Operations Research (Master's Degree Course)

5. Duality

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Duality: An informal introduction

A simple LP in canonical form:

$$\begin{array}{rrcrcr} \max z = & 4x_1 & + & 2x_2 & & \\ & 3x_1 & + & x_2 & \leq & 7 \\ & x_1 & + & x_2 & \leq & 3 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

Input data:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \quad c' = \begin{bmatrix} 4 & 2 \end{bmatrix}.$$

Remind that:

- every row of A represents the coefficients of an inequality;
- every element of b represents the right end side of the corresponding inequality;
- every element of c represents the unit value (cost) of the corresponding variable.

The optimal solution is clearly $x_1 = 2$, $x_2 = 1$, and its value is $\mathbf{z = 10}$.

Let's do an exercise: We define a **new problem** using the **same input data** but in a **"symmetrical" (dual)** way. Namely,

- we define the inequality coefficients using the **columns** of A instead of its **rows**;
- we use the **costs** as right hand sides and the **right hand sides** as the costs;
- we flip the **inequality signs**;
- we **minimize** instead of maximizing.

Using variables π_1 and π_2 for the new problem (the **dual**) we get:

$$\begin{array}{rrcrcr} \min w = & 7\pi_1 & + & 3\pi_2 & & \\ & 3\pi_1 & + & \pi_2 & \geq & 4 \\ & \pi_1 & + & \pi_2 & \geq & 2 \\ & \pi_1 & , & \pi_2 & \geq & 0. \end{array}$$

Easy optimal solution: $\pi_1 = 1$, $\pi_2 = 1$ of value **w = 10**.

Something curious: two totally different problems have (obviously) different solutions but the two solutions have the **same value!** We will see that this is a general property.

Duality emerges in many scientific areas (Mathematics, Physics, Electrical Engineering, Control theory, Economics). It gives two different points of view of looking at the same object.

In many cases (including linear programming) duality corresponds to an **involution**, a function that is its own inverse (i.e., such that $f(f(x)) = x$). Note indeed that, if we apply to the dual the same transformation that produced it, its dual coincides with the original problem.

Algebraic view of the Simplex algorithm

$$\begin{aligned} \min c'x & \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Optimality criterion:

$$\underbrace{\begin{array}{|c|c|} \hline & B \\ \hline \end{array}}_{A} \qquad \underbrace{\begin{array}{|c|c|} \hline [y_{ij}] & I \\ \hline \end{array}}_{Y}$$

- **Base:** $\mathcal{B} = \{A_{\beta(1)}, \ \ldots \ , A_{\beta(m)}\} \leftrightarrow B = [A_{\beta(i)}]$;
- **Simplex algorithm** $\longrightarrow$ Y **= tableau** ($m \times n$ matrix);
- **Algebraic interpretation:**

$$\begin{array}{|c|} \hline B^{-1} \\ \hline \end{array} \; \underbrace{\begin{array}{|c|c|} \hline & B \\ \hline \end{array}}_{A} = \begin{array}{|c|c|} \hline [y_{ij}] & I \\ \hline \end{array} \iff Y = B^{-1}A$$

- $c_\beta = [c_{\beta(i)}]$ (costs corresponding to basic variables of Y);

- $z_j = \sum_{i=1}^{m} y_{ij} c_{\beta(i)}$:

$$\underbrace{\begin{array}{|c|} \hline \\ \hline \end{array}}_{c'_\beta} \; \underbrace{\begin{array}{|c|c|} \hline [y_{ij}] & I \\ \hline \end{array}}_{Y} = \begin{array}{|c|c|} \hline z_j & z_j = c_j \\ \hline \notin \mathcal{B} & \in \mathcal{B} \end{array} \Leftrightarrow z' = c'_\beta Y = c'_\beta B^{-1} A$$

- **Relative costs** (row 0 of the tableau): $\overline{c}' = c' - z'$
- **Optimality criterion:** $\overline{c}' \geq 0 \ \Leftrightarrow c' - c'_\beta B^{-1} A \geq 0$.

Dual of an LP in standard form

- The LP duality immediately emerges from the optimality criterion:
- **Optimality criterion:** $c' - (c'_\beta B^{-1})A \geq 0$, where:
  - c', A input data;
  - $c'_\beta B^{-1}$ unknown ($\leftrightarrow$ optimal BFS).
- By defining $\pi' = c'_\beta B^{-1}$, we see that π' **is a feasible solution to**

$$\pi' A \leq c'$$

where π' is a vector of m variables corresponding to the selection of the optimal basis.

- Let us "invent" a new objective function $\max \pi' b$. We get the **Dual problem:**

$$\begin{array}{rcl} \max \pi' b & & \\ \pi' A & \leq & c' \\ \pi & \lesseqqgtr & 0 \end{array}$$

- We will show that

$\mathcal{B}$ is an optimal BFS for the LP

$\Updownarrow$

$\pi' = c'_\beta B^{-1}$ is an optimal solution to the dual.

Example:

$$\begin{array}{rrrrrrrrr} \min\ z = & 2 & x_1 & + & 3 & x_2 & + & x_3 & & \\ & 2 & x_1 & + & & x_2 & & & = & 1 \\ & & x_1 & & & & + & x_3 & = & 4 \\ & & x_1, & & & x_2, & & x_3 & \geq & 0 \end{array}$$

$$\begin{array}{c|c|ccc|} & & x_1 & x_2 & x_3 \\ \hline -z & -7 & -5 & 0 & 0 \\ \hline x_2 & 1 & 2 & 1 & 0 \\ x_3 & 4 & 1 & 0 & 1 \\ \hline \end{array} \ \longrightarrow \ \begin{array}{c|c|ccc|} & & x_1 & x_2 & x_3 \\ \hline -z & -\frac{9}{2} & 0 & \frac{5}{2} & 0 \\ \hline x_1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ x_3 & \frac{7}{2} & 0 & -\frac{1}{2} & 1 \\ \hline \end{array}$$

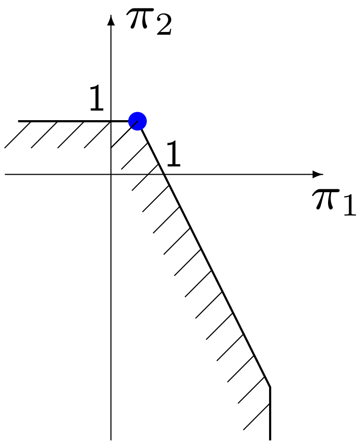
A $\qquad\qquad$ Y

$$B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}, \qquad B^{-1}A = Y\ ;$$

$c'_\beta Y = [2 \quad 1]\ Y = [2 \quad \frac{1}{2} \quad 1];$

$c' - z' = [2 \quad 3 \quad 1] - c'_\beta Y = [0 \quad \frac{5}{2} \quad 0];$

$$\begin{array}{rrrrrr} \max\ w = & \pi_1 & + & 4\ \pi_2 & & \\ & 2\ \pi_1 & + & \pi_2 & \leq & 2 \\ & \pi_1 & & & \leq & 3 \\ & & & \pi_2 & \leq & 1 \\ & \pi_1, & & \pi_2 & \geq & 0 \end{array}$$



Current tableau: $\pi' = c'_\beta B^{-1} = [\frac{1}{2} \quad 1]$ ($\bullet$)

Dual of an LP in general form

$$\min c'x$$

$$\begin{array}{rcll} a_i'x & = & b_i & i \in M \\ a_i'x & \geq & b_i & i \in \overline{M} \\ x_j & \geq & 0 & j \in N \\ x_j & \gtreqless & 0 & j \in \overline{N} \end{array}$$

By introducing: $\begin{cases} x_i^s & \forall\ i \in \overline{M}(x_i^s \geq 0) \\ x_j^+,\ x_j^- & \forall\ j \in \overline{N}(x_j^+,\ x_j^- \geq 0) \end{cases}$

we get an equivalent LP: $\min \hat{c}'\hat{x}$

$$\begin{array}{rcl} \hat{A}\hat{x} & = & b \\ \hat{x} & \geq & 0 \end{array}$$

$$\hat{A} = \begin{array}{cccc} N & \overline{N} & \overline{M} & \\ \left[\begin{array}{c|cc|c} & & & 0 \\ A_j & A_j & -A_j & \\ \hline & & & -I \end{array}\right] & & & \begin{array}{c} M \\ \\ \overline{M} \end{array} \end{array}$$

$$\hat{x}' = \begin{array}{|c|c|c|} \hline x_j & (x_j^+,\ x_j^-) & x_i^s \\ \hline \end{array}$$

$$\hat{c}' = \begin{array}{|c|c|c|} \hline c_j & (c_j,\ -c_j) & 0 \\ \hline \end{array}$$

Dual of an LP in general form (cont'd)

- **Optimality criterion:** an optimal solution $\hat{x}_0$ corresponds to a base $\hat{\mathcal{B}} = \{\hat{A}_{\hat{\beta}(1)}, \dots, \hat{A}_{\hat{\beta}(m)}\}$ ($\leftrightarrow$ $\hat{B} = [\hat{A}_{\hat{\beta}(i)}]$) s.t. $\hat{c}' - \hat{z}' = \hat{c}' - (\hat{c}'_{\hat{\beta}}\hat{B}^{-1})\hat{A} \geq 0$
- By defining $\pi' = \hat{c}'_{\hat{\beta}}\hat{B}^{-1}$, π is a feasible solution to $\pi'\hat{A} \leq \hat{c}'$:

$$\begin{array}{c} \boxed{} \\ \pi' \end{array} \quad \begin{array}{c|c|c|c|c} & N & \overline{N} & \overline{M} & \\ \hline & A_j & A_j \quad -A_j & 0 & M \\ & & & -I & \overline{M} \\ \hline \leq & c_j & (c_j, \ -c_j) & 0 & \end{array} \quad \leq$$

- Hence π is a feasible solution to:

$$\begin{array}{rclcrclcl} \pi' A_j & \leq & c_j \Rightarrow\searrow & & \bullet \ \ \pi' A_j & \leq & c_j & , & j \in N \\ -\pi' A_j & \leq & -c_j \rightarrow & \Rightarrow & \bullet \ \ \pi' A_j & = & c_j & , & j \in \overline{N} \\ -\pi_i & \leq & 0 & \Rightarrow & \bullet \ \ \pi_i & \geq & 0 & , & i \in \overline{M} \\ \text{nothing for } i \in M & & & \Rightarrow & \bullet \ \ \pi_i & \gtreqless & 0 & , & i \in M \end{array}$$

- Constraints of a new problem (**Dual problem**). Original LP = **Primal problem**.
- We will adopt the objective function $\max \pi' b$.

