Research Presentation for CGG interview

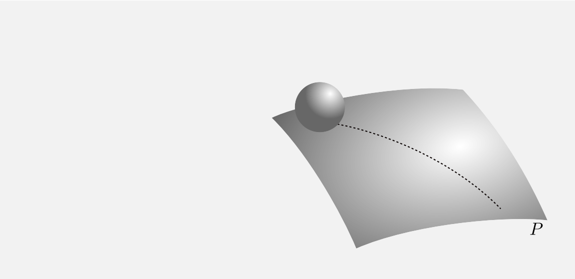
Bowei Zhao

joint work with Renato Feres

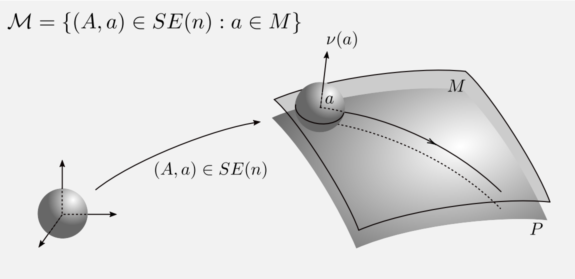
Introduction

- Billiard systems are broadly conceived as a class of mechanical/geometric dynamical systems whose trajectories consist of continuous, piecewise free (or geodesic) motion in the interior of some configuration manifold M
- My research deals with two types of mechanic systems.
- Billiard systems: systems involving elastic collisions. Billiards are discrete time dynamical systems. Specifically no-slip billiard which can be traced to Richard Garwin's model to explain the bouncing behaviour of a a Wham-O Super Ball
- Non-holonomic systems: these are mechanical systems involving rolling. These are continuous time dynamical systems defined by differential equations. They are not defined in terms of collisions between bodies.

Rolling ball: basic setup

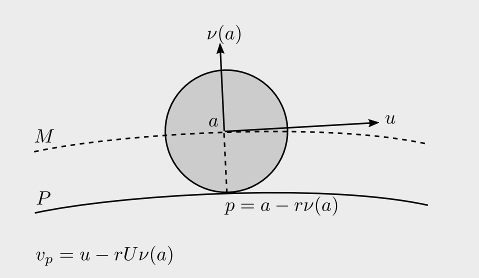


Rolling ball



Rolling ball: constraints

- Ball is in tangential contact with P: $\mathcal{M} = \{(A, a) \in SE(n) : a \in M\}$.
- No-slip constraint: velocity of contact point is zero.



- Velocity of point of contact: $V_x(g, \xi) = u_\xi - rU_\xi \nu(a)$.
- So the no-slip constraint equation is

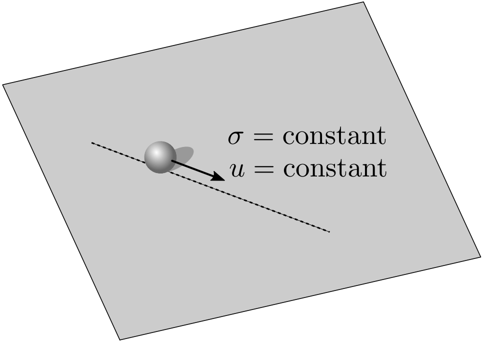
$$\boxed{u_\xi = rU_\xi \nu(a)}$$

- The no-slip constraint defines a vector subbundle $\mathfrak{R}$ of $T\mathcal{M}$ (rolling subbundle)

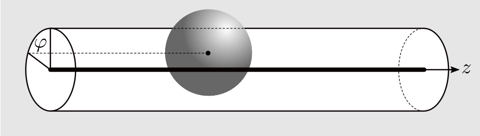
Elementary examples

Rolling on a flat plane:

- Let $P = \mathbb{R}^2 \subset \mathbb{R}^3$.
- Then $M = \mathbb{R}^2$, $\mathbb{S} = 0 \implies u(t), \sigma(t)$ are constant.
- Ball rolls with constant linear velocity and constant tangential spin.



