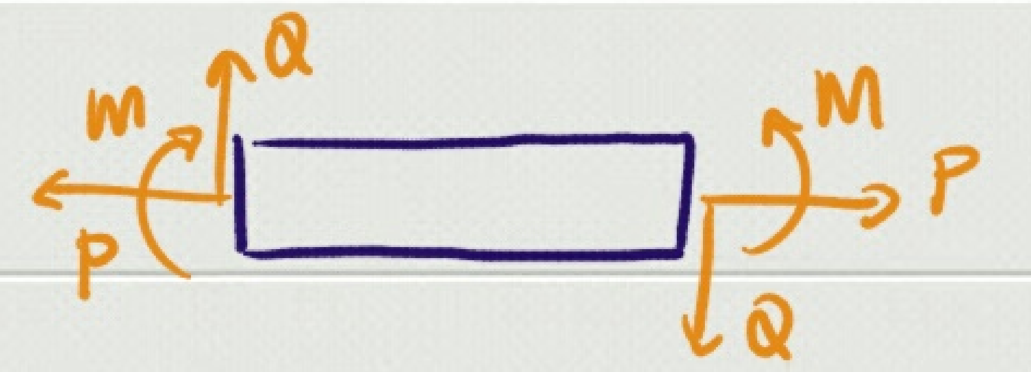
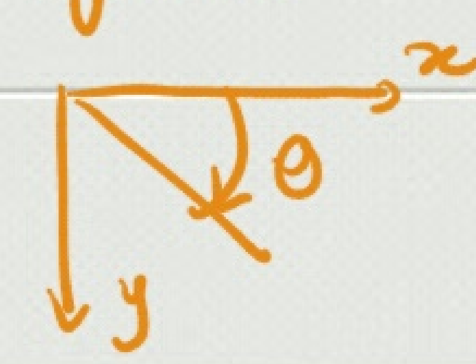
Mech 0013 Summary - Mechanics of Solids and Structures.

