

# Operations Research (Master's Degree Course)

## 5. Duality

Silvano Martello

*DEI "Guglielmo Marconi", Università di Bologna, Italy*



This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivs 3.0 Unported License.

Based on a work at <http://www.editrice-esculapio.com>

## Duality: An informal introduction

A simple LP in canonical form:

$$\begin{array}{rclcl} \max z = & 4x_1 & + & 2x_2 & \\ & 3x_1 & + & x_2 & \leq 7 \\ & x_1 & + & x_2 & \leq 3 \\ & x_1 & , & x_2 & \geq 0 \end{array}$$

Input data:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \quad c' = [4 \quad 2].$$

Remind that:

- every row of  $A$  represents the coefficients of an inequality;
- every element of  $b$  represents the right end side of the corresponding inequality;
- every element of  $c$  represents the unit value (cost) of the corresponding variable.

The optimal solution is clearly  $x_1 = 2$ ,  $x_2 = 1$ , and its value is  $z = 10$ .

Let's do an exercise: We define a **new problem** using the **same input data** but in a **“symmetrical” (dual)** way. Namely,

- we define the inequality coefficients using the **columns** of  $A$  instead of its **rows**;
- we use the **costs** as right hand sides and the **right hand sides** as the costs;
- we flip the **inequality signs**;
- we **minimize** instead of maximizing. ■

Using variables  $\pi_1$  and  $\pi_2$  for the new problem (the **dual**) we get:

$$\begin{aligned} \min w = & 7\pi_1 + 3\pi_2 \\ & 3\pi_1 + \pi_2 \geq 4 \\ & \pi_1 + \pi_2 \geq 2 \\ & \pi_1, \pi_2 \geq 0. \end{aligned} \quad \blacksquare$$

Easy optimal solution:  $\pi_1 = 1, \pi_2 = 1$  of value  **$w = 10$** . ■

Something curious: two totally different problems have (obviously) different solutions but the two solutions have the **same value!** We will see that this is a general property. ■

Duality emerges in many scientific areas (Mathematics, Physics, Electrical Engineering, Control theory, Economics). It gives two different points of view of looking at the same object. ■

In many cases (including linear programming) duality corresponds to an **involution**, a function that is its own inverse (i.e., such that  $f(f(x)) = x$ ). ■ Note indeed that, if we apply to the dual the same transformation that produced it, its dual coincides with the original problem. ■

## Algebraic view of the Simplex algorithm

$$\begin{aligned} \min \quad & c'x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

### Optimality criterion:

- **Base:**  $\mathcal{B} = \{A_{\beta(1)}, \dots, A_{\beta(m)}\} \leftrightarrow B = [A_{\beta(i)}];$
- **Simplex algorithm**  $\longrightarrow Y = \text{tableau}$  ( $m \times n$  matrix);
- **Algebraic interpretation:**

	$B$
--	-----

$[y_{ij}]$	$I$
------------	-----

$A \qquad Y$

$B^{-1}$
----------

	$B$
--	-----

 $=$ 

$[y_{ij}]$	$I$
------------	-----

 $\iff Y = B^{-1}A$ 

$A$

- $c_{\beta} = [c_{\beta(i)}]$  (costs corresponding to basic variables of  $Y$ );

- $z_j = \sum_{i=1}^m y_{ij} c_{\beta(i)}$ :
 

$c'_{\beta}$
--------------

$[y_{ij}]$	$I$
------------	-----

 $=$ 

$z_j$	$z_j = c_j$
-------	-------------

 $\iff z' = c'_{\beta} Y = c'_{\beta} B^{-1} A$ 

$Y$

- **Relative costs** (row 0 of the tableau):  $\bar{c}' = c' - z'$
- **Optimality criterion:**  $\bar{c}' \geq 0 \iff c' - c'_{\beta} B^{-1} A \geq 0.$

## Dual of an LP in standard form

- The LP duality immediately emerges from the optimality criterion:■
- **Optimality criterion:**  $c' - (c'_\beta B^{-1})A \geq 0$ , where:■
  - $c'$ ,  $A$  input data;■
  - $c'_\beta B^{-1}$  unknown ( $\leftrightarrow$  optimal BFS).■
- By defining  $\pi' = c'_\beta B^{-1}$ , we see that  $\pi'$  is a feasible solution to

$$\pi' A \leq c'$$

where  $\pi'$  is a vector of  $m$  variables corresponding to the selection of the optimal basis.■

- Let us “invent” a new objective function  $\max \pi' b$ . We get the **Dual problem:**

$$\begin{array}{ll} \max & \pi' b \\ & \pi' A \leq c' \\ & \pi \geq 0 \end{array}$$

- We will show that

**$\mathcal{B}$  is an optimal BFS for the LP**



**$\pi' = c'_\beta B^{-1}$  is an optimal solution to the dual.**

**Example:**  $\min z = \begin{array}{rcl} 2 & x_1 & + \\ 2 & x_1 & + \\ & x_1 & + \\ & x_1, & \end{array} \begin{array}{rcl} 3 & x_2 & + \\ & x_2 & \\ & & \\ & x_2, & \end{array} \begin{array}{rcl} x_3 & & \\ & & \\ & & \\ & x_3 & \\ & x_3 & \end{array} \begin{array}{rcl} & & \\ = & 1 & \\ = & 4 & \\ \geq & 0 & \end{array}$

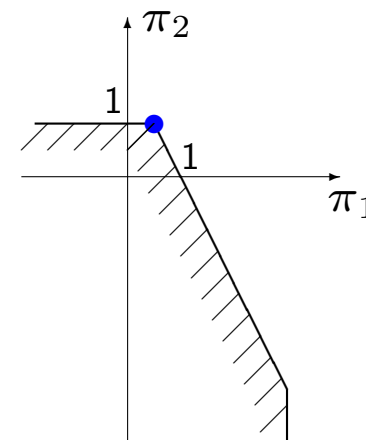
		$x_1$	$x_2$	$x_3$			$x_1$	$x_2$	$x_3$	
$-z$	$-7$	$-5$	$0$	$0$		$-z$	$-\frac{9}{2}$	$0$	$\frac{5}{2}$	$0$
$x_2$	$1$	$2$	$1$	$0$	$\rightarrow$	$x_1$	$\frac{1}{2}$	$1$	$\frac{1}{2}$	$0$
$x_3$	$4$	$1$	$0$	$1$		$x_3$	$\frac{7}{2}$	$0$	$-\frac{1}{2}$	$1$
$A$						$Y$				

$$B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}, \quad B^{-1}A = Y;$$

$$c'_\beta Y = [2 \ 1] Y = [2 \ \frac{1}{2} \ 1];$$

$$c' - z' = [2 \ 3 \ 1] - c'_\beta Y = [0 \ \frac{5}{2} \ 0];$$

$$\max w = \begin{array}{rcl} \pi_1 & + & 4 \pi_2 \\ 2 \pi_1 & + & \pi_2 \leq 2 \\ \pi_1 & & \leq 3 \\ & \pi_2 & \leq 1 \\ \pi_1, & \pi_2 & \geq 0 \end{array}$$



Current tableau:  $\pi' = c'_\beta B^{-1} = [\frac{1}{2} \ 1] (\bullet)$

## Dual of an LP in general form

$$\min c'x$$

$$a'_i x = b_i \quad i \in M$$

$$a'_i x \geq b_i \quad i \in \overline{M}$$

$$x_j \geq 0 \quad j \in N$$

$$x_j \leq 0 \quad j \in \overline{N}$$

By introducing:

$$\begin{cases} x_i^s & \forall i \in \overline{M} (x_i^s \geq 0) \\ x_j^+, x_j^- & \forall j \in \overline{N} (x_j^+, x_j^- \geq 0) \end{cases}$$

we get an equivalent LP:  $\min \hat{c}'\hat{x}$

$$\begin{aligned} \hat{A}\hat{x} &= b \\ \hat{x} &\geq 0 \end{aligned}$$

$$\hat{A} = \begin{array}{c|cc|c} & N & \overline{N} & \overline{M} \\ \hline & \boxed{A_j} & \boxed{A_j} \quad \boxed{-A_j} & \begin{array}{c} 0 \\ -I \end{array} \\ \hline & & & \begin{array}{c} M \\ \overline{M} \end{array} \end{array}$$

$$\hat{x}' = \begin{array}{|c|c|c|} \hline x_j & (x_j^+, x_j^-) & x_i^s \\ \hline \end{array}$$

$$\hat{c}' = \begin{array}{|c|c|c|} \hline c_j & (c_j, -c_j) & 0 \\ \hline \end{array}$$

