Operations Research (Master's Degree Course)

6. Integer Linear Programming

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Integer Linear Programming (ILP)

- In many real-world situations the variables cannot assume fractional values: assignment problem (candidates to be assigned to duties), transportation problems (x_j = number of vehicles to be used on route j), ...
- $A(m \times n)$, $b(m)$, $c(n)$ integer (like for LP); **ILP in standard form**:

$$\begin{aligned} \min\ & c'x \\ Ax &= b \\ x &\geq 0 \\ x &\text{ integer} \end{aligned}$$

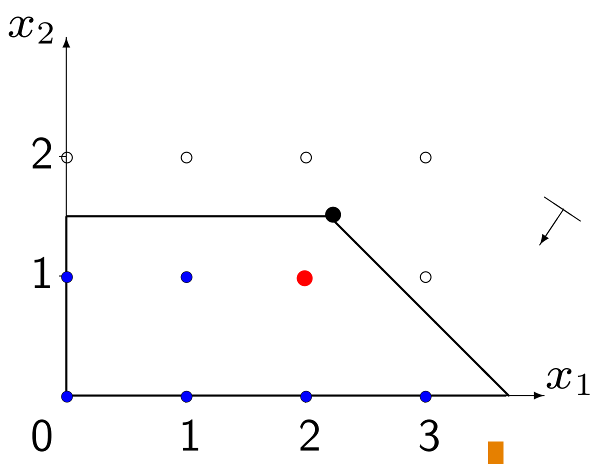
- Canonical form, general form, transformations: like for LP (integer slack and surplus variables).
- By removing the integrality constraint we obtain the **continuous relaxation of the ILP**.
- Let $z(\text{ILP})$ and $z(\text{LP})$ be the solution values of an ILP and of its continuous relaxation:
  - **Property** $z(\text{LP}) \leq z(\text{ILP})$

    Proof We look for a minimum in a larger feasible set. $\square$
  - We say that $z(\text{LP})$ is a **lower bound** on $z(\text{ILP})$.
  - $\Rightarrow$ **If $z(\text{LP})$ corresponds to an integer point $x \in R^n$ then x is optimal for the ILP.**

Geometrical interpretation

- The constraints

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \end{aligned}$$ define a polyhedron (like in LP):

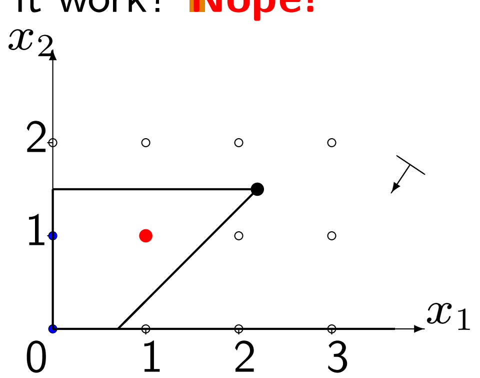


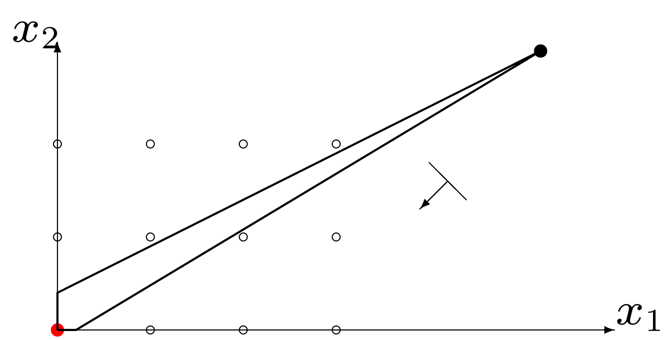
- The integrality constraint x integer imposes that the **feasible region** only consists of the points • with integer coordinates within the polyhedron.
- The **optimal solution** • is the best among such points.
- By removing the integrality constraint (**Continuous Relaxation**), the LP provides a **fractional** solution (•) and a lower bound on the optimal ILP solution.
- Could we solve ILP by rounding the fractional solution provided by the LP?

A naïve Algorithm for ILP

- **procedure LIGABUE**:
  begin
  find the optimal solution x to the continuous relaxation LP of the ILP;
  if LP is impossible **then** ILP is impossible (**comment:** certainly true)
  else
  if LP is unbounded **then** ILP is unbounded (**comment:** "normally" true)
  else
  if x is integer **then** $x^* := x$ (**comment:** certainly true)
  else round each fractional x_j to its closest integer
  end.

- Does it work? **Nope!**





- rounded points can all be unfeasible;
- integer and continuous solution can be very "far" from each other.
- Are there cases where it works, i.e., the LP relaxation always has an integer solution?

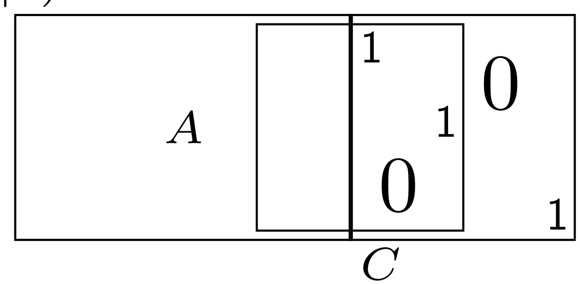
Unimodularity

- An integer square matrix B is **unimodular (UM)** if $\det(B) = \pm 1$.
- An integer rectangular $m \times n$ matrix A is **totally unimodular (TUM)** if every non-singular square submatrix of A is UM.
- **Property 1** *If A is TUM then the vertices of $\{x : Ax = b, x \geq 0\}$ are integer $\forall$ integer b.*

  Proof B = matrix corresponding to a base of A. $\Rightarrow$ basic solution: $x_\beta = B^{-1}b = \dfrac{B^a}{\det(B)}b$,
  with B^a = adjoint of B ($b^a_{ij} = (-1)^{i+j} \cdot (j,i)$-minor).
  A is TUM $\Rightarrow$ B is UM $\Rightarrow$ x is integer. $\square$
- **Property 2** *If A is TUM then the vertices of $\{x : Ax \leq b, x \geq 0\}$ are integer $\forall$ integer b.*

  Proof It is enough to show that if A is TUM then $(A|I)$ is TUM.

  Let C be a square non-singular submatrix of $(A|I)$:

  Permute the rows of C so that $\widetilde{C} = \left(\begin{array}{c|c} B & 0 \\ \hline D & I \end{array}\right)$:

  $\det(\widetilde{C}) = \det(B) \Rightarrow \det(C) = \pm\det(\widetilde{C}) = \pm 1$. $\square$
- Hence: **if A is TUM then the simplex algorithm solves ILP.**
- **Sufficient conditions** exist to check if a matrix is TUM.

General methods for ILP

Two main methods: **cutting planes** and **branch-and-bound** (combined in **branch-and-cut**).

