# Bin packing problems

## 23rd Belgian Mathematical Optimization Workshop

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The field is still very active:

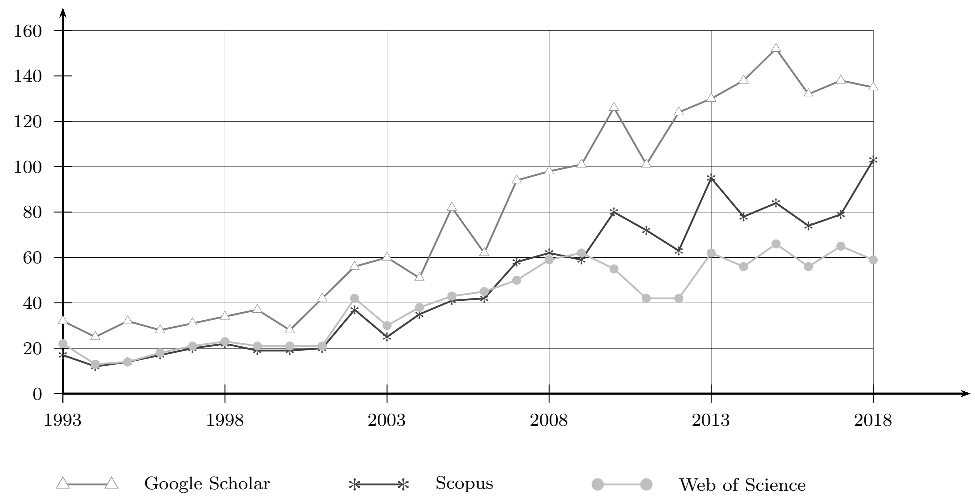


Figure 1: Number of papers dealing with bin packing and cutting stock problems, 1993-2018

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**Real world** applications in
packing trucks with a given weight limit,

assigning commercials to station breaks

allocating memory in computers,

subproblems in more complex optimization problems ...

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- *Integer Linear Program* (ILP) for the **BPP** (Martello and Toth, 1990)

$$\min \quad \sum_{i=1}^{u} y_i \tag{1}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} w_j x_{ij} \le c y_i \quad (i = 1, \ldots, u), \tag{2}$$

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The BPP (and hence the CSP) has been proved to be $\mathcal{NP}$-hard in the strong sense (Garey and Johnson, 1979: transformation from 3-Partition).

# Upper and lower bounds

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- We will normally refer to the **BPP** (unless otherwise specified).
- **Worst-case performance**
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  - **Off-line algorithms:** all items are known in advance, and are available for sorting, preprocessing, grouping, etc.
- Many other (less relevant) families:
  - semi on-line,
  - bounded space,
  - open-end,
  - conservative,
  - re-pack,
  - dynamic,
  - ...

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- Other algorithms: **Worse-Fit** (**WF**, leave the largest residual space), Any-Fit, Almost Any-Fit, Bounded space, Next-$k$-Fit, Harmonic-Fit, Refined First-Fit, Modified Harmonic-Fit, ...

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  In other words, we could solve PARTITION in polynomial time! $\square$

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  smallest real number $\overline{r}^{\infty}(A) > 1$ such that, for some positive integer $k$, $A(I)/OPT(I) \le \overline{r}^{\infty}(A)$ for all instances $I$ satisfying $OPT(I) \ge k$.
- $\overline{r}^{\infty}(FFD) = \overline{r}^{\infty}(BFD) = \frac{11}{9}$.
- Impressive number of results, of mostly theoretical relevance (see surveys).
- "History" of the $11/9$ ratio:
  - Johnson (1974): $FFD(I) \le \frac{11}{9} OPT(I)$ **+ 4** $\forall I$. **Proof: 100 pages**;
  - Baker (1985): $FFD(I) \le \frac{11}{9} OPT(I)$ **+ 3** $\forall I$. **Proof: 20 pages**;
  - Yue (1991): $FFD(I) \le \frac{11}{9} OPT(I)$ **+ 1** $\forall I$. **Proof: 10 pages**.

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**MFFD** (Modified FFD): Try to pack pairs of items with size in $(c/6, c/3]$ into bins containing a single item of size $> c/2$.

**B2F** (Best Two Fit): Fill one bin at a time, in greedy way; when no further item fits into the current bin, if the bin contains more than one item, try to replace the smallest item in the bin with a pair of unpacked items with size $\ge c/6$.

**CFB** (combined FFD–B2F): run both B2F and FFD and take the better packing.

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- Improved by Karmarkar and Karp, 1982: **Fully Polynomial-Time Approximation Scheme**; time complexity polynomial (linear) both in $n$ and $\frac{1}{\varepsilon}$.

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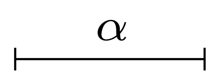


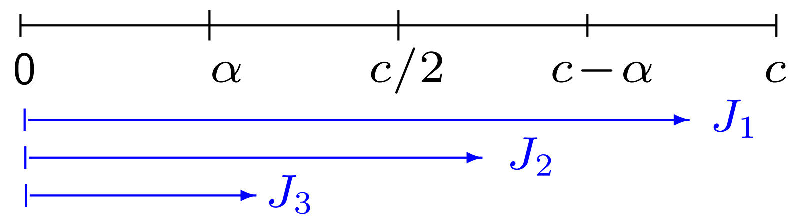

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  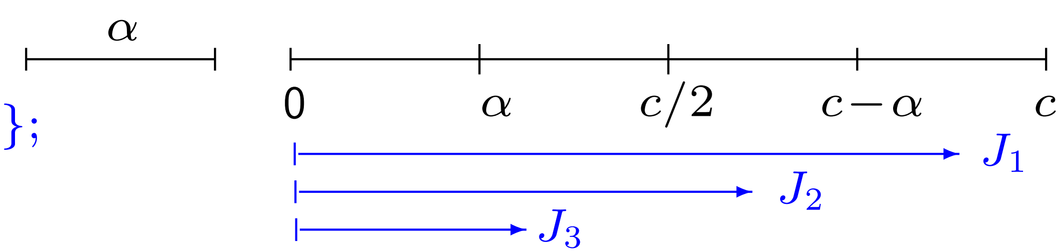
  

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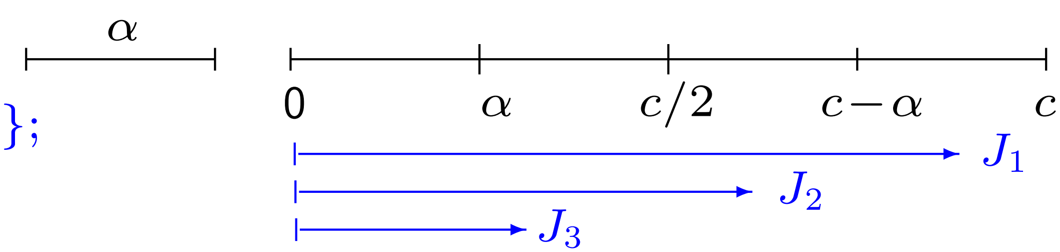


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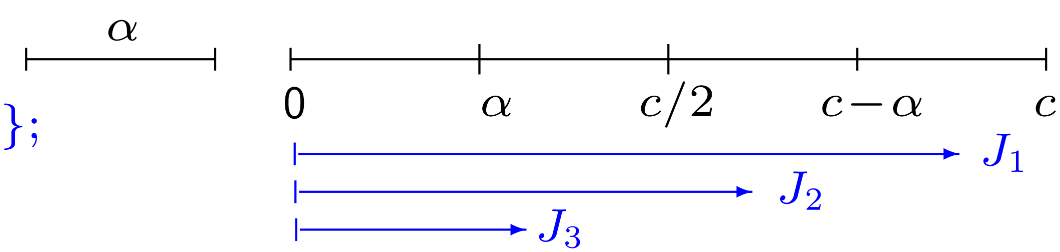


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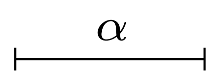


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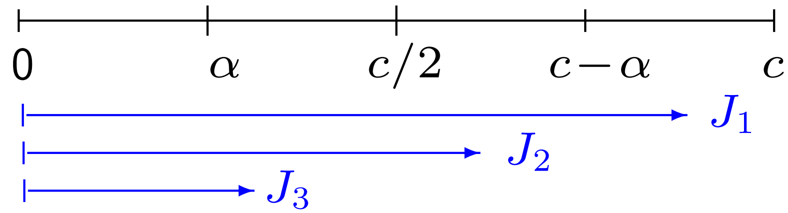


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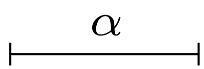


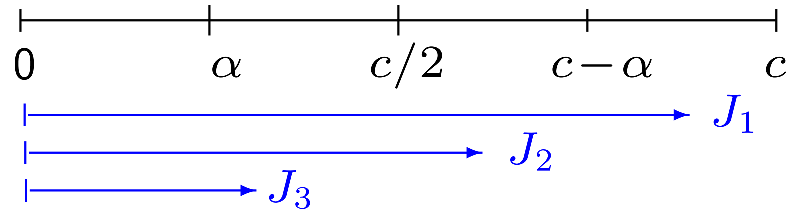


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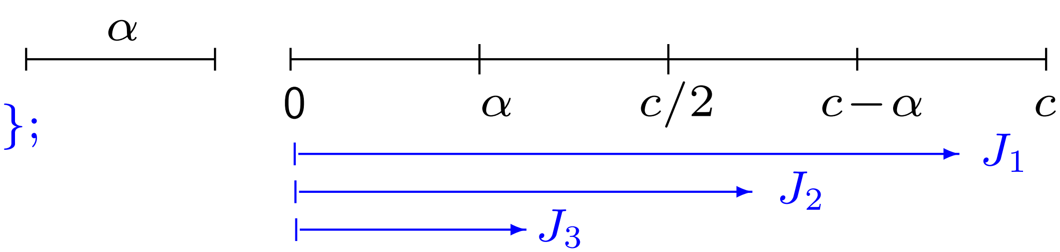


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  $L(\alpha) = |J_1| + |J_2| + \max\left(0, \left\lceil \frac{\sum_{j \in J_3} w_j - (|J_2| c - \sum_{j \in J_2} w_j)}{c} \right\rceil\right)$ is a valid lower bound.

  The **overall bound** $L_2 = \max\{L(\alpha) : 0 \leq \alpha \leq c/2, \alpha \text{ integer}\}$

  (1) can be computed in $O(n \log n)$ time;

  (2) has **WCPR equal to** $\frac{2}{3}$.

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- Methods to improve on a lower bound value (Dell'Amico and Martello, Alvim et al., Haouari and Gharbi, Jarboui et al.)

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- Ideas generalized to a
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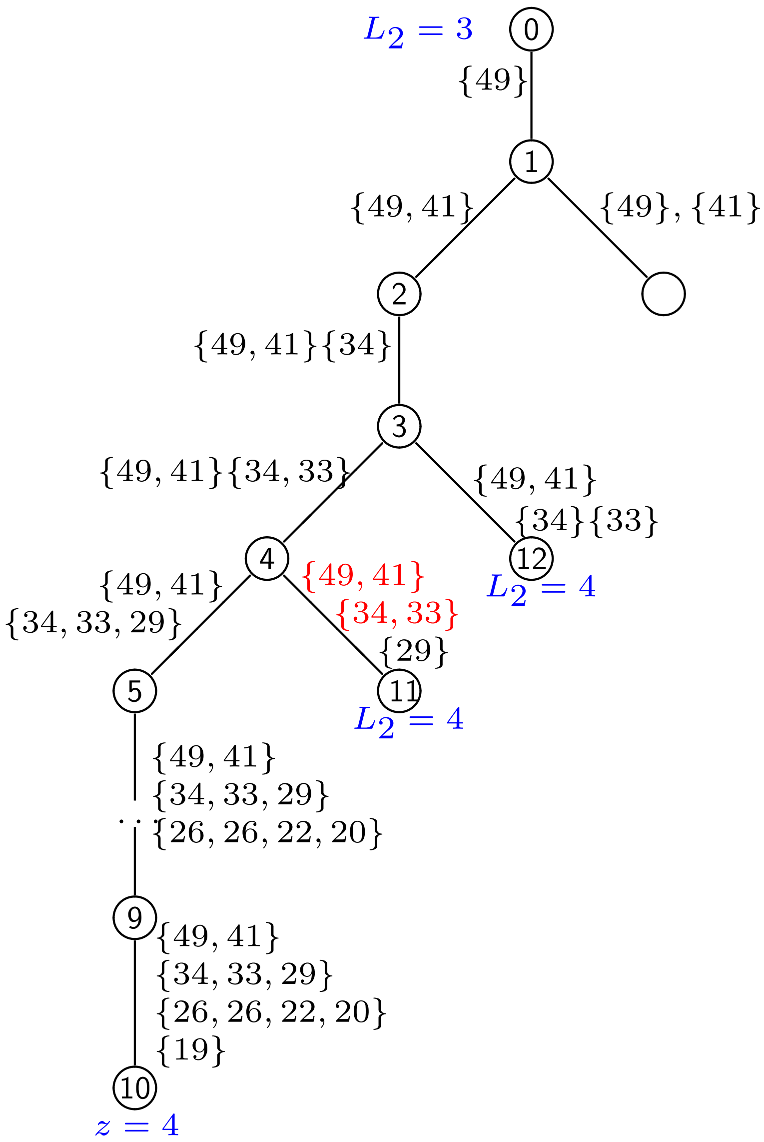

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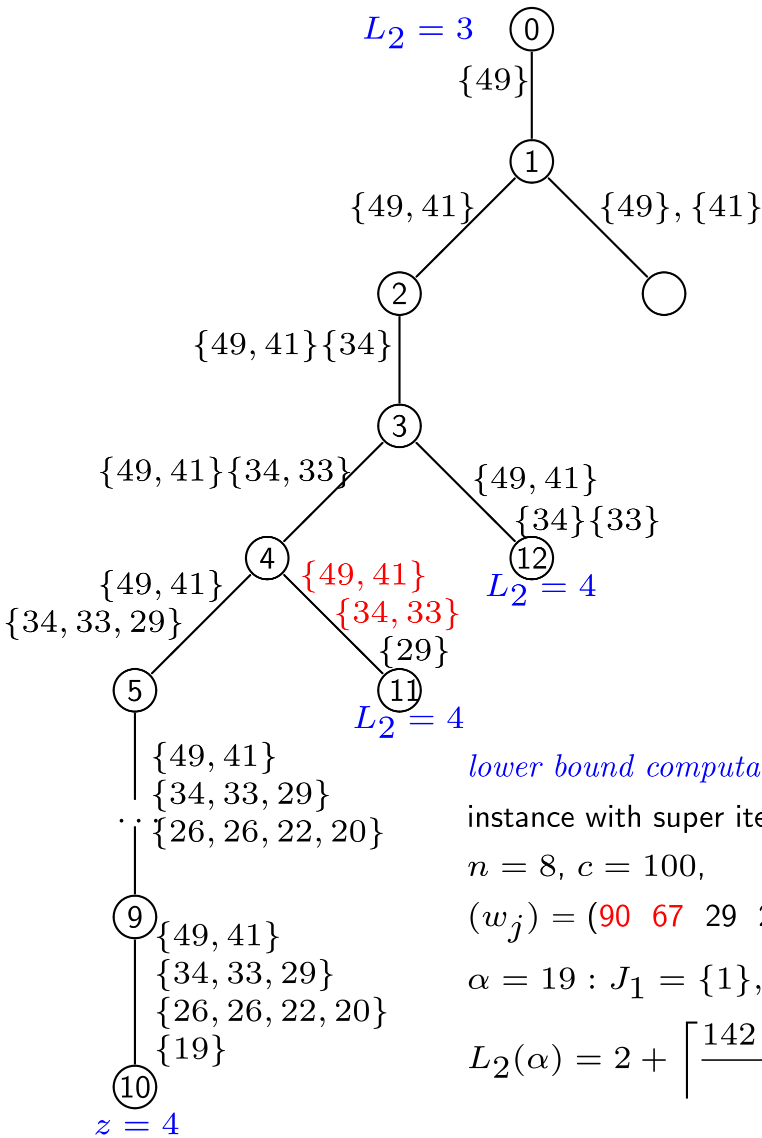


