Assignments, Matchings, and the European Roots of Combinatorial Optimization

EURO Gold Medal 2018 Laureate Lecture

Silvano Martello

DEI "Guglielmo Marconi", Alma Mater Studiorum Università di Bologna, Italy

1. The Fifties: Birth of Combinatorial Optimization

2. The Assignment Problem and Kőnig's Matching Theorem

3. The First Egerváry's Theorem (1931) and Strong Duality

4. Jacobi (1851) and the Hungarian method

5. Open shop, satellite communications,
   and the Second Egerváry's Theorem

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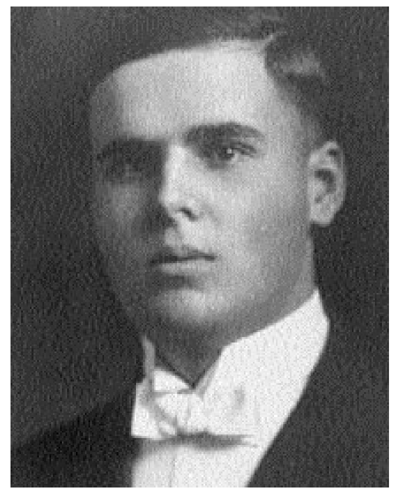

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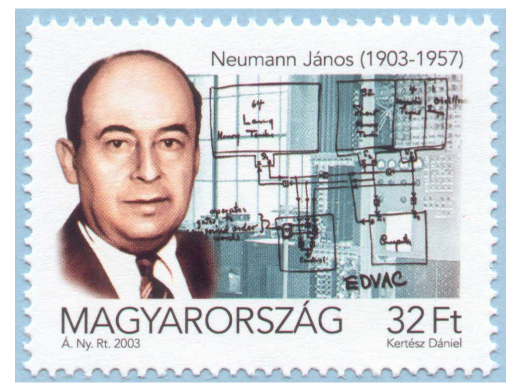

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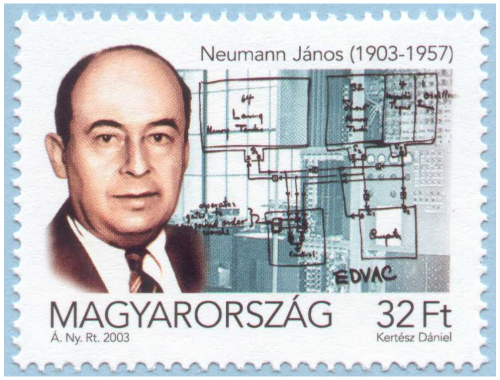

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The Fifties: Specific results

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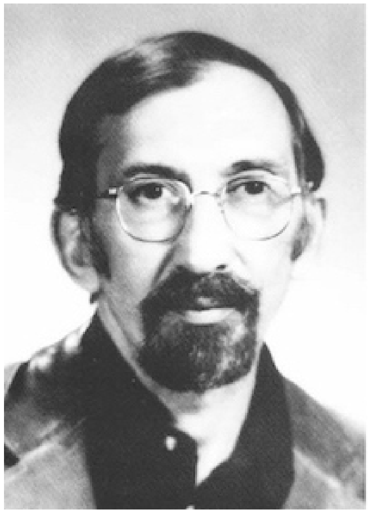


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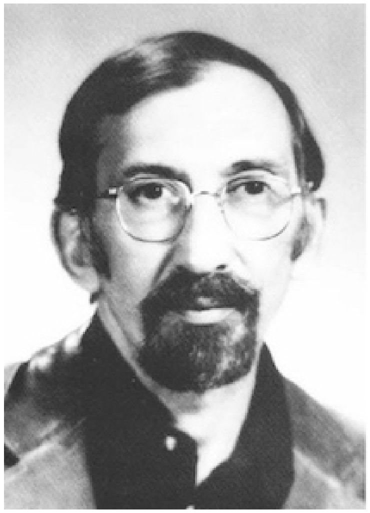


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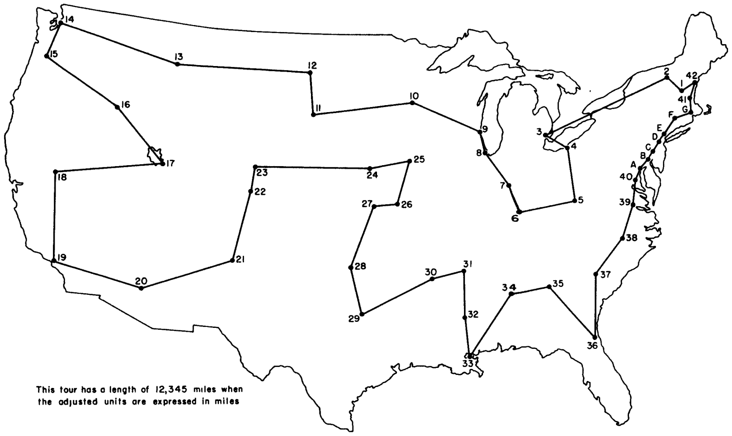