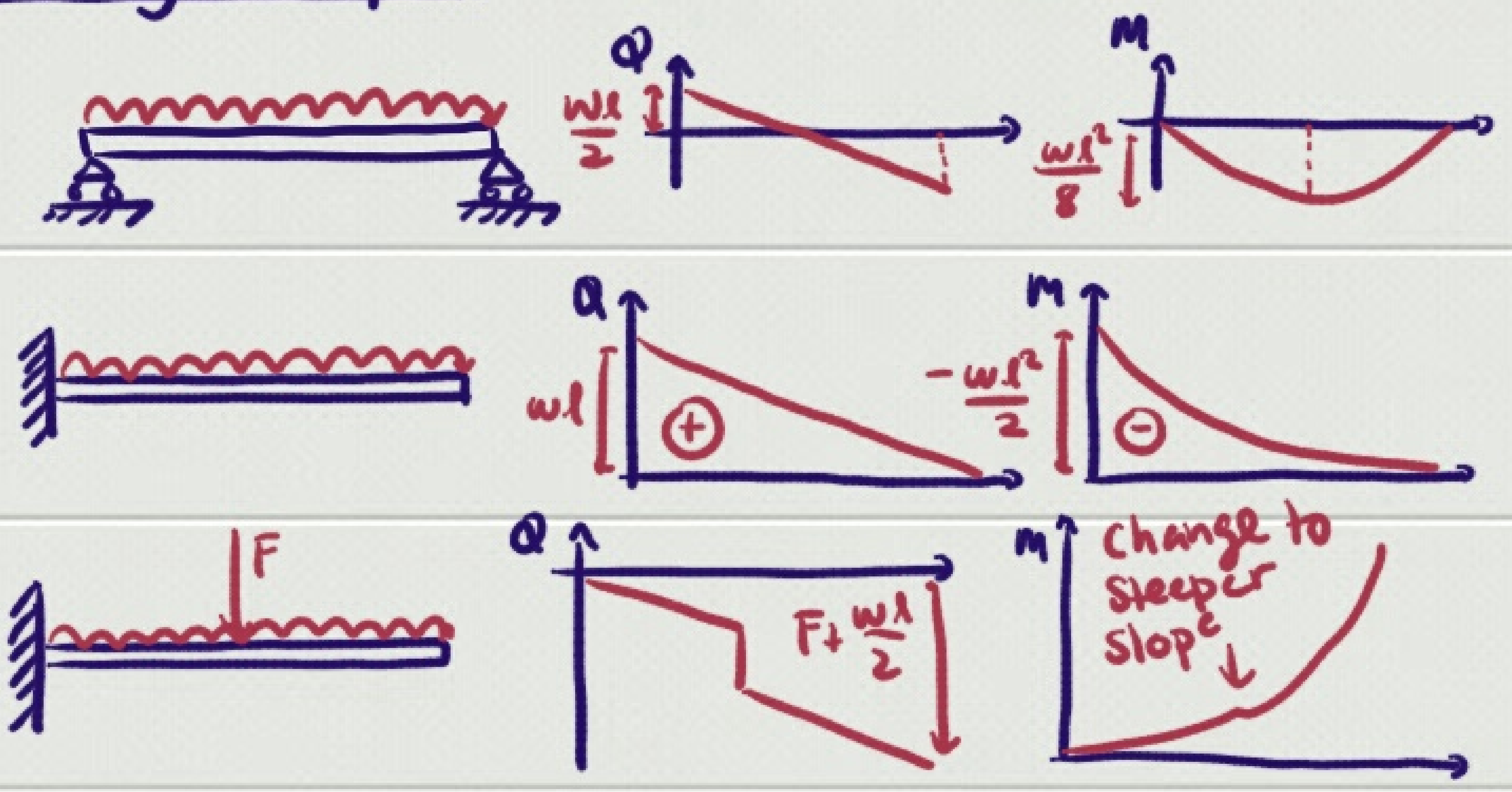
External Actions → Equilibrium Eqns 1
↕
Internal Actions → Equilibrium Eqns 2
↕
Stresses → Constitutive Laws
↕
Strains → Compatibility Relations
↕
Deformations
↕
Displacements



Reference System:



Loading Examples



Load ↪ Shear force

$w = -\frac{dQ}{dx}$; $Q = -\int w \cdot dx$

$Q = \frac{dM}{dx}$; $M = \int Q\,dx$

Longitudinal Strain $\varepsilon = \frac{\text{elongation}}{\text{initial length}} = \frac{h}{R}$ — h: Distance from neutral axis; R: Radius of curvature

$\sigma = E\varepsilon \Rightarrow \sigma_{max} = E\frac{h_{max}}{R}$

From Moment Equilibrium : $M = \frac{EI}{R}$, hence $\sigma_{max} = \frac{Mh_{max}}{I}$

For small deformations: $ds \simeq dx = -R \cdot d\theta$

$\Rightarrow \frac{1}{R} = -\frac{d\theta}{dx}$ ⟶ $M = -EI\frac{d\theta}{dx}$

Integrate ↑ ↓ Differentiate

deflection: y

slope: $\theta = \frac{dy}{dx}$

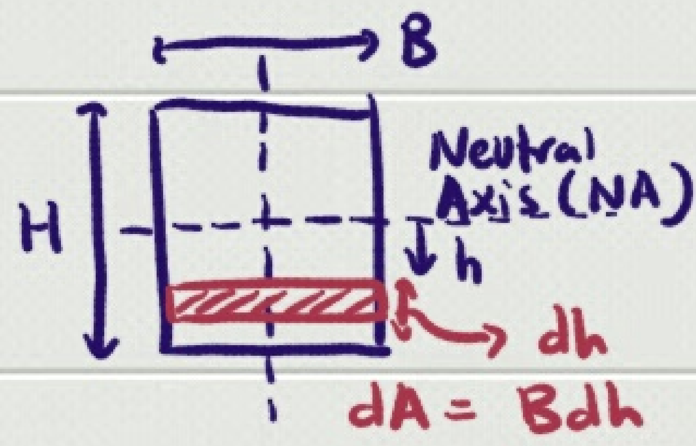
bending moment: $M = -EI\frac{d\theta}{dx} = -EI\frac{d^2y}{dx^2}$