Primal-Dual relationships

Primal				Dual		
(P) $\min c'x$				(D) $\max \pi' b$		
$a_i' x$	$=$	b_i	$i \in M$	π_i	$\gtreqless$	0
$a_i' x$	$\geq$	b_i	$i \in \overline{M}$	π_i	$\geq$	0
x_j	$\geq$	0	$j \in N$	$\pi' A_j$	$\leq$	c_j
x_j	$\gtreqless$	0	$j \in \overline{N}$	$\pi' A_j$	$=$	c_j

constraint	$\longleftrightarrow$	variable
variable	$\longleftrightarrow$	constraint
row	$\longleftrightarrow$	column
column	$\longleftrightarrow$	row
"strong" constraint	$\longleftrightarrow$	"weak" constraint
"weak" constraint	$\longleftrightarrow$	"strong" constraint
costs	$\longleftrightarrow$	right handside
right handside	$\longleftrightarrow$	costs

Example. **Primal:**

$$
\begin{array}{lcccccccc}
\min z = & 2x_1 & + & x_2 & + & x_3 & & \\
 & x_1 & & & + & x_3 & = & 4 \\
 & & & x_2 & + & 2x_3 & \geq & 2 \\
 & x_1 & , & x_2 & & & \geq & 0 \\
 & & & & & x_3 & \lesseqgtr & 0
\end{array}
$$

	0	2	1	1	-1	0
		x_1	x_2	x_3^+	x_3^-	x_2^s
$-z$	-10	0	0	-3	3	1
x_1	4	1	0	1	-1	0
x_2	2	0	1	2	-2	-1

		x_1	x_2	x_3^+	x_3^-	x_2^s
$-z$	-7	0	$\frac{3}{2}$	0	0	$-\frac{1}{2}$
x_1	3	1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
x_3^+	1	0	$\frac{1}{2}$	1	-1	$-\frac{1}{2}$

		x_1	x_2	x_3^+	x_3^-	x_2^s
$-z$	-4	1	1	0	0	0
x_2^s	6	2	-1	0	0	1
x_3^+	4	1	0	1	-1	0

Sol: $z = 4$; $x_1 = x_2 = 0$, $x_3 = 4$.

Dual:

$$
\begin{array}{rcccccccl}
\max w = & 4\pi_1 & + & 2\pi_2 & & & & & \\
& \pi_1 & & & \le & 2 & \text{(redundant)} & & \\
& & & \pi_2 & \le & 1 & \Uparrow & & \\
& \pi_1 & + & 2\pi_2 & = & 1 & \Longrightarrow & & \\
& \pi_1 & & & \gtreqless & 0 & & & \\
& & & \pi_2 & \ge & 0 & & &
\end{array}
$$

$$
\begin{array}{rccccccccccc}
\max w = & 4\pi_1^+ & - & 4\pi_1^- & + & 2\pi_2 & & & & & \\
\downarrow & & & & & & & & & & \\
-\min(-w) = & -4\pi_1^+ & + & 4\pi_1^- & - & 2\pi_2 & & & & & \\
& & & & & \pi_2 & + & \pi_1^s & = & 1 \\
& \pi_1^+ & - & \pi_1^- & + & 2\pi_2 & & & = & 1 \\
& \pi_1^+ & , & \pi_1^- & , & \pi_2 & , & \pi_1^s & \ge & 0
\end{array}
$$

	0	-4	4	-2	0
		π_1^+	π_1^-	π_2	π_1^s
$+w$	4	0	0	6	0
π_1^s	1	0	0	1	1
π_1^+	1	1	-1	2	0

Optimal solution: $w = 4$; $\pi_1 = 1$, $\pi_2 = 0$. (**Observation:** $w = z$).

Strong duality theorem

Theorem *If an* LP *has a finite optimal solution then*

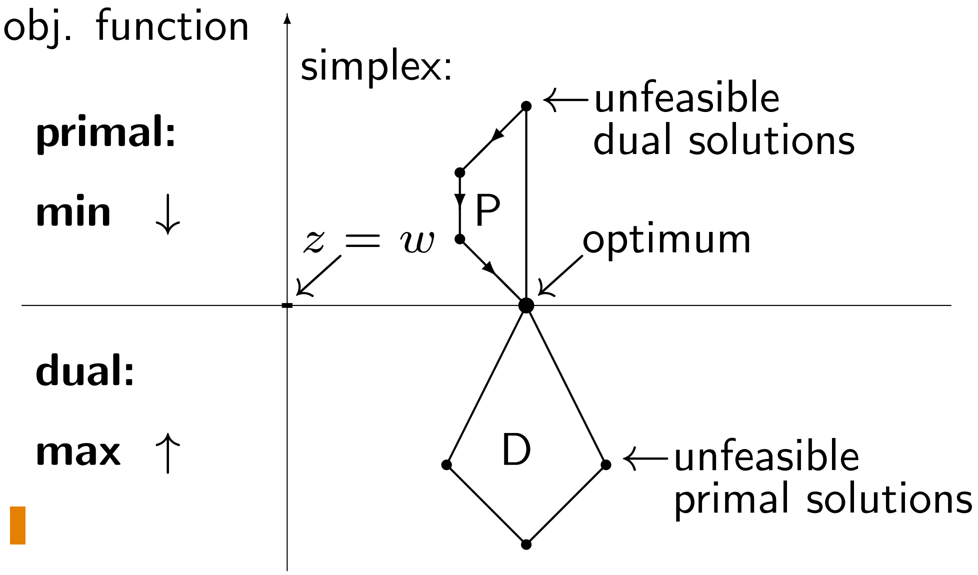
1. *its dual has a finite optimal solution;*

2. *the two solutions have the same value.*

Proof 1. x = feasible primal solution, π = feasible dual solution.

Dual constraints: $c' \geq \pi' A \Longrightarrow$(primal constraints)$\Longrightarrow c'x \geq \pi' A x = \pi' b$. **Weak duality:**

cost in the primal $\geq$ cost in the dual! $\Longrightarrow$ the dual cannot have an unbounded solution.



The dual has the feasible solution $\pi' = c'_\beta B^{-1}$ (with B = optimal primal basis) $\Longrightarrow$ it has finite optimal solution.

2. Solution $\pi' = c'_\beta B^{-1}$ has cost $\pi' b = c'_\beta B^{-1} b = c'_\beta x_\beta$ (= optimal primal cost). $\square$

Dual of the dual

Theorem *The dual of the dual is the primal.*

Proof Re-write the dual in primal form:

$$
\begin{array}{lrcll}
\min\ (-b')\pi & & & & \\
& (-A'_j)\pi & \geq & -c_j & j \in N \\
& (-A'_j)\pi & = & -c_j & j \in \overline{N} \\
& \pi_i & \geq & 0 & i \in \overline{M} \\
& \pi_i & \gtreqless & 0 & i \in M
\end{array}
$$

Define its dual using variables x:

$$
\begin{array}{lrcll}
\max\ x'(-c) & & & & \\
& x_j & \geq & 0 & j \in N \\
& x_j & \gtreqless & 0 & j \in \overline{N} \\
& (-a'_i)x & \leq & -b_i & i \in \overline{M} \\
& (-a'_i)x & = & -b_i & i \in M
\end{array}
$$

This is the primal. $\square$

Possible categories of a primal-dual pair

dual$\rightarrow$ **primal**$\downarrow$	finite optimum	unbounded	infeasible
finite optimum	**1**	**NO**	**NO**
unbounded	**NO**	*NO*	**3**
infeasible	**NO**	**3**	**2**