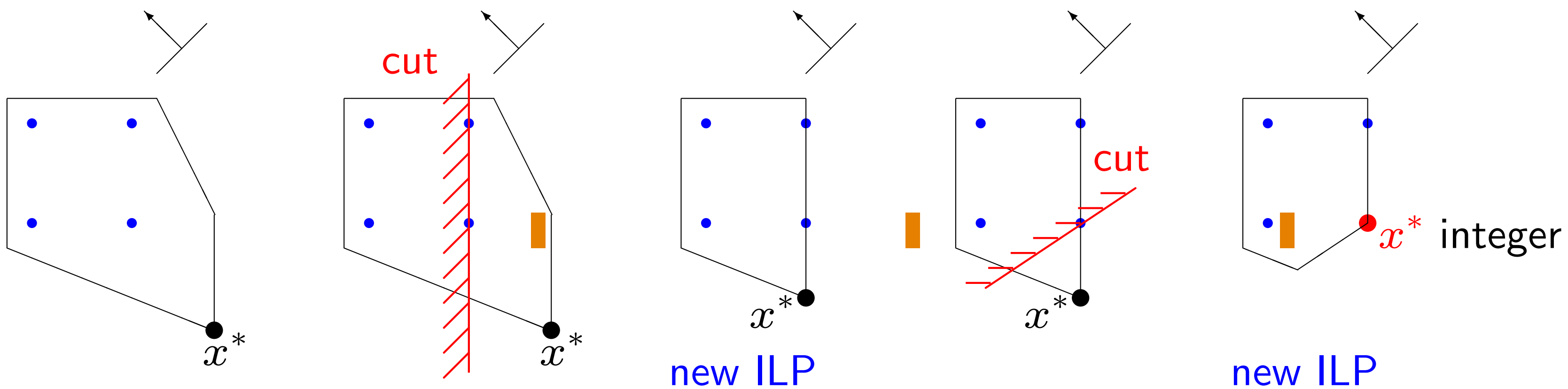
Cutting plane algorithms

Basic scheme (suppose the continuous relaxation is bounded and non-empty):

1. solve the continuous relaxation, and let x^* be the optimal solution;
2. if x^* is an integer point, then the problem is solved. Otherwise
3. **add** to the ILP a **linear constraint (cut)** which:

   (i) eliminates a part of the feasible region **containing x^*, but**

   (ii) does not eliminate any feasible integer solution;
4. solve the continuous relaxation of the new ILP ($\rightarrow$ new x^*) and go to 2.



Cutting plane algorithms (cont'd): Gomory cuts (1958)

- $\forall \; y \in R^1$, the **integer part** of y is $\lfloor y \rfloor = \max$ integer $q : q \le y$.
- Y = final LP tableau; $\mathcal{B}$ = optimal base ($\leftrightarrow B$; $x_{\beta(0)} = (-z)$);
- $\forall \; i (i = 0, \ldots, m)$ we have: $$x_{\beta(i)} + \sum_{A_j \notin \mathcal{B}} y_{ij} x_j = y_{i0} \qquad (\alpha)$$
- $x \ge 0 \Rightarrow \displaystyle\sum_{A_j \notin \mathcal{B}} \lfloor y_{ij} \rfloor x_j \le \sum_{A_j \notin \mathcal{B}} y_{ij} x_j \Rightarrow \underbrace{x_{\beta(i)} + \sum_{A_j \notin \mathcal{B}} \lfloor y_{ij} \rfloor x_j}_{\text{integer } \forall \text{ integer } x \implies} \le y_{i0}$

$$\implies x_{\beta(i)} + \sum_{A_j \notin \mathcal{B}} \lfloor y_{ij} \rfloor x_j \le \lfloor y_{i0} \rfloor \qquad (\beta)$$

- $(\alpha) - (\beta)$: $\displaystyle\sum_{A_j \notin \mathcal{B}} (y_{ij} - \lfloor y_{ij} \rfloor) x_j \ge (y_{i0} - \lfloor y_{i0} \rfloor)$.
- The **fractional part** of y_{ij} is $f_{ij} = y_{ij} - \lfloor y_{ij} \rfloor$; $0 \le f_{ij} < 1$.
- $\displaystyle\sum_{A_j \notin \mathcal{B}} f_{ij} x_j \ge f_{i0}$ (**Gomory cut** corresponding to row i).
- Multiply by -1 and add a slack variable: $\displaystyle -\sum_{A_j \notin \mathcal{B}} f_{ij} x_j + s = -f_{i0}$.

Cutting plane algorithms: Gomory cuts (cont'd)

- **Theorem** *By adding to a final* LP *tableau the cut*

$$-\sum_{A_j \notin \mathcal{B}} f_{ij} x_j + s = -f_{i0},$$

  1. *no feasible integer point is eliminated;*
  2. *the resulting tableau contains a base which is:*
     (i) *primal infeasible, if y_{i0} is not integer;*
     (ii) *dual feasible.*

  Proof

  1. the cut was obtained by just imposing the integrality constraints to the LP.

  2.(i) s is a new basic variable which, when added to $\mathcal{B}$, gives a base whose solution includes $s = -f_{i0} \leq 0$, i.e., it is primal infeasible, if y_{i0} is not integer;

  2.(ii) in row 0, the column corresponding to s has 0 $\Rightarrow$ the solution remains dual feasible. □

- It is thus convenient to continue with the dual simplex algorithm.
- Note that 2. $\Rightarrow$ if y_{i0} is not integer, the current solution is outside the resulting feasible region.
- The selected row i is called the **generating row**.
- Up to 1958 it was widely considered **impossible** to solve ILP through the Simplex algorithm.
- In the 70s and the 80s the Gomory cuts have been considered impractical, and "abandoned".
- Starting from the late 90s their usefulness has been rediscovered:
  modern Branch-and-cut solvers include Gomory cuts.

Cutting plane algorithms: Gomory cuts (cont'd)

procedure GOMORY:
begin
  remove the integrality constraints from the ILP thus obtaining an LP;
  call TWO_PHASE for LP, and let Y be the final tableau;
  if $infeasible$ = false **and** $unbounded$ = false **then**
    begin
      $feasible$:= true; $k := 0$ (**comment:** cut counter);
      while $\exists$ fractional y_{i0} **and** $feasible$ = true **do**
        begin
          select an $i : y_{i0}$ is fractional, and set $k := k + 1$;
          add the equation $-\sum_{A_j \notin \mathcal{B}} f_{ij} x_j + s_k = -f_{i0}$ to the tableau;
          call DUAL_SIMPLEX;
          if $infeasible$ = true **then** $feasible$:= false (**comment:** unbounded dual, impossible ILP)
        end
    end
end.

Convergence: If there is no degeneration, each iteration considers a base that is different from the previous ones. It can be proved that, even in case of degeneration, the method converges if:
 (i) the generating row is selected as the first fractional row;
(ii) the dual simplex pivot is selected according to a method analogous to Bland's rule.

Example:

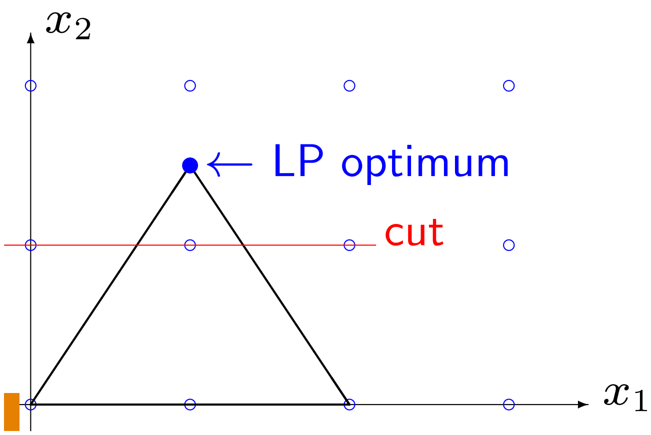
$$\begin{array}{rrcll}
\min & & - x_2 & & \\
3x_1 & +2x_2 & \le & 6 & \rightarrow \quad 3x_1 + 2x_2 + x_3 = 6 \\
-3x_1 & +2x_2 & \le & 0 & \rightarrow \quad -3x_1 + 2x_2 + x_4 = 0 \\
x_1 & , \; x_2 & \ge & 0 \text{ integer} & \qquad\qquad x_3, x_4 \ge 0 \text{ integer}
\end{array}$$

		x_1	x_2	x_3	x_4
$-z$	0	0	-1	0	0
x_3	6	3	2	1	0
x_4	0	-3	2	0	1

		x_1	x_2	x_3	x_4
$-z$	0	$-\frac{3}{2}$	0	0	$\frac{1}{2}$
x_3	6	6	0	1	-1
x_2	0	$-\frac{3}{2}$	1	0	$\frac{1}{2}$

		x_1	x_2	x_3	x_4
$-z$	$\frac{3}{2}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$
x_1	1	1	0	$\frac{1}{6}$	$-\frac{1}{6}$
x_2	$\frac{3}{2}$	0	1	$\frac{1}{4}$	$\frac{1}{4}$

generating row :	**0** (fractional)	**1** (integer)	**2** (fractional)
cut :	$\frac{1}{4}x_3 + \frac{1}{4}x_4 \ge \frac{1}{2}$	$\frac{1}{6}x_3 + \frac{5}{6}x_4 \ge 0$	$\frac{1}{4}x_3 + \frac{1}{4}x_4 \ge \frac{1}{2}$
	$\Downarrow$	$\Downarrow$	$\Downarrow$
by substitution :	$x_2 \le 1$	$x_1 - x_2 \ge -\frac{1}{2}$	$x_2 \le 1$