## Dual of an LP in general form (cont'd)

- **Optimality criterion:** an optimal solution  $\hat{x}_0$  corresponds to a base  $\hat{B} = \{\hat{A}_{\hat{\beta}(1)}, \dots, \hat{A}_{\hat{\beta}(m)}\}$  ( $\leftrightarrow \hat{B} = [\hat{A}_{\hat{\beta}(i)}]$ ) s.t.  $\hat{c}' - \hat{z}' = \hat{c}' - (\hat{c}'_{\hat{\beta}} \hat{B}^{-1}) \hat{A} \geq 0$  ■
- By defining  $\pi' = \hat{c}'_{\hat{\beta}} \hat{B}^{-1}$ ,  $\pi$  is a feasible solution to  $\pi' \hat{A} \leq \hat{c}'$ : ■

$$\begin{array}{c}
 \boxed{\pi'} \quad \begin{array}{c|c|c|c}
 N & \overline{N} & \overline{M} & \\
 \hline
 A_j & A_j & -A_j & \begin{array}{c} 0 \\ -I \end{array} \\
 \hline
 \end{array} \begin{array}{c} M \\ \overline{M} \end{array} \leq \\
 \leq \begin{array}{c|c|c}
 c_j & (c_j, -c_j) & 0 \\
 \hline
 \end{array}
 \end{array}$$

- Hence  $\pi$  is a feasible solution to:

$$\begin{array}{ll}
 \bullet \quad \pi' A_j \leq c_j, & j \in N \blacksquare \\
 \pi' A_j \leq c_j \Rightarrow \swarrow \\
 -\pi' A_j \leq -c_j \rightarrow \Rightarrow \bullet \quad \pi' A_j = c_j, & j \in \overline{N} \blacksquare \\
 -\pi_i \leq 0 \Rightarrow \bullet \quad \pi_i \geq 0, & i \in \overline{M} \blacksquare \\
 \text{nothing for } i \in M \Rightarrow \bullet \quad \pi_i \geq 0, & i \in M \blacksquare
 \end{array}$$

- Constraints of a new problem (**Dual problem**). Original LP = **Primal problem**.
- We will adopt the objective function  $\max \pi' b$ . ■



## Primal-Dual relationships

Primal				Dual			
(P) $\min c'x$				(D) $\max \pi'b$			
$a'_i x$	$=$	$b_i$	$i \in M$	$\pi_i$	$\begin{matrix} \geq \\ \leq \end{matrix}$	$0$	
$a'_i x$	$\geq$	$b_i$	$i \in \overline{M}$	$\pi_i$	$\geq$	$0$	
$x_j$	$\geq$	$0$	$j \in N$	$\pi' A_j$	$\leq$	$c_j$	
$x_j$	$\begin{matrix} \geq \\ \leq \end{matrix}$	$0$	$j \in \overline{N}$	$\pi' A_j$	$=$	$c_j$	
constraint		$\longleftrightarrow$	variable				
variable		$\longleftrightarrow$	constraint				
row		$\longleftrightarrow$	column				
column		$\longleftrightarrow$	row				
“strong” constraint		$\longleftrightarrow$	“weak” constraint				
“weak” constraint		$\longleftrightarrow$	“strong” constraint				
costs		$\longleftrightarrow$	right handside				
right handside		$\longleftrightarrow$	costs				

**Example. Primal:**  $\min z = 2x_1 + x_2 + x_3$

$$\begin{array}{rcll} x_1 & & + x_3 & = 4 \\ & x_2 & + 2x_3 & \geq 2 \\ x_1, x_2 & & & \geq 0 \\ & x_3 & & \leq 0 \end{array}$$

	0	2	1	1	-1	0
		$x_1$	$x_2$	$x_3^+$	$x_3^-$	$x_2^s$
$-z$	-10	0	0	-3	3	1
$x_1$	4	1	0	1	-1	0
$x_2$	2	0	1	2	-2	-1

		$x_1$	$x_2$	$x_3^+$	$x_3^-$	$x_2^s$
$-z$	-7	0	$\frac{3}{2}$	0	0	$-\frac{1}{2}$
$x_1$	3	1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
$x_3^+$	1	0	$\frac{1}{2}$	1	-1	$-\frac{1}{2}$

		$x_1$	$x_2$	$x_3^+$	$x_3^-$	$x_2^s$
$-z$	-4	1	1	0	0	0
$x_2^s$	6	2	-1	0	0	1
$x_3^+$	4	1	0	1	-1	0

Sol:  $z = 4; x_1 = x_2 = 0, x_3 = 4.$

$$\begin{array}{rclcl}
 \text{Dual: } \max w = & 4\pi_1 & + & 2\pi_2 & \\
 & \pi_1 & & & \leq 2 \quad (\text{redundant}) \\
 & & \pi_2 & & \leq 1 \quad \uparrow \\
 & \pi_1 & + & 2\pi_2 & = 1 \quad \Rightarrow \\
 & \pi_1 & & & \geq 0 \\
 & & \pi_2 & & \geq 0
 \end{array}$$

$$\begin{array}{rclcl}
 \max w = & 4\pi_1^+ & - & 4\pi_1^- & + & 2\pi_2 \\
 \downarrow & & & & & \\
 -\min(-w) = & -4\pi_1^+ & + & 4\pi_1^- & - & 2\pi_2 \\
 & & & & \pi_2 & + & \pi_1^s = 1 \\
 & \pi_1^+ & - & \pi_1^- & + & 2\pi_2 & = & 1 \\
 & \pi_1^+ & , & \pi_1^- & , & \pi_2 & , & \pi_1^s \geq 0
 \end{array}$$

		0	-4	4	-2	0
		$\pi_1^+$	$\pi_1^-$	$\pi_2$	$\pi_1^s$	
$+w$	4	0	0	6	0	
$\pi_1^s$	1	0	0	1	1	
$\pi_1^+$	1	1	-1	2	0	

Optimal solution:  $w = 4$ ;  $\pi_1 = 1$ ,  $\pi_2 = 0$ . (**Observation:**  $w = z$ ). ■

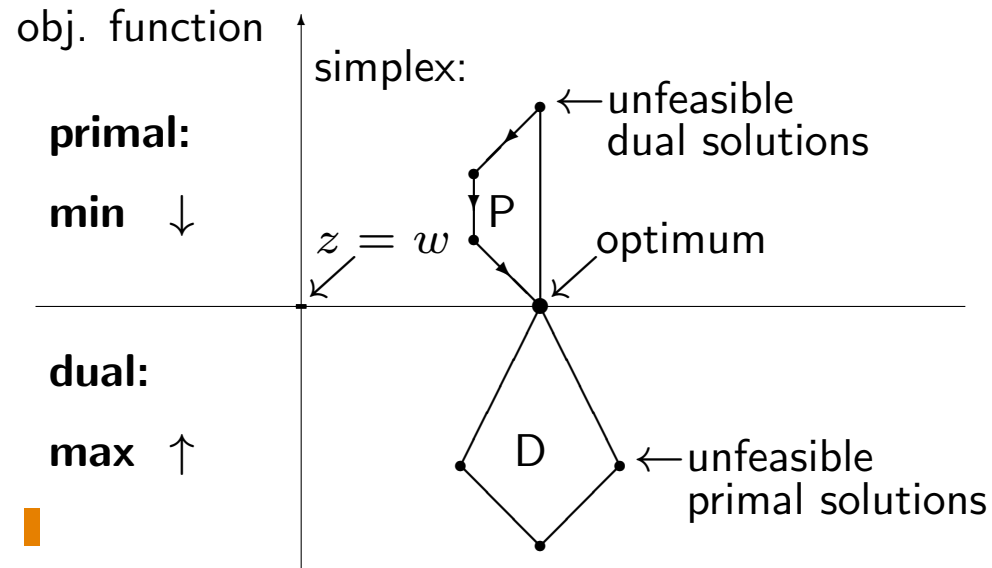
## Strong duality theorem

**Theorem** If an LP has a finite optimal solution then

1. its dual has a finite optimal solution;
2. the two solutions have the same value.

**Proof 1.**  $x$  = feasible primal solution,  $\pi$  = feasible dual solution.

Dual constraints:  $c' \geq \pi' A \implies$  (primal constraints)  $\implies c'x \geq \pi' Ax = \pi' b$ . **Weak duality:**   
**cost in the primal  $\geq$  cost in the dual!**  $\implies$  the dual cannot have an unbounded solution.



The dual has the feasible solution  $\pi' = c'_\beta B^{-1}$  (with  $B$  = optimal primal basis)  $\implies$  it has finite optimal solution.