*lower bound computation at node 11*:

instance with super items:

$n = 8$, $c = 100$,

$(w_j) = (90 \;\; 67 \;\; 29 \;\; 26 \;\; 26 \;\; 22 \;\; 20 \;\; 19\ )$:

$\alpha = 19 : J_1 = \{1\}, J_2 = \{2\}, J_3 = \{3, \ldots, 8\};$

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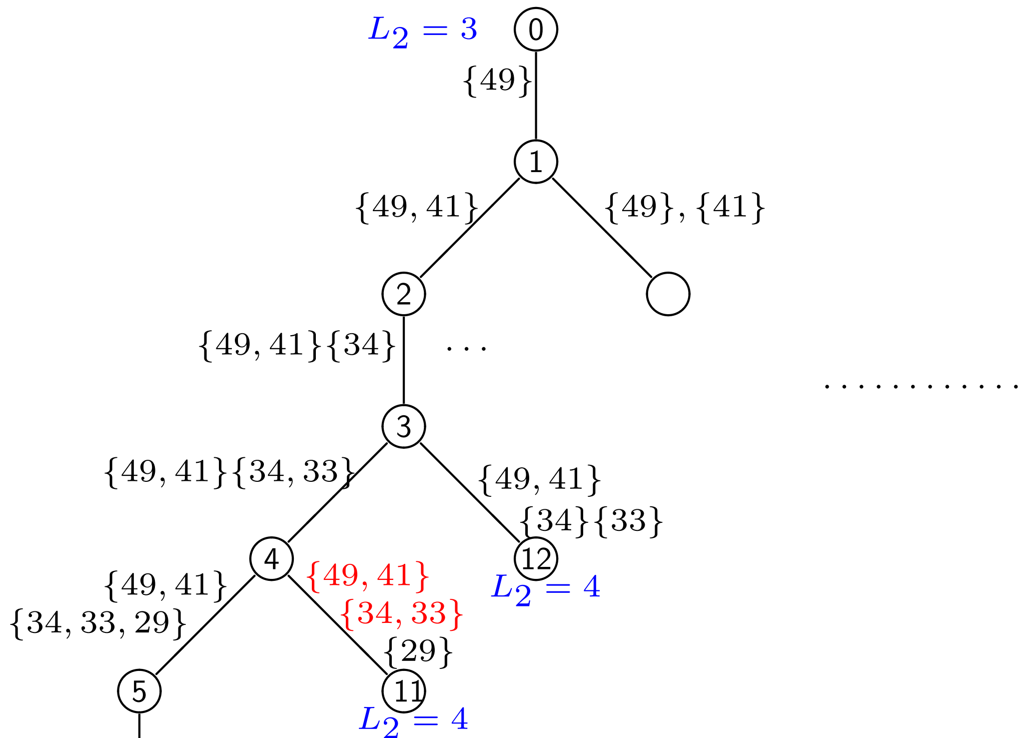


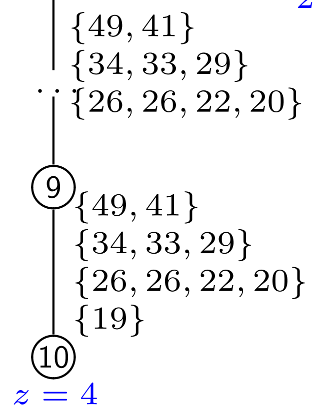


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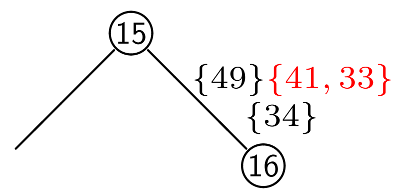


Reduction (super item 41+33=74):
$\{41, 33, 26\}$ removed from the instance
FFD on the residual instance:
$\{49, 29, 22\}$
$\{34, 26, 20, 19\}$
optimal

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- For the BPP and the CSP, all Branch-and-Price (-and-Cut) algorithms are based on the *set covering formulation* and the solution of its continuous relaxation through column generation (seminal work by Gilmore and Gomory).

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- The resulting ILP model has $O(mc)$ variables and $O(c)$ constraints.

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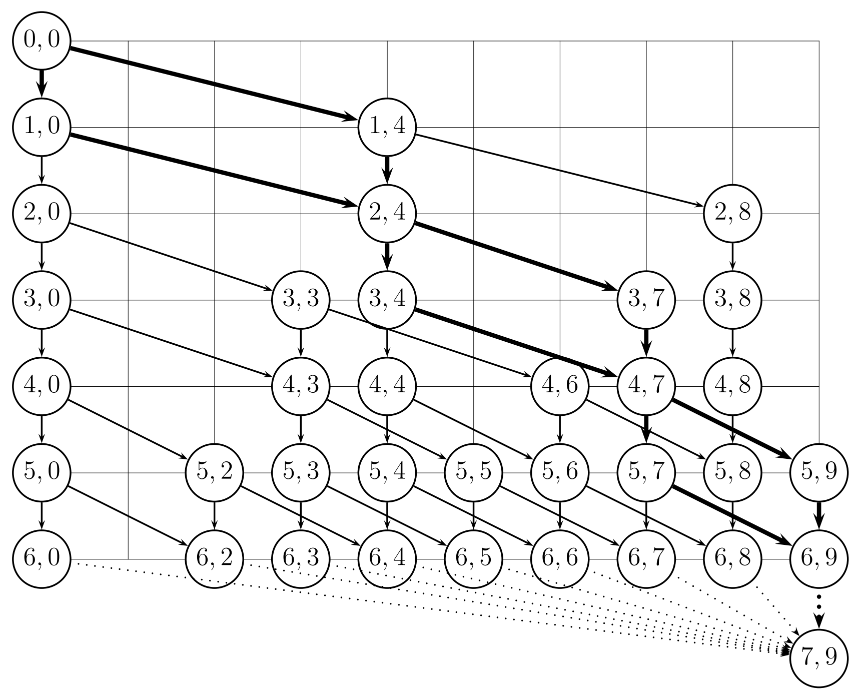
**Figure 1** DP-flow graph construction for Example 1



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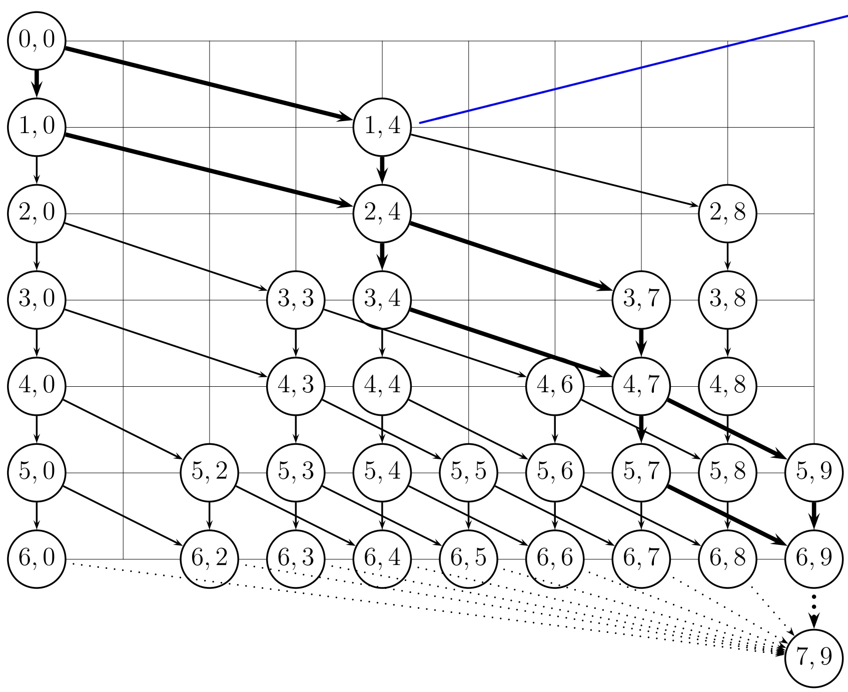

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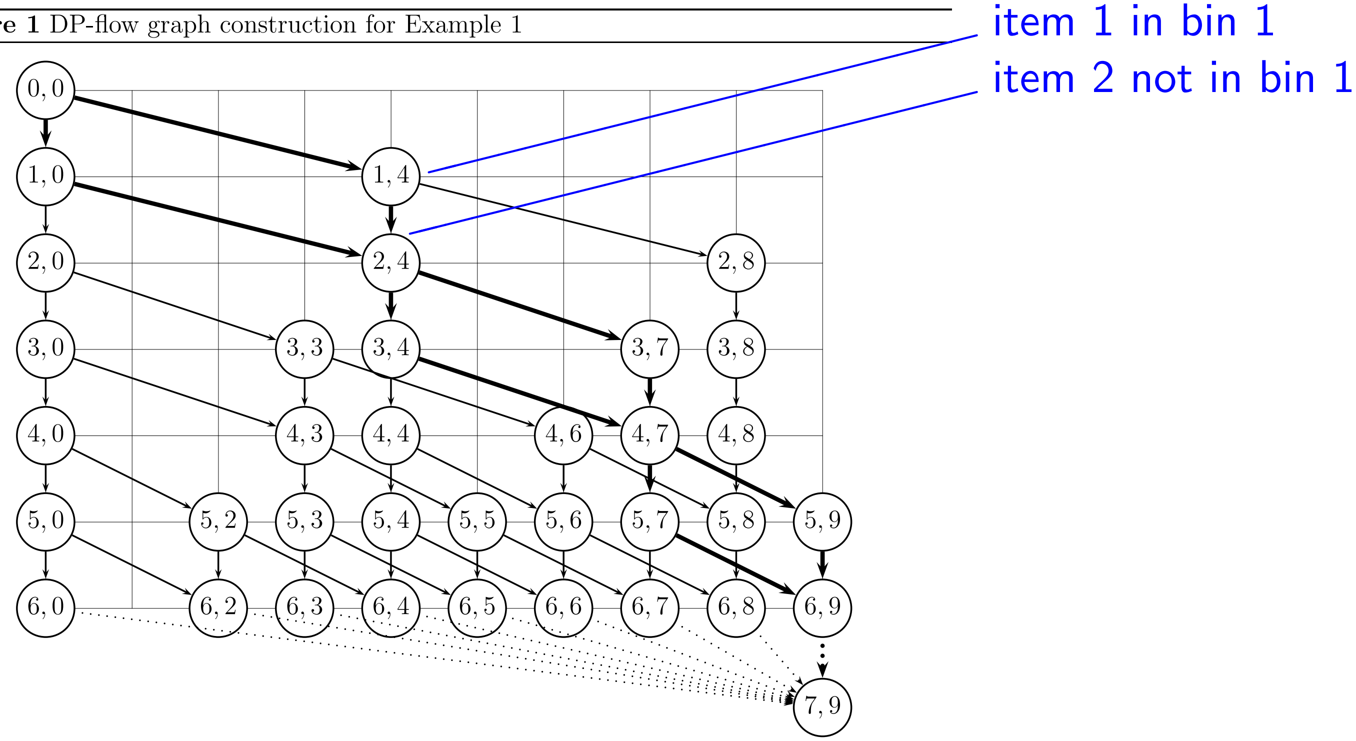

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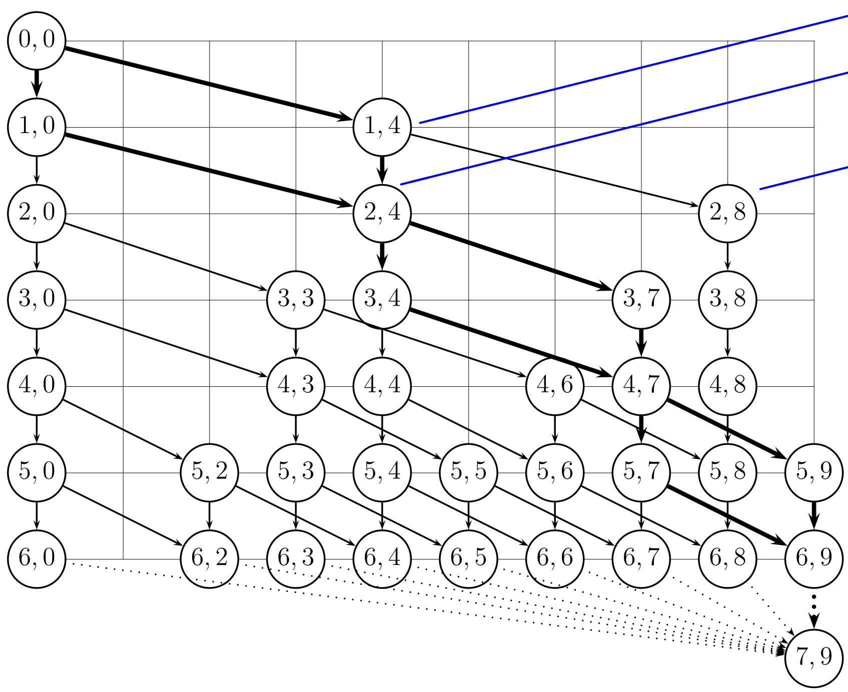