ILP formulation,
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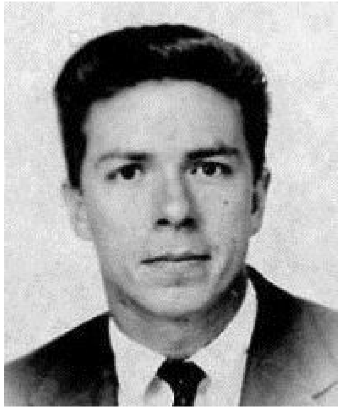

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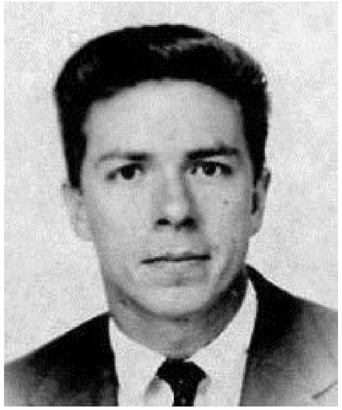

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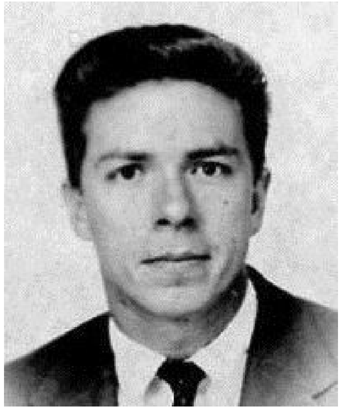

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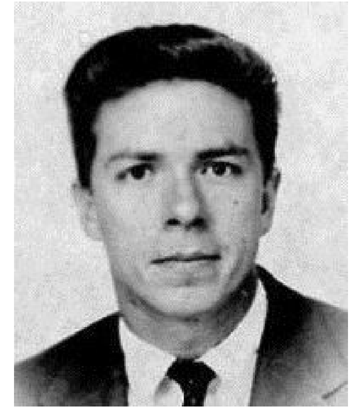

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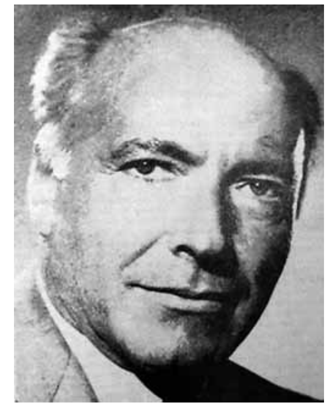
D. Kőnig, 1916  **J. Egerváry, 1931** 

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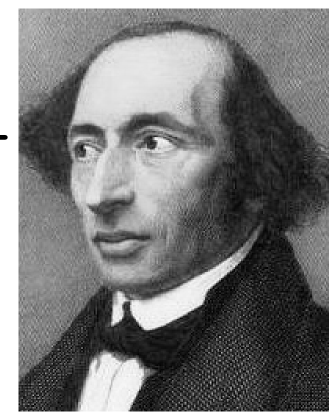
Frobenius, 1917,1912  **. . . Jacobi, 1851** 

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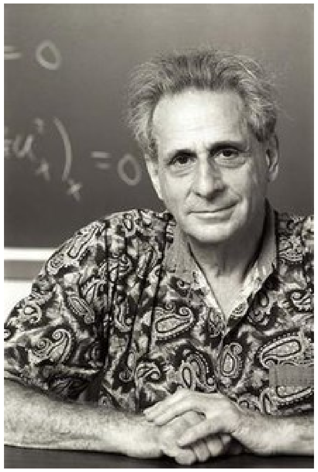
R.C. Prim  **J.B. Kruskal, Jr.** 

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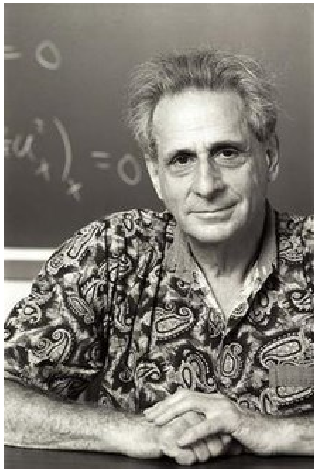

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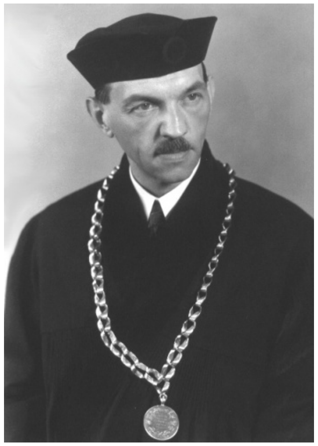
V. Jarník, 1930



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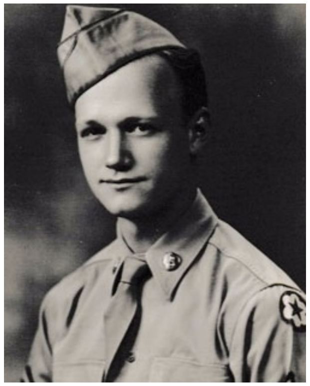

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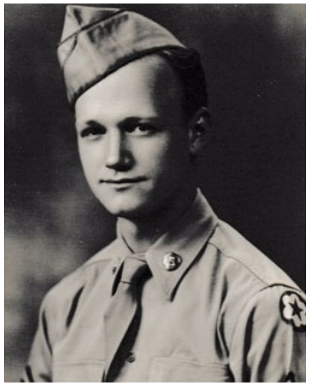