2. Solution  $\pi' = c'_\beta B^{-1}$  has cost  $\pi' b = c'_\beta B^{-1} b = c'_\beta x_\beta$  (= optimal primal cost).  $\square$

## Dual of the dual

**Theorem** *The dual of the dual is the primal.*

**Proof** Re-write the dual in primal form:

$$\begin{aligned} \min \quad & (-b')\pi \\ & (-A'_j)\pi \geq -c_j \quad j \in N \\ & (-A'_j)\pi = -c_j \quad j \in \overline{N} \\ & \pi_i \geq 0 \quad i \in \overline{M} \\ & \pi_i \leq 0 \quad i \in M \end{aligned}$$

Define its dual using variables  $x$ :

$$\begin{aligned} \max \quad & x'(-c) \\ & x_j \geq 0 \quad j \in N \\ & x_j \leq 0 \quad j \in \overline{N} \\ & (-a'_i)x \leq -b_i \quad i \in \overline{M} \\ & (-a'_i)x = -b_i \quad i \in M \end{aligned}$$

This is the primal.  $\square$

## Possible categories of a primal-dual pair

dual→ primal↓	finite optimum	unbounded	infeasible
finite optimum	1	NO	NO
unbounded	NO	<u>NO</u>	3
infeasible	NO	3	2

Entries '1' and 'NO': already proved

- If the primal is unbounded ( $z \rightarrow -\infty$ ) then the dual cannot have a solution  $w \rightarrow +\infty$  ( $\Leftarrow c'x \geq \pi'b$ )  $\Rightarrow$  NO. We show that cases 2 and 3 are possible:

$$\begin{array}{lcl}
 \cdot \min & \begin{array}{l} x_1 \\ x_1 + x_2 \geq 1 \\ -x_1 - x_2 \geq 1 \\ x_1, x_2 \geq 0 \end{array} & \longrightarrow \\
 & \text{(impossible primal)} & \\
 \cdot \min & \begin{array}{l} x_1 \\ x_1 + x_2 \geq 1 \\ -x_1 - x_2 \geq 1 \\ x_1, x_2 \geq 0 \end{array} & \longrightarrow \\
 & \text{(impossible primal)} & 
 \end{array}
 \qquad
 \begin{array}{lcl}
 \max & \begin{array}{l} \pi_1 + \pi_2 \\ \pi_1 - \pi_2 = 1 \\ \pi_1 - \pi_2 = 0 \\ \pi_1, \pi_2 \geq 0 \end{array} & \\
 & \text{(impossible dual)} & \Rightarrow \text{2} \\
 \max & \begin{array}{l} \pi_1 + \pi_2 \\ \pi_1 - \pi_2 \leq 1 \\ \pi_1 - \pi_2 \leq 0 \\ \pi_1, \pi_2 \geq 0 \end{array} & \\
 & \text{(unbounded dual)} & \Rightarrow \text{3}
 \end{array}$$

## The dual of the diet problem

$$\begin{aligned} \text{Primal : } \min \quad & c'x \\ & Ax \geq r, \\ & x \geq 0. \end{aligned}$$

$c_j$  = cost of 1 unit of the  $j$ th food;

$x_j$  = quantity of the  $j$ th food to buy;

$a_{ij}$  = quantity of the  $i$ th nutrient in the  $j$ th food;

$r_i$  = requirement of the  $i$ th nutrient. ■

$$\begin{aligned} \text{Dual : } \max \quad & \pi' r \\ & \pi' A \leq c', \quad \blacksquare \\ & \pi' \geq 0. \end{aligned}$$

**Interpretation:** A pill-maker wants to enter the market with  $m$  pills, one per nutrient:

$\pi_i$  = selling price for one unit of the  $i$ th nutrient; ■

$\pi' r$  = cost of the diet; ■

$\sum_{i=1}^m \pi_i a_{ij} \leq c_j \iff$  the cost in pill form of all nutrients in the  $j$ th food  
cannot exceed the cost of the  $j$ th food. ■

## Farkas' Lemma (1894)

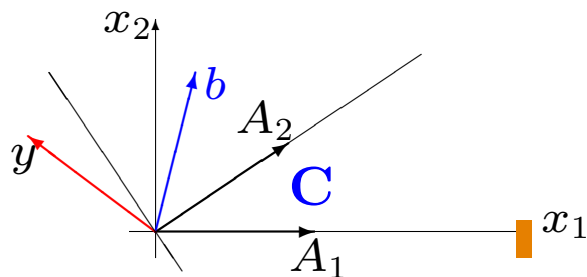
**Lemma** The system (I) :  $\{Ax = b, x \geq 0\}$  is impossible iff (if and only if) the system (II) :  $\{y'A \leq 0, b'y > 0\}$  is possible. ■

**Proof**

$$\begin{array}{ll}
 \text{(P)} \quad \min & 0'x \\
 & Ax = b \\
 & x \geq 0
 \end{array}
 \iff
 \begin{array}{ll}
 \text{(D)} \quad \max & b'y \\
 & y'A \leq 0 \\
 & y \geq 0
 \end{array}
 \quad \blacksquare$$

- (D) is possible ( $\Leftarrow y = 0$ ); ■ hence
- (P) is impossible iff (D) is unbounded, i.e., ■  
(I) is impossible iff (D) is unbounded. ■
- (P) cannot have a solution of value  $> 0$ , so ■  
if (D) has a solution with  $b'y > 0$ , ■ then it must be  $b'y \rightarrow +\infty$  from which ■
- (D) is unbounded iff (II) is possible. ■ □

**Geometric interpretation:**  $C$  = cone generated, in  $R^m$ , by vectors  $A_j$ ;  $b$  = vector in  $R^m$ :



- (I)  $\Leftrightarrow b \in C$ ; ■
- (II)  $\Leftrightarrow \exists$  a vector  $y \in R^m$  such that ■  
its angle with  $b$  is no more than  $90^\circ$ , but ■  
its angle with all vectors in  $C$  is more than  $90^\circ$ . ■



## Complementary slackness

**Theorem** A pair of solutions  $x, \pi$ , respectively feasible for a primal-dual pair is optimal if and only if

$$u_i = \pi_i(a'_i x - b_i) = 0 \quad \forall i, \quad (\alpha)$$

$$v_j = (c_j - \pi' A_j)x_j = 0 \quad \forall j \quad (\beta)$$

### Proof

- $u_i \geq 0 \quad \forall i$  (if  $a'_i x = b_i \Rightarrow u_i = 0$ , if  $a'_i x \geq b_i \Rightarrow \pi_i \geq 0 \Rightarrow u_i \geq 0$ );
- $v_j \geq 0 \quad \forall j$  (similarly).
- $U = \sum_{i=1}^m u_i \geq 0$ ;
- $V = \sum_{j=1}^n v_j \geq 0$ .
- $(U = V = 0)$  if and only if  $(\alpha)$  and  $(\beta)$  hold.
- $U + V = \sum_{i=1}^m \pi_i \left( \sum_{j=1}^n a_{ij} x_j - b_i \right) + \sum_{j=1}^n (c_j - \sum_{i=1}^m \pi_i a_{ij}) x_j = c'x - \pi'b$ , from which
- $(\alpha)$  and  $(\beta)$  hold  $\Leftrightarrow \pi'b = c'x \Leftrightarrow x$  and  $\pi$  are optimal.  $\square$

## Complementary slackness (cont'd)

- Complementary slackness implies that
  - $\forall j : \pi' A_j < c_j$  in the optimal solution to the dual
    - we must have  $x_j = 0$  in the optimal solution to the primal, and conversely
  - $\forall j : x_j > 0$  in the optimal solution to the primal,
    - we must have  $\pi' A_j = c_j$  in the optimal solution to the dual.
- Similar relationships between primal constraints and dual variables.
- Example (see previous slides in this chapter):**

$$\begin{array}{rcll}
 \min z = & 2x_1 & + & x_2 & + & x_3 \\
 & x_1 & & & + & x_3 & = & 4 & \rightarrow & \text{1st } (\alpha) \text{ constraint satisfied} \\
 & & & x_2 & + & 2x_3 & \geq & 2 \\
 & x_1 & , & x_2 & & & \geq & 0 \\
 & & & & & x_3 & \geq & 0
 \end{array}$$

Solution:  $z = 4, x_3 = 4, x_1 = x_2 = 0 \rightarrow$  1st and 2nd ( $\beta$ ) constraints satisfied

2nd ( $\alpha$ ) constraint:  $\pi_2(x_2 + 2x_3 - 2) = 0 \Rightarrow \pi_2 = 0$

3rd ( $\beta$ ) constraint:  $(1 - (\pi_1 + 2\pi_2))x_3 = 0 \Rightarrow \underbrace{\pi_1 = 1}_{\uparrow}$

optimal dual solution.

## Complementary slackness (cont'd)

- Example (see slides in Chapter 2):

$$\begin{array}{rcll}
 \min z = & -x_1 & -2x_2 & \\
 & x_1 & +x_2 & +x_3 = 2 \\
 & x_1 & +3x_2 & +x_4 = 3 \\
 & & 3x_2 & +x_5 = 2 \\
 & x_1 & , & x_2 , x_3 , x_4 , x_5 \geq 0
 \end{array}$$

Standard form  $\Rightarrow$  all ( $\alpha$ ) constraints satisfied.

Solution:  $z = -\frac{5}{2}$ ,  $x_3 = x_4 = 0 \rightarrow$  3rd and 4th ( $\beta$ ) constraints satisfied

$$x_1 = \frac{3}{2}, x_2 = \frac{1}{2}, x_5 = \frac{1}{2}$$

$$\begin{array}{rcll}
 \text{(D)} \max w = & 2\pi_1 & +3\pi_2 & +2\pi_3 \\
 & \pi_1 & +\pi_2 & \leq -1 \\
 & \pi_1 & +3\pi_2 & +3\pi_3 \leq -2 \\
 & \pi_1 & & \leq 0 \\
 & & \pi_2 & \leq 0 \\
 & & & \pi_3 \leq 0 \\
 & \pi_1 & , & \pi_2 , \pi_3 \leq 0
 \end{array}$$

1st, 2nd and 5th ( $\beta$ ) constraints:

$$(-1 - \pi_1 - \pi_2)\frac{3}{2} = 0, \quad (-2 - \pi_1 - 3\pi_2 - 3\pi_3)\frac{1}{2} = 0, \quad (0 - \pi_3)\frac{1}{2} = 0$$

The last one gives  $\pi_3 = 0$ . The resulting  $2 \times 2$  linear system gives  $\pi_1 = \pi_2 = -\frac{1}{2}$ .