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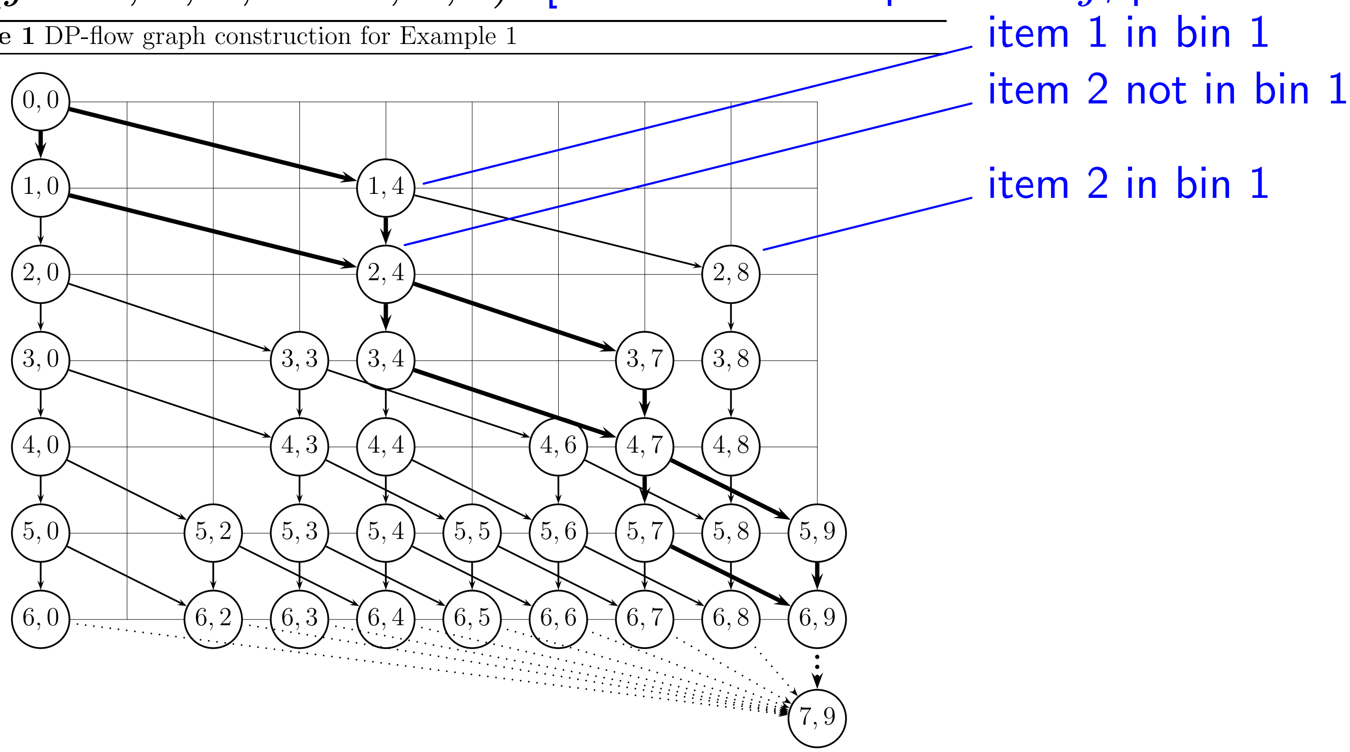


- Network Flow-type model to minimize the number of paths. $O(nc)$ variables and constraints.

# Arc-flow formulation

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- Valério de Carvalho (1999) anticipated DP-flow (but Wolsey (1977) anticipated everybody): vertically shrunk the DP graph: states with the same partial bin filling $\longrightarrow$ single state:

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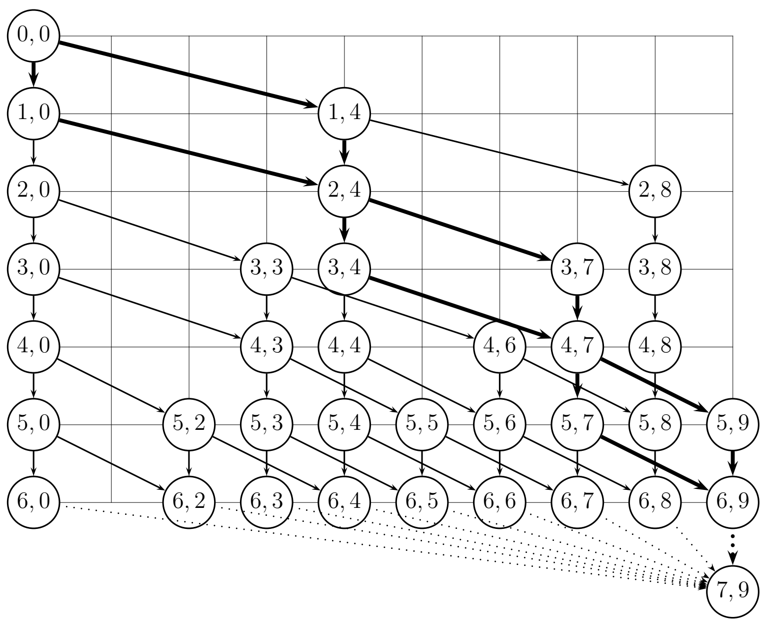

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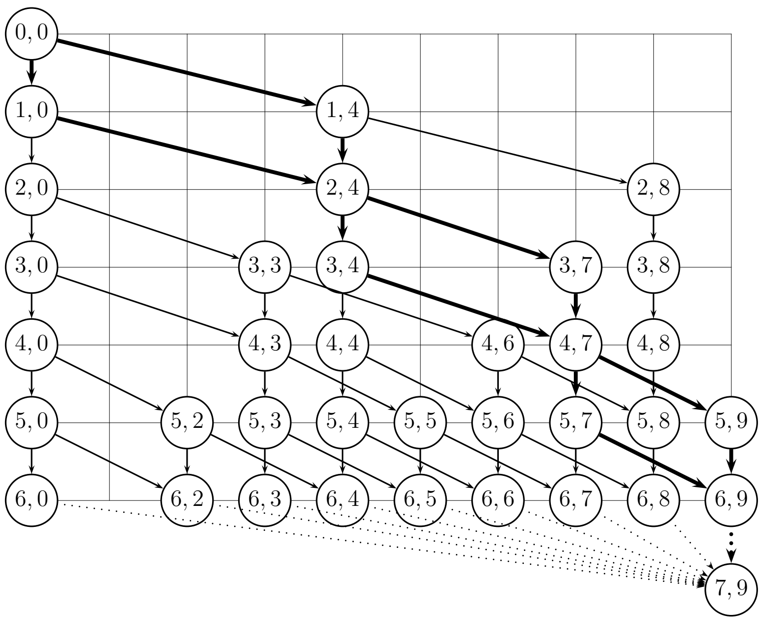
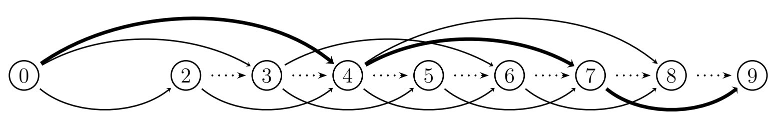



**Figure 2** Arc-flow representation of the graph of Figure 1



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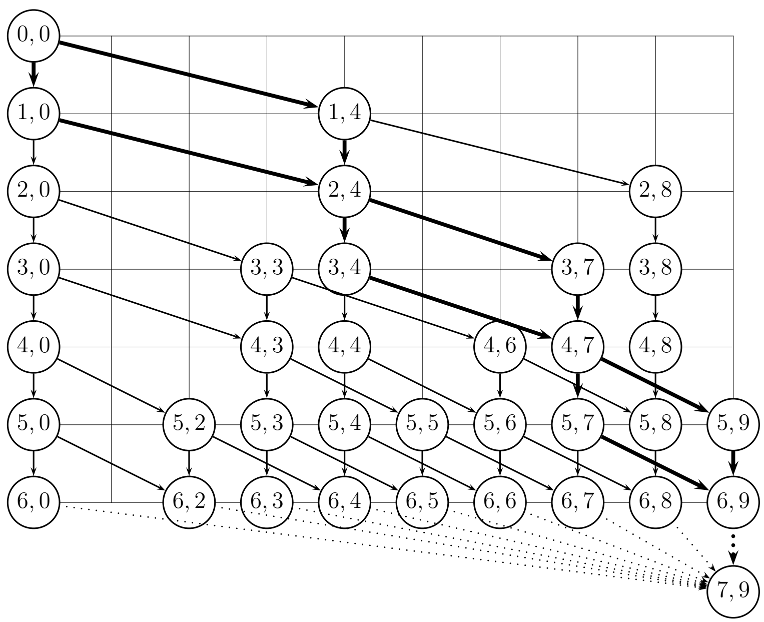


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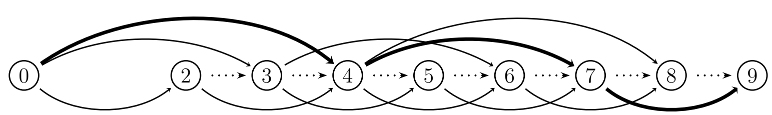


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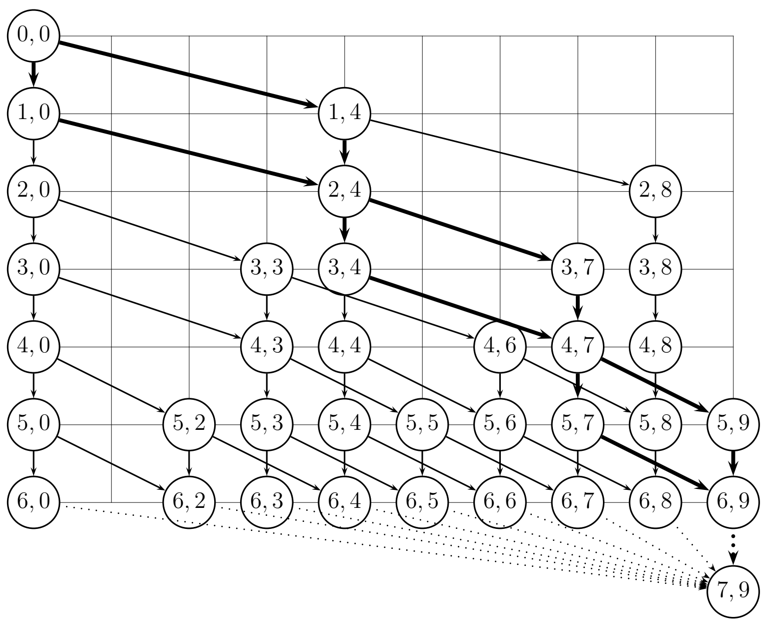


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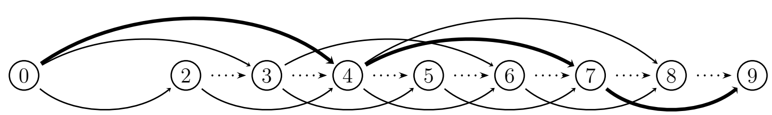


- **CSP** modeled as a network flow problem;
- Brandão and Pedroso (2016): alternative arc-flow formulation, very effective code VPSOLVER.

# Computer codes and the BPPLIB

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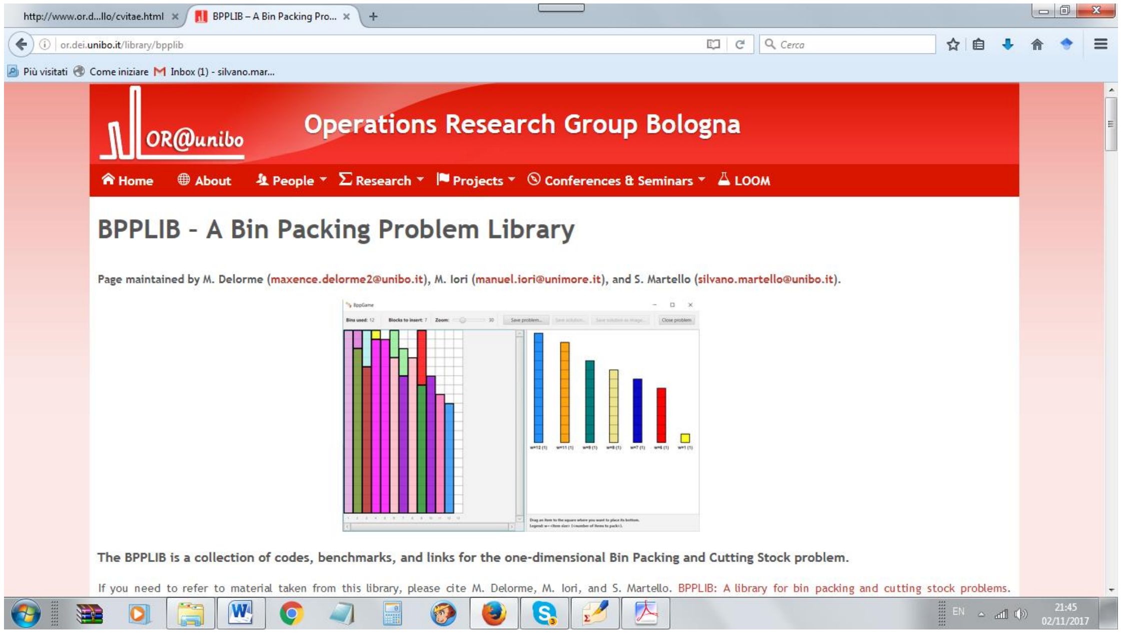

# Computer codes and the BPPLIB


You will find (from this talk):
MTP (branch-and-bound, Fortran)
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Other codes, benchmarks, links, BibTeX file, interactive visual solver.