Entries '**1**' and '**NO**': already proved

- If the primal is unbounded ($z \rightarrow -\infty$) then the dual cannot have a solution $w \rightarrow +\infty$ ($\Leftarrow c'x \geq \pi' b$) $\Rightarrow$ *NO*. We show that cases **2** and **3** are possible:

$$
\begin{array}{llllll}
\cdot \ \min & x_1 & & & & \\
& x_1 & + & x_2 & \geq & 1 \\
- & x_1 & - & x_2 & \geq & 1 \\
& x_1 & , & x_2 & \gtrless & 0 \\
\multicolumn{6}{c}{\text{(impossible primal)}}
\end{array}
\longrightarrow
\begin{array}{llllll}
\max & \pi_1 & + & \pi_2 & & \\
& \pi_1 & - & \pi_2 & = & 1 \\
& \pi_1 & - & \pi_2 & = & 0 \\
& \pi_1 & , & \pi_2 & \geq & 0 \\
\multicolumn{6}{c}{\text{(impossible dual)}}
\end{array}
\Rightarrow \mathbf{2}
$$

$$
\begin{array}{llllll}
\cdot \ \min & x_1 & & & & \\
& x_1 & + & x_2 & \geq & 1 \\
- & x_1 & - & x_2 & \geq & 1 \\
& x_1 & , & x_2 & \geq & 0 \\
\multicolumn{6}{c}{\text{(impossible primal)}}
\end{array}
\longrightarrow
\begin{array}{llllll}
\max & \pi_1 & + & \pi_2 & & \\
& \pi_1 & - & \pi_2 & \leq & 1 \\
& \pi_1 & - & \pi_2 & \leq & 0 \\
& \pi_1 & , & \pi_2 & \geq & 0 \\
\multicolumn{6}{c}{\text{(unbounded dual)}}
\end{array}
\Rightarrow \mathbf{3}
$$

The dual of the diet problem

$$\text{Primal}: \min \; c'x$$
$$Ax \geq r,$$
$$x \geq 0.$$

c_j = cost of 1 unit of the jth food;

x_j = quantity of the jth food to buy;

a_{ij} = quantity of the ith nutrient in the jth food;

r_i = requirement of the ith nutrient.

$$\text{Dual}: \max \; \pi' r$$
$$\pi' A \leq c',$$
$$\pi' \geq 0.$$

Interpretation: A pill-maker wants to enter the market with m **pills**, one per nutrient:

π_i = selling price for one unit of the ith nutrient;

$\pi' r$ = cost of the diet;

$\sum_{i=1}^{m} \pi_i a_{ij} \leq c_j \iff$ the cost in pill form of all nutrients in the jth food cannot exceed the cost of the jth food.

Farkas' Lemma (1894)

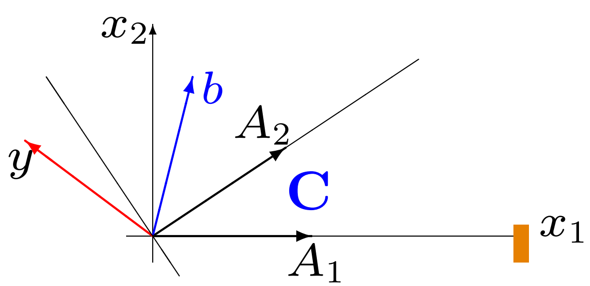
Lemma *The system* (I) $:\{Ax = b,\ x \geq 0\}$ *is impossible iff* (if and only if)
the system (II) $:\{y'A \leq 0,\ b'y > 0\}$ *is possible.*

Proof

$$
\text{(P)}\ \min\ \begin{array}{llll} 0' & x & & \\ A & x & = & b \\ & x & \geq & 0 \end{array}
\qquad \Longleftrightarrow \qquad
\text{(D)}\ \max\ \begin{array}{llll} b' & y & & \\ y' & A & \leq & 0 \\ y & \gtreqless & 0 & \end{array}
$$

- (D) is possible ($\Leftarrow y = 0$); hence
- (P) is impossible iff (D) is unbounded, i.e.,
  (I) is impossible iff (D) is unbounded.
- (P) cannot have a solution of value > 0, so
  if (D) has a solution with $b'y > 0$, then it must be $b'y \to +\infty$ from which
- (D) is unbounded iff (II) is possible. $\square$

Geometric interpretation: $\mathbf{C}$ = cone generated, in R^m, by vectors A_j; b = vector in R^m:



(I) $\Leftrightarrow b \in C$;

(II) $\Leftrightarrow \exists$ a vector $y \in R^m$ such that
its angle with b is no more than $90°$, but
its angle with all vectors in $\mathbf{C}$ is more than $90°$.

Complementary slackness

Theorem *A pair of solutions* x, π*, respectively feasible for a primal-dual pair is optimal if and only if*

$$\begin{array}{lllllll} u_i & = & \pi_i(a_i'x - b_i) & = & 0 & \forall\ i, & (\alpha) \\ v_j & = & (c_j - \pi' A_j)x_j & = & 0 & \forall\ j & (\beta) \end{array}$$

Proof

- $u_i \geq 0\ \forall\ i$ (if $a_i'x = b_i \Rightarrow u_i = 0$, if $a_i'x \geq b_i \Rightarrow \pi_i \geq 0 \Rightarrow u_i \geq 0$);
- $v_j \geq 0\ \forall\ j$ (similarly).
- $U = \sum_{i=1}^{m} u_i \geq 0$;
- $V = \sum_{j=1}^{n} v_j \geq 0$.
- $(U = V = 0)$ if and only if (α) and (β) hold.
- $U + V = \sum_{i=1}^{m} \pi_i(\sum_{j=1}^{n} a_{ij}x_j - b_i) + \sum_{j=1}^{n}(c_j - \sum_{i=1}^{m} \pi_i a_{ij})x_j = c'x - \pi' b$, from which
- (α) and (β) hold $\Leftrightarrow \pi' b = c'x \Leftrightarrow x$ and π are optimal. $\square$

Complementary slackness (cont'd)

- Complementary slackness implies that
  - $\forall j \ : \ \pi' A_j < c_j$ in the optimal solution to the dual

    we must have $x_j = 0$ in the optimal solution to the primal, and conversely
  - $\forall j \ : \ x_j > 0$ in the optimal solution to the primal,

    we must have $\pi' A_j = c_j$ in the optimal solution to the dual.
- Similar relationships between primal constraints and dual variables.
- **Example (see previous slides in this chapter):**

$$\begin{array}{rccccccccl}
\min z = & 2x_1 & + & x_2 & + & x_3 & & & & \\
 & x_1 & & & + & x_3 & = & 4 & \rightarrow & \text{1st } (\alpha) \text{ constraint satisfied} \\
 & & & x_2 & + & 2x_3 & \geq & 2 & & \\
 & x_1 & , & x_2 & & & \geq & 0 & & \\
 & & & & & x_3 & \lesseqgtr & 0 & &
\end{array}$$

Solution: $z = 4$, $x_3 = 4$, $x_1 = x_2 = 0 \rightarrow$ 1st and 2nd (β) constraints satisfied

2nd (α) constraint: $\pi_2(x_2 + 2x_3 - 2) = 0 \quad \Rightarrow \pi_2 = 0$

3rd (β) constraint: $(1 - (\pi_1 + 2\pi_2))x_3 = 0 \Rightarrow \underbrace{\pi_1 = 1}_{\uparrow}$

optimal dual solution.

Complementary slackness (cont'd)

- **Example (see slides in Chapter 2):**

$$
\begin{array}{rrrrrrrrrrrrl}
\min z = - & x_1 & - & 2x_2 & & & & & & & & \\
 & x_1 & + & x_2 & + & x_3 & & & & & = & 2 \\
 & x_1 & + & 3x_2 & & & + & x_4 & & & = & 3 \\
 & & & 3x_2 & & & & & + & x_5 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

Standard form $\Rightarrow$ all (α) constraints satisfied.

Solution: $z = -\frac{5}{2}$, $x_3 = x_4 = 0 \rightarrow$ 3rd and 4th (β) constraints satisfied

$x_1 = \frac{3}{2}$, $x_2 = \frac{1}{2}$, $x_5 = \frac{1}{2}$

$$
\begin{array}{lrrrrrrl}
\text{(D)} \quad \max w = & 2\pi_1 & + & 3\pi_2 & + & 2\pi_3 & & \\
 & \pi_1 & + & \pi_2 & & & \leq & -1 \\
 & \pi_1 & + & 3\pi_2 & + & 3\pi_3 & \leq & -2 \\
 & \pi_1 & & & & & \leq & 0 \\
 & & & \pi_2 & & & \leq & 0 \\
 & & & & & \pi_3 & \leq & 0 \\
 & \pi_1 & , & \pi_2 & , & \pi_3 & \gtreqless & 0 \; .
\end{array}
$$

1st, 2nd and 5th (β) constraints:

$(-1 - \pi_1 - \pi_2)\frac{3}{2} = 0, \quad (-2 - \pi_1 - 3\pi_2 - 3\pi_3)\frac{1}{2} = 0, \quad (0 - \pi_3)\frac{1}{2} = 0$

The last one gives $\pi_3 = 0$. The resulting 2×2 linear system gives $\pi_1 = \pi_2 = -\frac{1}{2}$.

Optimal dual solution: $\pi = (-\frac{1}{2}, -\frac{1}{2}, 0)$, of value $-\frac{5}{2}$.