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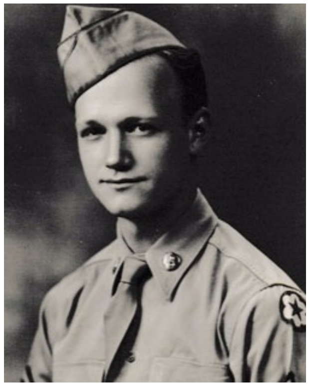

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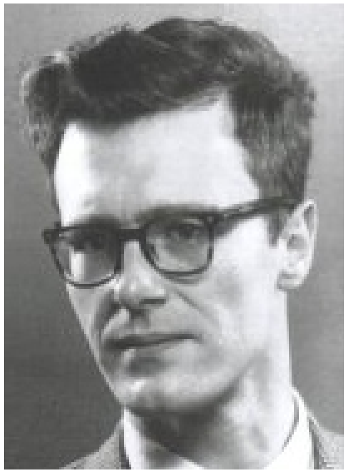
L.R. Ford, Jr.  **E.W. Dijkstra**  **et al.**

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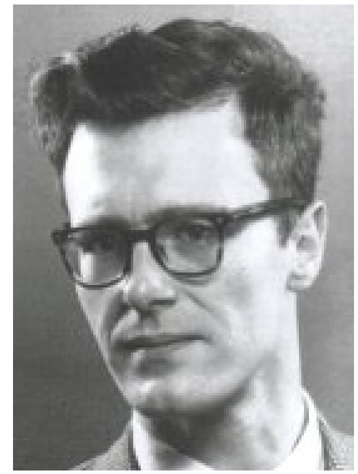
L.R. Ford, Jr.  **E.W. Dijkstra**  **et al.**

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 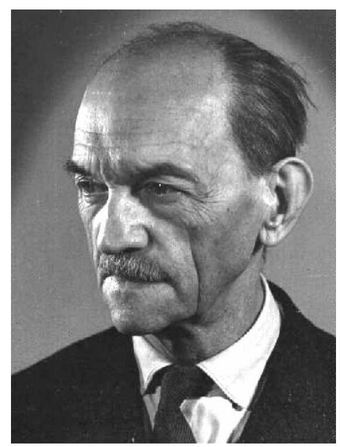

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Ailsa Land  Alison Doig 

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- Given a **weighted bipartite graph** $G = (U, V; E)$ with $|U| = |V| = n$ and c_{ij} = cost of edge $(i, j) \in E$, find a perfect matching of maximum value.

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PSYCHOMETRIKA—VOL. 15, NO. 3
SEPTEMBER, 1950

THE PROBLEM OF CLASSIFICATION OF PERSONNEL*

ROBERT L. THORNDIKE
TEACHERS COLLEGE, COLUMBIA UNIVERSITY

The personnel classification problem arises in its pure form when all job applicants must be used, being divided among a number of job categories. The use of tests for classification involves problems of two types: (1) problems concerning the design, choice, and weighting of tests into a battery, and (2) problems of establishing the optimum administrative procedure of using test results for assignment. A consideration of the first problem emphasizes the desirability of using simple, factorially pure tests which may be expected to have a wide range of validities for different job categories. In the use of test results for assignment, an initial problem is that of expressing predictions of success in different jobs in comparable score units. These units should take account of predictor validity and of job importance. Procedures are described for handling assignment either in terms of daily quotas or in terms of a stable predicted yield.

The past decade, and particularly the war years, have witnessed a great concern about the classification of personnel and a vast expenditure of effort presumably directed towards this end. In all branches of the military establishment were found "general classification" tests or test batteries planned to serve a classification function. Since the war the number of published test batteries designed for differential prediction has rapidly multiplied. It seems timely, therefore, to look into the problem of the classification of personnel to see what the concept means, what issues it raises with respect to the theory of measurement, and what problems it presents with respect to the practical operation of a testing program.

It must be indicated that much of the present discussion represents an examination of concepts, a raising of questions, and an offering of intuitive suggestions, rather than a presentation of mathematically established answers. The defining of questions represents a first step in answering them. It is hoped that clarification of the problems and issues in the following pages may stimulate others to solve them.

Personnel classification, as the term is used here, is best de-

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R.L. Thorndike:

"**Given**: A set of N vacancies to be filled, and N individuals to be used in filling them, **Required**: To assign the individuals to the jobs in such a way that the average success of all the individuals in all the jobs to which they are assigned will be a maximum."

There are, as has been indicated, a finite number of permutations in the assignment of men to jobs. When the classification problem as formulated above was presented to a mathematician, he pointed to this fact and said that from the point of view of the mathematician there was no problem. Since the number of permutations was finite, one had only to try them all and choose the best. He dismissed the problem at that point. This is rather cold comfort to the psychologist, however, when one considers that only ten men and ten jobs mean over three and a half million permutations. Trying out all the permutations may be a mathematical solution to the problem, it is not a practical solution.

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- Today, the fastest supercomputer on earth (93 Petaflops) cannot solve a 25×25 AP through enumeration in less than one century.