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Number of **literature** instances solved in less than **10 minutes**

| Set | # inst. | BISON | BELOV | ARCFLOW | VPSOLVER |
|---|---|---|---|---|---|
| Falkenauer U | 74 | 50 | **74** | **74** | **74** |
| Falkenauer T | 80 | 47 | **80** | **80** | **80** |
| Scholl 1 | 323 | 290 | **323** | **323** | **323** |
| Scholl 2 | 244 | 234 | **244** | 231 | 242 |
| Scholl 3 | 10 | 3 | **10** | 0 | **10** |
| Wäscher | 17 | 10 | **17** | 4 | 13 |
| Schwerin 1 | 100 | **100** | **100** | **100** | **100** |
| Schwerin 2 | 100 | 63 | **100** | **100** | **100** |
| Hard28 | 28 | 0 | **28** | 26 | 26 |
| Total | 976 | 797 | **976** | 938 | 968 |

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## Number of **random** instances solved in less than **10 minutes**

| $n$ | # inst. | BISON | BELOV | ARCFLOW | VPSOLVER |
|---|---|---|---|---|---|
| 50 | 165 | **165** | **165** | **165** | **165** |
| 100 | 271 | 261 | **271** | **271** | **271** |
| 200 | 359 | 299 | **359** | **359** | **359** |
| 300 | 393 | 269 | **393** | **393** | **393** |
| 400 | 425 | 250 | **425** | **425** | **425** |
| 500 | 414 | 212 | **414** | **414** | **414** |
| 750 | 433 | 217 | **433** | 431 | **433** |
| 1000 | 441 | 200 | **441** | 434 | **441** |
| Total | 2901 | 1873 | **2901** | 2892 | **2901** |

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| $n$ | $\overline{c}$ | BISON | BELOV | ARCFLOW | VPSOLVER |
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| 201 | 2500 | 0 (1.0) | **50** (0.0) | 16 (0.7) | 47 (0.1) |
| 402 | 10000 | 0 (1.0) | 1 (1.0) | 0 (1.0) | 6 (0.9) |
| 600 | 20000 | - | (1.0) | - | 0 (1.0) |
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| Time | inst. | 6.0 (1998) | 7.0 (1999) | 8.0 (2002) | 9.0 (2003) | 10.0 (2006) | 11.0 (2007) | 12.1 (2009) | 12.6.0 (2013) |
|---|---|---|---|---|---|---|---|---|---|
| 10 minutes | 20 | 13 [366] | 10 [420] | 5 [570] | 17 [268] | 19 [162] | **20** [65] | 19 [117] | **20** [114] |
| 60 minutes | 20 | 16 [897] | 15 [1210] | 15 [2009] | **20** [343] | **20** [186] | **20** [65] | 19 [267] | **20** [114] |

# Two-Dimensional Packing Problems: Definitions

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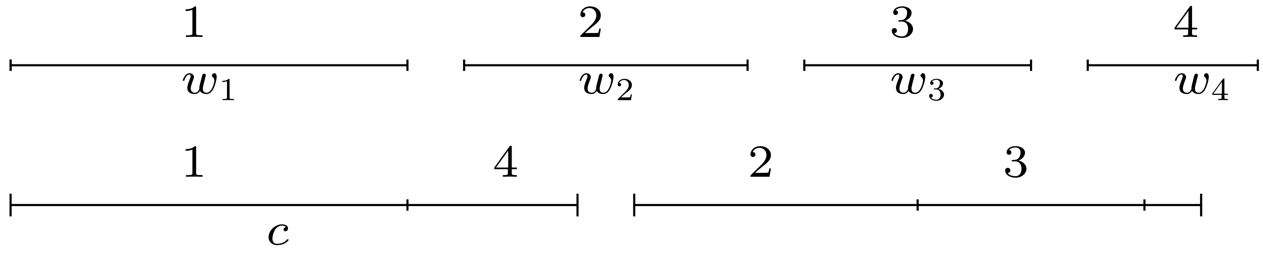
  pack a set of segments (*items*) into the minimum number of identical large segments (*bins*):



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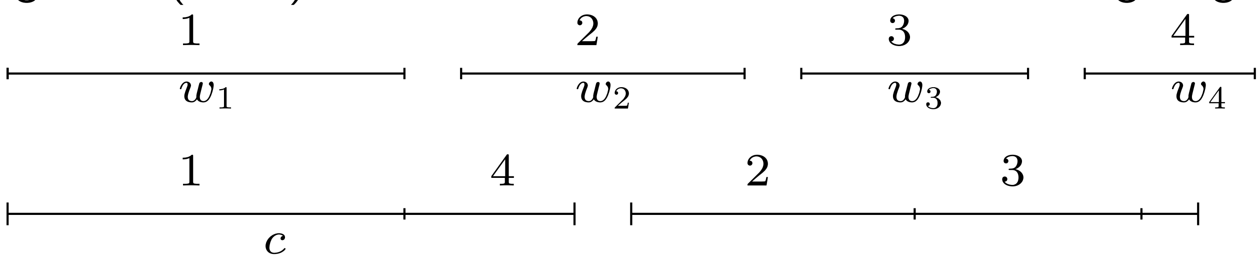


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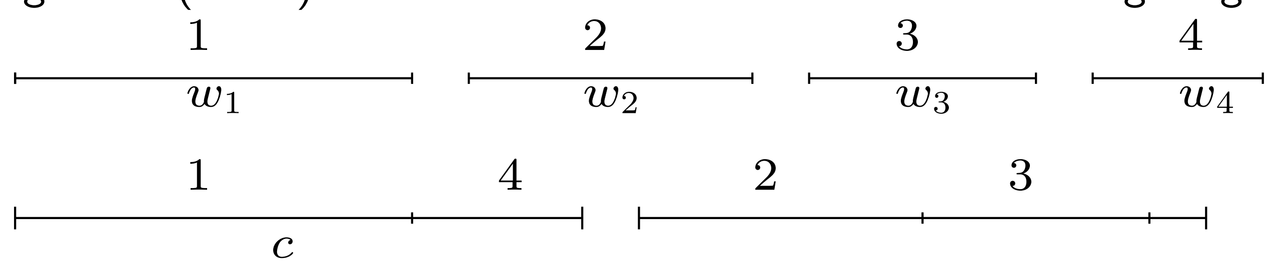


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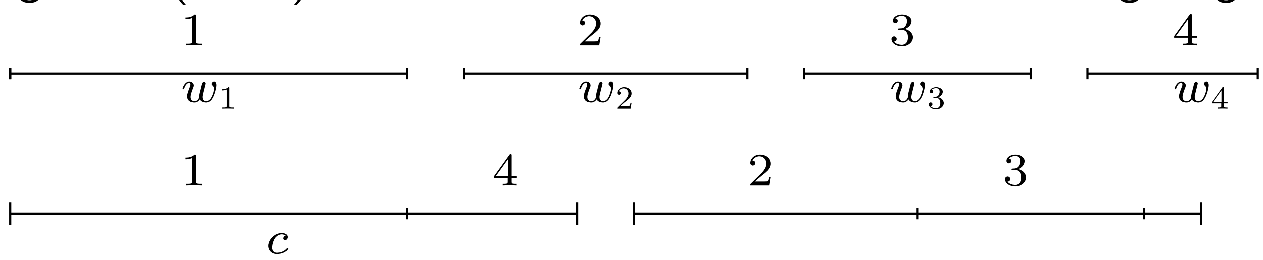


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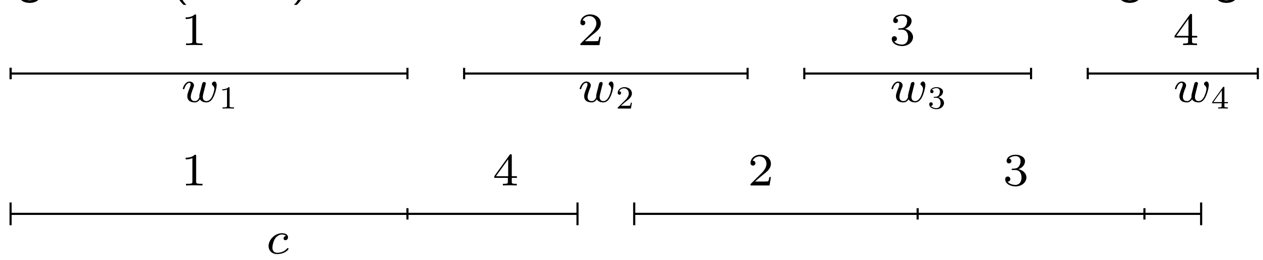


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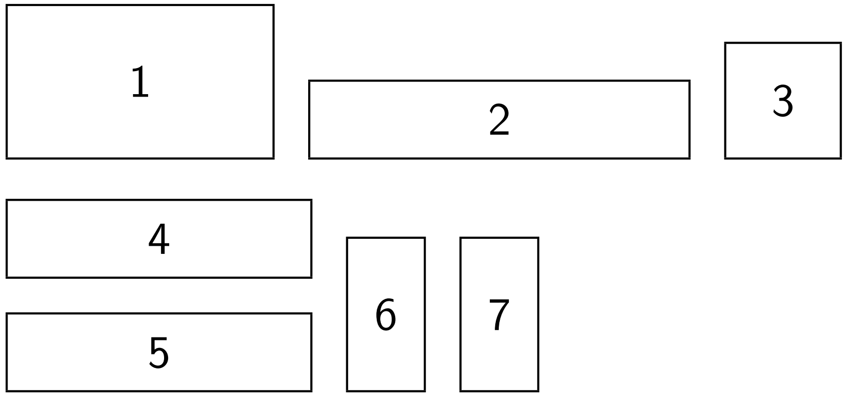
such that the height to which the strip is filled is minimized.
(Also called **1.5-dimensional packing**.)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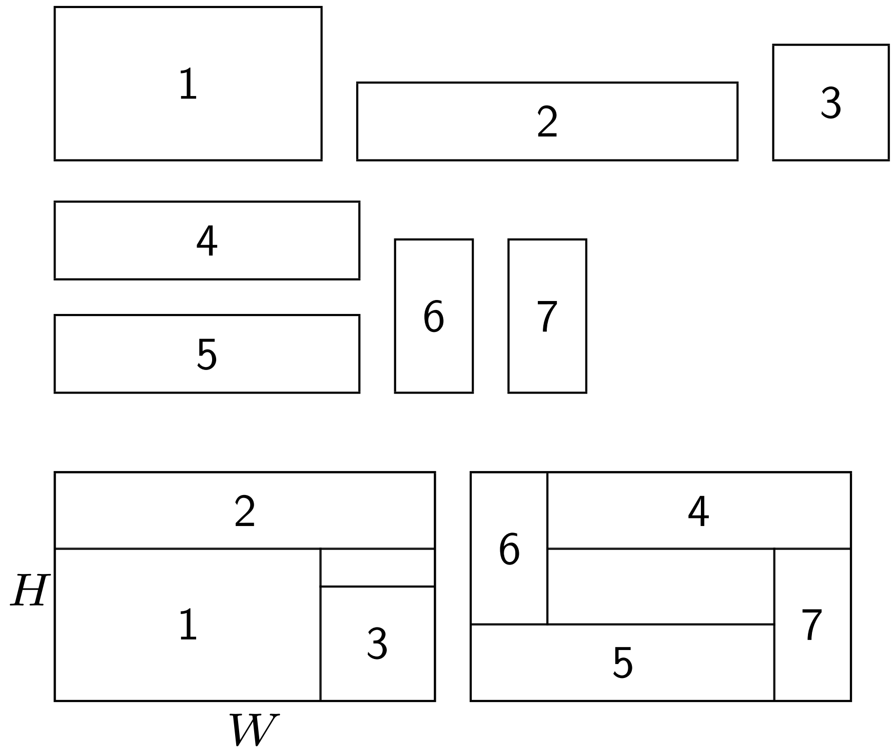
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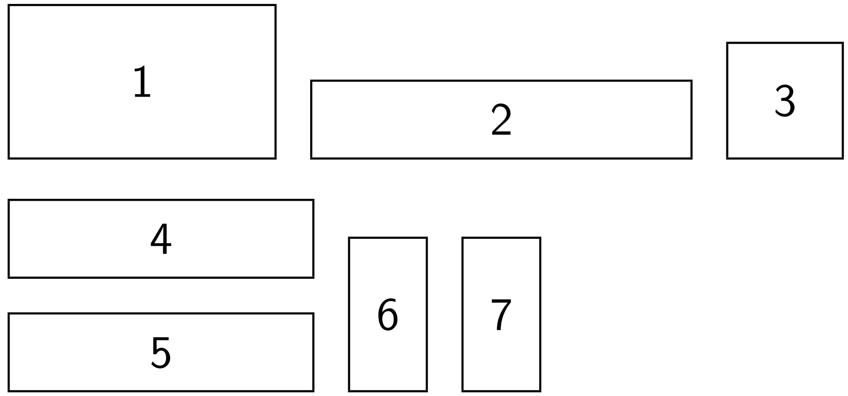
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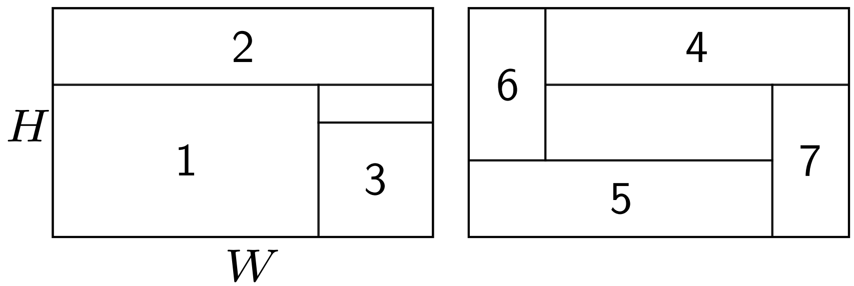
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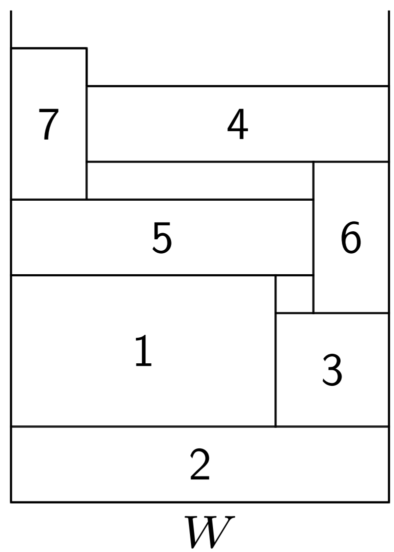
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## Complexity

- Both the 2BPP and the 2SPP are special cases of the BPP;
- both are **strongly $\mathcal{NP}$-hard**.

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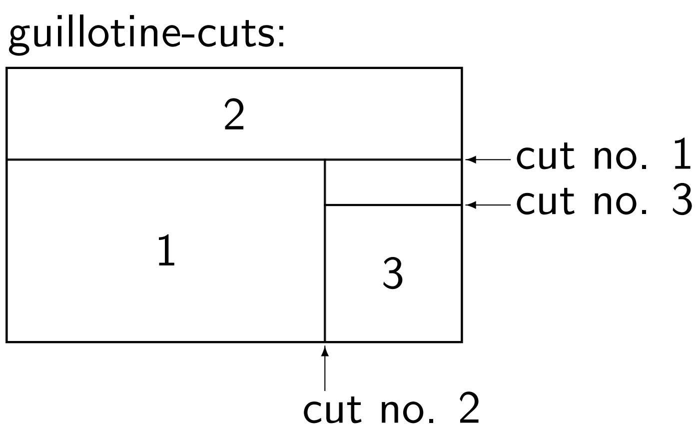

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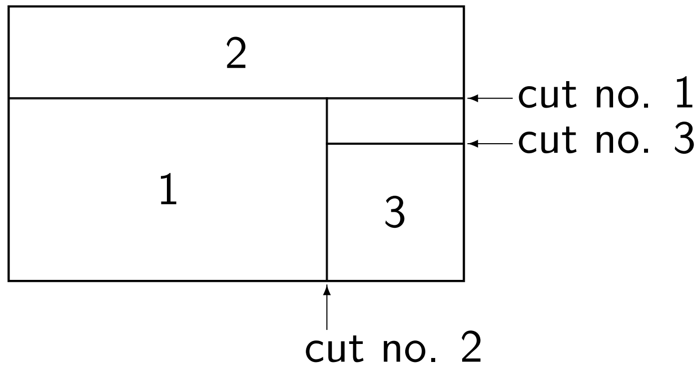
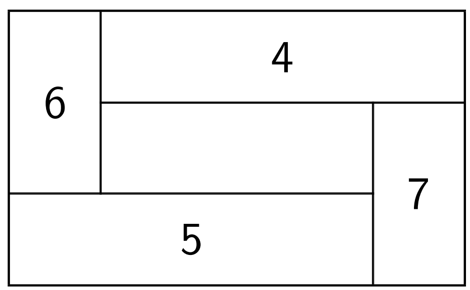



non guillotine-cuts:



## Variants

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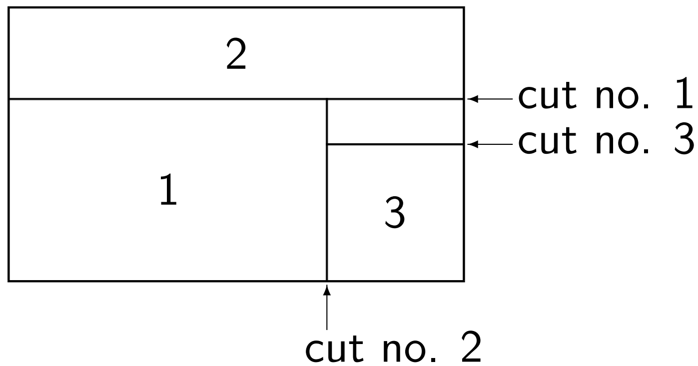


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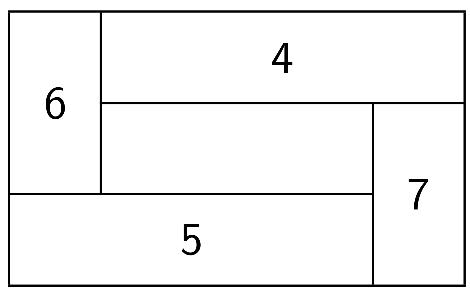


- additional constraints: limit on the number of cuts per bin (2,3).