shear force: $Q = \frac{dM}{dx} = -EI\frac{d^3y}{dx^3}$

load dist: $w = -\frac{dQ}{dx} = EI\frac{d^4y}{dx^4}$

Second Moment of Area

$I = \int_A h^2\,dA$



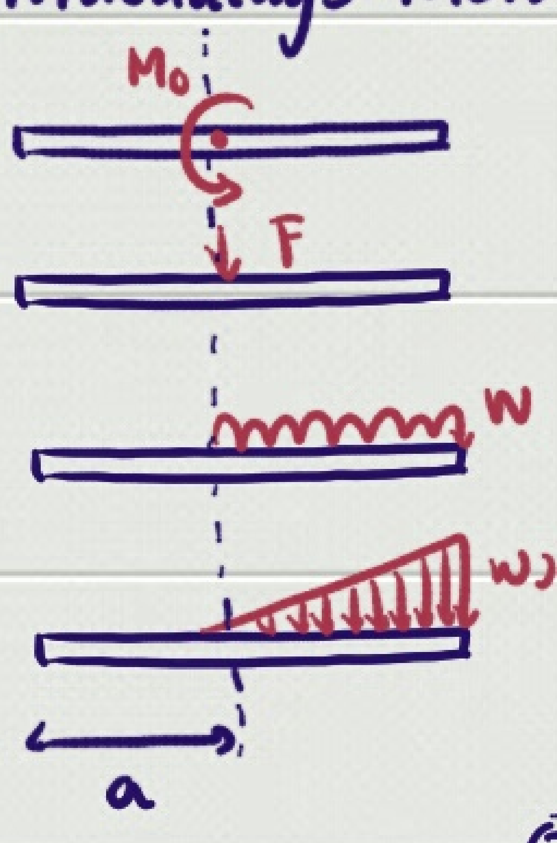
For Rectangular Cross section:

$I_z = \int_{-H/2}^{H/2} h^2 B\,dh = \left[\frac{1}{3}Bh^3\right]_{-H/2}^{H/2} = \frac{BH^3}{12}$

Circular Cross Section:

$I_z = \frac{\pi R^4}{4}$

Macaulay's Method : $M =$



$- M_0\langle x-a\rangle^0$

$- F\langle x-a\rangle^1$

$- \frac{w}{2}\langle x-a\rangle^2$

$- \frac{w}{6}\langle x-a\rangle^3$

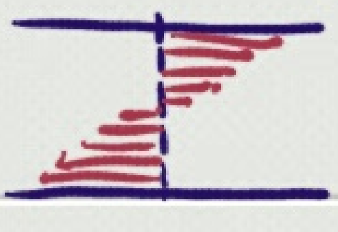
Collapse Mechanisms.

↳ Stress in elastic regime
- returns to original state

Stress in plastic regime
- permanent distortion
- plastic collapse ⟷ plastic bending

① Yield Bending Moment : $M_y = \sigma_y\frac{ab^2}{6}$ (elastic limit σ_y)

