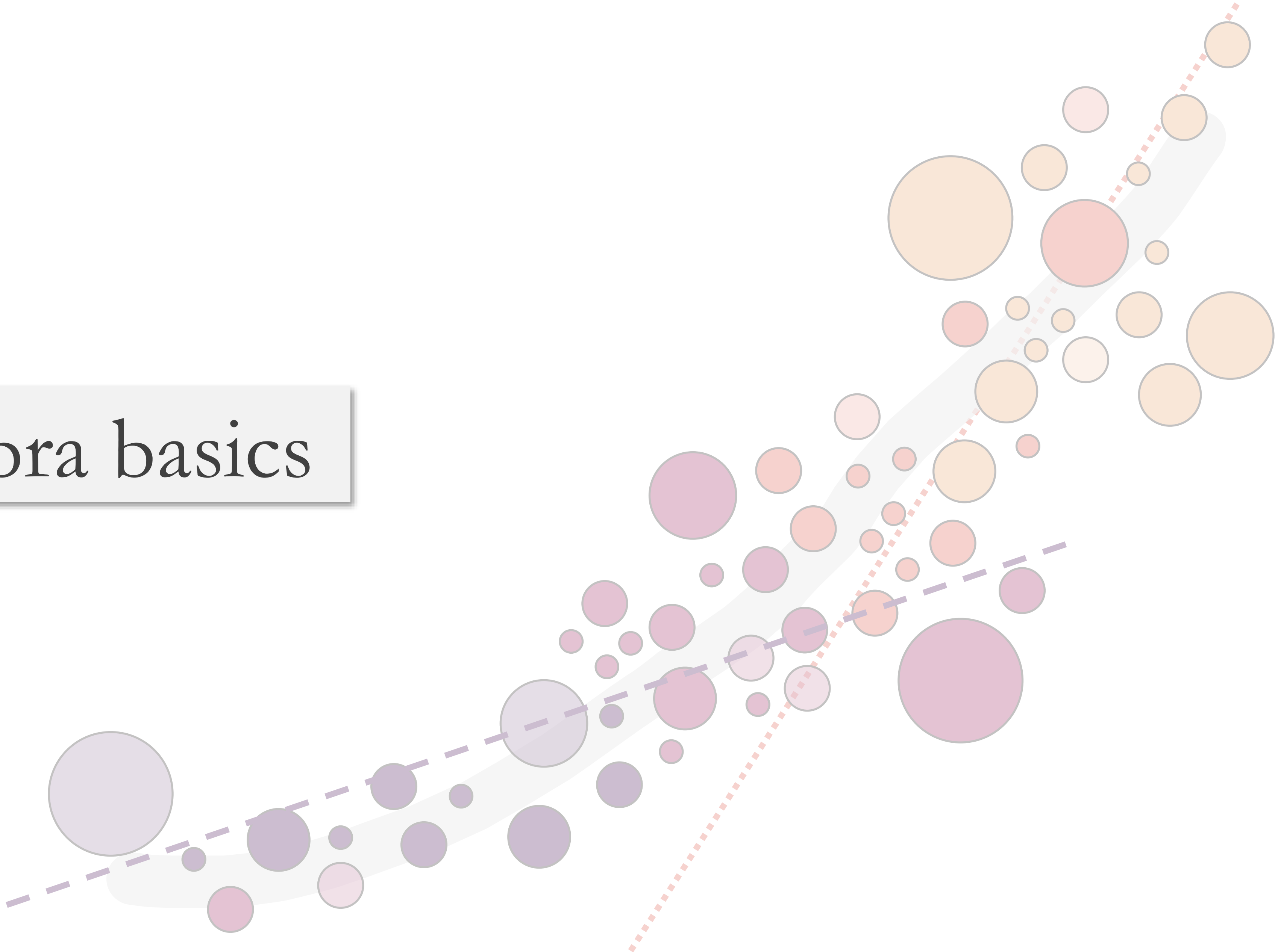
Linear algebra basics



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Working with vectors

A 3D column vector

$$\mathbf{x} = x = \vec{x} = \begin{bmatrix} 2.0 \\ 5.0 \\ 7.0 \end{bmatrix}$$

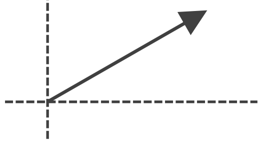
A 4D row vector

$$\mathbf{y} = \begin{bmatrix} 3.0 & -5.0 & 11.0 & 2.0 \end{bmatrix}$$

$\mathbf{x} \in \mathbb{R}^3$ dimension

Geometric interpretation

$[x \quad y]$



Algebraic interpretation

$[x_1 \quad x_2 \quad x_3]$

An ordered list of number

Transpose

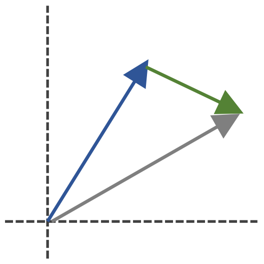
$$\mathbf{x}^{\mathbf{T}} = \begin{bmatrix} 2.0 & 5.0 & 7.0 \end{bmatrix}$$

Vector addition

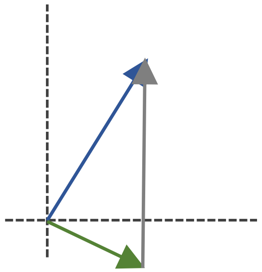
$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 9 \end{bmatrix}$$

Two vectors can be added together only if they have the same dimensionality and the same orientation.

$\mathbf{w} = \mathbf{u} + \mathbf{v}$



$\mathbf{w} = \mathbf{u} - \mathbf{v}$



Unit vector

$$\|\mathbf{v}\| = 1 \qquad \hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

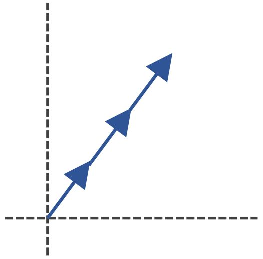
Any non unit vector has an associated unit vector.

Scalar multiplication

$$\lambda = 2.0$$

$$\lambda\mathbf{x} = \begin{bmatrix} 4.0 \\ 10.0 \\ 14.0 \end{bmatrix}$$

$\mathbf{w} = 3\mathbf{u}$



Vector magnitude (norm)

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Euclidean distance