Optimal dual solution:  $\pi = (-\frac{1}{2}, -\frac{1}{2}, 0)$ , of value  $-\frac{5}{2}$ .

## Dual information in the tableau

- $Y = \text{tableau } (m \times n)$ , base  $\mathcal{B} (\leftrightarrow B = [A_{\beta(i)}])$ , costs  $c_\beta$ :  $z' = c'_\beta Y = c'_\beta B^{-1} A = \pi' A$ .
- Suppose the initial and final tableaux are  $(\tilde{B} \leftrightarrow \text{optimal final basis})$ :

0	$c'$		
$b$	$I$		$\tilde{B}$

$\underbrace{\hspace{10em}}_A$

Initial tableau

$-z_0$	$c_j - \pi_j$		
$x_0$	$\tilde{B}^{-1}$		$I$

$\underbrace{\hspace{10em}}_{\text{initial basis}} \quad \underbrace{\hspace{10em}}_{\tilde{\mathcal{B}}}$

Final tableau

- The final tableau satisfies the **optimality criterion**:

$$\bar{c}_j = c_j - z_j = c_j - \pi' A_j \geq 0 \quad \forall j,$$

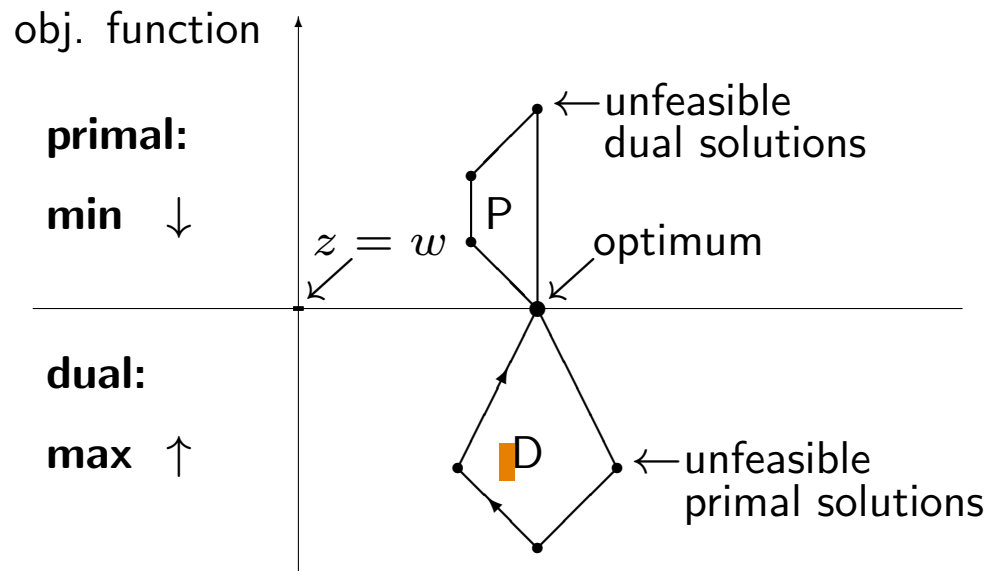
with  $\pi = \text{optimal dual solution}$ . Hence

- in row 0, in the columns corresponding to the initial basis, we have  $\bar{c}_j = c_j - \pi_j$ , from which we get the optimal dual solution  $\pi_j = c_j - \bar{c}_j$ .
- If the initial basis is provided by the artificial variables of Phase 1 (cost  $c_j = 0$ ), then  $\pi_j = -\bar{c}_j$ .
- In the final tableau the columns of the initial base contain the inverse of the optimal base.




## Dual simplex algorithm

- (Primal) Simplex: maintain a feasible primal solution, and move it towards dual feasibility.
- Dual simplex: maintain a feasible dual solution, and move it towards primal feasibility.




1. We start with a base that is **dual feasible** ( $y_{0j} \geq 0 \ j = 1, \dots, n$ ) but **primal infeasible** ( $\exists \ y_{i0} < 0, \ i \geq 1$ ), e.g., ( $r1:=r1-r2, \ r0:=r0-r1, \ r2:=r2-r1$ )

	$x_1$	$x_2$	$x_3$	$x_4$			$x_1$	$x_2$	$x_3$	$x_4$	
0	1	0	$-\frac{1}{2}$	0		$\rightarrow$	2	0	0	$\frac{1}{2}$	1
2	2	1	-2	0		$x_1$	-2	1	0	-1	-1
4	1	1	-1	1		$x_2$	6	0	1	0	2

## Dual simplex algorithm (cont'd)

2. We select for **pivoting a row**  $i$  corresponding to a  $y_{i0} < 0$ .
3. We select the **pivot**  $y_{is}$  among the  $y_{ij} < 0$  (the pivoting must make  $y_{i0}$  positive):
  - the pivoting must obtain 0 in  $y_{0s}$  through  $\tilde{y}_{0j} := \underbrace{y_{0j}}_{\geq 0} - \underbrace{\left(\frac{y_{0s}}{y_{is}}\right)}_{\leq 0 (\Leftarrow y_{0s} \geq 0, y_{is} < 0)} y_{ij}$
  - and  $\tilde{y}_{0j}$  must remain  $\geq 0$  ( $j = 1, \dots, n$ )  $\Rightarrow \frac{y_{0j}}{y_{ij}} \leq \frac{y_{0s}}{y_{is}} \forall j : y_{ij} < 0$ , so
  - **pivot selected as**  $\max_{j: y_{ij} < 0} \left\{ \frac{y_{0j}}{y_{ij}} \right\}$

		$x_1$	$x_2$	$x_3$	$x_4$		$x_1$	$x_2$	$x_3$	$x_4$			
$-z$	2	0	0	$\frac{1}{2}$	1		$-z$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	feasible
$x_1$	-2	1	0	$\textcircled{-1}$	-1		$x_3$	2	-1	0	1	1	$\Downarrow$
$x_2$	6	0	1	0	2		$x_2$	6	0	1	0	2	optimal

- **Note:** This choice also ensures the minimum cost increase. Indeed

$$-z = \tilde{y}_{00} = y_{00} - \overbrace{\left(\frac{y_{0s}}{y_{is}}\right)}^{\text{cost increase } (\geq 0)} \underbrace{y_{i0}}_{< 0}$$

$\underbrace{\hspace{1.5cm}}_{\max \leq 0}$

## Dual simplex algorithm (cont'd)

**procedure DUAL\_SIMPLEX:**

**comment:** we are given a tableau such that  $y_{0j} \geq 0 \ \forall j \geq 1$  and  $\exists i > 0 : y_{i0} < 0$ ;

**begin**

*optimal* := false;

*infeasible* := false;

**while** *optimal* = false **and** *infeasible* = false **do**

**if**  $y_{i0} \geq 0$  for  $i = 1, \dots, m$  **then** *optimal* = true

**else**

**begin**

select an  $i > 0 : y_{i0} < 0$  ;

**if**  $y_{ij} \geq 0$  for  $j = 1, \dots, n$  **then** *infeasible* := true (**comment:** unbounded dual)

**else**  $\theta := \max_{j>0: y_{ij}<0} \left\{ \frac{y_{0j}}{y_{ij}} \right\} = \frac{y_{0s}}{y_{is}}$ , and perform a pivoting on  $y_{is}$

**end**

**end.**

No “two-phase” method is needed: the dual simplex algorithm is normally used when

- we are given the optimal tableau of a primal LP;
- one or more constraints are added so the solution becomes infeasible;
- we want the new optimal solution without starting from scratch.