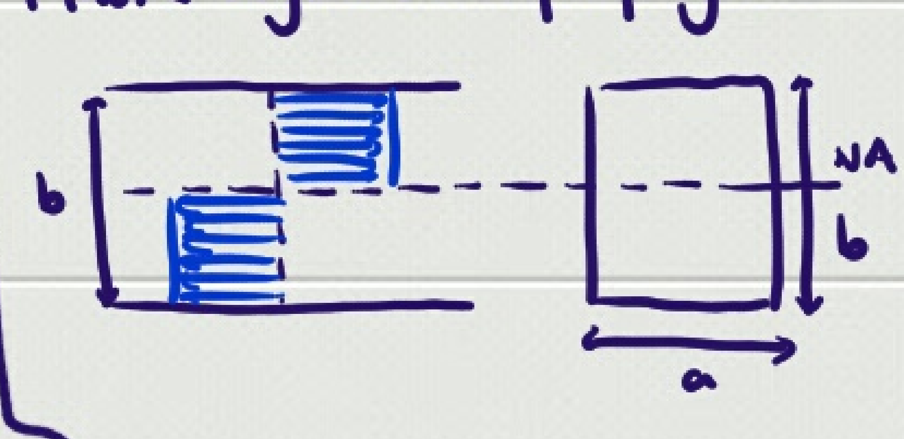
↳ At elastic moment, all fibres are still in the elastic condition.



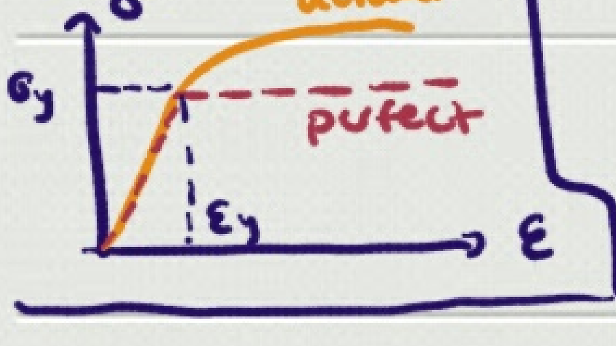
③ Plastic Bending Moment

Plastic Bending Moment → $M_p = \sigma_y\frac{ab^2}{4}$

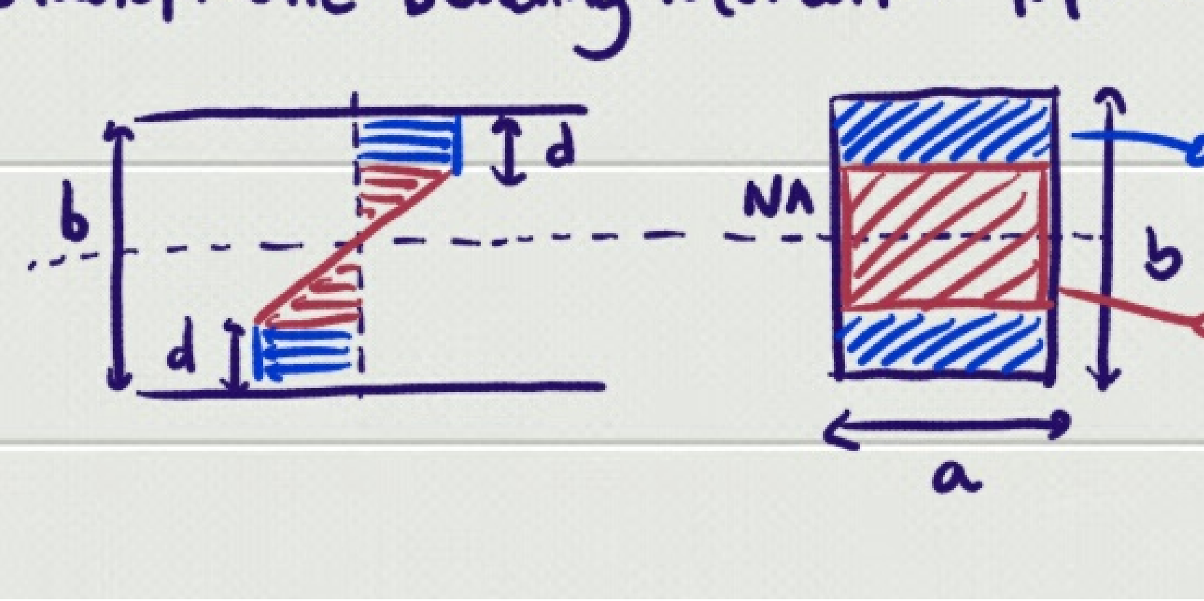
Plastic region has propagated



*Assuming perfectly elasto-plastic:



② Elastoplastic Bending Moment : $M = \frac{\sigma_y ab^2}{6}\left[1+2\frac{d}{b}\left(1-\frac{d}{b}\right)\right]$



Shape factor : Ratio of yielding bending moment to plastic bending moment

$$f = \frac{M_p}{M_y}$$

Neutral Axis with Asymmetry :

- Elastic Neutral Axis corresponds to centroid of section (center of gravity)

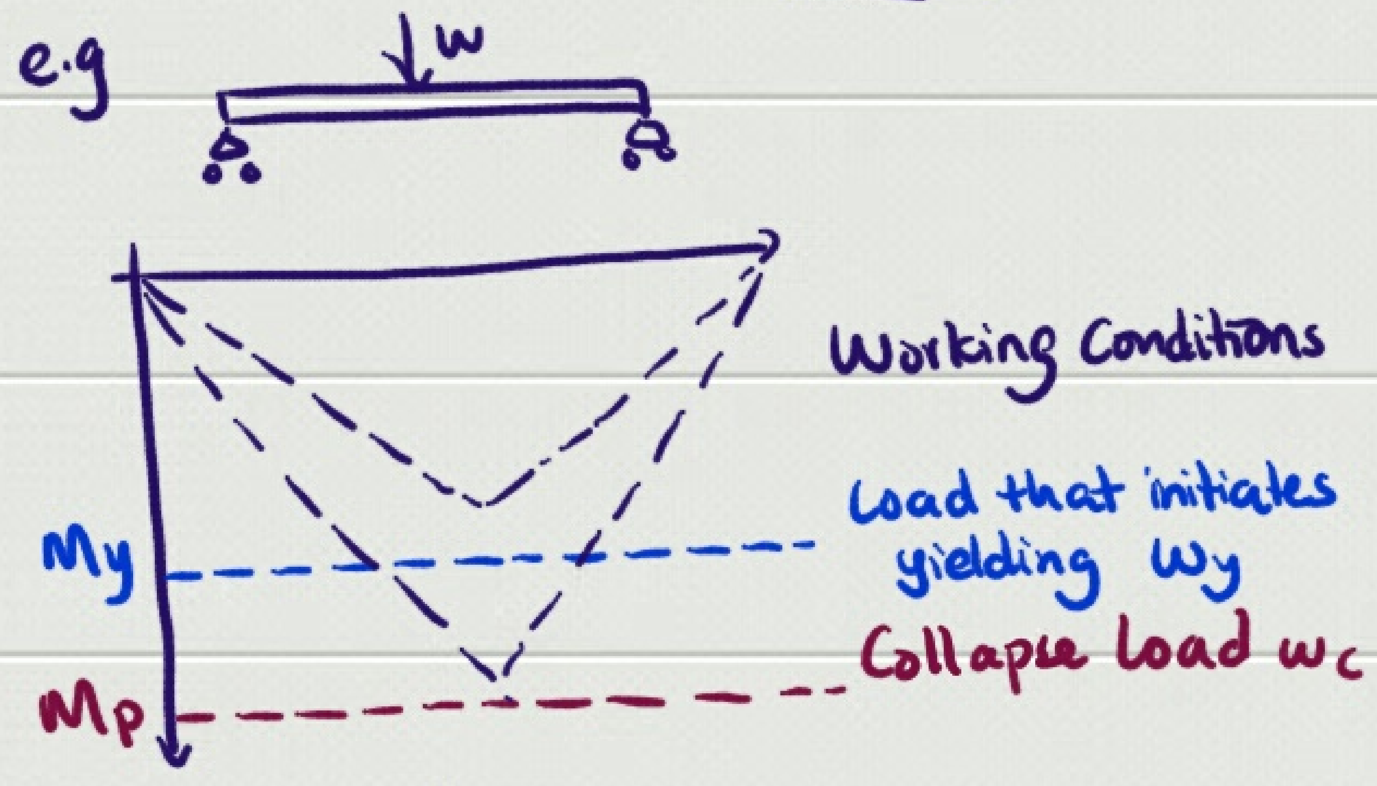
Find with first moment of area : $\int_A h \cdot dA = 0$