Selecting row 0:

$$-\frac{1}{4}x_3 - \frac{1}{4}x_4 + s_1 = -\frac{1}{2}$$

		x_1	x_2	x_3	x_4	s_1
$-z$	$\frac{3}{2}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0
x_1	1	1	0	$\frac{1}{6}$	$-\frac{1}{6}$	0
x_2	$\frac{3}{2}$	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0
s_1	$-\frac{1}{2}$	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	1

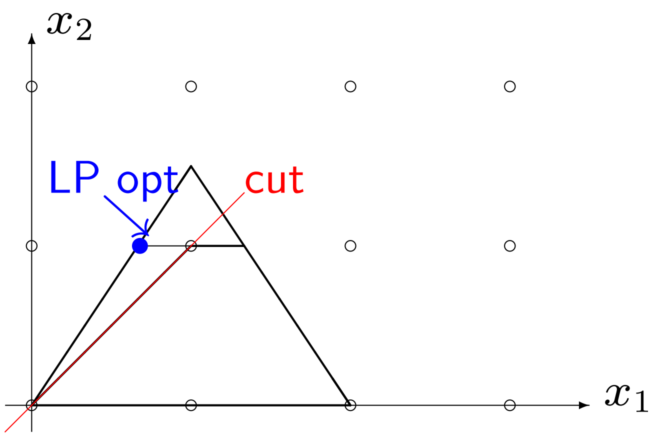
		x_1	x_2	x_3	x_4	s_1
$-z$	1	0	0	0	0	1
x_1	$\frac{2}{3}$	1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$
x_2	1	0	1	0	0	1
x_3	2	0	0	1	1	-4

generating row : $\mathbf{1}$ $(\mathbf{0}, \mathbf{2}, \mathbf{3}$ integer)

cut : $\frac{2}{3}x_4 + \frac{2}{3}s_1 \geq \frac{2}{3}$

$$\Downarrow$$

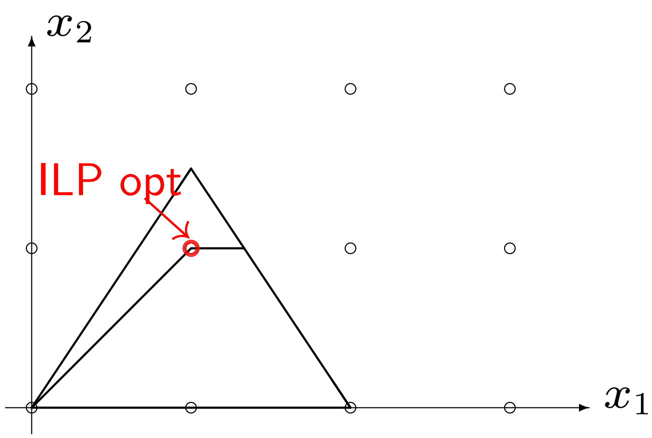
by substitution : $x_1 \geq x_2$



cut equation: $-\frac{2}{3}x_4 - \frac{2}{3}s_1 + s_2 = -\frac{2}{3}$

		x_1	x_2	x_3	x_4	s_1	s_2
$-z$	1	0	0	0	0	1	0
x_1	$\frac{2}{3}$	1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$	0
x_2	1	0	1	0	0	1	0
x_3	2	0	0	1	1	-4	0
s_2	$-\frac{2}{3}$	0	0	0	$-\frac{2}{3}$	$-\frac{2}{3}$	1

		x_1	x_2	x_3	x_4	s_1	s_2
$-z$	1	0	0	0	0	1	0
x_1	1	1	0	0	0	1	$-\frac{1}{2}$
x_2	1	0	1	0	0	1	0
x_3	1	0	0	1	0	-5	$\frac{3}{2}$
x_4	1	0	0	0	1	1	$-\frac{3}{2}$



Optimal integer solution: $x_1 = x_2 = 1$; $z = -1$.

Example:

$$
\begin{array}{rcrcrcl}
\min & x_1 & + & x_2 & & & \\
& 6x_1 & + & x_2 & \leq & 4 & \rightarrow \quad 6x_1 + x_2 + x_3 = 4 \\
& 3x_1 & & & \geq & 1 & \rightarrow \quad 3x_1 - x_4 = 1 \\
& x_1 & , & x_2 & \geq & 0 \text{ integer} & \quad x_3, x_4 \geq 0 \text{ integer}
\end{array}
$$

Phase 1:

	0	1	0	0	0	0
		x_1^a	x_1	x_2	x_3	x_4
$-\xi$	-1	0	-3	0	0	1
x_2	4	0	6	1	1	0
x_1^a	1	1	3	0	0	-1

		x_1^a	x_1	x_2	x_3	x_4
$-\xi$	0	1	0	0	0	0
x_2	2	-2	0	1	1	2
x_1	$\frac{1}{3}$	$\frac{1}{3}$	1	0	0	$-\frac{1}{3}$

Phase 2:

		0	1	1	0	0
			x_1	x_2	x_3	x_4
$-z$		$-\frac{7}{3}$	0	0	-1	$-\frac{5}{3}$
x_2		2	0	1	1	2
x_1		$\frac{1}{3}$	1	0	0	$-\frac{1}{3}$

		x_1	x_2	x_3	x_4
$-z$	$-\frac{1}{3}$	0	1	0	$\frac{1}{3}$
x_3	2	0	1	1	2
x_1	$\frac{1}{3}$	1	0	0	$-\frac{1}{3}$

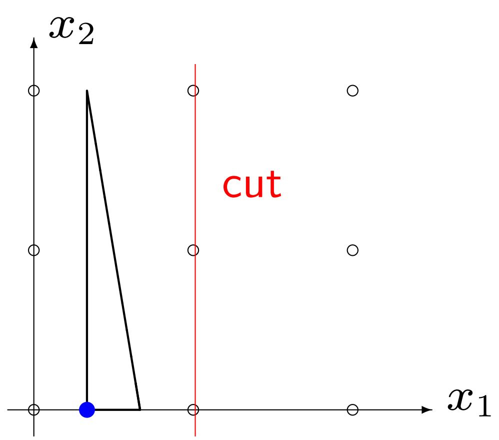
Generating row **0**: cut $\frac{1}{3}x_4 \geq \frac{2}{3}$ $(\Rightarrow -\frac{1}{3}x_4 + s_1 = -\frac{2}{3})$

		x_1	x_2	x_3	x_4	s_1
$-z$	$-\frac{1}{3}$	0	1	0	$\frac{1}{3}$	0
x_3	2	0	1	1	2	0
x_1	$\frac{1}{3}$	1	0	0	$-\frac{1}{3}$	0
s_1	$-\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	1

		x_1	x_2	x_3	x_4	s_1
$-z$	-1	0	1	0	0	1
x_3	-2	0	1	1	0	6
x_1	1	1	0	0	0	-1
x_4	2	0	0	0	1	-3

unlimited dual $\Rightarrow$

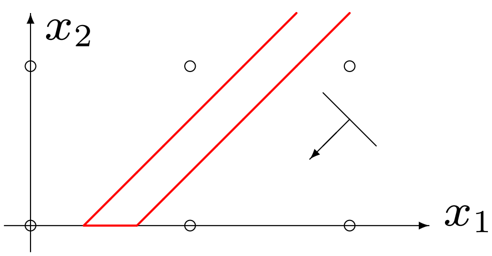
impossible primal.