Dual information in the tableau

- Y = tableau $(m \times n)$, base $\mathcal{B}$ ($\leftrightarrow B = [A_{\beta(i)}]$), costs c_β: $z' = c'_\beta Y = c'_\beta B^{-1} A = \pi' A$.
- Suppose the initial and final tableaux are ($\widetilde{B} \leftrightarrow$ optimal final basis):

0	c'		
b	I		$\widetilde{B}$

$\underbrace{\qquad\qquad\qquad}_{A}$

Initial tableau

$-z_0$	$c_j - \pi_j$		
x_0	$\widetilde{B}^{-1}$		I

initial basis $\qquad \widetilde{\mathcal{B}}$

Final tableau

- The final tableau satisfies the optimality criterion:

$$\overline{c}_j = c_j - z_j = c_j - \pi' A_j \geq 0 \quad \forall\, j,$$

with π = optimal dual solution. Hence

- in row 0, in the columns corresponding to the initial basis, we have $\overline{c}_j = c_j - \pi_j$,

  from which we get the optimal dual solution $\pi_j = c_j - \overline{c}_j$.
- If the initial basis is provided by the artificial variables of Phase 1 (cost $c_j = 0$), then $\pi_j = -\overline{c}_j$.
- In the final tableau the columns of the initial base contain the inverse of the optimal base.

Dual information in the tableau (cont'd)

- **Example (see previous slides in this chapter): initial and final tableaux**

		x_1	x_2	x_3^+	x_3^-	x_2^s
	0	2	1	1	-1	0
$-z$	-10	0	0	-3	3	1
x_1	4	1	0	1	-1	0
x_2	2	0	1	2	-2	-1

		x_1	x_2	x_3^+	x_3^-	x_2^s
$-z$	-4	1	1	0	0	0
x_2^s	6	2	-1	0	0	1
x_3^+	4	1	0	1	-1	0

$\pi_1 = 2 - 1 = 1$, $\pi_2 = 1 - 1 = 0$.

- **Example (see previous slides in Chapter 4): initial and final tableaux**

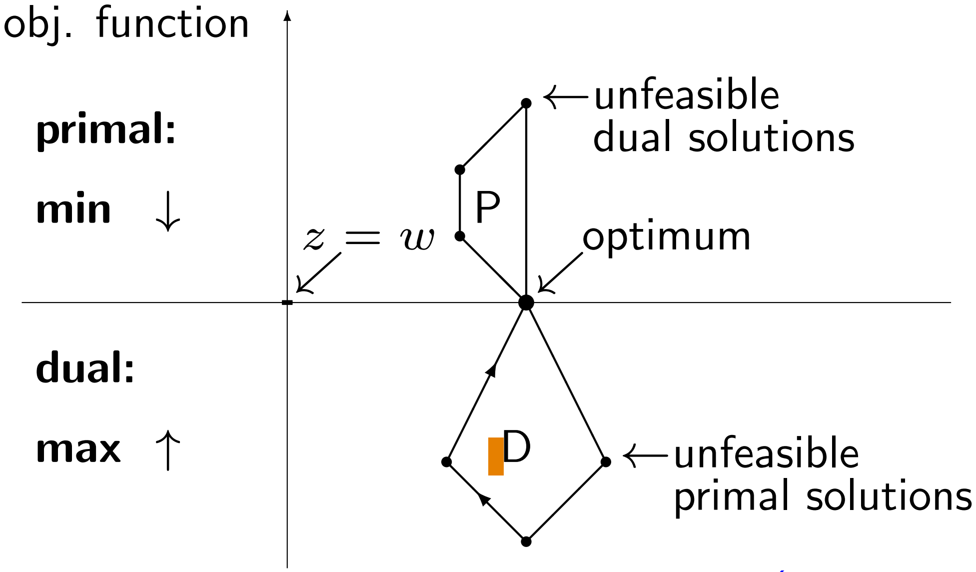
	x_1^a	x_2^a	x_1	x_2	x_3	s_1
0	1	1	0	0	0	0
5	1	0	1	2	0	1
6	0	1	0	1	2	0

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
$-z$	$-\frac{7}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{5}{4}$	0	0	$\frac{1}{4}$
x_2	$\frac{5}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$
x_3	$\frac{7}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	1	$-\frac{1}{4}$

$\pi_1 = 0 - \frac{1}{4} = -\frac{1}{4}$, $\pi_2 = 0 - (-\frac{1}{2}) = \frac{1}{2}$ (the true cost of an artificial variable is 0).

Dual simplex algorithm

- (Primal) Simplex: maintain a feasible primal solution, and move it towards dual feasibility.
- Dual simplex: maintain a feasible dual solution, and move it towards primal feasibility.



1. We start with a base that is **dual feasible** $(y_{0j} \geq 0 \;\; j = 1, \dots, n)$ but **primal infeasible** $(\exists \; y_{i0} < 0, i \geq 1)$, e.g., (r1:=r1−r2, r0:=r0−r1, r2:=r2−r1)

$$
\begin{array}{c|cccc}
 & x_1 & x_2 & x_3 & x_4 \\
\hline
0 & 1 & 0 & -\frac{1}{2} & 0 \\
\hline
2 & 2 & 1 & -2 & 0 \\
4 & 1 & 1 & -1 & 1
\end{array}
\quad \rightarrow \quad
\begin{array}{c|c|cccc}
 & & x_1 & x_2 & x_3 & x_4 \\
-z & 2 & 0 & 0 & \frac{1}{2} & 1 \\
\hline
x_1 & -2 & 1 & 0 & -1 & -1 \\
x_2 & 6 & 0 & 1 & 0 & 2
\end{array}
$$

Dual simplex algorithm (cont'd)

2. We select for **pivoting a row** i corresponding to a $y_{i0} < 0$.

3. We select the **pivot** y_{is} among the $y_{ij} < 0$ (the pivoting must make y_{i0} positive):

   – the pivoting must obtain 0 in $\tilde{y}_{0s}$ through $\tilde{y}_{0j} := \underbrace{y_{0j}}_{\geq 0} - \underbrace{\left(\frac{y_{0s}}{y_{is}}\right)}_{\leq 0 (\Leftarrow y_{0s} \geq 0,\; y_{is} < 0)} y_{ij} \quad \forall j$

   – and $\tilde{y}_{0j}$ must remain ≥ 0 $(j = 1, \ldots, n) \Rightarrow \frac{y_{0j}}{y_{ij}} \leq \frac{y_{0s}}{y_{is}} \; \forall \; j \; : \; y_{ij} < 0$, so

   – **pivot selected as** $\max\limits_{j : y_{ij} < 0} \left\{ \frac{y_{0j}}{y_{ij}} \right\}$

		x_1	x_2	x_3	x_4
$-z$	2	0	0	$\frac{1}{2}$	1
x_1	-2	1	0	-1	-1
x_2	6	0	1	0	2

		x_1	x_2	x_3	x_4	
$-z$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	feasible
x_3	2	-1	0	1	1	$\Downarrow$
x_2	6	0	1	0	2	optimal

- **Note:** This choice also ensures the minimum cost increase. Indeed

$$-z = \tilde{y}_{00} = y_{00} \overbrace{- \underbrace{\left(\frac{y_{0s}}{y_{is}}\right)}_{\max \leq 0} \underbrace{y_{i0}}_{<0}}^{\text{cost increase } (\geq 0)}$$

Dual simplex algorithm (cont'd)

procedure DUAL_SIMPLEX:
comment: we are given a tableau such that $y_{0j} \geq 0 \ \forall j \geq 1$ and $\exists \ i > 0 \ : \ y_{i0} < 0$;
begin
  $optimal :=$ false;
  $infeasible :=$ false;
  while $optimal =$ false **and** $infeasible =$ false **do**
    if $y_{i0} \geq 0$ for $i = 1, \ldots, m$ **then** $optimal =$ true
    else
      begin
        select an $i > 0 \ : y_{i0} < 0$;
        if $y_{ij} \geq 0$ for $j = 1, \ldots, n$ **then** $infeasible :=$ true (**comment:** unbounded dual)
        else $\vartheta := \max\limits_{j>0: y_{ij}<0} \left\{ \dfrac{y_{0j}}{y_{ij}} \right\} = \dfrac{y_{0s}}{y_{is}}$, and perform a pivoting on y_{is}
      end
end.

No "two-phase" method is needed: the dual simplex algorithm is normally used when

- we are given the optimal tableau of a primal LP;
- one or more constraints are added so the solution becomes infeasible;
- we want the new optimal solution without starting from scratch.