## Dual simplex algorithm (cont'd)

SIMPLEX	PRIMAL	DUAL
we move	from BFS to BFS	from unfeasible base to unfeasible base
$z$ starts from a value	higher than the optimum	lower than the optimum
the value of $z$	decreases	increases
we first select the variable that	the column, i.e., enters the base	the row, i.e. leaves the base
then we select the variable that	the row, i.e., leaves the base	the column, i.e., enters the base
pivot defined by a	minimum among positive values	maximum among negative values

The dual simplex algorithm can be seen as the primal simplex algorithm executed on the dual problem, but operating on the primal tableau:

(P)

0	$c'$
$b$	$A$

(D)

0	$\hat{b}'$
$c$	$\hat{A}'$

## Dual simplex algorithm (cont'd)

- Example :  $\min z = -x_1 - 4x_2$   

$$\begin{aligned} x_1 + x_2 + s_1 &= 2 \\ x_1 + 3x_2 + s_2 &= 3 \\ 3x_2 + s_3 &= 2 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

### Primal Simplex:

		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$
$-z$	0	-1	-4	0	0	0
$s_1$	2	1	1	1	0	0
$s_2$	3	1	3	0	1	0
$s_3$	2	0	3	0	0	1

		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$
$-z$	$\frac{8}{3}$	-1	0	0	0	$\frac{4}{3}$
$s_1$	$\frac{4}{3}$	1	0	1	0	$-\frac{1}{3}$
$s_2$	1	1	0	0	1	-1
$x_2$	$\frac{2}{3}$	0	1	0	0	$\frac{1}{3}$

		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$
$-z$	$\frac{11}{3}$	0	0	0	1	$\frac{1}{3}$
$s_1$	$\frac{1}{3}$	0	0	1	-1	$\frac{2}{3}$
$x_1$	1	1	0	0	1	-1
$x_2$	$\frac{2}{3}$	0	1	0	0	$\frac{1}{3}$

Optimal

- **To obtain a dual tableau:** (i) execute, on the first primal tableau,  $\text{row } 0 := \text{row } 0 + 4 \cdot (\text{row } 1)$ ; (ii) perform a pivoting on  $y_{12}$ :

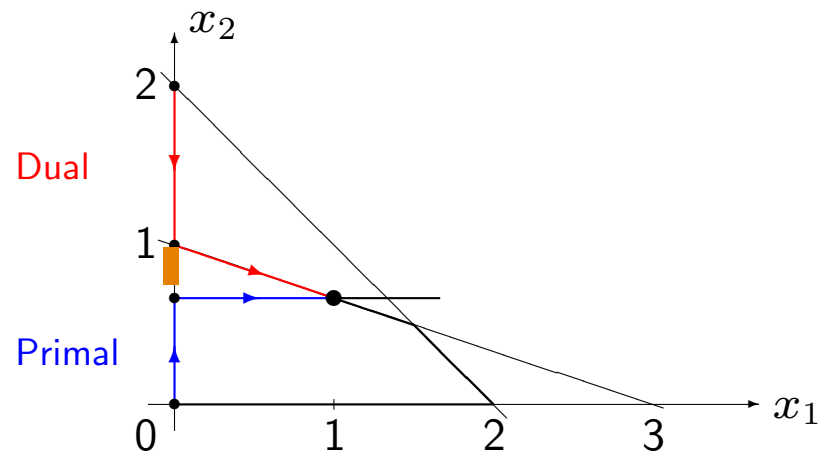
- **Dual Simplex:**

		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$
$-z$	8	3	0	4	0	0
$x_2$	2	1	1	1	0	0
$s_2$	-3	-2	0	-3	1	0
$s_3$	-4	-3	0	-3	0	1

		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$
$-z$	4	$\frac{1}{3}$	0	0	$\frac{4}{3}$	0
$x_2$	1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0
$s_1$	1	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	0
$s_3$	-1	$\ominus 1$	0	0	-1	1

		$x_1$	$x_2$	$s_1$	$s_2$	$s_3$
$-z$	$\frac{11}{3}$	0	0	0	1	1
$x_2$	$\frac{2}{3}$	0	1	0	0	$\frac{1}{3}$
$s_1$	$\frac{1}{3}$	0	0	1	-1	$\frac{2}{3}$
$x_1$	1	1	0	0	1	-1

Optimal



## Sensitivity analysis

- Industrial problem: the input data ( $A$ ,  $b$ , and  $c$ ) are frequently uncertain (market surveys, managerial evaluations, ...);
- practically impossible to evaluate all alternatives.■
- **Sensitivity analysis:** method to evaluate the result of the variation of **a single** input data; ■  
⇒ definition of a **confidence interval** such that:  
*if the value is modified within the interval, the optimal basis remains optimal.*■
- Important: this does NOT imply that the solution does not change:  
The new solution ( $x$ ) and its value need to be recomputed.■
- Two main cases: effect on the optimal solution of a change in
  - the right-hand side of a constraint;
  - the objective function coefficient for a variable;■using these results, it is possible to determine the corresponding confidence intervals.
- We will not see how to determine the confidence intervals: they are **provided by any software for linear programming.**■
- The analysis becomes more complex when more values are changed at the same time:  
it is frequently better to solve the LP with the new data.

## Changing the right-hand side of a constraint

- $i$ th constraint:  $b_i$  changed to  $\bar{b}_i$ . ■
- **Remind:**  $b$  is only used to compute the values of the basic variables (**column 0**) as

$$x'_\beta = B^{-1}b$$

while the relative costs do not vary; ■

- let  $\hat{b}' = (b_1, b_2, \dots, \bar{b}_i, \dots, b_m)$ : ■

**the current optimal basis remains optimal for all  $\bar{b}_i$  values per for which the basic solution remains feasible, i.e., if  $B^{-1}\hat{b} \geq 0$ .** ■

## Changing the objective function coefficient for a variable

- cost of the  $k$ th variable changes from  $c_k$  to  $\bar{c}_k$ . ■
- **Remind:**  $c$  is only used to compute the relative costs (**row 0**) as

$$c' - c'_\beta B^{-1}A$$

- **Observe:** the current solution value can only change if  $x_k \in \mathcal{B}$ ; ■  
changing the cost of a single variable modifies ALL relative costs; ■
- let  $\hat{c}' = (c_1, c_2, \dots, \bar{c}_k, \dots, c_n)$ : ■
- **the current optimal basis remains optimal if  $\hat{c}' - \hat{c}'_\beta B^{-1}A \geq 0$ .** ■

## Sensitivity analysis (cont'd)

- Example of slide 1 in standard form:

$$\begin{array}{rcllclcl} \min & -4x_1 & - & 2x_2 & & & \\ & 3x_1 & + & x_2 & + & x_3 & = 7 \\ & x_1 & + & x_2 & & + & x_4 = 3 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 \geq 0 \end{array}$$

- The simplex algorithm produces the optimal solution  $x_1 = 2, x_2 = 1$ . From the final tableau:

$$Y = \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \text{ from which } B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

- Second right-end side changed from 3 to 4:

$$\hat{b} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \Rightarrow x_\beta = B^{-1}\hat{b} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix} \geq 0. \text{ The optimal basis does not change.}$$

- Second right-end side changed from 3 to 2:

$$\hat{b} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \Rightarrow x_\beta = B^{-1}\hat{b} = \begin{bmatrix} 3 \\ -\frac{1}{2} \end{bmatrix} \not\geq 0. \text{ The current basis is no longer optimal.}$$

## Sensitivity analysis (cont'd)

- From the  $A$  matrix of the standard form,

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \Rightarrow B^{-1}A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} A = \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

- Coefficient of  $x_2$  changed from  $-2$  to  $-3$ :

$$\hat{c}' = \begin{bmatrix} -4 & -3 & 0 & 0 \end{bmatrix}, \hat{c}'_{\beta} = \begin{bmatrix} -4 & -3 \end{bmatrix}, \hat{c}'_{\beta} B^{-1}A = \begin{bmatrix} -4 & -3 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}.$$

$$\hat{c}' - \hat{c}'_{\beta} B^{-1}A = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{5}{2} \end{bmatrix} \geq 0. \quad \text{The current basis remains optimal.}$$

- Coefficient of  $x_2$  changed from  $-2$  to  $-5$ :

$$\hat{c}' = \begin{bmatrix} -4 & -5 & 0 & 0 \end{bmatrix}, \hat{c}'_{\beta} = \begin{bmatrix} -4 & -5 \end{bmatrix}, \hat{c}'_{\beta} B^{-1}A = \begin{bmatrix} -4 & -5 & \frac{1}{2} & -\frac{11}{2} \end{bmatrix}.$$

$$\hat{c}' - \hat{c}'_{\beta} B^{-1}A = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & \frac{11}{2} \end{bmatrix} \not\geq 0, \quad \text{The current basis is no longer optimal.}$$



## Shadow prices

- Frequently used, with various definition, in Economics (cost benefit analysis);
- in Linear programming, consider a model in a production context:

$$\mathbf{max} \ f(x) \text{ (profit function) s.t. } a'_i x \leq b_i (i = 1, \dots, m);$$

- $b_i$  = available quantity of the  $i$ th resource. In industrial practice, frequently  $b_i$  is not a hard limit, but it can be set by management with some flexibility.
- **Problem:** given the current optimal solution, is it convenient to increase a certain bound  $b_i$  (by also considering its cost)?
- **Definition:** the *shadow price* for resource  $i$  is the increase of the objective function value the could be obtained by increasing the available amount  $b_i$  of the resource, provided the new value lies inside the confidence interval of the sensitivity analysis.
- Remind the sensitivity analysis of the example of slide 1:
$$\begin{array}{rclcl} \max z = & 4x_1 & + & 2x_2 & \\ & 3x_1 & + & x_2 & \leq 7 \\ & x_1 & + & x_2 & \leq 3 \\ & x_1 & , & x_2 & \geq 0 \end{array}$$
- Optimal solution  $x_1 = 2, x_2 = 1$  of value 10.  
Sensitivity analysis: if  $b_2 := 4$ , the current basis,  $\mathcal{B} = \{A_1, A_2\}$ , remains optimal;  
optimal solution  $x_1 = \frac{3}{2}, x_2 = \frac{5}{2}$  of value 11: shadow price for the second resource = 1.
- Similarly: shadow price for the first resource = 1.