- Plastic Neutral Axis corresponds to center of area

Section with highest bending moment will reach fully plastic state first ⇒ can not react to further increases of moment: loses capability of reacting to rotations and curvature can increase indefinitely

↳ Forms a PLASTIC HINGE

e.g



Safety Factor $k_s = \frac{W_Y}{W_{max}}$ → W_Y: Load that initiates yielding; W_{max}: Max working Load

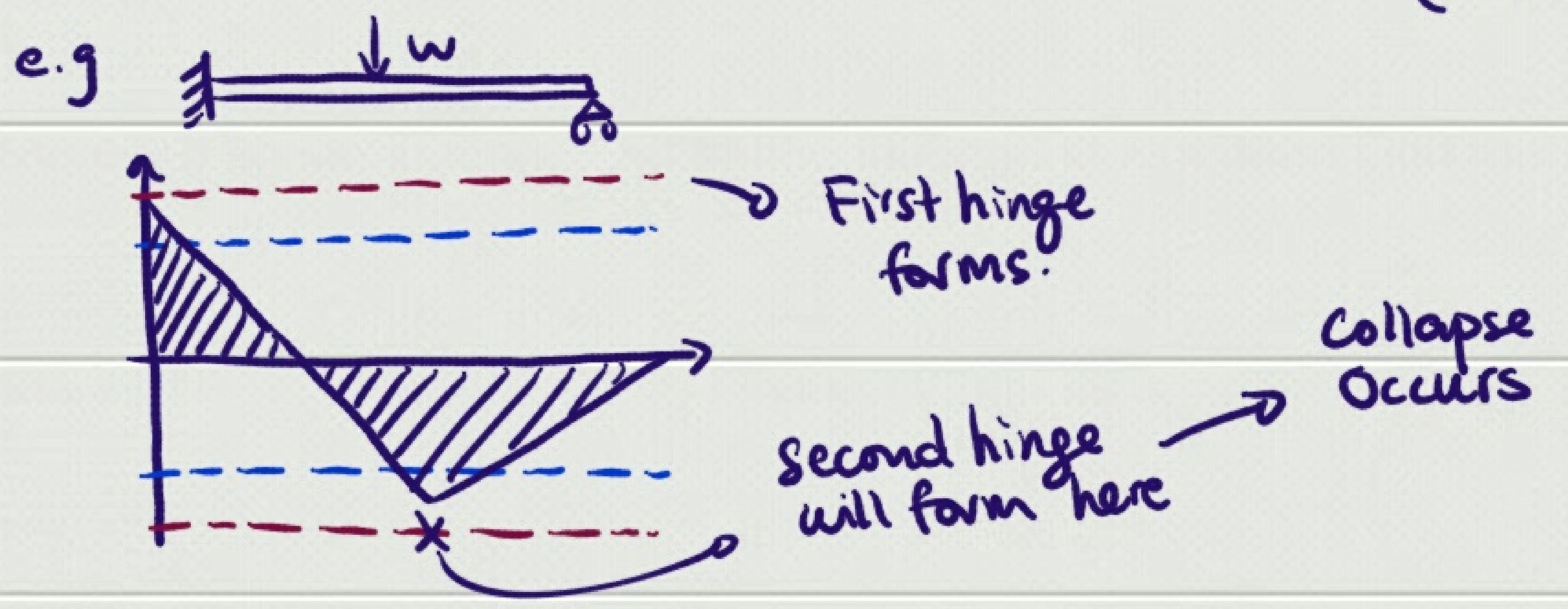
Load Factor $k_L = \frac{W_c}{W_{max}}$ → W_c: Critical Load that produces collapse