Dual simplex algorithm (cont'd)

SIMPLEX	**PRIMAL**	**DUAL**
we move	from BFS to BFS	from unfeasible base to unfeasible base
z starts from a value	higher than the optimum	lower than the optimum
the value of z	decreases	increases
we first select the variable that	the column, i.e., enters the base	the row, i.e. leaves the base
then we select the variable that	the row, i.e., leaves the base	the column, i.e., enters the base
pivot defined by a	minimum among positive values	maximum among negative values

The dual simplex algorithm can be seen as the primal simplex algorithm executed on the dual problem, but operating on the primal tableau:

(P)

0	c'
b	A

(D)

0	$\hat{b}'$
c	$\hat{A}'$

Dual simplex algorithm (cont'd)

- Example :
$$
\begin{array}{rcccccccccccl}
\min z = - & x_1 & - & 4x_2 & & & & & & & & \\
 & x_1 & + & x_2 & + & s_1 & & & & & = & 2 \\
 & x_1 & + & 3x_2 & & & + & s_2 & & & = & 3 \\
 & & & 3x_2 & & & & & + & s_3 & = & 2 \\
 & x_1 & , & x_2 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}
$$

- **Primal Simplex:**

		x_1	x_2	s_1	s_2	s_3
$-z$	0	-1	-4	0	0	0
s_1	2	1	1	1	0	0
s_2	3	1	3	0	1	0
s_3	2	0	3	0	0	1

		x_1	x_2	s_1	s_2	s_3
$-z$	$\frac{8}{3}$	-1	0	0	0	$\frac{4}{3}$
s_1	$\frac{4}{3}$	1	0	1	0	$-\frac{1}{3}$
s_2	1	1	0	0	1	-1
x_2	$\frac{2}{3}$	0	1	0	0	$\frac{1}{3}$

		x_1	x_2	s_1	s_2	s_3
$-z$	$\frac{11}{3}$	0	0	0	1	$\frac{1}{3}$
s_1	$\frac{1}{3}$	0	0	1	-1	$\frac{2}{3}$
x_1	1	1	0	0	1	-1
x_2	$\frac{2}{3}$	0	1	0	0	$\frac{1}{3}$

Optimal

- **To obtain a dual tableau:** (i) execute, on the first primal tableau, row 0 := row 0 + 4·(row 1);
  (ii) perform a pivoting on y_{12}:

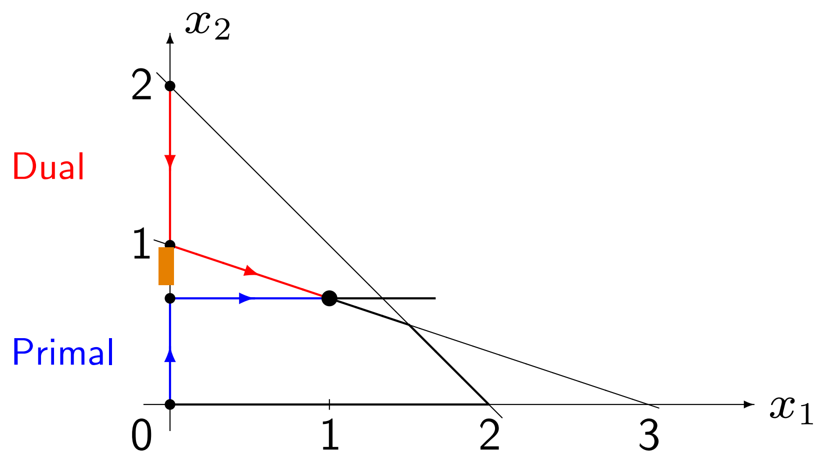
- **Dual Simplex:**

		x_1	x_2	s_1	s_2	s_3
$-z$	8	3	0	4	0	0
x_2	2	1	1	1	0	0
s_2	-3	-2	0	-3	1	0
s_3	-4	-3	0	-3	0	1

		x_1	x_2	s_1	s_2	s_3
$-z$	4	$\frac{1}{3}$	0	0	$\frac{4}{3}$	0
x_2	1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0
s_1	1	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	0
s_3	-1	-1	0	0	-1	1

		x_1	x_2	s_1	s_2	s_3
$-z$	$\frac{11}{3}$	0	0	0	1	1
x_2	$\frac{2}{3}$	0	1	0	0	$\frac{1}{3}$
s_1	$\frac{1}{3}$	0	0	1	-1	$\frac{2}{3}$
x_1	1	1	0	0	1	-1

Optimal



Sensitivity analysis

- Industrial problem: the input data (A, b, and c) are frequently uncertain (market surveys, managerial evaluations, ...);
- practically impossible to evaluate all alternatives.
- **Sensitivity analysis:** method to evaluate the result of the variation of **a single** input data;

  $\Longrightarrow$ definition of a **confidence interval** such that:

  if the value is modified within the interval, the optimal basis remains optimal.
- Important: this does NOT imply that the solution does not change:

  The new solution (x) and its value need to be recomputed.
- Two main cases: effect on the optimal solution of a change in
  - the right-hand side of a constraint;
  - the objective function coefficient for a variable;

  using these results, it is possible to determine the corresponding confidence intervals.
- We will not see how to determine the confidence intervals: they are **provided by any software for linear programming**.
- The analysis becomes more complex when more values are changed at the same time: it is frequently better to solve the LP with the new data.

Changing the right-hand side of a constraint

- ith constraint: b_i changed to $\overline{b}_i$.
- **Remind:** b is only used to compute the values of the basic variables (**column 0**) as

$$x'_\beta = B^{-1}b$$

  while the relative costs do not vary;
- let $\hat{b}' = (b_1, b_2, \ldots, \overline{b}_i, \ldots, b_m)$:

  the current optimal basis remains optimal for all $\overline{b}_i$ values per for which the basic solution remains feasible, i.e., if $B^{-1}\hat{b} \geq 0$.

Changing the objective function coefficient for a variable

- cost of the kth variable changes from c_k to $\overline{c}_k$.
- **Remind:** c is only used to compute the relative costs (**row 0**) as

$$c' - c'_\beta B^{-1}A$$

- **Observe:** the current solution value can only change if $x_k \in \mathcal{B}$;

  changing the cost of a single variable modifies ALL relative costs;
- let $\hat{c}' = (c_1, c_2, \ldots, \overline{c}_k, \ldots, c_n)$:
- **the current optimal basis remains optimal if $\hat{c}' - \hat{c}'_\beta B^{-1}A \geq 0$.**

Sensitivity analysis (cont'd)

- Example of slide 1 in standard form:

$$\begin{array}{rrrrrrrrrl} \min & -4x_1 & - & 2x_2 & & & & & & \\ & 3x_1 & + & x_2 & + & x_3 & & & = & 7 \\ & x_1 & + & x_2 & & & + & x_4 & = & 3 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}$$

- The simplex algorithm produces the optimal solution $x_1 = 2$, $x_2 = 1$. From the final tableau:

$$Y = \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}, \text{ from which } B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

- Second right-end side changed from 3 to 4:

$$\hat{b} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \Longrightarrow x_\beta = B^{-1}\hat{b} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix} \geq 0.$$ **The optimal basis does not change.**

- Second right-end side changed from 3 to 2:

$$\hat{b} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \Longrightarrow x_\beta = B^{-1}\hat{b} = \begin{bmatrix} 3 \\ -\frac{1}{2} \end{bmatrix} \not\geq 0.$$ **The current basis is no longer optimal.**

Sensitivity analysis (cont'd)

- From the A matrix of the standard form,

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \Longrightarrow B^{-1}A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} A = \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

- Coefficient of x_2 changed from -2 to -3:

$$\hat{c}' = \begin{bmatrix} -4 & -3 & 0 & 0 \end{bmatrix}, \hat{c}'_\beta = \begin{bmatrix} -4 & -3 \end{bmatrix}, \hat{c}'_\beta B^{-1}A = \begin{bmatrix} -4 & -3 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}.$$

$\hat{c}' - \hat{c}'_\beta B^{-1}A = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{5}{2} \end{bmatrix} \geq 0.$ **The current basis remains optimal.**

- Coefficient of x_2 changed from -2 to -5:

$$\hat{c}' = \begin{bmatrix} -4 & -5 & 0 & 0 \end{bmatrix}, \hat{c}'_\beta = \begin{bmatrix} -4 & -5 \end{bmatrix}, c'_\beta B^{-1}A = \begin{bmatrix} -4 & -5 & \frac{1}{2} & -\frac{11}{2} \end{bmatrix}.$$

$\hat{c}' - \hat{c}'_\beta B^{-1}A = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{11}{2} \end{bmatrix} \ngeq 0,$ **The current basis is no longer optimal.**

Shadow prices

- Frequently used, with various definition, in Economics (cost benefit analysis);
- in Linear programming, consider a model in a production context:

$$\textbf{max } f(x) \text{ (profit function) } \textbf{s.t. } a_i' x \le b_i (i = 1, \dots, m);$$

- b_i = available quantity of the ith resource. In industrial practice, frequently b_i is not a hard limit, but it can be set by management with some flexibility.
- **Problem:** given the current optimal solution, is it convenient to increase a certain bound b_i (by also considering its cost)?
- **Definition: the *shadow price* for resource i is the increase of the objective function value the could be obtained by increasing the available amount b_i of the resource, provided the new value lies inside the confidence interval of the sensitivity analysis.**
- Remind the sensitivity analysis of the example of slide 1:

$$\begin{array}{rrcrcr} \max z = & 4x_1 & + & 2x_2 & & \\ & 3x_1 & + & x_2 & \le & 7 \\ & x_1 & + & x_2 & \le & 3 \\ & x_1 & , & x_2 & \ge & 0 \end{array}$$

- Optimal solution $x_1 = 2$, $x_2 = 1$ of value 10.
  Sensitivity analysis: if $b_2 := 4$, the current basis, $\mathcal{B} = \{A_1, A_2\}$, remains optimal;
  optimal solution $x_1 = \frac{3}{2}$, $x_2 = \frac{5}{2}$ of value 11: shadow price for the second resource $= 1$.
- Similarly: shadow price for the first resource $= 1$.