- **Observation:**

When the continuous relaxation of the ILP is unbounded, "normally" the ILP is unbounded as well, but in very particular cases it can be impossible.

Example:

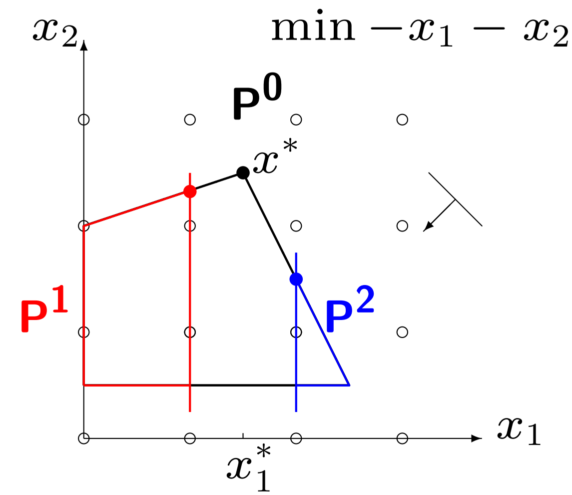


Branch-and-bound (Land and Doig, 1960)

1. $\mathbf{P^0}$ = problem to be solved;
2. solve the continuous relaxation (x^* = optimal solution);
3. if x^* is an integer point, then the problem is solved. Otherwise
4. select a fractional component, x_j^*, of x^* and impose

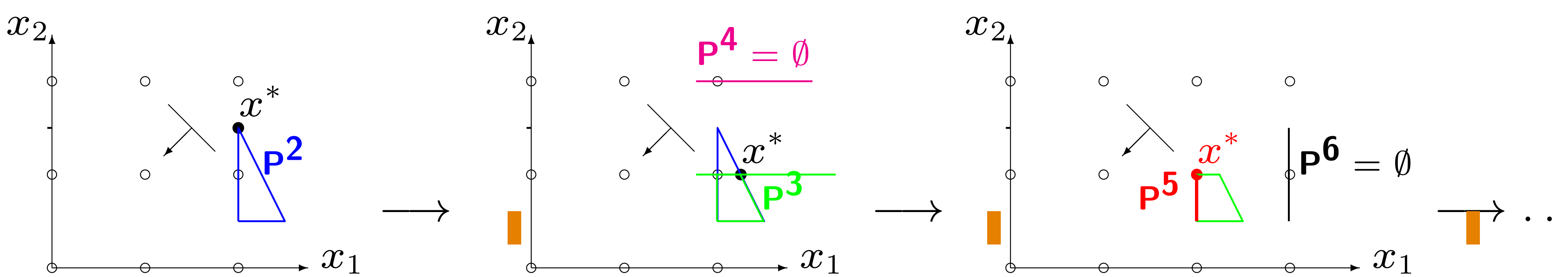
   two **mutually exclusive** and **exhaustive** constraints (**Branching**):

   $$x_j \le \lfloor x_j^* \rfloor \quad \textbf{or} \quad x_j \ge \lfloor x_j^* \rfloor + 1 \quad (\lfloor a \rfloor = \text{largest integer} \le a);$$

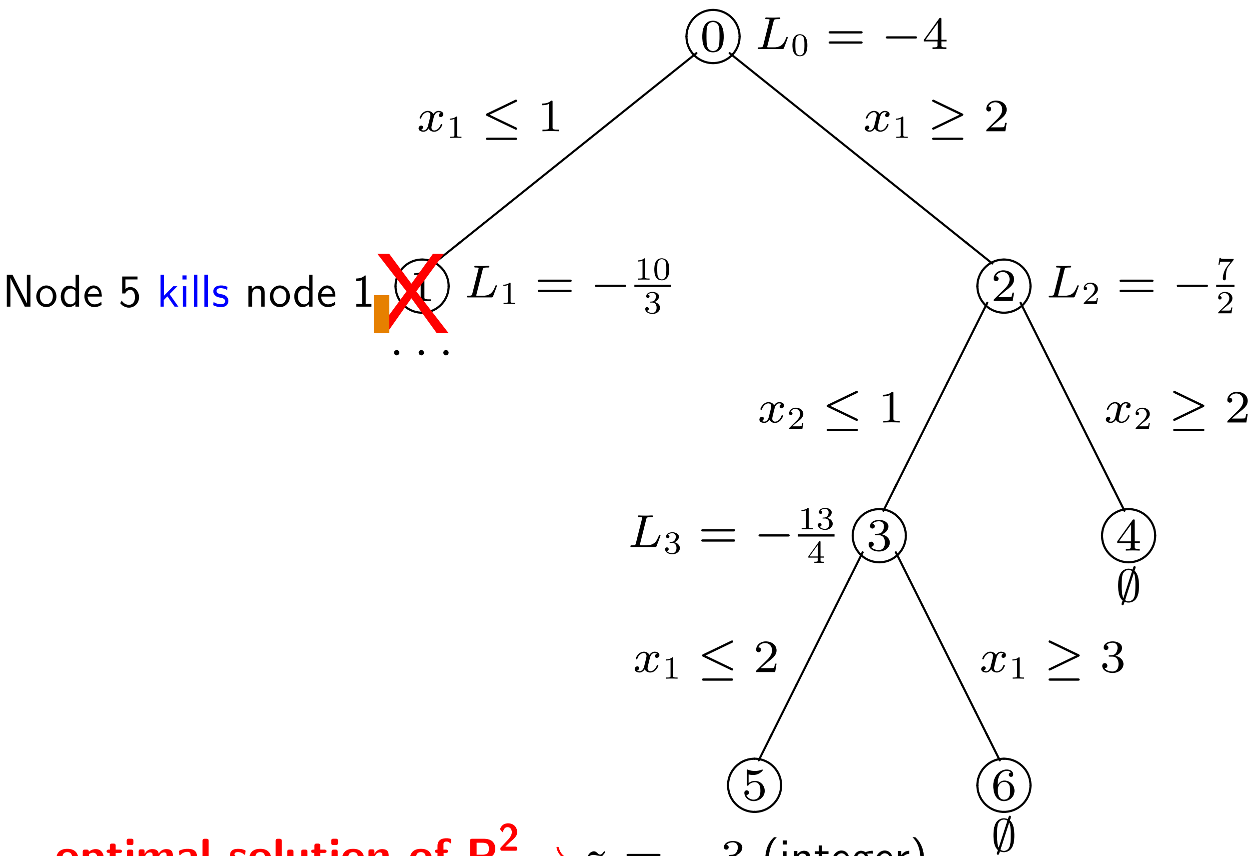


5. we obtain two new ILP problems: $\mathbf{P^1} = \mathbf{P^0}$ & $(x_j \le \lfloor x_j^* \rfloor)$,

   $\mathbf{P^2} = \mathbf{P^0}$ & $(x_j \ge \lfloor x_j^* \rfloor + 1)$;
6. **solution of** $\mathbf{P^0}$ = best solution between that of $\mathbf{P^1}$ and that of $\mathbf{P^2}$;
7. solve $\mathbf{P^1}$ and $\mathbf{P^2}$ in the same way (recursively). Example: take $\mathbf{P^2}$:



- Let L_i = solution value of the continuous relaxation of $\mathbf{P}^i$ (**Lower bound**);
  Representation through **branch-decision tree:**



Terminology:

- node
  branch
- 0 root
- 4, 5, 6 leaves
- 2 parent of 3 and 4
- 3, 4 children of 2
- 2 ancestor of 3, 4, 5 e 6
- 3, 4, 5, 6 descendants of 0 and 2

optimal solution of $\mathbf{P}^2$ $\rightarrow z = -3$ (integer)

- We should now find in the same way the optimal solution of **$\mathbf{P}^1$**, **BUT** (**Bounding**):
- the solution of $\mathbf{P}^i$ cannot have a smaller value than L_i (computed on a larger feasible region);
- as c is integer, $\lceil L_i \rceil$ is a **Lower bound** for problem $\mathbf{P}^i$ ($\lceil a \rceil$ = smallest integer $\geq a$);
- if we have already found an integer solution of value $z \leq \lceil L_i \rceil$, **we don't need to solve $\mathbf{P}^i$**.
- Solution of $\mathbf{P}^0$ = best integer solution found when no new branching is possible.

