Hungarian algorithm: Theoretical bases

The Hungarian algorithm is recognized as a predecessor of the **primal-dual method** for linear programming, designed one year later by Dantzig, Ford and Fulkerson.

1. **Initialize** with any (u_i) and (v_j) satisfying $u_i + v_j \le c_{ij}$ $(i, j = 1, \ldots, n)$;
2. find a maximum matching M (König) in the subgraph $\mathbf{G^0 = (U,V;E^0)}$ of $G = (U, V; E)$ that only contains the edges of E that satisfy $u_i + v_j = c_{ij}$ (i.e., such that $\overline{c}_{ij} = 0$);
3. **if** M is perfect **then** it has maximum weight $w(M) = \sum_{k=1}^{n}(u_k + v_k)$, hence **stop**;
4. **else** G^0 must contain (Hall) a subset $U' \subseteq U$ such that $|U'| > |F(U')|$: update the current covering system through (Egerváry)

$$\begin{cases} u_i := u_i + 1 \text{ for } i \in U'; \\ v_j := v_j - 1 \text{ for } j \in F(U'), \end{cases} \tag{12}$$

   thus keeping $u_i + v_j = c_{ij}$, but increasing the value of $\sum_{k=1}^{n}(u_k + v_k)$ by $|U'| - |F(U')| > 0$ and **go to 2** (possibly new edges satisfy $u_i + v_j = c_{ij}$).

Pseudo-polynomial time complexity, but the two 1s in (12) can be replaced by

$$\min\{u_i + v_j - c_{ij} \; : \; i \in U', j \in F(U')\}$$

$\Longrightarrow$ **Polynomial time complexity**