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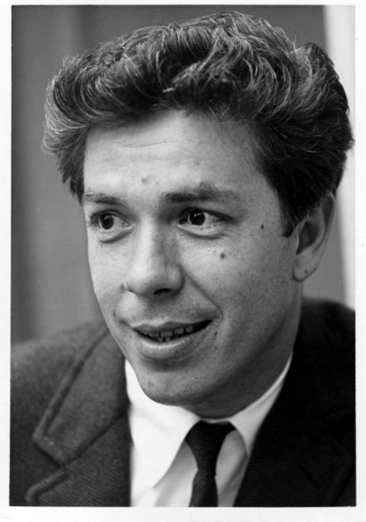

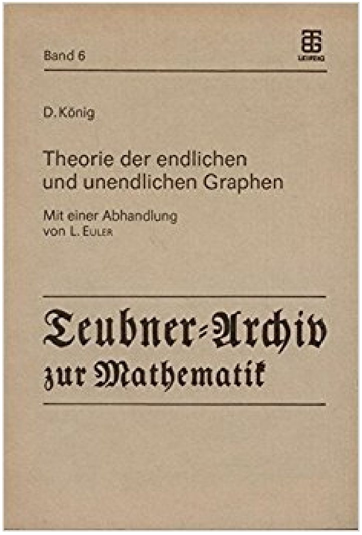

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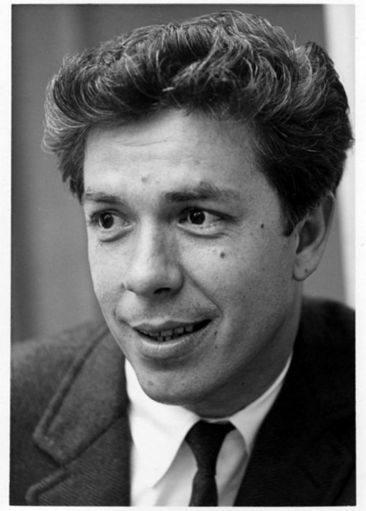

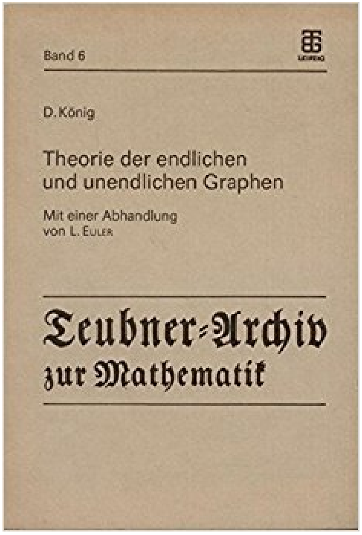

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- During WW2 he worked to help Jewish mathematicians.

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- The alternating path algorithm is the ancestor of many modern algorithms, including Ford-Fulkerson's max-flow min-cut.

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- He was however reluctant to contribute to the war effort and he always refused to hand over Hungarian Jews to German authorities.

- In 1944 Horty attempted to strike a secret deal with the Allies.

  The Germans invaded and took control of the country.

  Horthy was removed from power and a puppet government led by the Arrow Cross Party was established.

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- in October 1944, fearing to be ordered to move to the ghetto, Kőnig committed suicide.

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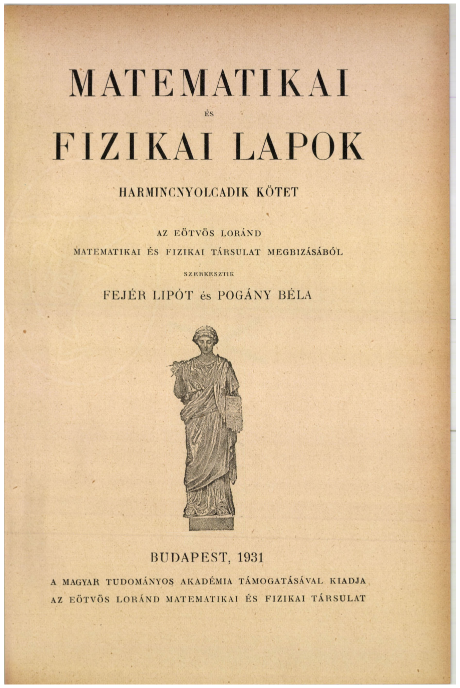


MATEMATIKAI

ÉS

FIZIKAI LAPOK

HARMINCNYOLCADIK KÖTET

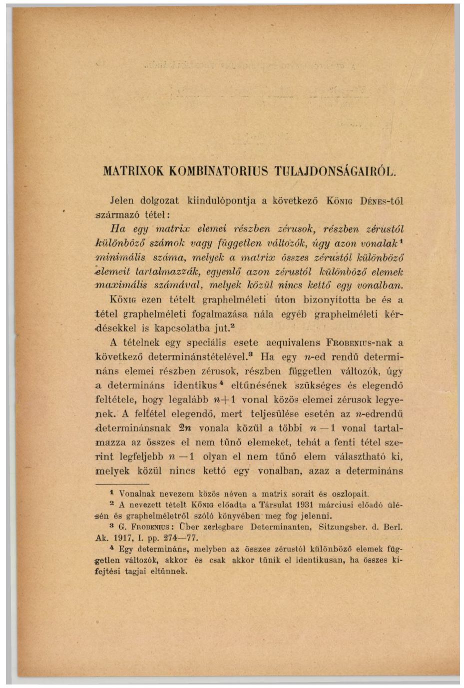
AZ EÖTVÖS LORÁND
MATEMATIKAI ÉS FIZIKAI TÁRSULAT MEGBIZÁSÁBÓL

SZERKESZTIK

FEJÉR LIPÓT és POGÁNY BÉLA

BUDAPEST, 1931

A MAGYAR TUDOMÁNYOS AKADÉMIA TÁMOGATÁSÁVAL KIADJA
AZ EÖTVÖS LORÁND MATEMATIKAI ÉS FIZIKAI TÁRSULAT



MATRIXOK KOMBINATORIUS TULAJDONSÁGAIRÓL.

