Operations Research (Master's Degree Course)

7.1 Problems on Graphs: Definitions

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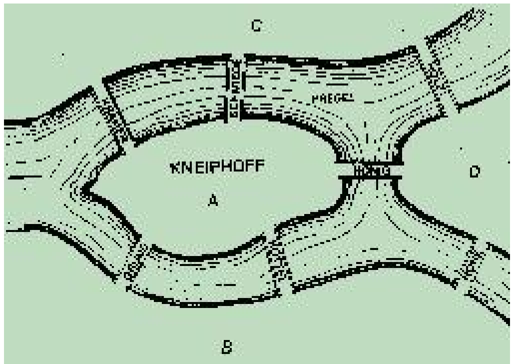
The birth of Graph Theory

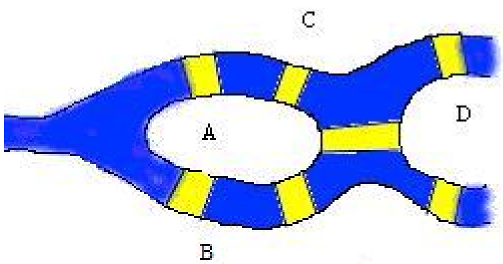
- Königsberg (East Prussia, today Kaliningrad (Russia)) in the 18th century:



- **Problem:** does there exist a walk that starts and ends in the same location, and crosses every bridge exactly once?

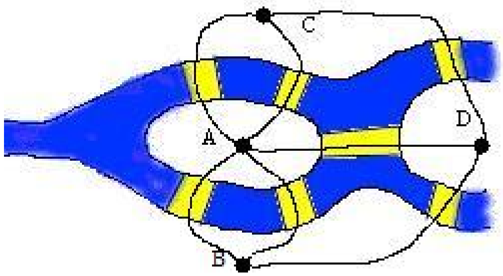
- Leonhard Euler (Saint Petersburg Academy of Sciences).

- Analysis technique developed by Euler (1736): main elements of the problem:

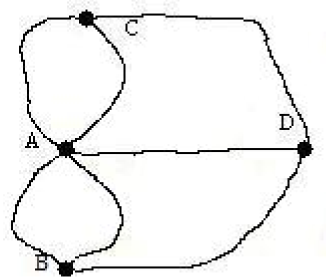


- The solution does not emerge yet.

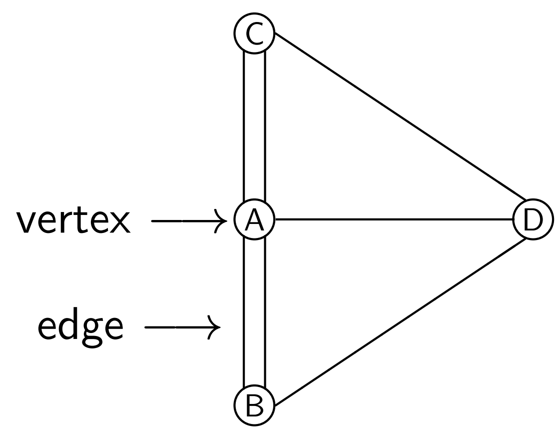
  Next step: associate a “point” with each land mass, and a “line” with each bridge:



- Eliminate everything else:



$\Longrightarrow$

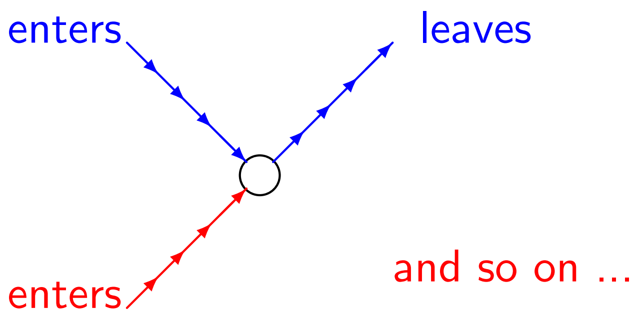


Model (Graph)

- **Equivalent problem on the model:**
  does there exist a closed walk that crosses each edge exactly once?

- **Euler's reasoning on the model:**

  Consider any vertex: if the walk exists then one must enter along an edge and leave along a different edge (one or more times).

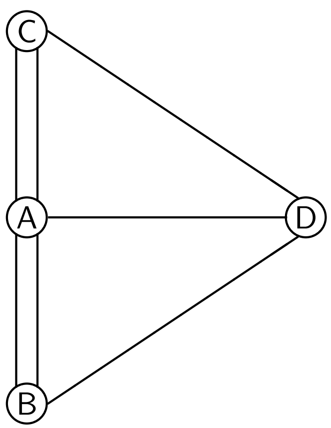


- **Conclusion:**

  the walk can only exists if **each vertex** has an **even number of edges** touching it.

  This **elementary property was not evident** in the real system (the map).

- **Solution to the Königsberg problem:** The walk does not exist.



Let us think once more about the models

- **A model** is a simplified representation of a real system;
- the model is **designed** to answer **specific questions** on the system;
- the model allows us to **better understand** reality.

Historical evolution of Graph Theory

- **18th Century:** recreational mathematics (games, riddles, ...);
- **19th Century:** first applications:
  - Kirchhoff's (1845) laws of electric circuits: **electric circuit = graph**;
  - theory of molecular diagrams: **molecule structure = graph**.
- **20th Century:** Development of a rich **theoretical apparatus**;

  availability of **powerful computers**;

  many applications: telecommunications, electricity, construction, mechanics, biology, economy, sociology, computer science ...