Shadow prices (cont'd)

- Why do we use the term **"prize"** for a profit increase?
- When a shadow price is positive, the cost of the resource increase must be considered.
- For example, suppose the shadow price for resource i is 3: if the unit **cost** for increasing the resource is **lower than 3** then increasing it would be **convenient**, otherwise it would not.
- **Shadow price** $=$ maximum price we are willing to pay for each additional unit of a resource.
- **Duality** offers an immediate way to determine the shadow prices.
- Let us resume the example:

$$\begin{array}{rrcrcr} \max z = & 4x_1 & + & 2x_2 & & \\ & 3x_1 & + & x_2 & \leq & 7 \\ & x_1 & + & x_2 & \leq & 3 \\ & x_1 & , & x_2 & \geq & 0 \end{array} \qquad \begin{array}{rrcrcr} \min w = & 7\pi_1 & + & 3\pi_2 & & \\ & 3\pi_1 & + & \pi_2 & \geq & 4 \\ & \pi_1 & + & \pi_2 & \geq & 2 \\ & \pi_1 & , & \pi_2 & \geq & 0. \end{array}$$

optimal dual solution: $\pi_1 = \pi_2 = 1$.

- **Remind:** The objective function of the dual is $\pi' b$;

  primal and dual optimal solutions have the same value.

- the objective function coefficient for π_i is b_i (the right-end side of primal constraint i). **Hence**
- **Property: the optimal value of dual variable π_i is the shadow price for primal constraint i.**

An example using commercial software

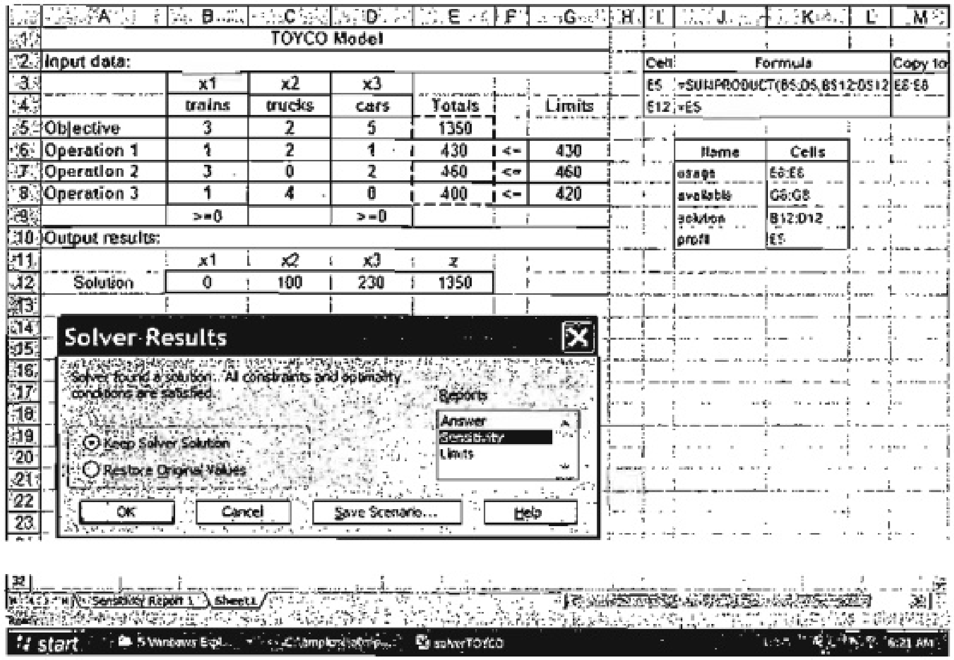


FIGURE 3.14
TORA sensitivity analysis for the TOYCO model

Sensitivity Analysis

Variable	CurrObjCoeff	MinObjCoeff	MaxObjCoeff	Reduced Cost
x1:	3.00	-infinity	7.00	4.00
x2:	2.00	0.00	10.00	0.00
x3:	5.00	2.33	infinity	0.00
Constraint	**Curr RHS**	**Min RHS**	**Max RHS**	**Dual Price**
1(<):	430.00	230.00	440.00	1.00
2(<):	460.00	440.00	860.00	2.00
3(<):	420.00	400.00	infinity	0.00

Primal-Dual algorithm

(P) $\min z = c'x$

$$\begin{aligned} Ax &= b \geq 0 \\ x &\geq 0 \end{aligned}$$

(D) $\max w = \pi' b$

$$\begin{aligned} \pi' A &\leq c' \\ \pi' &\gtrless 0 \end{aligned}$$

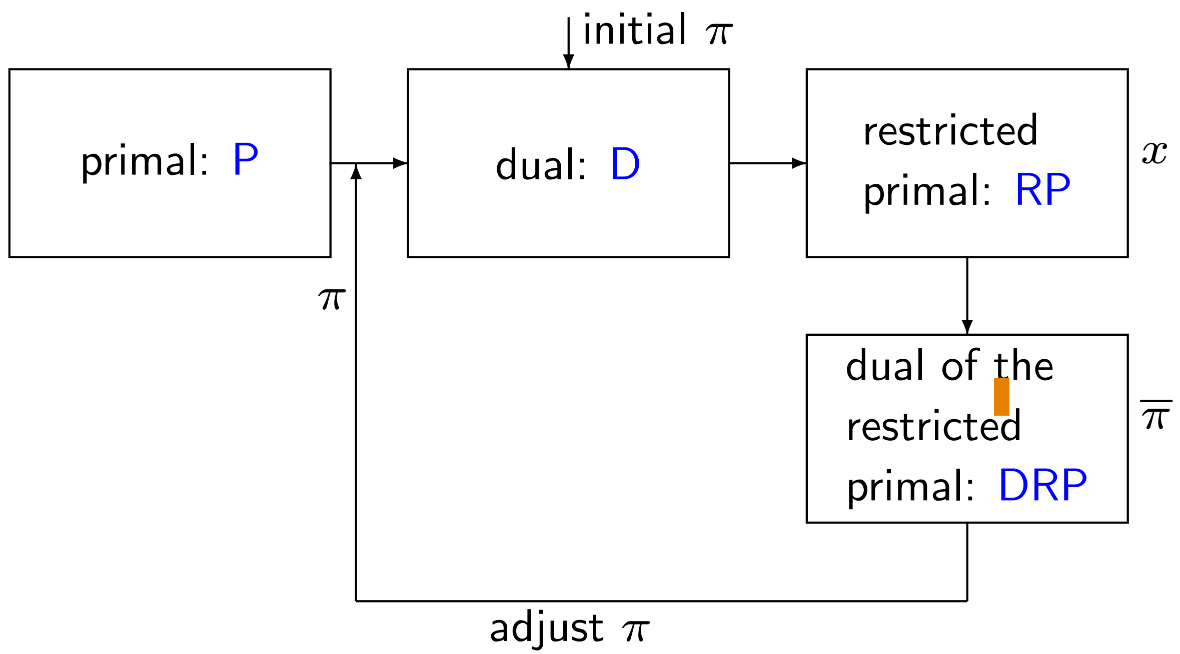
- **Complementary slackness:**

$$\begin{aligned} \pi_i(a_i'x - b_i) &= 0 \quad \forall\ i \qquad (\alpha) \quad \text{(satisfied)} \\ (c_j - \pi' A_j)x_j &= 0 \quad \forall\ j \qquad (\beta) \end{aligned}$$

- **Algorithm**: given a π feasible for D, look for an x feasible for P satisfying (β), i.e., such that:

$$x_j = 0 \ \ \forall j \ : \ c_j - \pi' A_j > 0$$

- **Approach:**



Primal-Dual algorithm (cont'd)

1. Inizialization (how to find an initial feasible π)

- If $c \geq 0$ then $\pi = 0$ is feasible.
- If $c \ngeq 0$: new variable x_{n+1} with cost $c_{n+1} = 0$;

  new constraint $x_1 + \cdots + x_n + x_{n+1} = b_{m+1} = M(\to +\infty)$:

$$
\begin{array}{rrcrcl}
(\mathrm{P}') & \min z = c'x & + & 0 & & \\
& Ax & & & = & b \\
& \displaystyle\sum_{j=1}^{n} x_j & + & x_{n+1} & = & b_{m+1} \\
& x & & & \geq & 0 \\
& & & x_{n+1} & \geq & 0
\end{array}
$$

Equivalent problem: (i) same objective function;

(ii) $\forall\ \overline{x}$ feasible for P, the new constraint is satisfied by setting $\overline{x}_{n+1} = b_{m+1} - \sum_{j=1}^{n} \overline{x}_j$.

$$
\begin{array}{rrcrcl}
(\mathrm{D}') & \max w = \pi'b & + & \pi_{m+1} b_{m+1} & & \\
& \pi' A_j & + & \pi_{m+1} & \leq & c_j \ (j = 1, \ldots, n) \\
& & & \pi_{m+1} & \leq & 0 \\
& \pi' & & & \gtreqless & 0
\end{array}
$$

Feasible solution: $\pi_i = 0 \ (i = 1, \ldots, m)$, $\pi_{m+1} = \min\{c_j\} \ (< 0)$

Primal-Dual algorithm (cont'd)

2. Primal $\rightarrow$ Dual $\rightarrow$ Restricted Primal

- π feasible for D: $J = \{j \ : \ \pi' A_j = c_j\}$:
- an x feasible for P is optimal if and only if $x_j = 0 \ \forall \ j \notin J$ $\Rightarrow$ we need an x (if any) such that:

$$\left\{ \begin{array}{rcl} Ax & = & b \\ x & \geq & 0 \\ x_j & = & 0 \ \forall \ j \notin J \end{array} \right. \text{, i.e., } \left\{ \begin{array}{rcl} \sum_{j \in J} a_{ij} x_j & = & b_i \ (i = 1, \ldots, m) \\ x_j & \geq & 0 \ \ (j \in J) \end{array} \right.$$

- to look for such x, let us "invent" a new LP:

$$\text{(RP)} \ \ \min \zeta = \sum_{i=1}^{m} x_i^a$$
$$\begin{array}{rcrcll} \sum_{j \in J} a_{ij} x_j & + & x_i^a & = & b_i & (i = 1, \ldots, m) \\ x_j & & & \geq & 0 & (j \in J) \\ & & x_i^a & \geq & 0 & (i = 1, \ldots, m) \end{array}$$

- **If the solution to RP is $\zeta_{\text{opt}} = 0$, then x and π solve P and D.**

$$
\text{Example 1 : (P)} \min z = \begin{array}{rrrcl} 2x_1 & +\ x_2 & +3x_3 & & \\ x_1 & +2x_2 & & = & 4 \\ x_1 & -\ x_2 & +\ x_3 & = & 1 \\ x_1 & ,\ x_2 & ,\ x_3 & \geq & 0 \end{array}
\qquad
\text{(D)} \max w = \begin{array}{rrcl} 4\pi_1 & +\pi_2 & & \\ \pi_1 & +\pi_2 & \leq & 2 \\ 2\pi_1 & -\pi_2 & \leq & 1 \\ & \pi_2 & \leq & 3 \\ \pi_1 & ,\ \pi_2 & \gtreqless & 0 \end{array}
$$

Feasible dual solution: $\pi_1 = \pi_2 = 1 \Rightarrow J = \{1, 2\}$.