## Shadow prices (cont'd)

- Why do we use the term “**prize**” for a profit increase? ■
- When a shadow price is positive, the cost of the resource increase must be considered. ■
- For example, suppose the shadow price for resource  $i$  is 3: if the unit **cost** for increasing the resource is **lower than 3** then increasing it would be **convenient**, otherwise it would not. ■
- **Shadow price** = maximum price we are willing to pay for each additional unit of a resource. ■
- **Duality** offers an immediate way to determine the shadow prices. ■
- Let us resume the example:

$$\begin{array}{rclcl}
 \max z = & 4x_1 & + & 2x_2 & \\
 & 3x_1 & + & x_2 & \leq 7 \\
 & x_1 & + & x_2 & \leq 3 \\
 & x_1 & , & x_2 & \geq 0
 \end{array}
 \qquad
 \begin{array}{rclcl}
 \min w = & 7\pi_1 & + & 3\pi_2 & \\
 & 3\pi_1 & + & \pi_2 & \geq 4 \\
 & \pi_1 & + & \pi_2 & \geq 2 \\
 & \pi_1 & , & \pi_2 & \geq 0.
 \end{array}$$

**optimal dual solution:**  $\pi_1 = \pi_2 = 1$ . ■

- **Remind:** The objective function of the dual is  $\pi' b$ ;  
     primal and dual optimal solutions have the same value. ■
- the objective function coefficient for  $\pi_i$  is  $b_i$  (the right-end side of primal constraint  $i$ ). **Hence**
- **Property: the optimal value of dual variable  $\pi_i$  is the shadow price for primal constraint  $i$ .** ■

## An example using commercial software

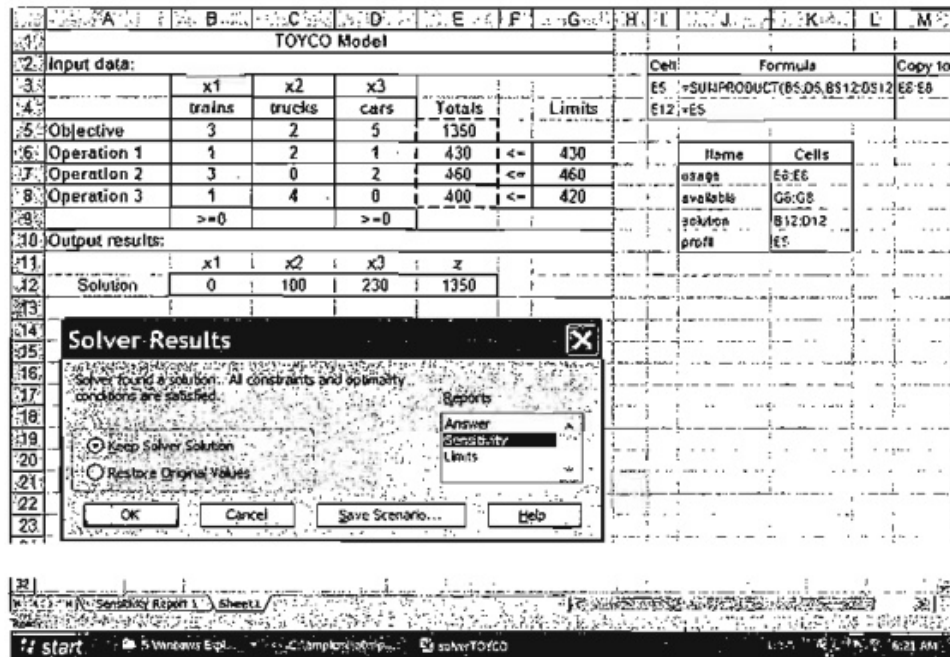


FIGURE 3.14  
TORA sensitivity analysis for the TOYCO model

***Sensitivity Analysis***				
Variable	CurrObjCoeff	MinObjCoeff	MaxObjCoeff	Reduced Cost
x1:	3.00	-infinity	7.00	4.00
x2:	2.00	0.00	10.00	0.00
x3:	5.00	2.33	infinity	0.00
Constraint	Curr RHS	Min RHS	Max RHS	Dual Price
1(<):	430.00	230.00	440.00	1.00
2(<):	460.00	440.00	860.00	2.00
3(<):	420.00	400.00	infinity	0.00

## Primal-Dual algorithm

$$(P) \min z = c'x$$

$$Ax = b \geq 0$$

$$x \geq 0$$

$$(D) \max w = \pi'b$$

$$\pi'A \leq c'$$

$$\pi' \geq 0$$

- Complementary slackness:**

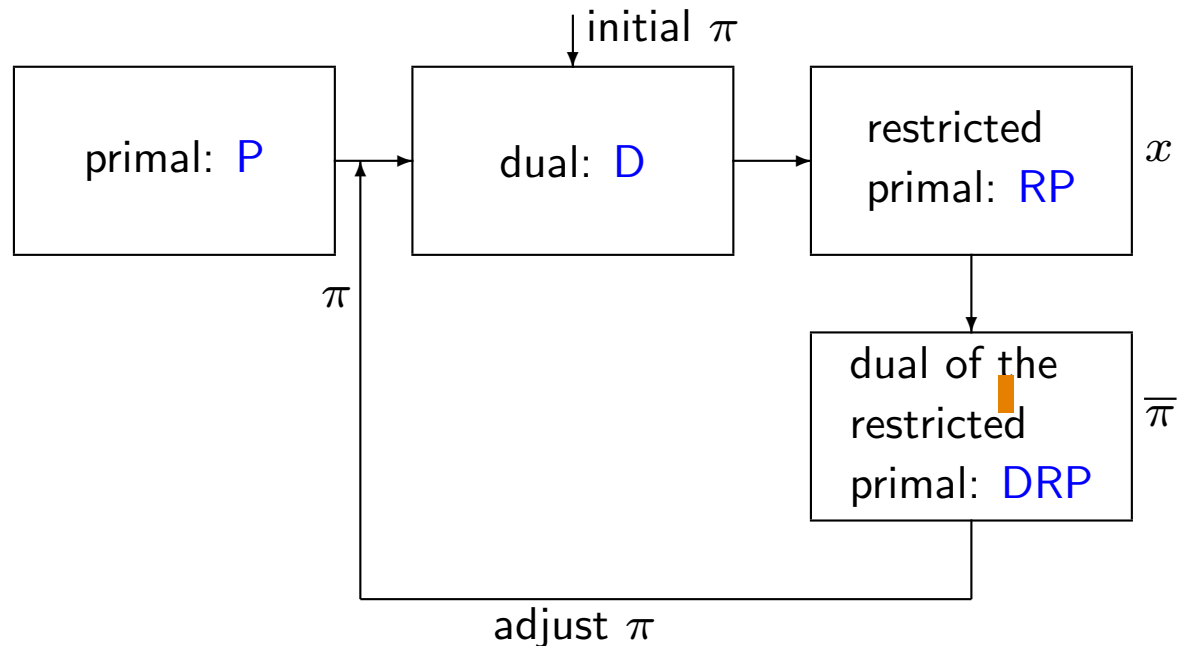
$$\pi_i(a'_i x - b_i) = 0 \quad \forall i \quad (\alpha) \quad (\text{satisfied})$$

$$(c_j - \pi'A_j)x_j = 0 \quad \forall j \quad (\beta)$$

- Algorithm:** given a  $\pi$  feasible for D, look for an  $x$  feasible for P satisfying  $(\beta)$ , i.e., such that:

$$x_j = 0 \quad \forall j : c_j - \pi'A_j > 0$$

- Approach:**



## Primal-Dual algorithm (cont'd)

### 1. Initialization (how to find an initial feasible $\pi$ )

- If  $c \geq 0$  then  $\pi = 0$  is feasible.
- If  $c \not\geq 0$ : new variable  $x_{n+1}$  with cost  $c_{n+1} = 0$ ;

new constraint  $x_1 + \cdots + x_n + x_{n+1} = b_{m+1} = M (\rightarrow +\infty)$ :

$$\begin{aligned}
 (\text{P}') \quad \min z &= c'x + 0 \\
 Ax &= b \\
 \sum_{j=1}^n x_j + x_{n+1} &= b_{m+1} \\
 x &\geq 0 \\
 x_{n+1} &\geq 0
 \end{aligned}$$

Equivalent problem: (i) same objective function;

(ii)  $\forall \bar{x}$  feasible for P, the new constraint is satisfied by setting  $\bar{x}_{n+1} = b_{m+1} - \sum_{j=1}^n \bar{x}_j$ .

$$\begin{aligned}
 (\text{D}') \quad \max w &= \pi' b + \pi_{m+1} b_{m+1} \\
 \pi' A_j + \pi_{m+1} &\leq c_j \quad (j = 1, \dots, n) \\
 \pi_{m+1} &\leq 0 \\
 \pi' &\geq 0
 \end{aligned}$$