Jelen dolgozat kiindulópontja a következő König Dénes-től származó tétel:

Ha egy matrix elemei részben zérusok, részben zérustól különböző számok vagy független változók, úgy azon vonalak[1] *minimális száma, melyek a matrix összes zérustól különböző elemeit tartalmazzák, egyenlő azon zérustól különböző elemek maximális számával, melyek közül nincs kettő egy vonalban.*

König ezen tételt graphelméleti úton bizonyította be és a tétel graphelméleti fogalmazása nála egyéb graphelméleti kérdésekkel is kapcsolatba jut.[2]

A tételnek egy speciális esete aequivalens Frobenius-nak a következő determinánstételével.[3] Ha egy n-ed rendű determináns elemei részben zérusok, részben független változók, úgy a determináns identikus[4] eltűnésének szükséges és elegendő feltétele, hogy legalább $n+1$ vonal közös elemei zérusok legyenek. A feltétel elegendő, mert teljesülése esetén az n-edrendű determinánsnak $2n$ vonala közül a többi $n-1$ vonal tartalmazza az összes el nem tűnő elemeket, tehát a fenti tétel szerint legfeljebb $n-1$ olyan el nem tűnő elem választható ki, melyek közül nincs kettő egy vonalban, azaz a determináns

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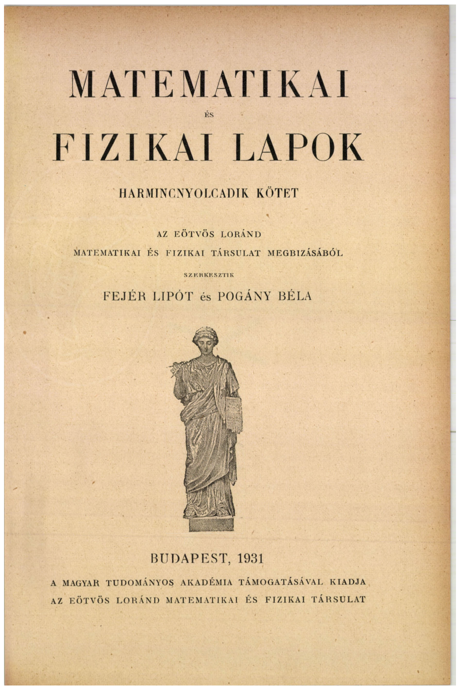

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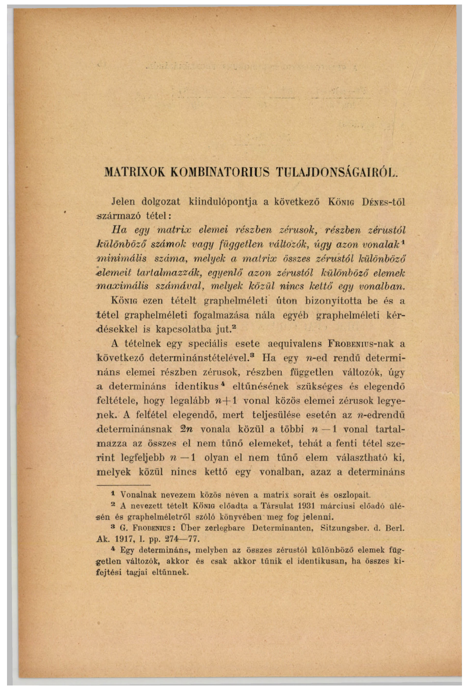

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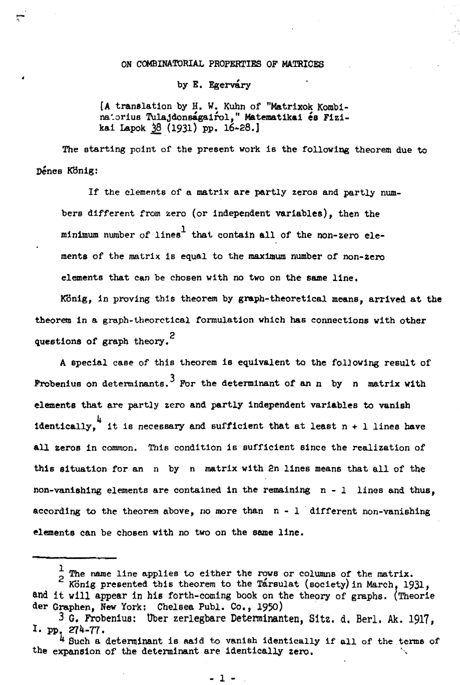

ON COMBINATORIAL PROPERTIES OF MATRICES

by E. Egerváry

[A translation by H. W. Kuhn of "Matrixok Kombinatorius Tulajdonságairól," Matematikai és Fizikai Lapok 38 (1931) pp. 16-28.]