How to add the constraints to the tableau

$x_i = a$ **fractional (in base)**:

1. $x_i \leq \lfloor a \rfloor \quad \rightarrow \quad x_i + s = \lfloor a \rfloor$

$$
\begin{array}{c|c|c|ccc|c|}
 & & & & & & s \\
\hline
 & -z_0 & \overline{c}_j & 0 & \dots & 0 & 0 \\
\hline
 & & & 1 & & \mathbf{0} & 0 \\
x_i & a & [y_{ij}] & & 1 & & 0 \\
 & & & \mathbf{0} & & 1 & 0 \\
\hline
s & \lfloor a \rfloor & 0 \quad \dots & 0 & 1 & \dots 0 & 1 \\
\hline
\end{array}
$$

subtract the row of $x_i \Rightarrow$

$$
\begin{array}{c|c|c|ccc|c|}
\hline
s & r & & 0 & \dots & 0 & 1 \\
\hline
\end{array}
$$

$r = \lfloor a \rfloor - a < 0$

The relative costs remain non-negative, the current solution becomes unfeasible $\Rightarrow$ dual simplex.

How to add the constraints to the tableau (cont'd)

$x_i = a$ **fractional (in base)**:

2. $x_i \geq \lfloor a \rfloor + 1 \quad \rightarrow \quad -x_i + s = -\lfloor a \rfloor - 1 = t$

				s
	$-z_0$	$\bar{c}_j$	$0 \quad \dots \quad 0$	0
x_i	a	$[y_{ij}]$	$\begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{matrix}$	$\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$
s	t	$0 \quad \dots$	$0 \quad -1 \quad \dots 0$	1

sum the row of $x_i \Rightarrow$

s	r		$0 \quad \dots \quad 0$	1

$$r = \underbrace{a - \lfloor a \rfloor}_{< 1} - 1 < 0$$

The relative costs remain non-negative, the current solution becomes unfeasible $\Rightarrow$ dual simplex.

Example: $\max \; z = \; x_1 + 4\,x_2$

$$
\begin{aligned}
x_1 + 3\,x_2 &\le 9 \\
2\,x_1 - x_2 &\ge 0 \\
x_1,\; x_2 &\ge 0, \text{ integer.}
\end{aligned}
$$

$$
\begin{aligned}
\min \; -z = - x_1 - 4\,x_2 & \\
x_1 + 3\,x_2 + x_3 &= 9 \\
-2\,x_1 + x_2 + x_4 &= 0 \\
x_1,\; x_2,\; x_3,\; x_4 &\ge 0, \text{ integer.}
\end{aligned}
$$

The simplex algorithm gives, in two pivoting operations (Dantzig rule),

		x_1	x_2	x_3	x_4
z	$\frac{81}{7}$	0	0	$\frac{9}{7}$	$\frac{1}{7}$
x_1	$\frac{9}{7}$	1	0	$\frac{1}{7}$	$-\frac{3}{7}$
x_2	$\frac{18}{7}$	0	1	$\frac{2}{7}$	$\frac{1}{7}$

Lower bound value $L = \lceil -\frac{81}{7} \rceil = -11.$

We select x_1 for branching:

1. $x_1 \le 1 \rightarrow x_1 + x_5 = 1.$

1. $x_1 \leq 1 \rightarrow x_1 + x_5 = 1$:

		x_1	x_2	x_3	x_4	x_5
z	$\frac{81}{7}$	0	0	$\frac{9}{7}$	$\frac{1}{7}$	0
x_1	$\frac{9}{7}$	1	0	$\frac{1}{7}$	$-\frac{3}{7}$	0
x_2	$\frac{18}{7}$	0	1	$\frac{2}{7}$	$\frac{1}{7}$	0
x_5	$-\frac{2}{7}$	0	0	$-\frac{1}{7}$	$\frac{3}{7}$	1

		x_1	x_2	x_3	x_4	x_5
z	9	0	0	0	4	9
x_1	1	1	0	0	0	1
x_2	2	0	1	0	1	2
x_3	2	0	0	1	-3	-7

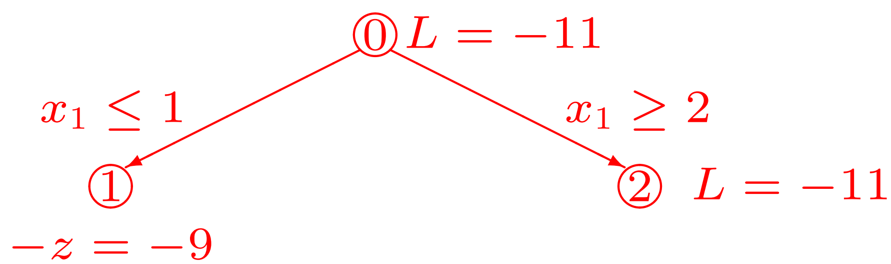
Integer solution of value $-9 > L$ (not provably optimal).

2. $x_1 \geq 2 \rightarrow -x_1 + x_6 = -2$:

		x_1	x_2	x_3	x_4	x_6
z	$\frac{81}{7}$	0	0	$\frac{9}{7}$	$\frac{1}{7}$	0
x_1	$\frac{9}{7}$	1	0	$\frac{1}{7}$	$-\frac{3}{7}$	0
x_2	$\frac{18}{7}$	0	1	$\frac{2}{7}$	$\frac{1}{7}$	0
x_6	$-\frac{5}{7}$	0	0	$\frac{1}{7}$	$-\frac{3}{7}$	1

		x_1	x_2	x_3	x_4	x_6
z	$\frac{34}{3}$	0	0	$\frac{4}{3}$	0	$\frac{1}{3}$
x_1	2	1	0	0	0	-1
x_2	$\frac{7}{3}$	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$
x_4	$\frac{5}{3}$	0	0	$-\frac{1}{3}$	1	$-\frac{7}{3}$

The LP solution (fractional) provides a lower bound $\lceil -\frac{34}{3} \rceil = -11$ for the current node.



The lower bound is better than the incumbent solution $\Rightarrow$ new branching.

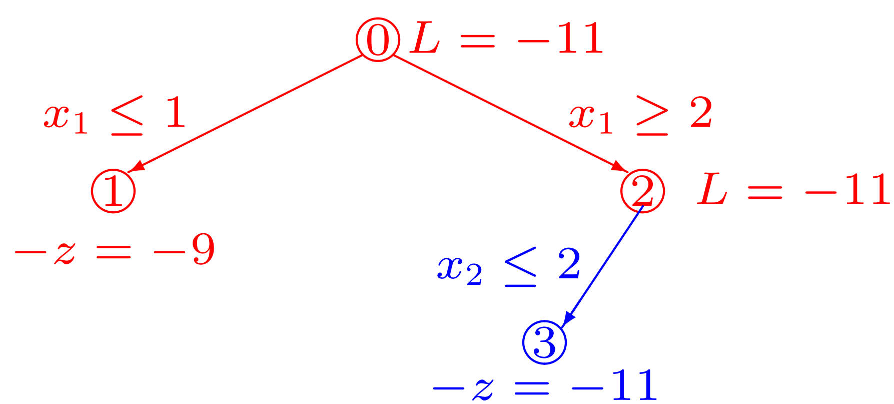
We select x_2 for branching:

3. $x_2 \leq 2 \rightarrow x_2 + x_7 = 2.$

3. $x_2 \le 2 \rightarrow x_2 + x_7 = 2$:

		x_1	x_2	x_3	x_4	x_6	x_7
z	$\frac{34}{3}$	0	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0
x_1	2	1	0	0	0	-1	0
x_2	$\frac{7}{3}$	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0
x_4	$\frac{5}{3}$	0	0	$-\frac{1}{3}$	1	$-\frac{7}{3}$	0
x_7	$-\frac{1}{3}$	0	0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	1

		x_1	x_2	x_3	x_4	x_6	x_7
z	11	0	0	1	0	0	1
x_1	3	1	0	1	0	0	-3
x_2	2	0	1	0	0	0	1
x_4	4	0	0	2	1	0	-7
x_6	1	0	0	1	0	1	-3



Optimal LP solution (integer) of value -11 = lower bound of the parent node

$\Rightarrow$ no need to explore the second child.