Find an x such that

$$
\begin{array}{rrrcl} x_1 & +2x_2 & & = & 4 \\ x_1 & -\ x_2 & & = & 1 \\ & & x_3 & = & 0 \\ x_1 & ,\ x_2 & & \geq & 0 \end{array}
\quad \Rightarrow \quad
\text{(RP)} \min \zeta = \begin{array}{rrrrcl} & & x_1^a & +x_2^a & & \\ x_1 & +2x_2 & +x_1^a & & = & 4 \\ x_1 & -\ x_2 & & +x_2^a & = & 1 \\ x_1 & ,\ x_2 & ,\ x_1^a & ,\ x_2^a & \geq & 0 \end{array}
$$

$$
\begin{array}{r|r|rrrr|}
 & 0 & 0 & 0 & 1 & 1 \\
 & & x_1 & x_2 & x_1^a & x_2^a \\
\hline
-\zeta & -5 & -2 & -1 & 0 & 0 \\
\hline
x_1^a & 4 & 1 & 2 & 1 & 0 \\
x_2^a & 1 & 1 & -1 & 0 & 1 \\
\hline
\end{array}
$$

		x_1	x_2	x_1^a	x_2^a
$-\zeta$	-3	0	-3	0	2
x_1^a	3	0	3	1	-1
x_1	1	1	-1	0	1

		x_1	x_2	x_1^a	x_2^a	
$-\zeta$	0	0	0	1	1	
x_2	1	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	Optimal
x_1	2	1	0	$\frac{1}{3}$	$\frac{2}{3}$	

Example 2 : (P)

$$\begin{array}{lrrrrrr} \min & x_1 & & +\ x_3 & & & \\ & x_1 & +2x_2 & & +x_4 & = & 5 \\ & & x_2 & +2x_3 & & = & 6 \\ & x_1 & ,\ x_2 & ,\ x_3 & ,\ x_4 & \geq & 0 \end{array}$$

(D)

$$\begin{array}{lrrrr} \max & 5\pi_1 & +6\pi_2 & & \\ & \pi_1 & & \leq & 1 \\ & 2\pi_1 & +\ \pi_2 & \leq & 0 \\ & & 2\pi_2 & \leq & 1 \\ & \pi_1 & & \leq & 0 \\ & \pi_1 & ,\ \pi_2 & \gtrless & 0 \end{array}$$

Feasible dual solution: $\pi_1 = \pi_2 = 0 \Rightarrow J = \{2, 4\}$.

$$
\begin{array}{lrrrrrr}
\text{(RP)} \min \zeta = & & & x_1^a & +x_2^a & & \\
& 2x_2 & +x_4 & +x_1^a & & = & 5 \\
& x_2 & & & +x_2^a & = & 6 \\
& x_2 & , x_4 & , x_1^a & , x_2^a & \geq & 0
\end{array}
$$

$$
\begin{array}{r|r|rrrr|}
 & 0 & 0 & 0 & 1 & 1 \\
 & & x_2 & x_4 & x_1^a & x_2^a \\
\hline
-\zeta & -11 & -3 & -1 & 0 & 0 \\
\hline
x_1^a & 5 & 2 & 1 & 1 & 0 \\
x_2^a & 6 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
$$

$$
\begin{array}{r|r|rrrr|}
 & & x_2 & x_4 & x_1^a & x_2^a \\
\hline
-\zeta & -\frac{7}{2} & 0 & \frac{1}{2} & \frac{3}{2} & 0 \\
\hline
x_2 & \frac{5}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\
x_2^a & \frac{7}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \\
\hline
\end{array}
$$

Optimal with $\zeta_{\text{opt}} > 0$.

Solution of the dual of RP: $\overline{\pi}_1 = -\frac{1}{2}$, $\overline{\pi}_2 = 1$.

Primal-Dual algorithm (cont'd)

3. Restricted Primal $\rightarrow$ Dual of the Restricted Primal $\rightarrow$ Dual

- If $\zeta_{\text{opt}} > 0$, we consider the dual of RP:

$$\begin{array}{llll} \text{(DRP)} \ \max & \pi' b & & \\ & \pi' A_j & \leq & 0 \quad (j \in J) \\ & \pi_i & \leq & 1 \quad (i = 1, \ldots, m) \\ & \pi_i & \gtreqless & 0 \quad (i = 1, \ldots, m) \end{array}$$

- RP $\Rightarrow$ solution $\overline{\pi}$ of DRP.

  We try again with a corrected π: $\pi^* = \pi + \vartheta \, \overline{\pi}$, with ϑ such that

  (a) π^* is feasible for D;

  (b) the solution value of the dual increases.

- **Determining** ϑ:

  From **(b)**: new solution value $= \pi^{*\prime} b = \pi' b + \vartheta \, \overline{\pi}' b$,

  where $\overline{\pi}' b = \zeta_{\text{opt}} > 0$ ($\Leftarrow$ RP and DRP have the same solution value)

  $\Rightarrow$ **it must be** $\vartheta > 0$.

- **Determining ϑ:**

  From **(a)**: it must be

$$\begin{aligned} \pi^{*\prime} A_j &\le c_j \quad (j = 1, \dots, n) \quad \Rightarrow \\ \pi' A_j + \vartheta \overline{\pi}' A_j &\le c_j \quad (j = 1, \dots, n). \end{aligned}$$

  If $\overline{\pi}' A_j \le 0 \; \forall \; j$ then ϑ can indefinitely increase $\Rightarrow$ new cost of D $\to +\infty \Rightarrow$ P is impossible.

  On the other hand, $\overline{\pi}$ is an optimal (hence feasible) solution to DRP $\Rightarrow$

  $\overline{\pi}' A_j \le 0 \; \forall \; j \in J$. Hence

  A. If $\zeta_{\text{opt}} > 0$ in RP, and the optimal solution to DRP satisfies $\overline{\pi}' A_j \le 0 \; \forall \; j \notin J$, then P is impossible.

  If instead $\exists \; j \notin J \; : \; \overline{\pi}' A_j > 0$, we must ensure that for all such j's we have

$$\pi' A_j + \vartheta \overline{\pi}' A_j \le c_j, \text{ i.e., } \vartheta \le \frac{c_j - \pi' A_j}{\overline{\pi}' A_j} \text{ hence}$$

  B. If $\zeta_{\text{opt}} > 0$ in RP and $\exists \; j \notin J \; : \; \overline{\pi}' A_j > 0$, then the maximum ϑ that ensures feasibility is

$$\vartheta_{\max} = \min_{j \notin J \; : \; \overline{\pi}' A_j > 0} \left\{ \frac{c_j - \pi' A_j}{\overline{\pi}' A_j} \right\}$$

  and the new cost is $w^* = w + \vartheta_{\max} \overline{\pi}' b > w$.

Example 2 (resumed) (DRP)

$$\begin{array}{rcrcl} \max 5\pi_1 & & +6\pi_2 & & \\ 2\pi_1 & + & \pi_2 & \leq & 0 \\ \pi_1 & & & \leq & 0 \\ \pi_1 & & & \leq & 1 \\ & & \pi_2 & \leq & 1 \\ \pi_1 & , & \pi_2 & \lesseqgtr & 0 \end{array}$$

Solution: $\overline{\pi}_1 = -\frac{1}{2}$, $\overline{\pi}_2 = 1$.

$\pi_1^* = 0 + \vartheta \cdot (-\frac{1}{2})$, $\pi_2^* = 0 + \vartheta \cdot 1$, with $\vartheta > 0$ and such that

$$\left.\begin{array}{rcrcl} \pi_1^* & & & \leq & 1 \\ 2\pi_1^* & + & \pi_2^* & \leq & 0 \\ & & 2\pi_2^* & \leq & 1 \\ \pi_1^* & & & \leq & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcll} -\dfrac{\vartheta}{2} & & & \leq & 1 & \text{(i)} \\ -\vartheta & + & \vartheta & \leq & 0 & \text{(ii)} \\ & & 2\vartheta & \leq & 1 & \\ -\dfrac{\vartheta}{2} & & & \leq & 0 & \text{(iii)} \end{array}$$

Certainly satisfied: (ii), (iii) $\Leftarrow J = \{2, 4\}$, and (i) $\Leftarrow \overline{\pi}' A_1 < 0$.