Elementary examples



Rolling on an infinite straight line in $\mathbb{R}^3$:

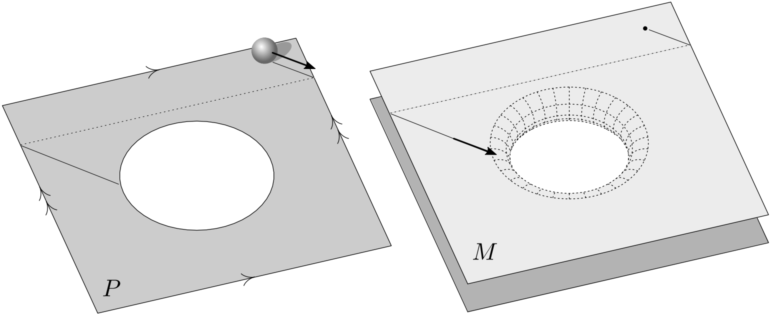
- Center of ball in polar coordinates: (φ, z); then

$$\varphi' = \text{ constant}; \quad z'' = -c\sigma, \quad \sigma' = kz'.$$

- Motion along the axis of cylinder:

$$z(t) = a_0 \cos(ct + \theta_0) + a_1 t + a_2.$$

Rolling Billiards



- M is the locus of the center points of the ball that rolls on a flat plate P

Definition (Rolling billiards)

We call the rolling motion on the 2-D plate, in the limit $r \to 0$, a rolling billiard.

Sinai Billiard Plate

- a submanifold of $\mathbb{T}^2 \times \mathbb{R}$
- P is a two-torus with a disc-shaped hole in the middle
- the ball does not lose contact with P, so the point of contact at any given time is 0.
- the associated pancake surface $\mathcal{N}$ is the boundary of the region in $\mathbb{T}^2 \times \mathbb{R}$ with points at distance no greater than r from P.

State of the system

- A state of the system at a given moment of time is the set of positions and velocities that uniquely specifies a trajectory through an initial value problem for Newton's differential equation.
- Each state consists of the position of the centre of the ball and three velocity components: two for the velocity of the centre of mass, which we call the centre velocity and one for the angular velocity component about the unit normal vector ν to $\mathcal{N}$, pointing outward.
- the third component is called (tangential) spin.

Equation of motion

- Assume a constraint force $F \in \mathfrak{R}^{\perp}$.
- Newton's equation:

$$\boxed{\frac{\nabla g'}{dt} = F}$$

- The constraint $g' \in \mathfrak{R}$ is used to determine F.

Dimension 3

- $J_a : T_a M \to T_a M$ rotation counterclockwise by $\pi/2$ (assume $J_a \nu(a) = 0$.)
- $\mathbb{S}_a : T_a M \to T_a M$ the *shape operator* at a

$$\mathbb{S}_a u = -D_u \nu \quad (D_u \text{ directional derivative at } a)$$

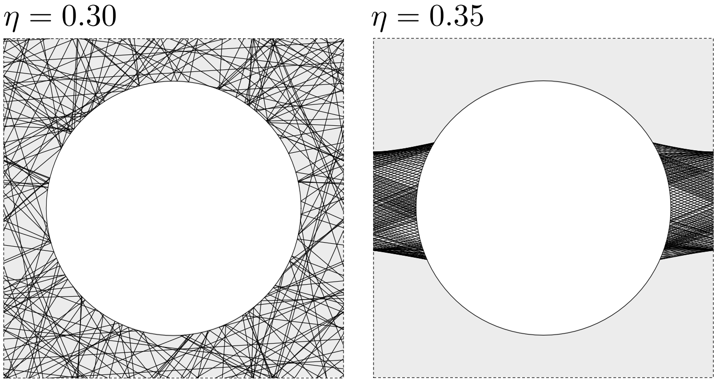
- Given $U \in \mathfrak{so}(3)$, define the *spin* $\sigma \in \mathbb{R}$ such that

$$\sigma = \frac{r\eta}{2} \operatorname{Tr}(U J_a^{\intercal}) \quad \text{where } \eta = \gamma / \sqrt{1 + \gamma^2}$$

