions approach: a logit in a choice set
- One set of parameters, no intercept
- Long data format (observation = choice in individual)

Mixing choosers and choices

The mixed conditional logit makes an interaction effect between choice-set variables and choice variables.