The starting point of the present work is the following theorem due to Dénes König:

If the elements of a matrix are partly zeros and partly numbers different from zero (or independent variables), then the minimum number of lines[1] that contain all of the non-zero elements of the matrix is equal to the maximum number of non-zero elements that can be chosen with no two on the same line.

König, in proving this theorem by graph-theoretical means, arrived at the theorem in a graph-theoretical formulation which has connections with other questions of graph theory.[2]

A special case of this theorem is equivalent to the following result of Frobenius on determinants.[3] For the determinant of an n by n matrix with elements that are partly zero and partly independent variables to vanish identically,[4] it is necessary and sufficient that at least n + 1 lines have all zeros in common. This condition is sufficient since the realization of this situation for an n by n matrix with 2n lines means that all of the non-vanishing elements are contained in the remaining n - 1 lines and thus, according to the theorem above, no more than n - 1 different non-vanishing elements can be chosen with no two on the same line.

[1] The name line applies to either the rows or columns of the matrix.

[2] König presented this theorem to the Társulat (society) in March, 1931, and it will appear in his forth-coming book on the theory of graphs. (Theorie der Graphen, New York: Chelsea Publ. Co., 1950)

[3] G. Frobenius: Über zerlegbare Determinanten, Sitz. d. Berl. Ak. 1917, I. pp. 274-77.

[4] Such a determinant is said to vanish identically if all of the terms of the expansion of the determinant are identically zero.

- 1 -

Kuhn translated Egerváry's paper, and published the translation as a Research Report of the George Washington University: *On Combinatorial Properties of Matrices*. It contains **2 theorems**.

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- In 1941 he became full professor at the T.U. of Budapest.
- in 1955 he became head of the Department of Mathematics.

First Egerváry's theorem

- Remind the **Duality of the Assignment problem** (**Fifties**):

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\begin{aligned}
(P)\ \max\quad & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\text{s.t.}\quad & \sum_{j=1}^{n} x_{ij} = 1 & (i = 1, 2, \ldots, n),\\
& \sum_{i=1}^{n} x_{ij} = 1 & (j = 1, 2, \ldots, n),\\
& x_{ij} \in \{0, 1\} & (i, j = 1, 2, \ldots, n).\\
(D)\ \min\quad & \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j & \\
\text{s.t.}\quad & u_i + v_j \geq c_{ij} & (i, j = 1, 2, \ldots, n).
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Strong Duality: $\min \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \; = \; \max \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$

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- **Covering system** = set of *lines* (rows and columns) that contain the ith row of an $n \times n$ matrix C with multiplicity λ_i and the jth column with multiplicity μ_j, and satisfy

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$$\min \sum_{k=1}^{n} (\lambda_k + \mu_k) = \max_{\varphi} \sum_{i=1}^{n} c_{i\varphi(i)}.$$

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- The revolt spread across Hungary and the Communist government quickly collapsed.
  A new democratic government was established.

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  In 1958, fearing to be imprisoned, Egerváry committed suicide.

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- Egerváry proved his theorem by giving an algorithm that iteratively adjusts the current (feasible, non optimal) λ_i and μ_j values so that

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- Using Egerváry's method and Kőnig's maximum matching algorithm, in the fall of 1953 Kuhn solved several 12×12 assignment problems **by hand**.

- Each of these examples took under two hours to solve.
  This must have been one of the last times when pencil and paper could beat the largest and fastest electronic computer in the world. (H.W. Kuhn, EURO XXIV, Lisbon, July 2010)

- The algorithm was christened the **Hungarian algorithm** in honor of these two mathematicians and published in two famous papers on *Naval Research Logistics Quarterly*:

  "The Hungarian method for the assignment problem" (1955)

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- **But this is not the end of the story.**
  A recent historical discovery: A posthumous paper written, prior to **1851 (!!!)** by one of the greatest mathematicians of all time.

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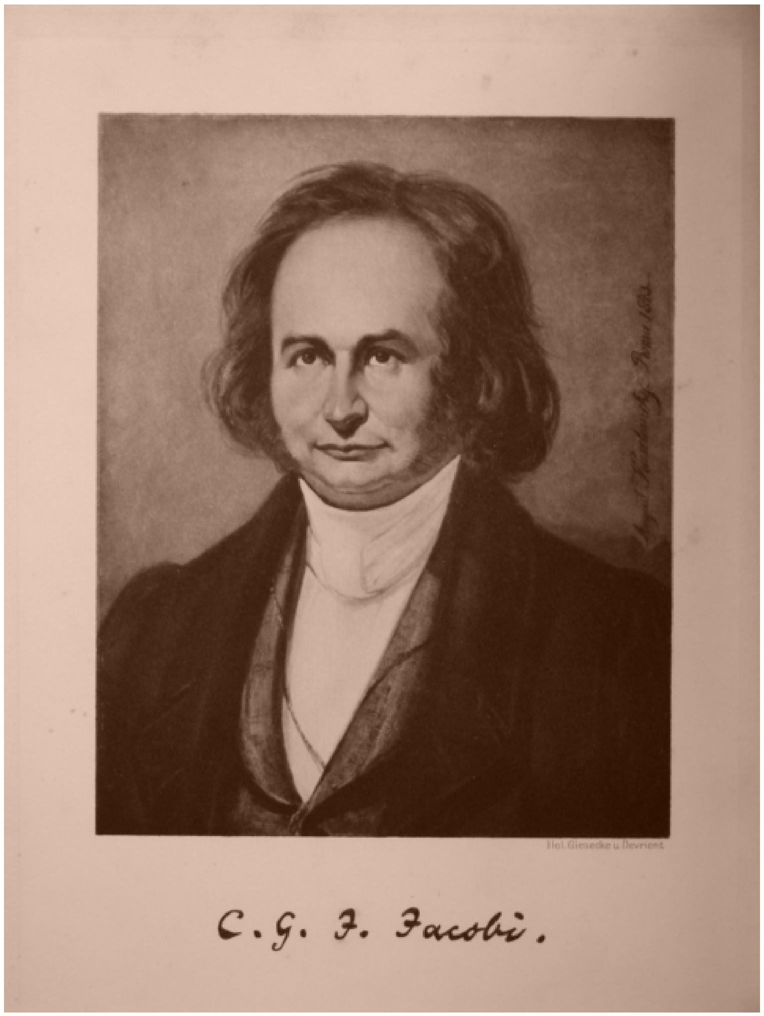