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_3 - v_3)^2}$$

Vector algebra

Commutative Property

$$\mathbf{u} + \mathbf{v} = (\mathbf{v} + \mathbf{u})$$

Additive Associative Property

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

Zero Property

$$\mathbf{u} + 0 = \mathbf{u} \Leftrightarrow \mathbf{u} - \mathbf{u} = 0$$

Unit Rule

$$1\mathbf{u} = \mathbf{u}$$

Vector Distributive Property

$$s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$$

Scalar Distributive Property

$$(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$$

Zero Multiplicative Property

$$0\mathbf{u} = 0$$

Let's practice all this in the first module of the following Grasple course (lesson 1 & 2).

Vector Dot Products

Vector *Dot* (also inner or *scalar*) Product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = [u_1 v_1 + u_2 v_2 + \cdots + u_k v_k] = \sum_{i=1}^{k} u_i v_i$$

Commutative Property

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

Associative Property

$$s(\mathbf{u} \cdot \mathbf{v}) = (s\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (s\mathbf{v})$$

Distributive Property

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$$

Unit Rule

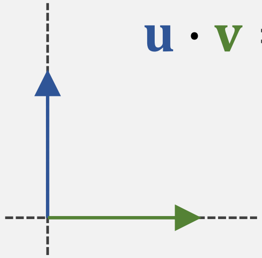
$$1\mathbf{u} = \mathbf{u}$$

Zero Property

$$\mathbf{u} \cdot 0 = 0$$

The dot product can be interpreted as a measure of similarity or mapping between two vectors.