Final optimal solution: $x_1 = 3$, $x_2 = 2$.

Exploration strategies in branch-and-bound

- Two main issues for designing a branch-and-bound algorithm:
  - explorations strategy (which node has to be explored next);
  - how to compute bounds (will be seen later).
- the branch-decision tree can be **binary** (two children per node)

  or **multiple** ($q > 2$ children per node, using q **exhaustive** conditions).
- the branching conditions are preferably **mutually exclusive**, but not necessarily.
- **Notation:**
  - z = value of the incumbent solution;
  - $L(P^k)$ = lower bound value for node P^k.

Exploration strategies in branch-and-bound (cont'd)

Depth-first:

- compute $L(P^0)$; generate the first child (P^1) of P^0, and compute $L(P^1)$;
- generate the first child of P^1, and so on. **Rule:**
- **Forward step:** generate one child of the last generated node, P^k, until
  - P^k is immediately solvable (e.g., integer LP solution), hence possibly update z, or
  - $L(P^k) \geq z$, or
  - P^k does not have feasible non-explored nodes. In such cases:
- **Backtracking:** Backtrack to the parent of P^k, say P^p, and,
  - if $L(P^p) < z$, generate the next child of $L(P^p)$, then its first child, and so on;
  - otherwise (or if all children of P^p have already been explored) backtrack to the parent of P^p;
- terminate when trying to backtrack from P^0.
- **Pros:**
  - small number of active nodes;
  - each node is parent or child of the previous node;
  - easy to implement;
  - feasible incumbent solutions are quickly produced.

Exploration strategies in branch-and-bound (cont'd)

Lowest-first (Highest-first for maximization problems):

- compute $L(P^0)$, and initialize $\Pi := \{P^0\}$ (active nodes);
- at each iteration:
  - remove the node with smallest lower bound from Π;
  - generate all its children, and compute their lower bounds;
  - if there are immediately solvable nodes, possibly update z;
  - add to Π the non immediately solvable nodes P^k for which $L(P^k) < z$;
- terminate when $\Pi = \emptyset$ or all nodes $P^k \in \Pi$ have $L(P^k) \geq z$.
- **Pros:**
  - the most promising node is explored first;
  - small number of nodes globally explored to obtain the solution;
  - no node is explored in vain (but in case of ties):

    if we explore P^r, with $L(P^r) > L(P^\ell)$, P^ℓ will have to be explored in any case.
- **Cons:**
  - higher computing time for exploring a node;
  - high number of active nodes;
  - no relationship between the current node and the previous node;
  - not easy to implement.

Exploration strategies in branch-and-bound (cont'd)

Depth-first (revisited):

- Same structure as depth-first, but
  - in the **forward step** all children of the current node are generated, their lower bounds are computed, and exploration continues with the child having minimum lower bound;
  - when **backtracking** the exploration continues with the non-explored child having minimum lower bound.
  - given two children, P^ℓ and P^r with $L(P^\ell) < L(P^r)$, P^ℓ will have to be explored in any case.

Breadth-first:

- all children of P^0 are generated;
- all children of all children of P^0 that are not non immediately solvable are generated, and so on.
- Rarely used;
- adopted when all (or a large subset of) feasible solutions have to be generated.

Mixed Integer Linear Programming

- In a general case:
  - a subset of the variables can only take **integer values**;
  - the other variables can take **fractional values**.
- **Example:** transportation problems in which the variables represent:
  - numbers of vehicles to be used on the various routes;
  - quantities of goods (in tons) to be loaded on the vehicles .
- **Mixed Integer Linear Programming (MILP):**

$$
\begin{array}{rcl}
\min c'x & & \\
Ax & = & b \\
x & \geq & 0 \\
x_j & & \text{integer } (j = 1, \dots, \overline{n}) \\
x_j & & \text{fractional } (j = \overline{n}+1, \dots, n)
\end{array}
$$

- In the branch-decision tree we branch on a fractional x_j $(j = 1, \dots, \overline{n})$.

0-1 (or Binary) Linear Programming

- Special case of ILP:
$$\begin{aligned} \min\, & c'x \\ Ax &= b \\ x_j &\in \{0,1\} \quad \forall\, j\,. \end{aligned}$$
- The simplest case occurs when A has just one row: $a'x = \bar{b}$:
- usually expressed in maximization canonical form (**0-1 Knapsack Problem (KP01)**):
$$\begin{aligned} \max \sum_{j=1}^{n} p_j x_j & \\ \sum_{j=1}^{n} w_j x_j &\leq c \\ x_j &\in \{0,1\} \quad (j = 1, \ldots, n)\,. \end{aligned}$$
  - n items, each having a value p_j (**profit**) and a **weight** w_j;
  - a container (**knapsack**) having **capacity** c;
  - select a subset of items having maximum profit and a total weight not exceeding c;
  - applications in *cargo loading*, *capital budgeting*, ...
  - widely studied because of its simple structure.
  - We will assume that p_j, w_j and c are positive integers, $w_j \leq c \;\forall\, j$, $\sum_{j=1}^{n} w_j > c$.

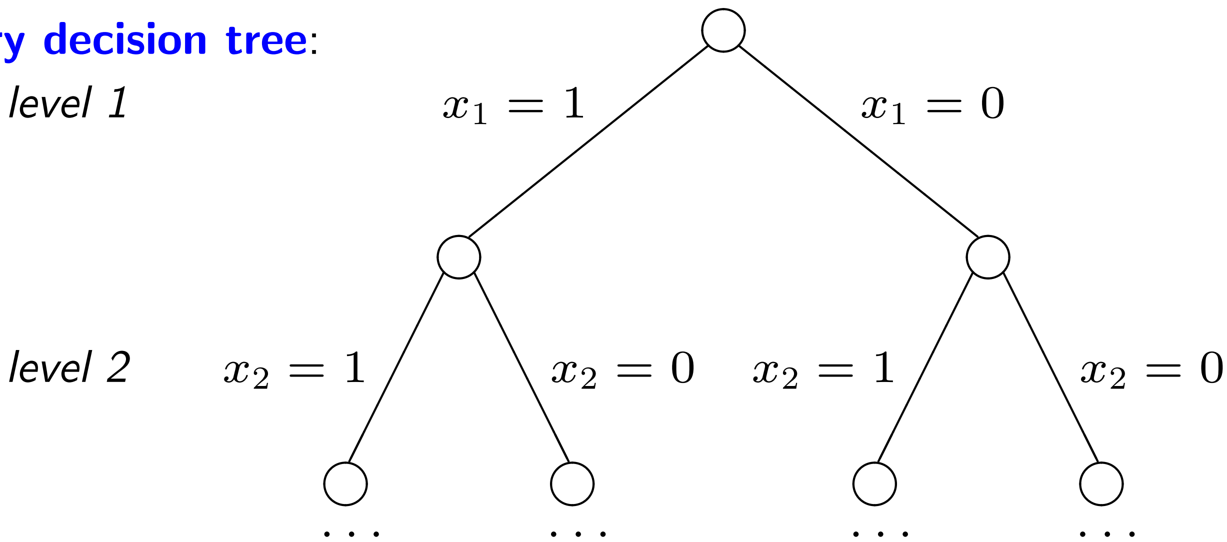
Branch-and-bound algorithm for the 0-1 Knapsack Problem

- It is convenient to preliminary sort the items by **non-increasing profit per unit weight**:

$$\frac{p_j}{w_j} \geq \frac{p_{j+1}}{w_{j+1}} \ \forall \ j$$

.