DE INVESTIGANDO ORDINE SYSTEMATIS AEQUATIONUM DIFFERENTIALIUM VULGARIUM CUJUSCUNQUE

AUCTORE

C. G. J. JACOBI,
PROF. ORD. MATH. REGIOM.

Borchardt Journal für die reine und angewandte Mathematik, Bd. 64 p. 297—320.

About the research of the order of a system of arbitrary ordinary differential equations (posthumous manuscript).

2.

De solutione problematis inaequalitatum, quo investigatio ordinis systematis aequationum differentialium quarumcunque innititur. Proposito schemate, definitur Canon. Dato Canone quocunque, invenitur simplicissimus.

Antecedentibus investigatio ordinis systematis aequationum differentialium vulgarium revocata est ad sequens problema inaequalitatum etiam per se tractatu dignum:

Problema.

Disponantur nn quantitates $h_k^{(i)}$ *quaecunque in schema Quadrati, ita ut habeantur n series horizontales et n series verticales, quarum quaeque est n terminorum. Ex illis quantitatibus eligantur n transversales, i. e. in seriebus horizontalibus simul atque verticalibus diversis positae, quod fieri potest* $1.2\ldots n$ *modis; ex omnibus illis modis quaerendus est is, qui summam n numerorum electorum suppeditet maximam.*

Dispositis quantitatibus $h_k^{(i)}$ in figuram quadraticam

$$\begin{matrix} h_1' & h_2' & \ldots & h_n' \\ h_1'' & h_2'' & \ldots & h_n'' \\ \cdot & \cdot & \cdot & \cdot \\ h_1^{(n)} & h_2^{(n)} & \ldots & h_n^{(n)}, \end{matrix}$$

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198 DE INVESTIGANDO ORDINE SYSTEMATIS AEQUATIONUM

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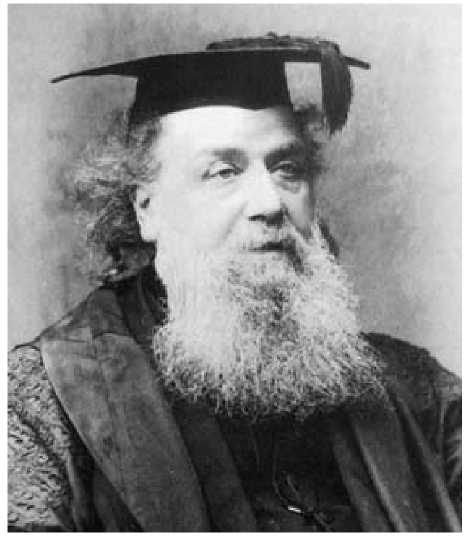
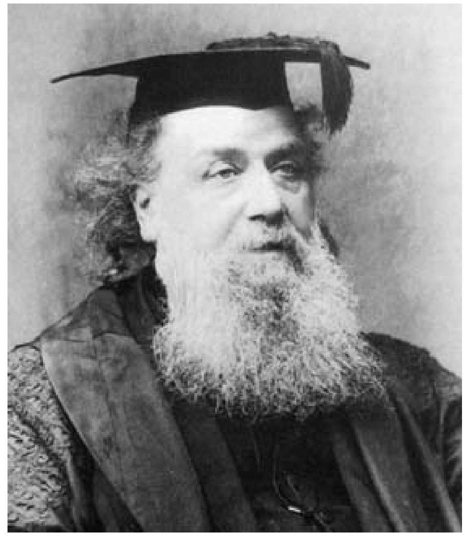
Note: Jacobi did not even have proper terminology for a "matrix"! The term was coined in same years by James J. Sylvester:



This homaloidal law has not been stated in the above commentary in its form of greatest generality. For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding to which may be termed determinants of the pth order. We have, then, the following proposition. The number of uncoevanescent determinants constituting a system of the pth order derived from a given matrix, n terms broad and m terms deep, may equal, but can never exceed the number

$$(n-p+1)(m-p+1).$$



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matrix (n.) late 14c., "uterus, womb," from Old French *matrice*.