Orthogonal Vectors Have a Zero Dot Product



$$\mathbf{u} \cdot \mathbf{v} = 0$$

Other kinds of products

$$\mathbf{u} \odot \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 20 \\ 14 \end{bmatrix}$$

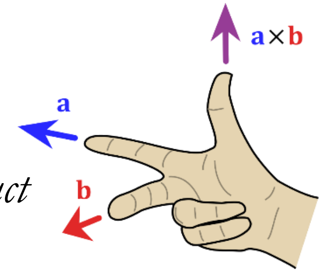
Hadamard multiplication

$$\mathbf{u}\mathbf{v}^{\mathbf{T}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} au & av \\ bu & bv \\ cu & cv \end{bmatrix}$$

Outer product

Let's practice all this in the first module of the following Grasple course (lesson 3).

Cross product



Working with matrices

dimension

$$\mathbf{X}_{3\times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

row × column

$$\mathbf{X}^T_{i,j} = \mathbf{X}_{j,i}$$

$$x_{12} = 2$$

Square matrix

$$\begin{bmatrix} 1 & 3 & 3 & 4 \\ 5 & 6 & 7 & 9 \\ 9 & 2 & 3 & 0 \\ 4 & 3 & 7 & 8 \end{bmatrix}$$

Upper triangle

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 8 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Symetric matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 8 & 5 & 6 \\ 3 & 5 & 1 & 7 \\ 4 & 6 & 7 & 8 \end{bmatrix}$$

Diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Identity matrix

$$\mathbf{I} = \mathbf{I}_{4\times 4} = \mathbf{I}(4) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix vector multiplication

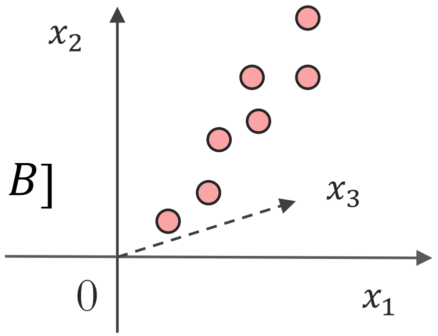
A matrix can be right-multiplied by a column vector but not a row vector, and it can be left-multiplied by a row vector but not a column vector.

$$\mathbf{X}\boldsymbol{\beta} = \boldsymbol{Y}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 5 & 6 & 7 \\ 9 & 2 & 3 \\ 4 & 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} X_{1,*} \cdot B \\ X_{2,*} \cdot B \\ X_{3,*} \cdot B \\ X_{4,*} \cdot B \end{bmatrix}$$

$$\mathbf{X}\boldsymbol{\beta} = \boldsymbol{Y}$$

$$\begin{bmatrix} 1 & 5 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & 9 & 4 \\ 3 & 6 & 2 & 3 \\ 3 & 7 & 3 & 7 \end{bmatrix} = \begin{bmatrix} X_{*,1} \cdot B & X_{*,2} \cdot B & X_{*,3} \cdot B & X_{*,4} \cdot B \end{bmatrix}$$



Matrix multiplication

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\underset{(n \times k)}{\mathbf{A}} \quad \underset{(k \times m)}{\mathbf{B}} \quad = \quad \underset{(n \times m)}{\mathbf{C}}$$

conformability

size of the resulting matrix

$$\sum_{i=1}^{k} a_{1i} bv_{i1} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

Two matrices are **conformable** for multiplication if the number of columns in the first matrix matches the number of rows in the second matrix.

Let's practice all this in the second module of the following Grasple course (lesson 1, 2 & 3).

Matrix inverse

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \Leftrightarrow det(\mathbf{A}) \neq 0$$

The inverse of a matrix A is another matrix that multiplies A to produce the identity matrix.

We need to "*cancel*" a matrix in order to solve problems that can be expressed in the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ and $\mathbf{b}$ are known quantities, and we want to solve for $\mathbf{x}$.

$$\mathbf{Ax} = \mathbf{b}$$
$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$$
$$\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}$$
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Matrix determinant

$$\det(\mathbf{X}) = |\mathbf{X}| = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

for 2 × 2 matrices only

Find the inverse of a *2 × 2 matrix*

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = det(\mathbf{X})^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Let's practice all this in the second module of the following Grasple course (lesson 4, 5 & 6).