- Then the equations of motion of the rolling ball can be reduced to:

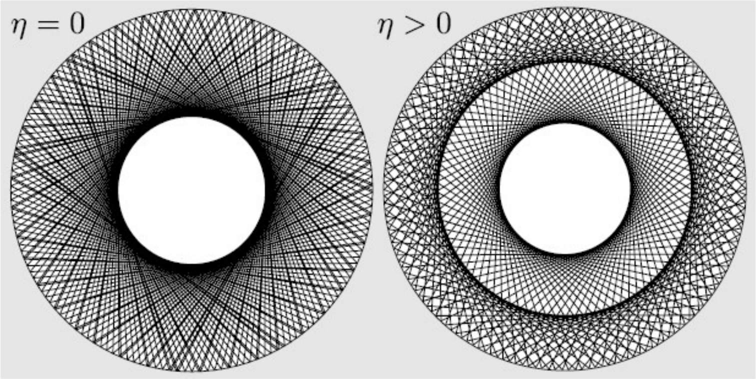
1. $u = a'$
2. $u' = -\eta \sigma J_a \mathbb{S}_a u + \langle u, \mathbb{S}_a u \rangle \nu(a)$
3. $\sigma' = \eta \langle J_a \mathbb{S}_a u, u \rangle$

Rolling on Sinai Billiard Systems



- For sufficiently small η, trajectories appear chaotic (result by Y. Sinai)
- For sufficiently large η the system shows strong stability and elliptic behaviour.

rolling on a disc shaped plate



- Here the flat plate P is a disc in $\mathbb{R}^3$ and $\mathcal{N}$ is a circular pancake with width $2r$, for a small value of r.
- When η is 0 , trajectories are nearly the same as the trajectories for the ordinary billiard on a disc and exhibit the characteristic caustic circle.
- For a positive η, caustics split into two concentric circles as seen on the right-hand side.

Main Results:Summary

- No-slip billiards arise from such rolling on submanifolds of Euclidean space under very general conditions.
- The rolling systems for general submanifolds of Euclidean space define one-parameter deformations of geodesic flows that depend on both the intrinsic and extrinsic submanifold geometry, the deformation parameter being the rolling ball's moment of inertia.
- The no-slip billiard system appears in the limit of the rolling motion as the radius of the ball approaches 0. We call these deformations of geodesic flows rolling flows on pancake manifolds.

No-slip billiard setup

- P the closure of a domain in $\mathbb{R}^k$ (extended billiard table)
- The configuration manifold of a ball moving freely in the interior of P is the subset $M = \{(x, A) \in SE(k) : x \in P\}$ of the Euclidean group $SE(k)$.
- Tangent vectors to M: $(u, S) \in T_x P \times \mathfrak{so}(k)$, where u is the center velocity and $S = \dot{A}A^{-1}$ is the angular velocity matrix, which lies in the Lie algebra $\mathfrak{so}(k)$ of the rotation group $SO(k)$.

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Definition

The kinetic energy Riemannian metric on M is given as follows: Let $\xi = (u_\xi, S_\xi)\,, \zeta = (u_\zeta, S_\zeta)$ be tangent to M at (x, A); then

$$\langle \xi, \zeta \rangle = m \left\{ \frac{(r\gamma)^2}{2} \operatorname{Tr}\left(S_\xi S_\zeta^\top \right) + u_\xi \cdot u_\zeta \right\}$$

No-slip billiard setup

The collision map at a boundary point $q = (x, A)$ of M is a linear map $C_q : T_qM \to T_qM$ that sends vectors pointing out of M to vectors pointing into it and satisfies the no-slip conditions.