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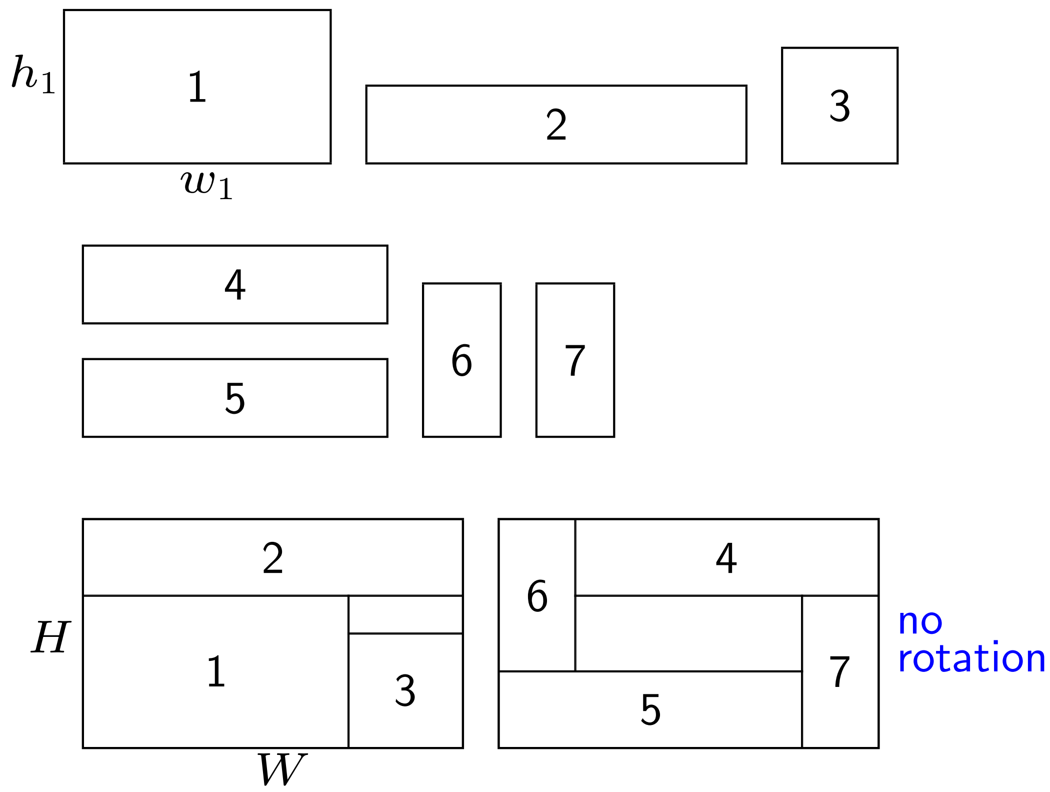

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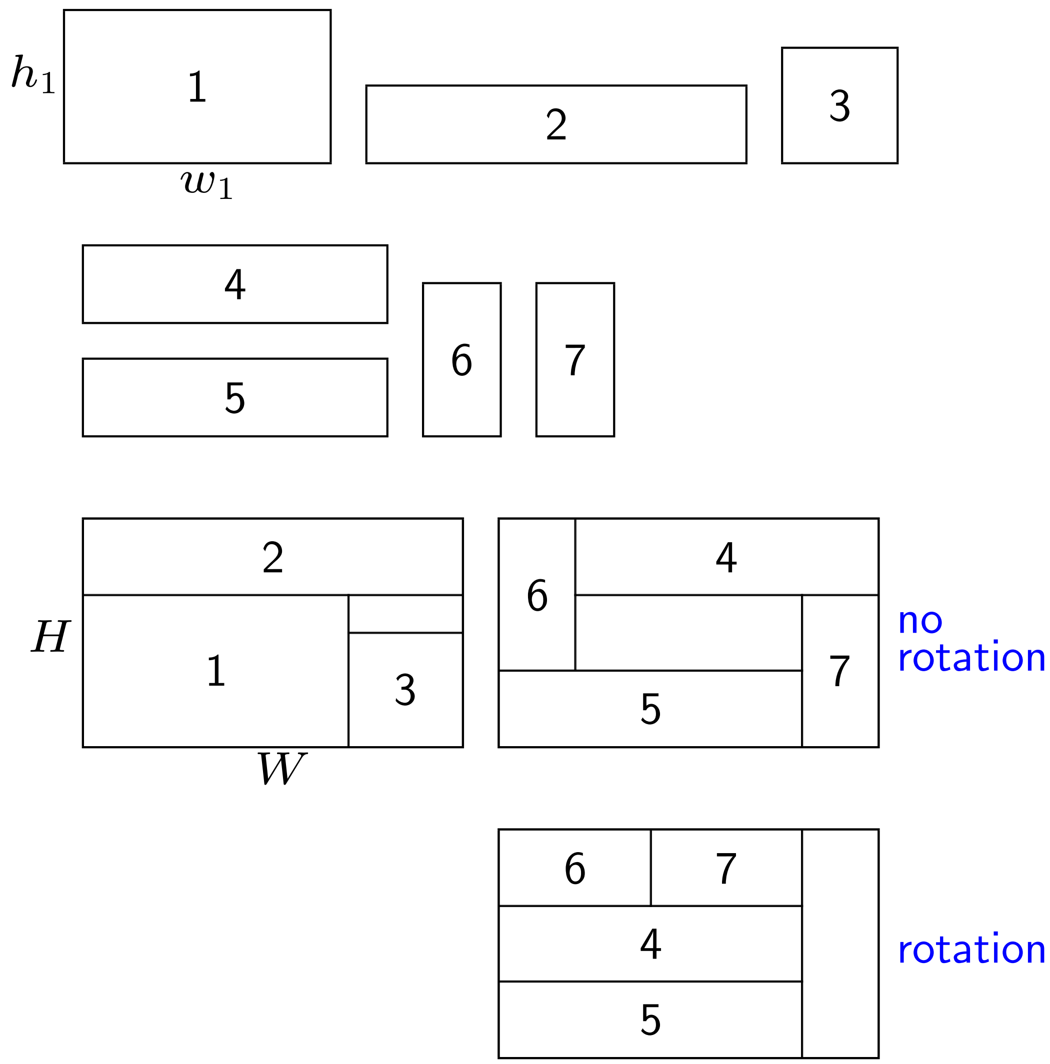
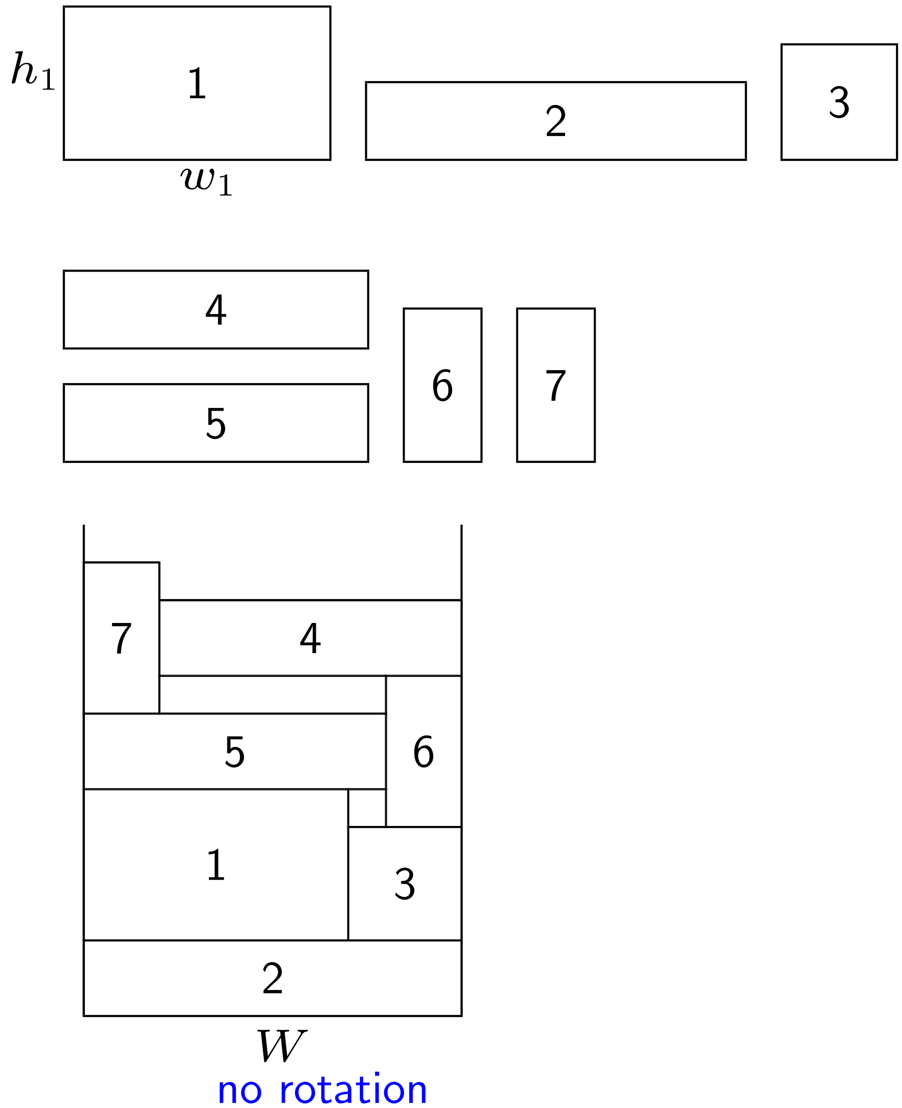
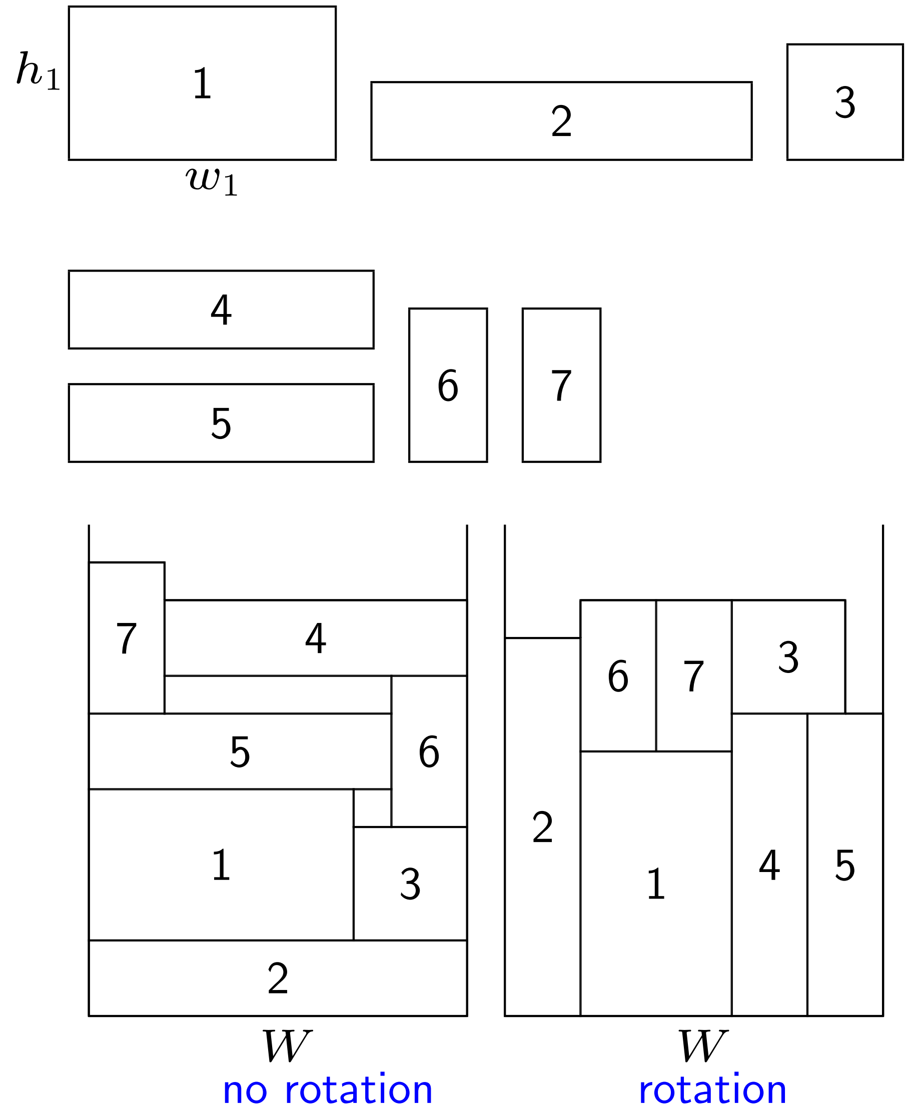