**Feasible solution:**  $\pi_i = 0$  ( $i = 1, \dots, m$ ),  $\pi_{m+1} = \min\{c_j\} (< 0)$

## Primal-Dual algorithm (cont'd)

### 2. Primal $\rightarrow$ Dual $\rightarrow$ Restricted Primal

- $\pi$  feasible for D:  $J = \{j : \pi' A_j = c_j\}$ :
- an  $x$  feasible for P is optimal if and only if  $x_j = 0 \forall j \notin J \Rightarrow$  we need an  $x$  (if any) such that:

$$\begin{cases} Ax = b \\ x \geq 0 \\ x_j = 0 \forall j \notin J \end{cases}, \text{ i.e., } \begin{cases} \sum_{j \in J} a_{ij} x_j = b_i \quad (i = 1, \dots, m) \\ x_j \geq 0 \quad (j \in J) \end{cases}$$

- to look for such  $x$ , let us “invent” a new LP:

$$\begin{aligned} \text{(RP)} \quad \min \zeta = & \sum_{i=1}^m x_i^a \\ & \sum_{j \in J} a_{ij} x_j + x_i^a = b_i \quad (i = 1, \dots, m) \\ & x_j \geq 0 \quad (j \in J) \\ & x_i^a \geq 0 \quad (i = 1, \dots, m) \end{aligned}$$

- If the solution to RP is  $\zeta_{\text{opt}} = 0$ , then  $x$  and  $\pi$  solve P and D.

$$\begin{aligned} \text{Example 1 : (P) } \min z &= 2x_1 + x_2 + 3x_3 \\ x_1 + 2x_2 &= 4 \\ x_1 - x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$\begin{aligned} \text{(D) } \max w &= 4\pi_1 + \pi_2 \\ \pi_1 + \pi_2 &\leq 2 \\ 2\pi_1 - \pi_2 &\leq 1 \\ \pi_2 &\leq 3 \\ \pi_1, \pi_2 &\geq 0 \end{aligned}$$

Feasible dual solution:  $\pi_1 = \pi_2 = 1 \Rightarrow J = \{1, 2\}$ . ■

Find an  $x$  such that

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 1 \\ x_3 &= 0 \\ x_1, x_2 &\geq 0 \end{aligned} \Rightarrow \begin{aligned} \text{(RP) } \min \zeta &= x_1^a + x_2^a \\ x_1 + 2x_2 + x_1^a &= 4 \\ x_1 - x_2 + x_2^a &= 1 \\ x_1, x_2, x_1^a, x_2^a &\geq 0 \end{aligned} \quad \blacksquare$$

		0	0	0	1	1
			$x_1$	$x_2$	$x_1^a$	$x_2^a$
$-\zeta$	-5	-2	-1	0	0	■
$x_1^a$	4	1	2	1	0	
$x_2^a$	1	①	-1	0	1	

		$x_1$	$x_2$	$x_1^a$	$x_2^a$
$-\zeta$	-3	0	-3	0	2
$x_1^a$	3	0	(3)	1	-1
$x_1$	1	1	-1	0	1

		$x_1$	$x_2$	$x_1^a$	$x_2^a$
$-\zeta$	0	0	0	1	1
$x_2$	1	0	1	$\frac{1}{3}$	$-\frac{1}{3}$
$x_1$	2	1	0	$\frac{1}{3}$	$\frac{2}{3}$

Optimal

Example 2 : (P) min  $x_1 + x_3$

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 5 \\ x_2 + 2x_3 &= 6 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(D) max  $5\pi_1 + 6\pi_2$

$$\begin{aligned} \pi_1 &\leq 1 \\ 2\pi_1 + \pi_2 &\leq 0 \\ 2\pi_2 &\leq 1 \\ \pi_1 &\leq 0 \\ \pi_1, \pi_2 &\geq 0 \end{aligned}$$

Feasible dual solution:  $\pi_1 = \pi_2 = 0 \Rightarrow J = \{2, 4\}$ .



$$\begin{aligned}
 \text{(RP) min } \zeta = & \quad x_1^a + x_2^a \\
 & 2x_2 + x_4 + x_1^a = 5 \\
 & \quad x_2 + x_2^a = 6 \\
 & x_2, x_4, x_1^a, x_2^a \geq 0
 \end{aligned}$$

	0	0	0	1	1
		$x_2$	$x_4$	$x_1^a$	$x_2^a$
$-\zeta$	-11	-3	-1	0	0
$x_1^a$	5	2	1	1	0
$x_2^a$	6	1	0	0	1

		$x_2$	$x_4$	$x_1^a$	$x_2^a$
$-\zeta$	$-\frac{7}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$	0
$x_2$	$\frac{5}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0
$x_2^a$	$\frac{7}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1

Optimal with  $\zeta_{\text{opt}} > 0$ .

Solution of the dual of RP:  $\bar{\pi}_1 = -\frac{1}{2}$ ,  $\bar{\pi}_2 = 1$ .

## Primal-Dual algorithm (cont'd)

### 3. Restricted Primal $\rightarrow$ Dual of the Restricted Primal $\rightarrow$ Dual

- If  $\zeta_{\text{opt}} > 0$ , we consider the dual of RP:

$$\begin{aligned} \text{(DRP)} \quad \max \quad & \pi' b \\ & \pi' A_j \leq 0 \quad (j \in J) \\ & \pi_i \leq 1 \quad (i = 1, \dots, m) \\ & \pi_i \geq 0 \quad (i = 1, \dots, m) \end{aligned}$$

- RP  $\Rightarrow$  solution  $\bar{\pi}$  of DRP.

We try again with a corrected  $\pi$ :  $\pi^* = \pi + \vartheta \bar{\pi}$ , with  $\vartheta$  such that

(a)  $\pi^*$  is feasible for D;

(b) the solution value of the dual increases.

- **Determining  $\vartheta$ :**

From (b): new solution value  $= \pi^{*'} b = \pi' b + \vartheta \bar{\pi}' b$ ,

where  $\bar{\pi}' b = \zeta_{\text{opt}} > 0$  ( $\Leftarrow$  RP and DRP have the same solution value)

$\Rightarrow$  it must be  $\vartheta > 0$ .

- **Determining  $\vartheta$ :**

From (a): it must be

$$\begin{aligned}\pi^{*'} A_j &\leq c_j \quad (j = 1, \dots, n) \Rightarrow \\ \pi' A_j + \vartheta \bar{\pi}' A_j &\leq c_j \quad (j = 1, \dots, n).\end{aligned}$$

If  $\bar{\pi}' A_j \leq 0 \quad \forall j$  then  $\vartheta$  can indefinitely increase  $\Rightarrow$  new cost of D  $\rightarrow +\infty \Rightarrow$  P is impossible.■

On the other hand,  $\bar{\pi}$  is an optimal (hence feasible) solution to DRP  $\Rightarrow$

$$\bar{\pi}' A_j \leq 0 \quad \forall j \in J. \quad \text{Hence} \blacksquare$$

**A. If  $\zeta_{\text{opt}} > 0$  in RP, and the optimal solution to DRP satisfies  $\bar{\pi}' A_j \leq 0 \quad \forall j \notin J$ , then P is impossible.■**

If instead  $\exists j \notin J : \bar{\pi}' A_j > 0$ , we must ensure that for all such  $j$ 's we have

$$\pi' A_j + \vartheta \bar{\pi}' A_j \leq c_j, \text{ i.e., } \vartheta \leq \frac{c_j - \pi' A_j}{\bar{\pi}' A_j} \blacksquare \text{ hence}$$

**B. If  $\zeta_{\text{opt}} > 0$  in RP and  $\exists j \notin J : \bar{\pi}' A_j > 0$ , then the maximum  $\vartheta$  that ensures feasibility is**

$$\vartheta_{\max} = \min_{j \notin J : \bar{\pi}' A_j > 0} \left\{ \frac{c_j - \pi' A_j}{\bar{\pi}' A_j} \right\}$$

**and the new cost is  $w^* = w + \vartheta_{\max} \bar{\pi}' b > w$ .■**

Example 2 (resumed) (DRP)

$$\begin{array}{rcll}
 \max & 5\pi_1 & +6\pi_2 & \\
 & 2\pi_1 & + \pi_2 & \leq 0 \\
 & \pi_1 & & \leq 0 \\
 & \pi_1 & & \leq 1 \\
 & & \pi_2 & \leq 1 \\
 & \pi_1 & , \pi_2 & \geq 0
 \end{array}$$

Solution:  $\bar{\pi}_1 = -\frac{1}{2}$ ,  $\bar{\pi}_2 = 1$ .

$\pi_1^* = 0 + \vartheta \cdot (-\frac{1}{2})$ ,  $\pi_2^* = 0 + \vartheta \cdot 1$ , with  $\vartheta > 0$  and such that

$$\left. \begin{array}{rcl}
 \pi_1^* & \leq & 1 \\
 2\pi_1^* + \pi_2^* & \leq & 0 \\
 2\pi_2^* & \leq & 1 \\
 \pi_1^* & \leq & 0
 \end{array} \right\} \Rightarrow \begin{array}{rcl}
 -\frac{\vartheta}{2} & \leq & 1 \quad \text{(i)} \\
 -\vartheta + \vartheta & \leq & 0 \quad \text{(ii)} \\
 2\vartheta & \leq & 1 \\
 -\frac{\vartheta}{2} & \leq & 0 \quad \text{(iii)}
 \end{array}$$

Certainly satisfied: (ii), (iii)  $\Leftarrow J = \{2, 4\}$ , and (i)  $\Leftarrow \bar{\pi}' A_1 < 0$ .