- **Binary decision tree**:



- **Exploration strategy**: at the first level $x_1 = 1$, $x_1 = 0$; at each iteration,
  if $\exists$ active node generated by $(x_j = 1)$, then branch from it
  otherwise branch from the last node generated by $(x_j = 0)$.

- $\Rightarrow$ the set of active nodes contains:
  at most one node generated by $(x_j = 1)$;
  one or more nodes generated by $(x_j = 0)$.

Branch-and-bound algorithm for the 0-1 Knapsack Problem (cont'd)

- **Upper bound:** continuous relaxation:

  $x_j \in \{0,1\}$ $(j = 1,\ldots,n)$ replaced by $0 \le x_j \le 1$ $(j = 1,\ldots,n)$.

- Solution of the continuous relaxation (Dantzig, 1957):

  consecutively insert the best element, taking a fraction of the first item, s, that does not fit:

  $$s := \min\{i \ : \sum_{j=1}^{i} w_j > c\} \text{(critical item)}; \quad \overline{c} := c - \sum_{j=1}^{s-1} w_j \text{(residual capacity)} :$$

  $$U := \left\lfloor \sum_{j=1}^{s-1} p_j + \overline{c}\,\frac{p_s}{w_s} \right\rfloor .$$

- **Example:**

  $$\begin{array}{rcl} n & = & 5 \\ p' & = & (\ 12,\ 12,\ 7,\ 6,\ 2) \\ w' & = & (\ 4,\ 5,\ 3,\ 3,\ 2) \\ c & = & 10 \\ U & = & \left\lfloor 12 + 12 + 1 \cdot \frac{7}{3} \right\rfloor = 26 \end{array}$$

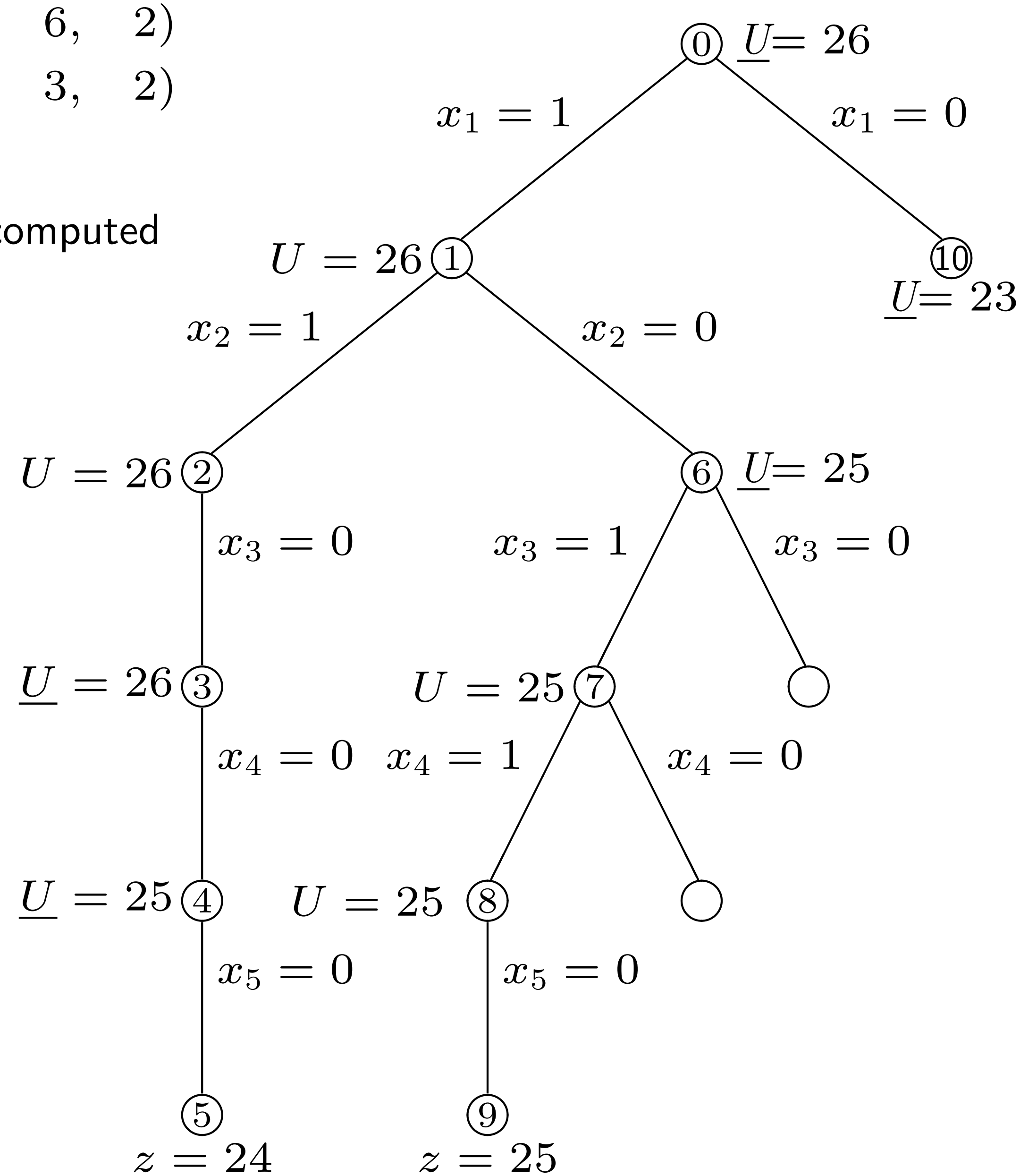
- **Note:** Upper bound of a node generated by $(x_j = 1)$ = upper bound of the parent node;

  $\forall$ node, upper bound of the left son $(x_j = 1) \ge$ upper bound of the right son $(x_j = 0)$.

Example:

$$\begin{array}{rcl} n & = & 5 \\ p' & = & (\ 12,\ 12,\ 7,\ 6,\ 2) \\ w' & = & (\ 4,\ 5,\ 3,\ 3,\ 2) \\ c & = & 10 \end{array}$$

Upper bound U underlined if it has to be computed



Branch-and-bound algorithm for the 0-1 Knapsack Problem (cont'd)

Implementation: k =current level;

$$x' = (1,\ 1,\ 0,\ 1,\ 0,\ \underbrace{1}_{k},\ -,\ -,\ -,\ -) = \text{current values in the tree;}$$

$$\left.\begin{array}{l}
k := k + 1 \text{ (next level);} \\
\textbf{if } w_k \le c - \sum_{j=1}^{k-1} w_j x_j \textbf{ then } x_k := 1 \\
\textbf{else} \\
\quad \textbf{begin} \\
\quad\quad x_k := 0;
\end{array}\right\} \textit{forward step}$$

U := upper bound for the current node;

while $U \le z$ (= incumbent solution value) **do**

begin

$$\left.\begin{array}{l}
k := \max\{j < k : x_j = 1\}; \\
x_k := 0;
\end{array}\right\} \textit{backtracking}$$

U := upper bound for the current node

end

end

Branch-and-bound algorithm for the 0-1 Knapsack Problem (cont'd)

Upper bound: A good upper bound should be **tight** (i.e., have a low value)

Improving the Dantzig bound:

- in the optimal solution the critical item s is either excluded or included:
- if $x_s = 0$ the bound is $B^1 = \left\lfloor \sum_{j=1}^{s-1} p_j + \overline{c}\,\frac{p_{s+1}}{w_{s+1}} \right\rfloor$;
- if $x_s = 1$ the bound is $B^2 = \left\lfloor \sum_{j=1}^{s} p_j - \underbrace{(w_s - \overline{c})}_{\downarrow} \frac{p_{s-1}}{\underbrace{w_{s-1}}} \right\rfloor$;

  missing capacity   worst ratio $\Rightarrow$ minimum loss
- **new bound:** $\overline{U} = \max(B^1, B^2)$.
- **Example:** $n = 5, \quad p' = (\; 15, \; 8, \; 8, \; 7, \; 5),$

  $w' = (\; 5, \; 3, \; 4, \; 4, \; 5), \quad c = 10.$

  U(Dantzig) $= 27$; $B^1 = 26, \quad B^2 = 25 \Rightarrow \overline{U} = 26.$
- $B^1 \leq U$ (obvious); easy to algebraically prove that $B^2 \leq U$; $\Rightarrow \; \overline{U} \leq U$
- $\overline{U}$ is tighter and requires few additional operations $\Rightarrow$ convenient.
- In general, compromise between tightness and computing time.

Branch-and-cut algorithms

- **Branch-and-cut** = (branch-and-bound) "+" (cutting-planes).
- Branch-and-bound algorithm which, at each decision node, generates cuts in an attempt to find an integer solution, or at least to improve the bound.
- Gomory cuts depend on the conditions imposed by the ancestor nodes;
- in branch-and-cut it is common to generate **weaker** cuts that are valid for the whole branch-decision tree (**global cuts**). The cuts are stored in a special data base (**cut pool**).
- At each branch-decision node:

  x^* := solution of the continuous relaxation of the current problem;

  while x^* not integer **and** iteration limit not reached **do**

      if the pool contains cuts that are violated by x^* **then**

          begin

              select one or more cuts not yet used, and add them to the current problem;

              x^* := solution of the continuous relaxation

          end

      else generate new cuts, and add them to the pool;
- Main difficulty: method for generating global cuts.
- Gomory local cuts are also frequently added.

Some considerations on time bounds

- **How many time (steps, iterations ...)** requires the branch-and-bound algorithm for KP01?
- **If we are lucky**, the first (leftmost) **n** branches will find the optimal solution,

  and the bounds will kill all other nodes: the algorithm will take a time proportional to **n**.
- **If we are unlucky**, the bounds will kill no node: time proportional to 2^n.
- We say that, **in the worst case**, the algorithm solves the problem in $O(2^n)$ time, or that the algorithm has **time complexity** $O(2^n)$.
- Other problems: consider a **graph** having **n vertices** (**Introduction**)
- <u>Problem</u>: find the **shortest path** that connects two vertices of a **graph**: there is an algorithm (studied in **Network Optimization**) that solves the problem in $O(n^2)$ time.
- <u>Problem</u>: find the **longest path** that connects two vertices of a graph: a branch-and-bound type algorithm (studied in **Network Optimization**) solves the problem in $O((n-1)!)$ time.
- <u>Problem</u>: find the **shortest tour** to deliver products from a depot to clients and return to the depot: a branch-and-bound type algorithm (studied in **Network Optimization**) solves the problem in $O((n-1)!)$ time.
- Great difference. For **n=100**: $100^2 \approx 10^4$, $2^{100} \approx 10^{30}$, $99! \approx 10^{156}$.
  (The number of atoms in the universe is estimated to be $\approx 10^{80}$.)
- Do we know faster algorithms for the longest path or the shortest tour in a graph? **No!**.
- These issues will be examined in the next section.

Software and freeware for LP and ILP

Three main difficulties for a practical implementation of the simplex algorithm:

1. **Numerical stability:**
   - floating point operations produce errors that can propagate;
   - example: $a := \frac{1}{3}(= 0.333\ldots)$; $b := 1 - 3 * a(= 0.000\ldots01)$; is b a valid pivot?

2. **CPU time:**
   - for large-size instances the tableau can include billions of entries, and pivoting operations can be prohibitive;
   - implementations adopt the **Revised Simplex Algorithm**, which only uses the inverse, B^{-1}, of the base sub-matrix:
     - column 0 of the tableau (rows 1–m) $= x_\beta = B^{-1}b$;
     - relative cost vector $= \overline{c}' = c' - c'_\beta B^{-1}A$;
     - jth column of the tableau $= B^{-1}A_j$.

3. **Sparsity:**
   - in real-world large instances matrix A is frequently very "sparse" (most values are 0);
   - special decomposition techniques used for efficiently handling B^{-1}.

Software and freeware for LP and ILP (cont'd)

Didactic software:

- Web page: `http://www.or.deis.unibo.it/staff_pages/martello/cvitae.html`
- $\rightarrow$ Courses $\rightarrow$ Didactic Tools. **Applets** to execute (student implementations):
  - Phase 1 and Phase 2 of the simplex algorithm;
  - Gomory algorithm;
  - Branch-and-bound algorithm for the 0-1 knapsack problem;
  - Dynamic programming algorithm for the 0-1 knapsack problem (to be seen later).

Commercial software:

- **CPLEX Optimizer** (ILOG $\rightarrow$ IBM)
- **Gurobi Optimizer** (Gurobi Optimization)
- **LINDO** and **LINGO** (LINDO Systems)
- **XPRESS Optimizer** (FICO)
- **MPL** (Maximal Software): high level language, independent on the platform (Windows, Unix, Mac, OSF), capable of interfacing with all solvers.

Demos

- All commercial softwares provide free *demos* and/or *student versions* (with a limit to the number of variables/constraints)

Excel (Microsoft) includes an LP/ILP solver (☹ not recommended)

Open source solvers (they can require a licence for commercial use):

- **lpsolve**: simplex, branch-and-bound;
  ANSI C; `http://sourceforge.net/projects/lpsolve`

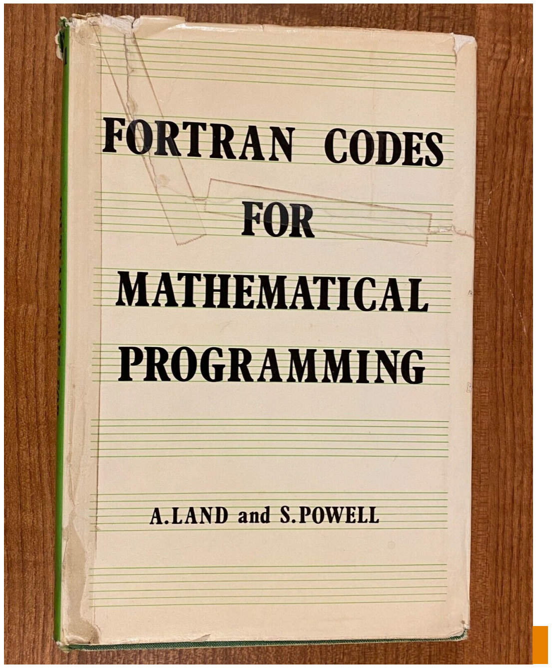
- **Clp** and **Cbc**: simplex, branch-and-cut;
  C++; `http://www.coin-or.org`

- **SCIP**: simplex, branch-cut-and-price + constraint programming;
  C callable library, very efficient; `http://scip.zib.de/`

- **GLPK**: simplex + interior point (to be defined later), branch-and-cut;
  GNU; ANSI C callable library; `https://www.gnu.org/software/glpk/`

Software and freeware for LP and ILP: The ancestor

In **1973**, Ailsa Land and Susan Powell published



This book provides a set of programs in the standard Fortran IV to solve linear, quadratic and discreet linear programming problems, together with a parametric facility for the linear case ... as well as providing a branch and bound algorithm for mixed integer programming.

Question: How were the programs provided?

Answer: The book contained the **listings!**