$h_1$
1
$w_1$
2
3
4
5
6
7
7
4
5
6
1
3
2
$W$
no rotation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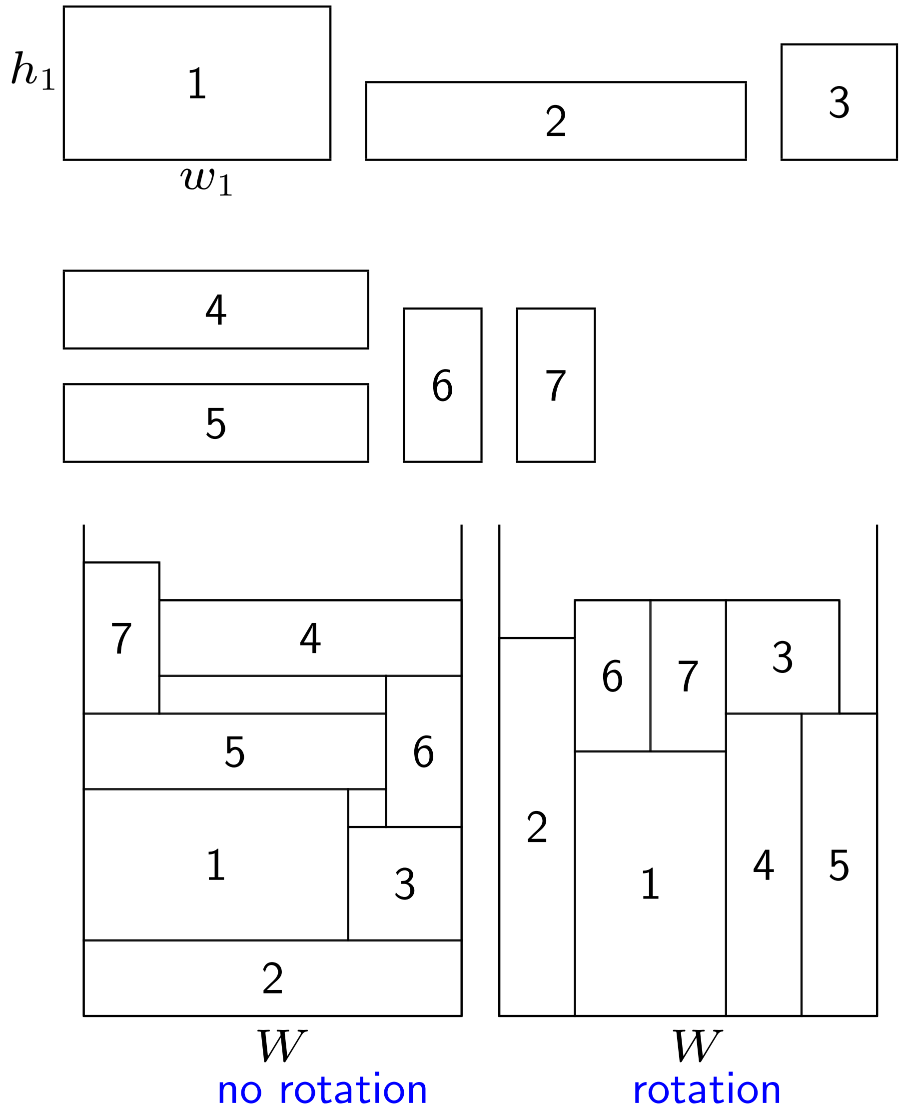


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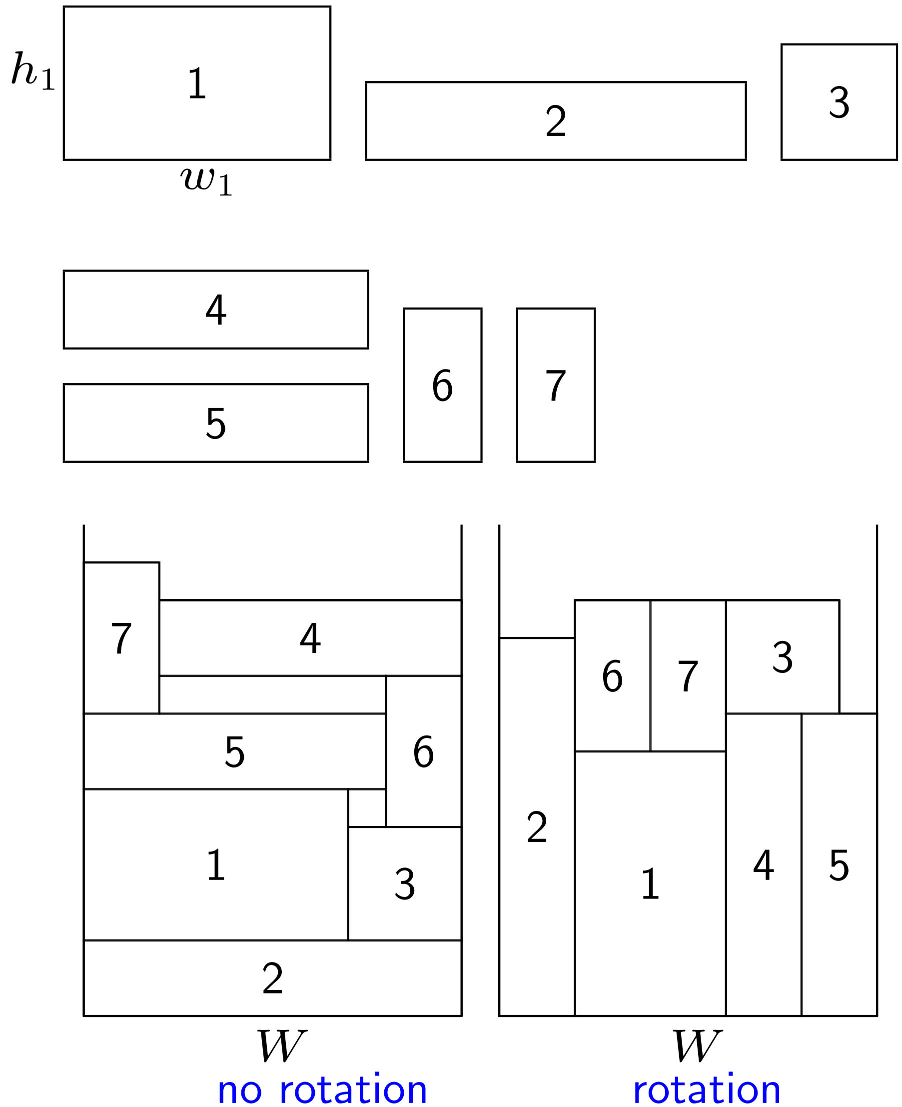


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- For two-dimensional bin (strip) packing problems most results concern the case: **no guillotine-cut required**, **no rotation allowed** (implicitly assumed in the following).

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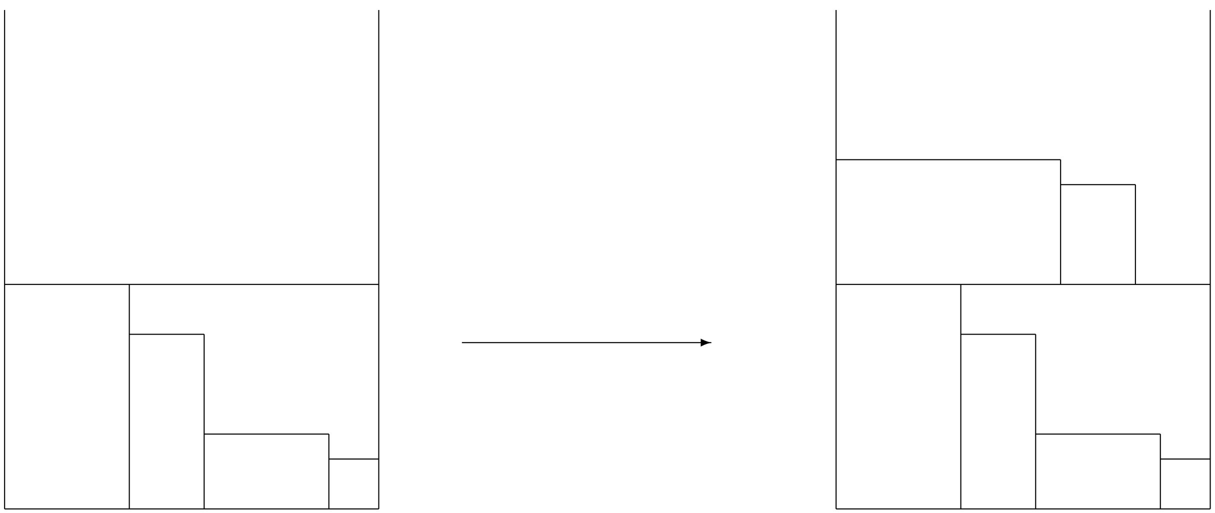

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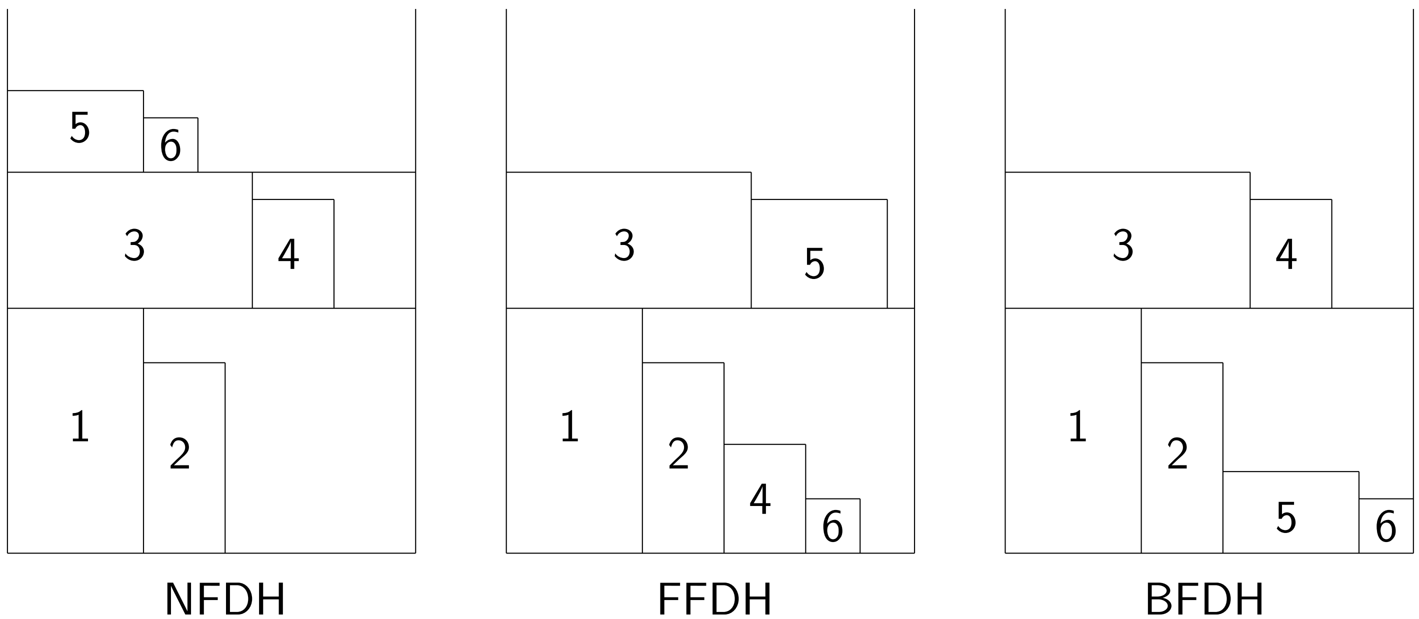

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- Both algorithms can be implemented so as to require $O(n \log n)$ time, through the appropriate data structures used for the 1BPP.

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- **Hybrid First-Fit (HFF,** Chung, Garey, and Johnson, 1982):
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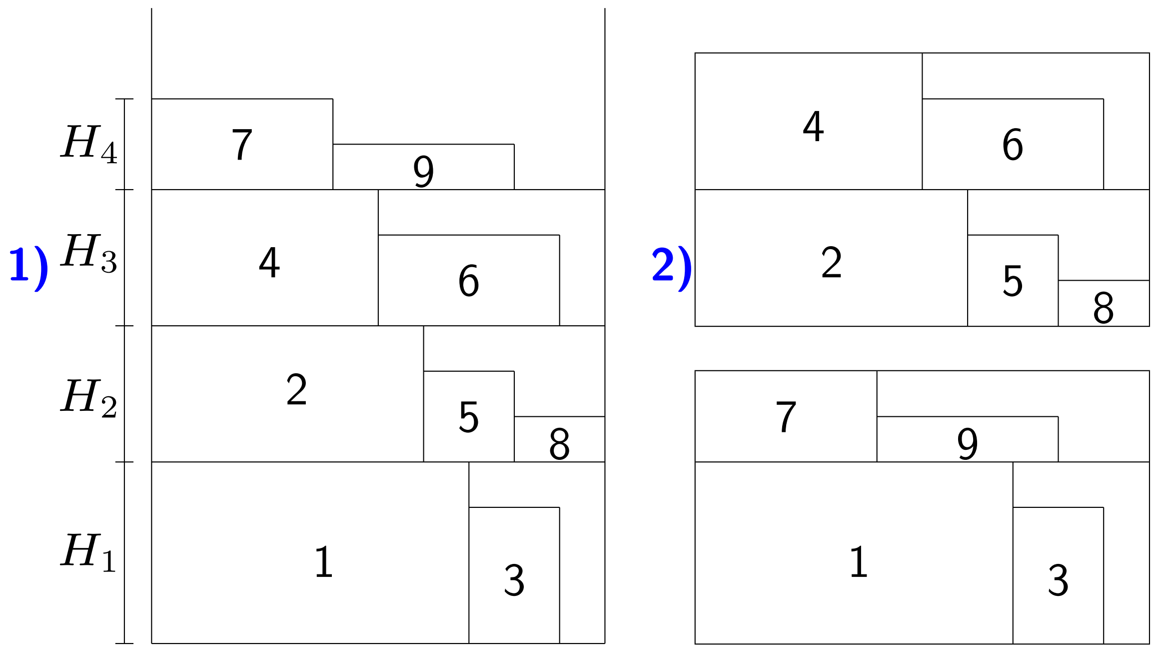

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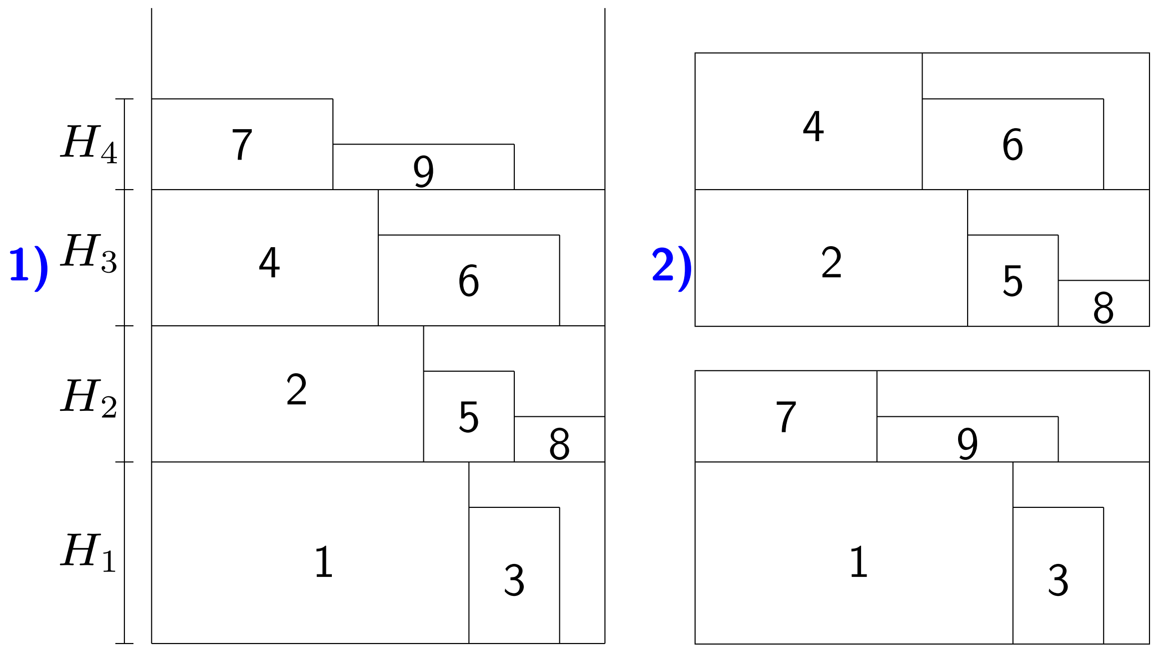