Terminology

- Graphs: **simple**: at most one edge between two vertices;
  multiple: more edges between two vertices (frequently transformable to simple ones);

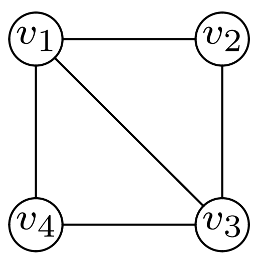
  we will always consider **simple graphs** (most results and practical applications).

undirected graphs

- $V = \{v_1, \ldots, v_n\}$ (*vertices*)
- $E = \{e_1, \ldots, e_m\}$ (*edges*)

  $e_i = (v_j, v_k) \equiv (v_k, v_j)$

  can be traversed in either direction
- $G = (V, E)$



$$V = \{v_1, v_2, v_3, v_4\}$$

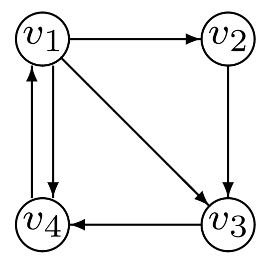
$$\begin{aligned} E &= \{e_1, e_2, e_3, e_4, e_5\} \\ &= \{(v_1, v_2), (v_2, v_3), (v_3, v_4), \\ &\quad (v_4, v_1), (v_1, v_3)\} \end{aligned}$$

directed graphs

- $V = \{v_1, \ldots, v_n\}$ (*vertices*)
- $A = \{a_1, \ldots, a_m\}$ (*arcs*)

  $a_i = (v_j, v_k) \not\equiv (v_k, v_j)$

  can be traversed in only one direction (v_j to v_k)
- $G = (V, A)$



$$V = \{v_1, v_2, v_3, v_4\}$$

$$\begin{aligned} A &= \{a_1, a_2, a_3, a_4, a_5, a_6\} \\ &= \{(v_1, v_2), (v_2, v_3), (v_3, v_4), \\ &\quad (v_4, v_1), (v_1, v_4), (v_1, v_3)\} \end{aligned}$$

Terminology (cont'd)

undirected graphs

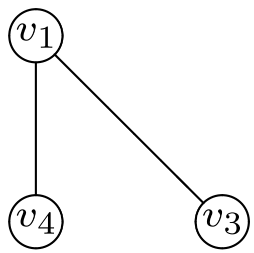
- $\Gamma(v) = \{v_j : (v, v_j) \in E\}$;

  $|\Gamma(v)|$ = **degree** of v;

- **subgraph** of $G = (V, E)$

  $G' = (V', E')$ with $V' \subseteq V$ and

  $E' \subseteq \{(v_i, v_j) \in E : v_i, v_j \in V'\}$;



directed graphs

- $\Gamma^+(v) = \{v_j : (v, v_j) \in A\}$;$\Gamma^-(v) = \{v_j : (v_j, v) \in A\}$;

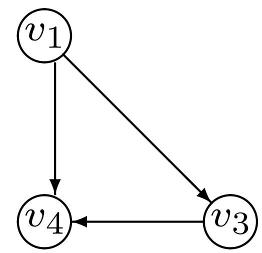
  $|\Gamma^+(v)|$ = **external degree** of v;

  $|\Gamma^-(v)|$ = **internal degree** of v;

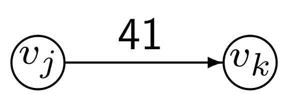
- **subgraph** of $G = (V, A)$:

  $G' = (V', A')$ with $V' \subseteq V$ and

  $A' \subseteq \{(v_i, v_j) \in A : v_i, v_j \in V'\}$;



- **Weighted graphs** (both directed and undirected):
  each arc/edge has an associated **weight** (or *cost, length, ...*)



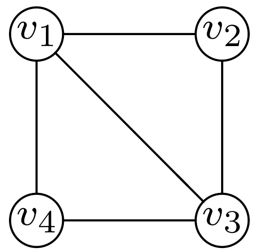
$w(v_j, v_k) = w(a_i) = w_{jk} = 41$

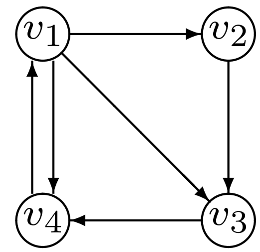
- A graph can (also) have a numerical value, called **capacity**, associated with each arc/edge;
  capacity = maximum amount of a certain facility that can (**flow**) along the arc/edge;
  examples: oil, gas, electricity, information; such graphs are usually called **Networks**.

Terminology (cont'd)

undirected graphs/networks | **directed graphs/networks**

- **Path** $=$ sequence of consecutive arcs/edges without repetition of vertices



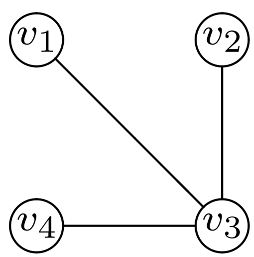


$\{v_1, v_4, v_3\}$ = path from v_1 to v_3 $\equiv$ path from v_3 to v_1 | $\{v_1, v_3, v_4\}$ = path from v_1 to v_4

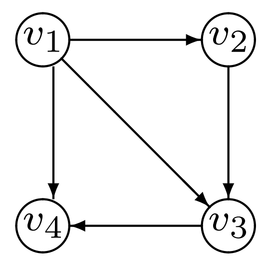
- **Circuit** or *cycle* $=$ path starting and ending on the same vertex

$\{v_1, v_4, v_3, v_1\}$ | $\{v_1, v_3, v_4, v_1\}$

- **Connected graph** (both directed and undirected) $\forall\ v_i, v_j \in V\ \exists$ path from v_i to v_j.



connected



not connected

An undirected graph is <u>not connected</u> iff it is formed by separate components.