Canon derivatus I.

	I	II	III	IV	V	VI	VII	l
I	11*	11	8	19	18	10	5	4
II	10	15*	14	13	18	21	17	7
III	8	13	17*	18	17	25	12	2
IV	4	11	14	25*	20	21	27	0
V	9	6	12	14	27*	22	34	4
VI	6	13	8	14	11	25*	22	5
VII	11	12	8	22	24	21	40*	0

Canon derivatus II.

	I	II	III	IV	V	VI	VII	l
I	11*	11	8	19	18	10	5	4
II	8	13*	12	11	16	19	15	5
III	6	11	15*	16	15	23	10	0
IV	4	11	14	25*	20	21	27	0
V	7	4	10	12	25*	20	32	2
VI	4	11	6	12	9	23*	20	3
VII	11	12	8	22	24	21	40*	0

Canon simplicissimus.

	I	II	III	IV	V	VI	VII	l
I	11*	11	8	19	18	10	5	4
II	7	12*	11	10	15	18	14	4
III	6	11	15*	16	15	23	10	0
IV	4	11	14	25*	20	21	27	0
V	6	3	9	11	24*	19	31	1
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E schemate proposito, addendo terminis serierum diversarum respective

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The Jacobi method replicates the patterns of the Hungarian Method. **It IS the Hungarian Algorithm!**

1. The Fifties: Birth of Combinatorial Optimization

2. The Assignment Problem and Kőnig's Matching Theorem

3. The First Egerváry's Theorem (1931) and Strong Duality

4. Jacobi (1851) and the Hungarian method

5. Open shop, satellite communications, and the Second Egerváry's Theorem

A result that has been "discovered" several times:

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- Famous result by **Garrett Birkhoff** on doubly stochastic matrices, published (**in Spanish**) by Garrett Birkhoff in **1946** and for which John von Neumann gave in **1953** an elegant proof.

UNIVERSIDAD NACIONAL DE TUCUMAN
FACULTAD DE CIENCIAS EXACTAS, PURAS Y APLICADAS

PUBLICACION N.º 408

REVISTA

SERIE A

MATEMATICAS Y FISICA TEORICA

Volumen 5 Nos. 1 y 2

*

TUCUMAN
REPUBLICA ARGENTINA
1946

Es evidente que cada matriz A que satisface (1) satisface también

$$(1') \qquad \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1, \text{ para todo } i, j = 1, \ldots, n.$$

Estas matrices son interesantes para la probabilidad,[1] y los cuadrados mágicos son múltiplos escalares de estas matrices.

Teorema. Si una matriz $n \times n$ A satisface (1′), entonces es una media aritmética de permutaciones.

Theorem: Every doubly stochastic matrix is a convex combination of permutation matrices.

Second Egerváry's theorem

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- a system of permutation matrices which contains the kth matrix, P^k, with **multiplicity** λ_k $(\lambda_k \geq 0 \ \forall \ k)$ is called a **diagonal covering system** for C if

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- **Problem:**

  find a diagonal covering system of minimum value $\displaystyle\sum_{k=1}^{n!} \lambda_k$.

- **Theorem:** $\min \sum_{k=1}^{n!} \lambda_k = \max(\max_i \sum_{j=1}^{n} c_{ij}, \max_j \sum_{i=1}^{n} c_{ij})(= c^*)$.

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  Proof:

  1. Define a **majorant** of C, i.e., a matrix C^* such that

$$c^*_{ij} \geq c_{ij} \quad \text{and} \quad \sum_{i=1}^{n} c^*_{ij} = \sum_{j=1}^{n} c^*_{ij} = c^* \quad (i, j = 1, 2, \ldots, n).$$

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- **The algorithms of the Seventies, in 1931!**

Egerváry's bust erected in the University gardens (1992, 2006).

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Thank you for your attention

Essential references

- P. Rózsa (1984). Jenő Egerváry (1891-1958), a great personality of Hungarian mathematical school. *Periodica Polytechnica*.
- T. Gallai (1986). Dénes Kőnig: A biographical sketch. In D. Kőnig, *Theory of Finite and Infinite Graphs*, Birkhäuser, Boston (English translation by R. McCoart).
- M. Dell'Amico, S. Martello (1996). Open shop, satellite communication and a theorem by Egerváry (1931). *Operations Research Letters*.
- A. Schrijver (2005). On the History of Combinatorial Optimization. *Handbooks in OR & MS*.
- A. Frank (2005). On Kuhn's Hungarian method - A tribute from Hungary, *Naval Res. Logistics*
- F. Ollivier et B. Sadik (2007). La borne de Jacobi pour une diffiété définie par un système quasi régulier. *Comptes Rendus Mathématique*.
- T. Rapcsák (2010). The life and works of Jenő Egerváry. *Central European Journal of Operations Research*.
- S. Martello (2010). Jenő Egerváry: from the origins of the Hungarian algorithm to satellite communication. *Central European Journal of Operations Research*.
- H.W. Kuhn (2012). A tale of three eras: The discovery and rediscovery of the Hungarian method. *European Journal of Operational Research*.
- R. Burkard, M. Dell'Amico, S. Martello (2009 and 2012). *Assignment Problems*, SIAM, Philadelphia, xx + 382 pages.