- If the heights are **normalized to 1**, $HFF(I) \leq \frac{17}{8} \cdot OPT(I) + 5 \;\; \forall \; I$

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  - pack on the shelf floor (left to right) and with the top edge on the ceiling (right to left).
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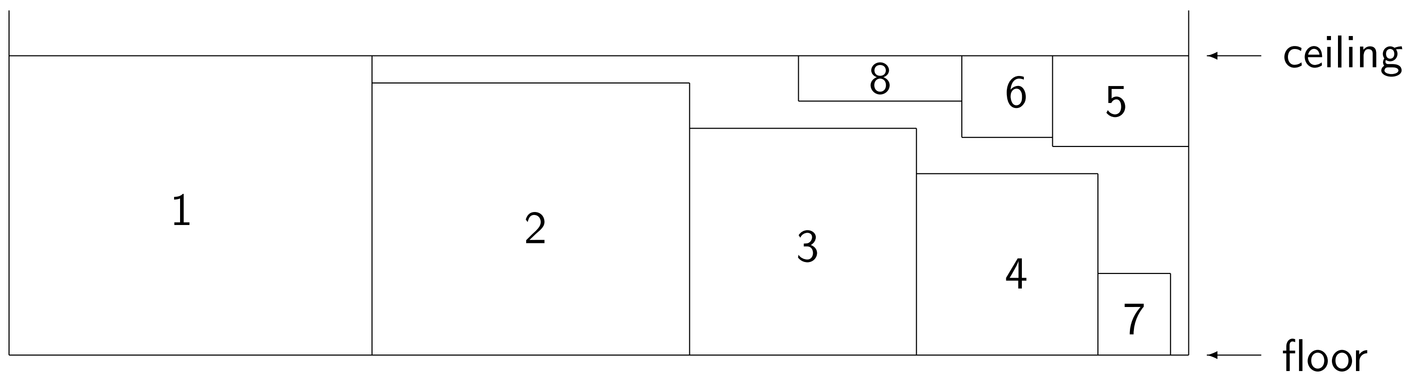

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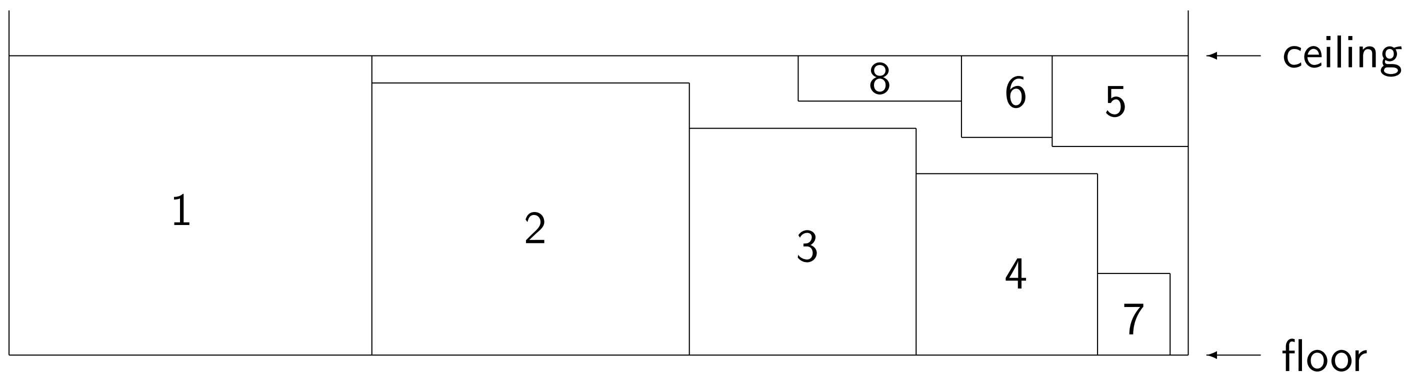


- **Knapsack packing** (Lodi, Martello, and Vigo, 1999): optimize the packing on the shelves by solving associated knapsack problems ($\mathcal{NP}$-hard).

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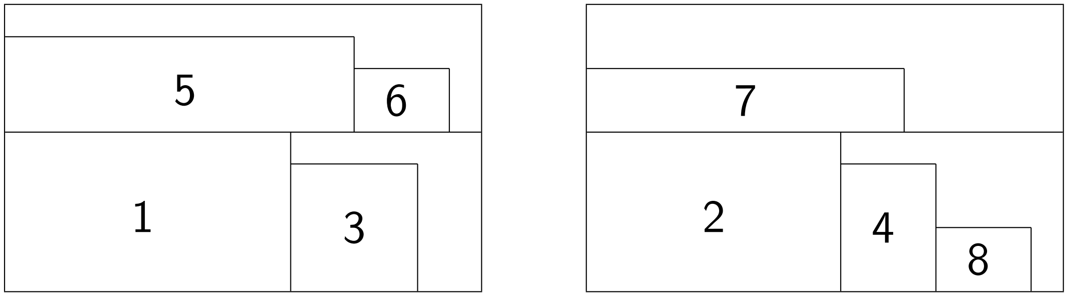

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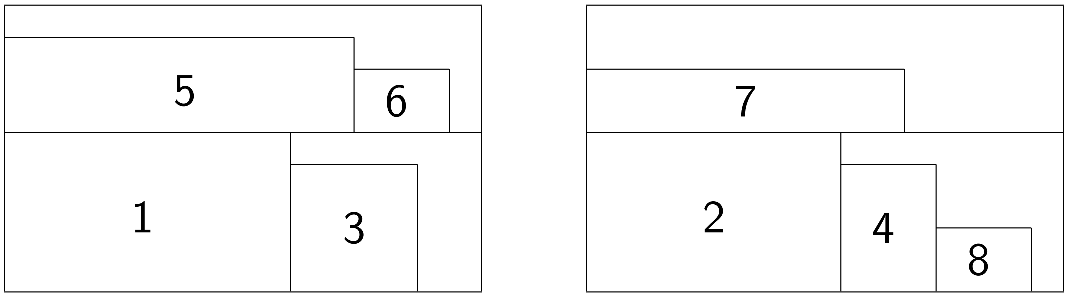


- Both $O(n \log n)$ time (Berkey and Wang, 1982).

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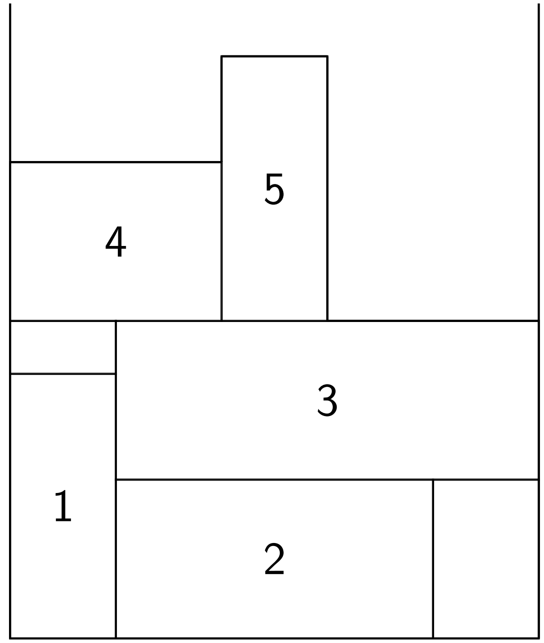

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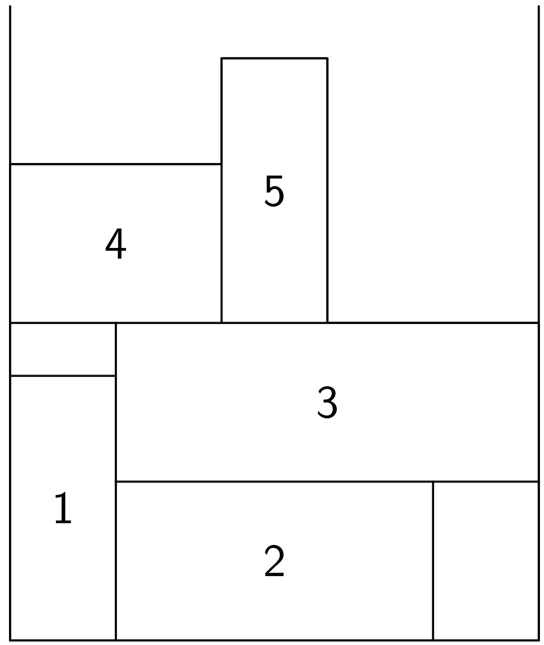


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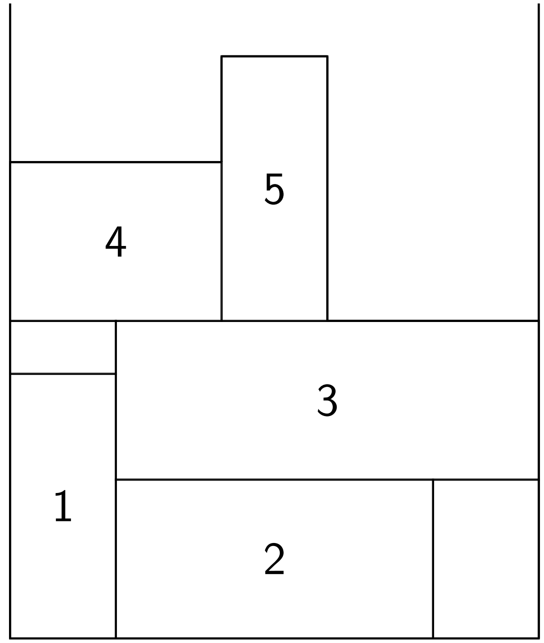


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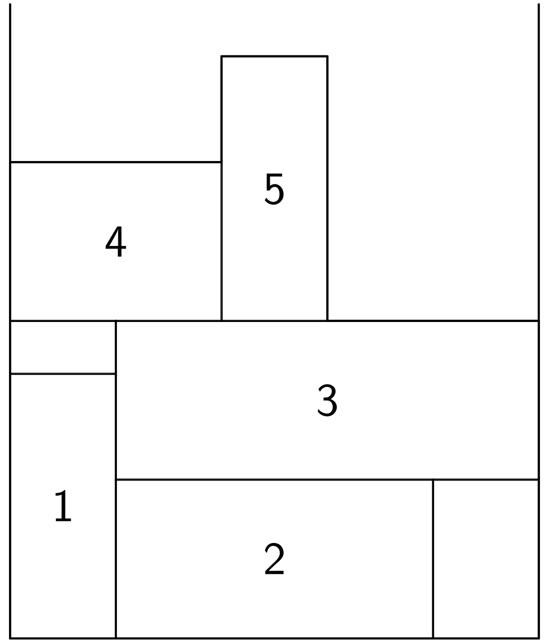
- Worst-case performance for the 2SPP (Baker, Coffman, and Rivest, 1980):
  - if no item ordering is used, then BL may be arbitrarily bad;

# One-phase algorithms (2BPP and 2SPP)

- Main non-shelf strategy:
  **Bottom-Left (BL)**: pack the current item in the lowest possible position, left justified.




- Complicated $O(n^2)$ time implementation (Chazelle).
- Worst-case performance for the 2SPP (Baker, Coffman, and Rivest, 1980):
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$$BL(I) \leq 3 \cdot OPT(I) \ \ \forall \ I \text{ (tight)}$$

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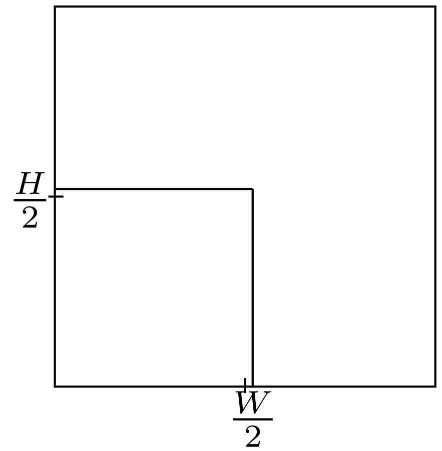

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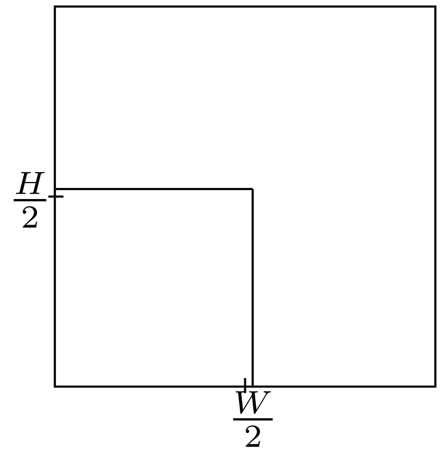


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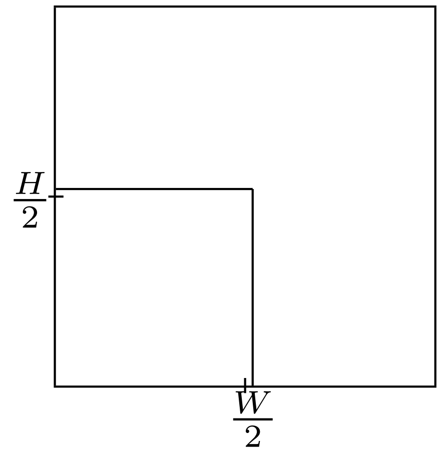


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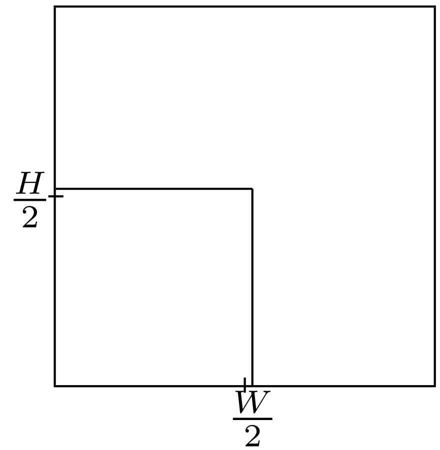


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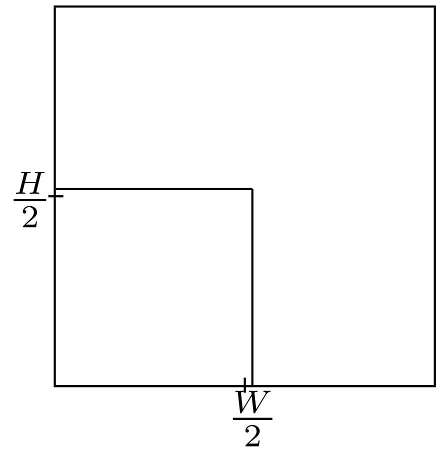


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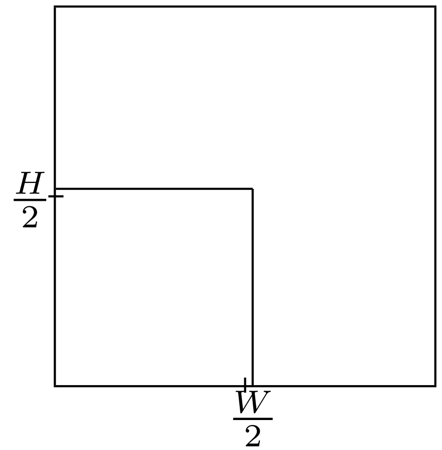


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- Other lower bounds derived from the from the (one-dimensional) BPP.