From the remaining one:  $\vartheta_{\max} = \frac{1}{2} \Rightarrow \begin{cases} \pi_1^* = -\frac{1}{4} \\ \pi_2^* = \frac{1}{2} \end{cases}$  (new feasible solution to D)

New cost of D:  $-\frac{5}{4} + \frac{6}{2} = \frac{7}{4}$  ( $= 0 + \frac{1}{2} \cdot \frac{7}{2}$ ); new  $J = \{2, 3\}$ .

New RP : (RP)  $\min \zeta =$

$$\begin{aligned} 2x_2 + x_1^a + x_2^a &= 5 \\ x_2 + 2x_3 + x_2^a &= 6 \\ x_2, x_3, x_1^a, x_2^a &\geq 0 \end{aligned}$$

	0	0	0	1	1
		$x_2$	$x_3$	$x_1^a$	$x_2^a$
$-\zeta$	-11	-3	-2	0	0
$x_1^a$	5	2	0	1	0
$x_2^a$	6	1	2	0	1

		$x_2$	$x_3$	$x_1^a$	$x_2^a$
$-\zeta$	$-\frac{7}{2}$	0	-2	$\frac{3}{2}$	0
$x_2$	$\frac{5}{2}$	1	0	$\frac{1}{2}$	0
$x_2^a$	$\frac{7}{2}$	0	2	$-\frac{1}{2}$	1

		$x_2$	$x_3$	$x_1^a$	$x_2^a$
$-\zeta$	0	0	0	1	1
$x_2$	$\frac{5}{2}$	1	0	$\frac{1}{2}$	0
$x_3$	$\frac{7}{4}$	0	1	$-\frac{1}{4}$	$\frac{1}{2}$

$\zeta_{\text{opt}} = 0$ :

optimal primal solution:  $x_1 = 0, x_2 = \frac{5}{2}, x_3 = \frac{7}{4}, x_4 = 0$ ;

optimal dual solution:  $\pi_1 = -\frac{1}{4}, \pi_2 = \frac{1}{2}; w = z = \frac{7}{4}$ .

If we had obtained  $\zeta_{\text{opt}} > 0$ , new iteration  $\text{RP} \rightarrow \text{DRP} \rightarrow \text{D}$ .

## Primal-Dual algorithm (cont'd)

procedure PRIMAL\_DUAL:

begin

for each  $b_i < 0$  do multiply the  $i$ th equation by  $-1$ ;

comment: we are given a feasible dual solution  $\pi$ ;

$optimal := infeasible := false$ ;

while  $optimal = false$  and  $infeasible = false$  do

begin

$J := \{j : \pi' A_j = c_j\}$ ;

call **SIMPLEX** for problem RP, and let  $\bar{\pi}$  be the solution to its dual (DRP);

if  $\zeta_{opt} = 0$  then  $optimal = true$

else

if  $\bar{\pi}' A_j \leq 0 \forall j \notin J$  then  $infeasible := true$

else  $\vartheta := \min_{j \notin J: \bar{\pi}' A_j > 0} \left\{ \frac{c_j - \pi' A_j}{\bar{\pi}' A_j} \right\}$ , and  $\pi := \pi + \vartheta \bar{\pi}$

end

end.

(1) P bounded, D bounded } algorithm;

(2) P impossible, D unbounded }

(3) P impossible, D impossible }

(4) P unbounded, D impossible }

$\Leftrightarrow \exists c_j < 0 \Rightarrow x_{n+1}, b_{m+1} \rightarrow D$  formally possible, but

if (3)  $\Rightarrow (\pi \rightarrow \infty) \Rightarrow \text{algorithm(P impossible)}$

if (4)  $\Rightarrow (x \rightarrow \infty) \Rightarrow \text{algorithm (P unbounded)}$

**Ex.:** min  $-x_1 - x_2$   

$$\begin{array}{rcl} x_1 - x_2 & \leq & 0 \\ x_1, x_2 & \geq & 0 \end{array} \rightarrow \begin{array}{rcl} x_1 - x_2 + x_3 & = & 0 \\ x_3 & \geq & 0 \end{array} \quad (\text{P unbounded})$$

$$\begin{array}{rcl} \min & -x_1 - x_2 & \\ & x_1 - x_2 + x_3 & = 0 \\ & x_1 + x_2 + x_3 + x_4 & = M \\ & x_1, x_2, x_3, x_4 & \geq 0 \end{array} \quad \begin{array}{rcl} \max & M\pi_2 & \\ & \pi_1 + \pi_2 & \leq -1 \\ & -\pi_1 + \pi_2 & \leq -1 \\ & \pi_1 + \pi_2 & \leq 0 \\ & \pi_2 & \leq 0 \\ & \pi_1, \pi_2 & \geq 0 \end{array}$$

$$\left. \begin{array}{l} \pi_1 = 0, \\ \pi_2 = \min\{c_j\} = -1 \\ J = \{1, 2\} \end{array} \right\} \Rightarrow \begin{array}{rcl} \text{(RP) min} & x_1^a + x_2^a & \\ & x_1 - x_2 + x_1^a & = 0 \\ & x_1 + x_2 + x_2^a & = M \\ & x_1, x_2, x_1^a, x_2^a & \geq 0 \end{array}$$


		0	0	0	1	1
			$x_1$	$x_2$	$x_1^a$	$x_2^a$
$-\zeta$	$-M$	$-2$	0	0	0	
$x_1^a$	0	1	$-1$	1	0	
$x_2^a$	$M$	1	1	0	1	

		$x_1$	$x_2$	$x_1^a$	$x_2^a$
$-\zeta$	$-M$	0	-2	2	0
$x_1$	0	1	-1	1	0
$x_2^a$	$M$	0	2	-1	1

		$x_1$	$x_2$	$x_1^a$	$x_2^a$
$-\zeta$	0	0	0	1	1
$x_1$	$\frac{M}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
$x_2$	$\frac{M}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Optimal solution:  $x_1 = x_2 = \frac{M}{2}$ ,  $z = w = -M$



**Ex.:** (P)  $\min x_1 - 2x_2$  (P, D impossible :  
 $x_1 - x_2 - x_3 = 1 \quad \pi_1 - \pi_2 \leq 1,$   
 $-x_1 + x_2 - x_4 = 1 \quad -\pi_1 + \pi_2 \leq -2)$    
 $x_1 + x_2 + x_3 + x_4 + x_5 = M \quad \leftarrow \text{initialization}$   
 $x_1, x_2, x_3, x_4, x_5 \geq 0$

(D)  $\max \pi_1 + \pi_2 + M\pi_3$   
 $\pi_1 - \pi_2 + \pi_3 \leq 1$   
 $-\pi_1 + \pi_2 + \pi_3 \leq -2$   
 $-\pi_1 + \pi_3 \leq 0$   
 $-\pi_2 + \pi_3 \leq 0$   
 $\pi_3 \leq 0$   
 $\pi_1, \pi_2, \pi_3 \geq 0$

feasible solution :  
 $\pi' = (0, 0, -2)$   
 $\Rightarrow J = \{2\}$

		0	0	1	1	1
			$x_2$	$x_1^a$	$x_2^a$	$x_3^a$
$-\zeta$	$-M - 2$	-1	0	0	0	
$x_1^a$	1	-1	1	0	0	
$x_2^a$	1	1	0	1	0	
$x_3^a$	$M$	1	0	0	1	

		$x_2$	$x_1^a$	$x_2^a$	$x_3^a$
$-\zeta$	$-M - 1$	0	0	1	0
$x_1^a$	2	0	1	1	0
$x_2$	1	1	0	1	0
$x_3^a$	$M - 1$	0	0	-1	1

$$\bar{\pi}' = (1, 0, 1)$$

$$\vartheta_{\max} = \min\{\frac{3}{2}, \frac{2}{1}\} = \frac{3}{2}$$

$$\pi' = (\frac{3}{2}, 0, -\frac{1}{2})$$

$$J = \{1, 2\}$$

		0	0	0	1	1	1
		$x_1$	$x_2$	$x_1^a$	$x_2^a$	$x_3^a$	
$-\zeta$	$-M - 2$	-1	-1	0	0	0	
$x_1^a$	1	①	-1	1	0	0	
$x_2^a$	1	-1	1	0	1	0	
$x_3^a$	$M$	1	1	0	0	1	

		$x_1$	$x_2$	$x_1^a$	$x_2^a$	$x_3^a$
$-\zeta$	$-M - 1$	0	-2	1	0	0
$x_1$	1	1	-1	1	0	0
$x_2^a$	2	0	0	1	1	0
$x_3^a$	$M - 1$	0	2	-1	0	1

		$x_1$	$x_2$	$x_1^a$	$x_2^a$	$x_3^a$
$-\zeta$	-2	0	0	0	0	1
$x_1$	$\frac{M+1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
$x_2^a$	2	0	0	1	1	0
$x_2$	$\frac{M-1}{2}$	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$

$$\overline{\pi}' = (1, 1, 0)$$

$$\overline{\pi}' A_j \leq 0 \quad \forall j \Rightarrow \text{P impossible.}$$