From the remaining one: $\vartheta_{\max} = \frac{1}{2} \Rightarrow \begin{cases} \pi_1^* = -\frac{1}{4} \\ \pi_2^* = \frac{1}{2} \end{cases}$ (new feasible solution to D)

New cost of D: $-\frac{5}{4} + \frac{6}{2} = \frac{7}{4}$ $(= 0 + \frac{1}{2} \cdot \frac{7}{2})$; new $J = \{2, 3\}$.

New RP : (RP) $\min \zeta =$

$$
\begin{array}{rrrrrl}
 & & x_1^a & +x_2^a & & \\
2x_2 & & +x_1^a & & = & 5 \\
x_2 & +2x_3 & & +x_2^a & = & 6 \\
x_2 & , x_3 & , x_1^a & , x_2^a & \geq & 0
\end{array}
$$

	0	0	0	1	1
		x_2	x_3	x_1^a	x_2^a
$-\zeta$	-11	-3	-2	0	0
x_1^a	5	2	0	1	0
x_2^a	6	1	2	0	1

		x_2	x_3	x_1^a	x_2^a
$-\zeta$	$-\frac{7}{2}$	0	-2	$\frac{3}{2}$	0
x_2	$\frac{5}{2}$	1	0	$\frac{1}{2}$	0
x_2^a	$\frac{7}{2}$	0	2	$-\frac{1}{2}$	1

		x_2	x_3	x_1^a	x_2^a
$-\zeta$	0	0	0	1	1
x_2	$\frac{5}{2}$	1	0	$\frac{1}{2}$	0
x_3	$\frac{7}{4}$	0	1	$-\frac{1}{4}$	$\frac{1}{2}$

$\zeta_{\text{opt}} = 0$:

optimal primal solution: $x_1 = 0, x_2 = \frac{5}{2}, x_3 = \frac{7}{4}, x_4 = 0$;

optimal dual solution: $\pi_1 = -\frac{1}{4}, \pi_2 = \frac{1}{2}$; $w = z = \frac{7}{4}$.

If we had obtained $\zeta_{\text{opt}} > 0$, new iteration RP$\rightarrow$DRP$\rightarrow$D.

Primal-Dual algorithm (cont'd)

procedure PRIMAL_DUAL:
begin
  for each $b_i < 0$ **do** multiply the ith equation by -1;
  comment: we are given a feasible dual solution π;
  $optimal := infeasible :=$ false;
  while $optimal =$ false **and** $infeasible =$ false **do**
    begin
      $J := \{j \; : \pi' A_j = c_j\}$;
      call SIMPLEX for problem RP, and let $\overline{\pi}$ be the solution to its dual (DRP);
      if $\zeta_{\text{opt}} = 0$ **then** $optimal =$ true
      else
        if $\overline{\pi}' A_j \leq 0 \; \forall j \notin J$ **then** $infeasible :=$ true
        else $\vartheta := \min\limits_{j \notin J : \overline{\pi}' A_j > 0} \left\{ \dfrac{c_j - \pi' A_j}{\overline{\pi}' A_j} \right\}$, and $\pi := \pi + \vartheta \overline{\pi}$
    end
end.

$$\left.\begin{array}{lll} (1) & \text{P bounded,} & \text{D bounded} \\ (2) & \text{P impossible,} & \text{D unbounded} \end{array}\right\} \text{algorithm;}$$

$$\left.\begin{array}{lll} (3) & \text{P impossible,} & \text{D impossible} \\ (4) & \text{P unbounded,} & \text{D impossible} \end{array}\right\} \Leftrightarrow \; \exists \; c_j < 0 \; \Rightarrow x_{n+1}, b_{m+1} \rightarrow \text{D formally possible, } \textbf{but}$$

if (3) $\Rightarrow (\pi \rightarrow \infty) \Rightarrow$ algorithm(P impossible)
if (4) $\Rightarrow (x \rightarrow \infty) \Rightarrow$ algorithm (P unbounded)

Ex.:

$$\begin{array}{lrcrcccrcrcrcrcl} \min & -x_1 & - & x_2 \\ & x_1 & - & x_2 & \leq & 0 & \rightarrow & x_1 & - & x_2 & + & x_3 & = & 0 & & \text{(P unbounded)} \\ & x_1 & , & x_2 & \geq & 0 & & & & & & x_3 & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcrcrcl} \min & -x_1 & - & x_2 \\ & x_1 & - & x_2 & + & x_3 & & & = & 0 \\ & x_1 & + & x_2 & + & x_3 & + & x_4 & = & M \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}
\qquad
\begin{array}{lrcrcl} \max & & & M\pi_2 \\ & \pi_1 & + & \pi_2 & \leq & -1 \\ & -\pi_1 & + & \pi_2 & \leq & -1 \\ & \pi_1 & + & \pi_2 & \leq & 0 \\ & & & \pi_2 & \leq & 0 \\ & \pi_1 & , & \pi_2 & \gtrless & 0 \end{array}$$

$$\left.\begin{array}{l} \pi_1 = 0, \\ \pi_2 = \min\{c_j\} = -1 \\ J = \{1, 2\} \end{array}\right\} \Rightarrow
\quad
\begin{array}{lrcrcrcrcl} \text{(RP)} \min & & & & & x_1^a & + & x_2^a \\ & x_1 & - & x_2 & + & x_1^a & & & = & 0 \\ & x_1 & + & x_2 & & & + & x_2^a & = & M \\ & x_1 & , & x_2 & , & x_1^a & , & x_2^a & \geq & 0 \end{array}$$

	0	0	0	1	1
		x_1	x_2	x_1^a	x_2^a
$-\zeta$	$-M$	-2	0	0	0
x_1^a	0	1	-1	1	0
x_2^a	M	1	1	0	1

		x_1	x_2	x_1^a	x_2^a
$-\zeta$	$-M$	0	-2	2	0
x_1	0	1	-1	1	0
x_2^a	M	0	2	-1	1

		x_1	x_2	x_1^a	x_2^a
$-\zeta$	0	0	0	1	1
x_1	$\frac{M}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
x_2	$\frac{M}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Optimal solution: $x_1 = x_2 = \frac{M}{2},\ z = w = -M$

Ex.: (P) $\min\; x_1 - 2x_2$ (P, D impossible : $\pi_1 - \pi_2 \le 1,\; -\pi_1 + \pi_2 \le -2$)

$$
\begin{array}{rcrcrcrcrcl}
x_1 & - & x_2 & - & x_3 & & & & & = & 1 \\
-x_1 & + & x_2 & & & - & x_4 & & & = & 1 \\
x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & M \\
x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \ge & 0
\end{array}
$$

$\leftarrow$ inizialization

(D) $\max\; \pi_1 + \pi_2 + M\pi_3$

$$
\begin{array}{rcrcrcr}
\pi_1 & - & \pi_2 & + & \pi_3 & \le & 1 \\
-\pi_1 & + & \pi_2 & + & \pi_3 & \le & -2 \\
-\pi_1 & & & + & \pi_3 & \le & 0 \\
& - & \pi_2 & + & \pi_3 & \le & 0 \\
& & & & \pi_3 & \le & 0 \\
\pi_1 & , & \pi_2 & , & \pi_3 & \lesseqgtr & 0
\end{array}
$$

feasible solution :

$\pi' = (0, 0, -2)$

$\Rightarrow J = \{2\}$

		0	1	1	1
	0	x_2	x_1^a	x_2^a	x_3^a
$-\zeta$	$-M-2$	-1	0	0	0
x_1^a	1	-1	1	0	0
x_2^a	1	1	0	1	0
x_3^a	M	1	0	0	1

		x_2	x_1^a	x_2^a	x_3^a
$-\zeta$	$-M-1$	0	0	1	0
x_1^a	2	0	1	1	0
x_2	1	1	0	1	0
x_3^a	$M-1$	0	0	-1	1

$\overline{\pi}' = (1, 0, 1)$

$\vartheta_{\max} = \min\{\frac{3}{2}, \frac{2}{1}\} = \frac{3}{2}$

$\pi' = (\frac{3}{2}, 0, -\frac{1}{2})$

$J = \{1, 2\}$

		0	0	1	1	1
	0	x_1	x_2	x_1^a	x_2^a	x_3^a
$-\zeta$	$-M-2$	-1	-1	0	0	0
x_1^a	1	1	-1	1	0	0
x_2^a	1	-1	1	0	1	0
x_3^a	M	1	1	0	0	1

		x_1	x_2	x_1^a	x_2^a	x_3^a
$-\zeta$	$-M-1$	0	-2	1	0	0
x_1	1	1	-1	1	0	0
x_2^a	2	0	0	1	1	0
x_3^a	$M-1$	0	2	-1	0	1

		x_1	x_2	x_1^a	x_2^a	x_3^a
$-\zeta$	-2	0	0	0	0	1
x_1	$\frac{M+1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
x_2^a	2	0	0	1	1	0
x_2	$\frac{M-1}{2}$	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$

$\overline{\pi}' = (1, 1, 0)$

$\overline{\pi}' A_j \leq 0 \ \forall \ j \Rightarrow$ P impossible.