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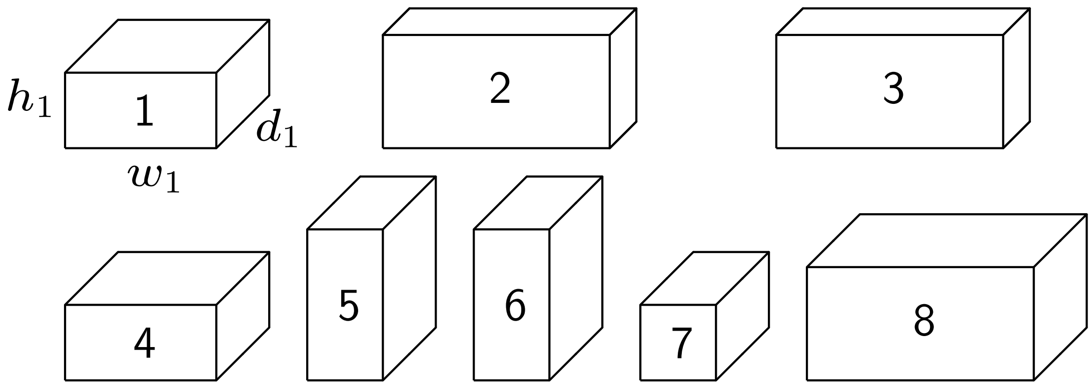

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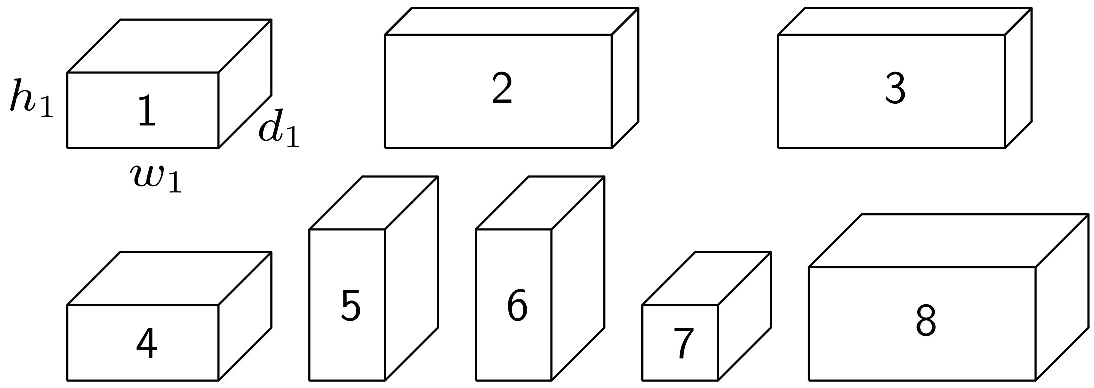


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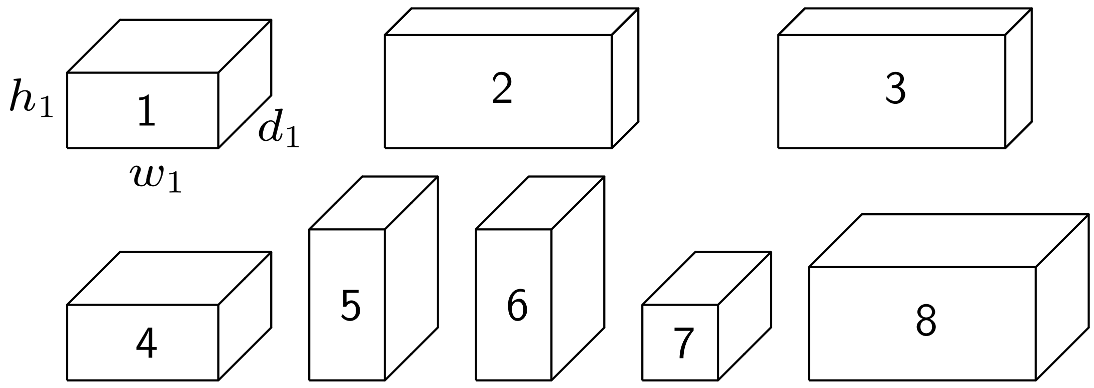


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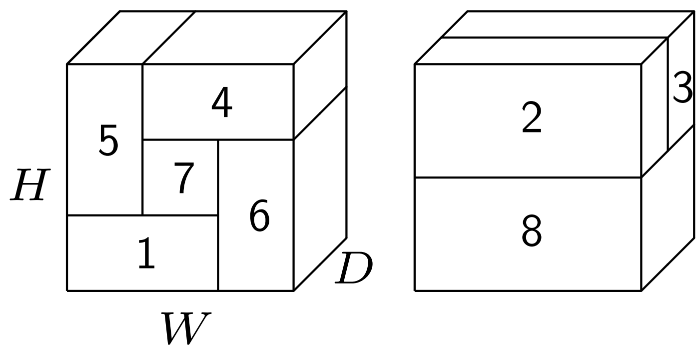

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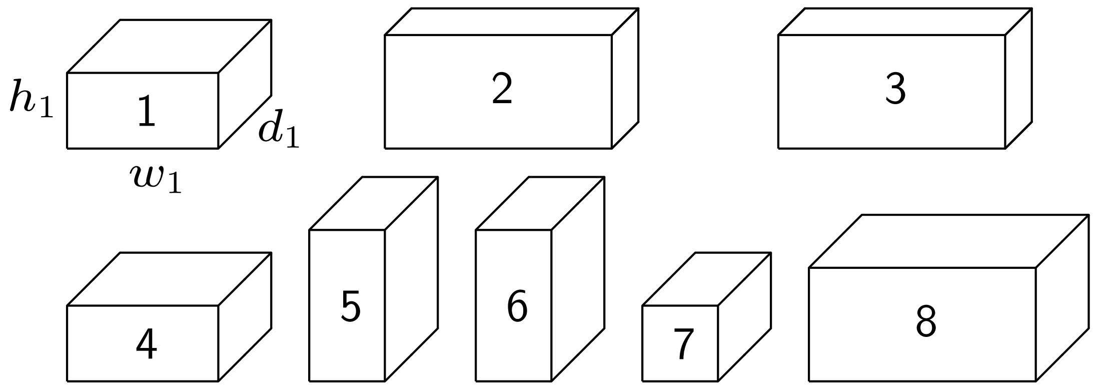


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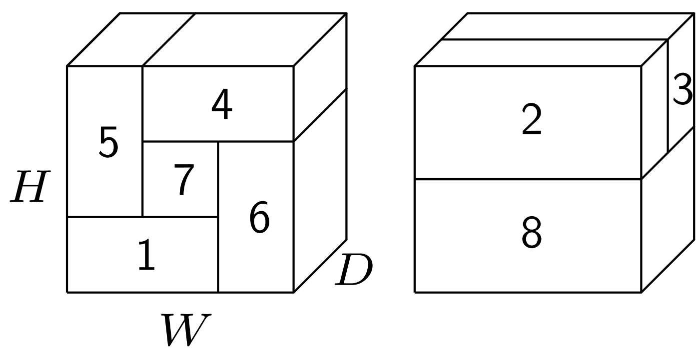


- ... and a single open-ended strip of **width** $W$**, depth** $D$, and **infinite height**, orthogonally pack all the boxes by minimizing the height to which the strip is filled (**Three-Dimensional Strip Packing Problem, 3SPP**).

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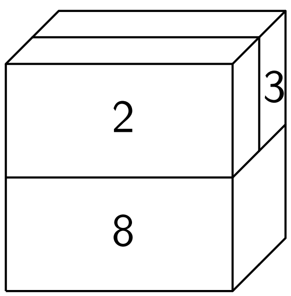


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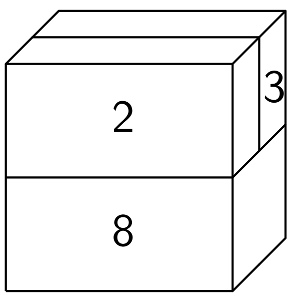


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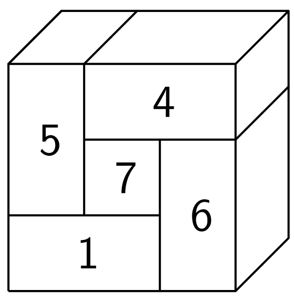


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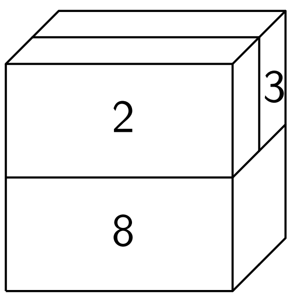


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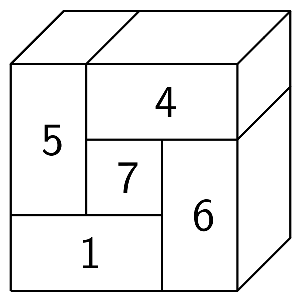


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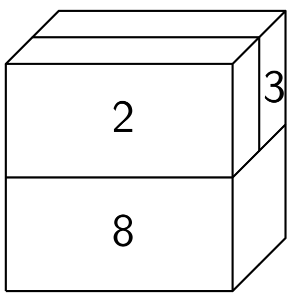


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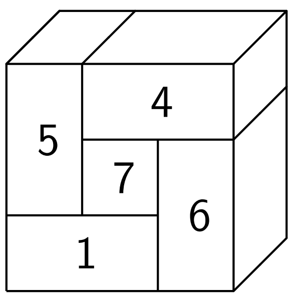


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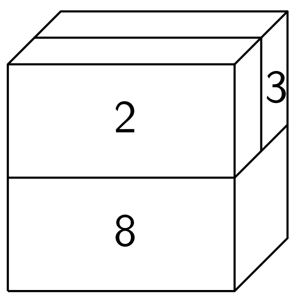
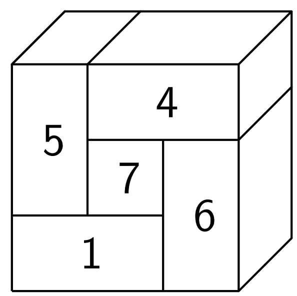
  - *Loading*: containers, vehicles (trucks, freight cars), pallets;
  - *Packaging design*: boxes, cases;
  - *Cutting*: foam rubber (arm-chair production).
- **Variants and additional constraints:**
  - Guillotine cuts:




guillotine-cuts

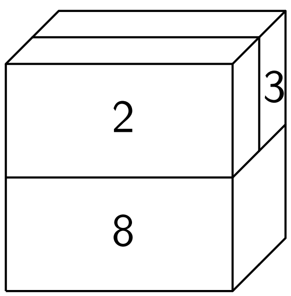



non guillotine-cuts

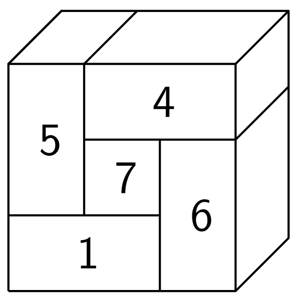
  - Boxes rotation;
  - Layers;
  - Limit on superposed weights;

## Three-dimensional packing problems (brief outline)

- **Complexity**: obviously NP-hard in the strong sense.
- **Applications:**
  - *Loading*: containers, vehicles (trucks, freight cars), pallets;
  - *Packaging design*: boxes, cases;
  - *Cutting*: foam rubber (arm-chair production).
- **Variants and additional constraints:**
  - Guillotine cuts:




guillotine-cuts




non guillotine-cuts

  - Boxes rotation;
  - Layers;
  - Limit on superposed weights;
  - Stability of the load . . .

# Essential Bibliography (Surveys and books)

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## I. One-dimensional bin packing problem


– M. Delorme, M. Iori, S. Martello (2016). Bin Packing and Cutting Stock Problems: Mathematical Models and Exact Algorithms, *European Journal of Operational Research*.

– E.G. Coffman, Jr, J. Csirik, G. Galambos, S. Martello, D. Vigo (2013). Bin packing approximation algorithms: Survey and classification
*Handbook of Combinatorial Optimization*, Springer.

– G. Wäscher, H. Haußner, and H. Schumann (2007). An improved typology of cutting and packing problems. *European Journal of Operational Research*.

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– H. Kellerer, U. Pferschy, D Pisinger (2004). *Knapsack problems*, Springer, Berlin.

– S. Martello, P. Toth (1990). *Knapsack Problems: Algorithms and Computer Implementations* (Ch. 6), John Wiley & Sons, Chichester-New York. **Free download @ my home page**.

## II. Two-dimensional bin packing problems

– A. Lodi, S. Martello, M. Monaci, D. Vigo (2014). Two-dimensional bin packing problems. In *Paradigms of Combinatorial Optimization: Problems and New Approaches*, ISTE and John Wiley & Sons.

– A. Lodi, S. Martello, M. Monaci (2002). Two-dimensional packing problems: A survey. *European Journal of Operational Research*. ($>$ $900$ citations on Google Scholar)

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– M. Iori, V. Loti de Lima, S. Martello, F.K. Miyazawa, M. Monaci (2019). *Two-dimensional Cutting and Packing: Problems and Solution Techniques* (in preparation).

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## III. Interactive visual solvers for packing problems


– M. Delorme, M. Iori, S. Martello (2018). BPPLIB: A library for bin packing and cutting stock problems. *Optimization Letters*.

– G. Costa, M. Delorme, M. Iori, E. Malaguti, S. Martello (2017). Training software for orthogonal packing problems. *Computers and Industrial Engineering*.

# It's been a long trip through packing

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# Thank you for your attention