Definition

The no-slip collision map can now be expressed as follows: (2)
$C_q(u, S) = \left(c_\beta u - \frac{s_\beta}{\gamma}(u \cdot \mathfrak{n}(x))\mathfrak{n}(x) + s_\beta \gamma r S \mathfrak{n}(x), S + \frac{s_\beta}{\gamma r}\mathfrak{n}(x) \wedge [u - rS\mathfrak{n}(x)]\right).$

Here $\wedge$ stands for the cross-product $(a, b) \in \mathbb{R}^k \times \mathbb{R}^k \mapsto a \wedge b \in \mathfrak{so}(k)$, defined by $u \mapsto (a \wedge b)u = (a \cdot u)b - (b \cdot u)a$ and $\mathfrak{n}(x)$ is the inward pointing unit normal vector to the boundary of P at x.

- Write $W := \mathcal{S}(x)$ and denote by Π_x the orthogonal projection from $\mathbb{R}^k$ to the tangent space of the boundary of P at $x \in \partial P$.
- Elements of $\mathfrak{so}(k)$ may be decomposed as

$$\mathcal{S} = \Pi_x \delta \Pi_x + \mathfrak{n}(x) \wedge W$$

- Then the effect of C_q is to map

$$\Pi_x \delta \Pi_x \mapsto \Pi_x \delta \Pi_x, \quad \mathfrak{n}(x) \mapsto -\mathfrak{n}(x), \quad \begin{pmatrix} \bar{u} \\ W \end{pmatrix} \mapsto \begin{pmatrix} c_\beta I & s_\beta I \\ s_\beta I & -c_\beta I \end{pmatrix} \begin{pmatrix} \bar{u} \\ W \end{pmatrix},$$

No-slip billiard setup

- No-slip billiards can be dfeined as the system whose orbits in the interior of P consist of straight line segments with constant u and constant
- At the boundary the ball undergoes a change of velocities according to the collision map C_q.
- When the mass distribution of the ball is entirely concentrated at the centre, $\gamma = 0$ and the collision map reduces to a transformation that decouples linear and angular velocities: the centre velocity u transforms according to the standard billiard reflection, and the components of the angular velocity contained in W switch sign while the other components remain the same.
- Many of the concepts for no-slip billiards will have their counterparts for rolling systems.

Rolling systems setup

- The rolling ball is now $(k+1)$ dimensional while the no-slip billiard ball is k-dimensional.
- The unit vector $\mathfrak{n}(x)$ is still the inward pointing unit normal vector at a boundary point x. Let $U \in \mathfrak{so}(k+1)$ represent the angular velocity matrix of the rolling ball.
- Let $\Pi_x^n, x \in P$, be the orthogonal projection from $\mathbb{R}^{k+1}$ to $T_x n$ and Π_x is the orthogonal projection to the tangent space to ∂P at a boundary point x.
- The parameters γ and η are defined just as in the no-slip billiard setting, but now they are associated to the mass distribution of a higher dimensional ball.
- $$S = \Pi_x^n U \Pi_x^n (x \in P), \quad \delta = r\eta S, \quad W = \mathcal{S}\mathfrak{n}(x)(x \in \partial P).$$

Main Result

Theorem

In the limit as the radius of the rolling ball goes to 0 , the velocity components of the ball immediately before and immediately after rounding the edge of the flat plate P at a boundary point x, are related by the linear map

$$\Pi_x \delta \Pi_x \mapsto \Pi_x \delta \Pi_x, \quad \mathfrak{n}(x) \mapsto -\mathfrak{n}(x), \quad \begin{pmatrix} \bar{u} \\ W \end{pmatrix} \mapsto \begin{pmatrix} c & s \\ s & -c \end{pmatrix} \begin{pmatrix} \bar{u} \\ W \end{pmatrix}$$

where $s = \sin(\pi\eta)$ *and* $c = \cos(\pi\eta)$.