For statically determinate structure, $k_L = f \cdot k_s$ → f: Shape factor

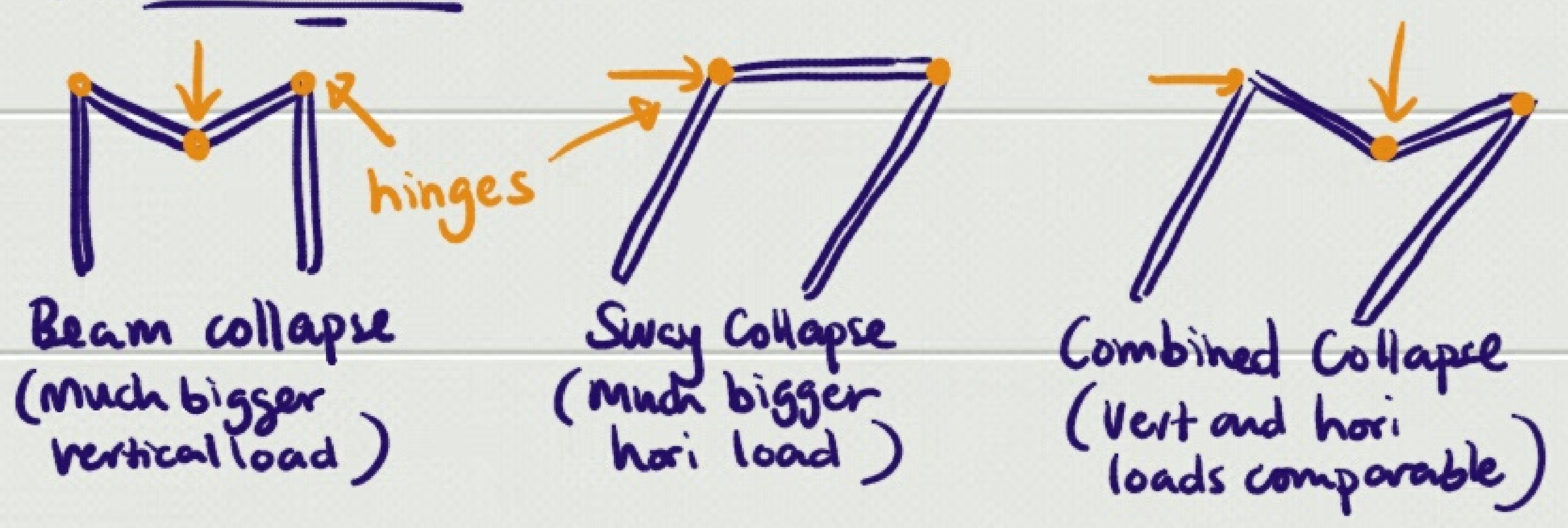
For statically indeterminate: $k_L \geq f \cdot k_s$ (more than one hinge is required)

↳ Only one hinge will give an alternative structure with higher degree of freedom.

e.g



PORTAL FRAMES



Virtual Work Principle

$$\sum_{i=1}^{n} F_i \cdot u_i = \sum_{j=1}^{m} M_{p_j} \cdot \theta_j$$

F_i: Applied Forces; u_i: Linear Displacement; M_{p_j}: Plastic Moment at hinges; θ_j: Angular displacement

The collapse mechanism of the frame is the one that requires lower critical forces.

Strain Energy

Energy stored by internal action into the Structure → $U = \frac{1}{2} P e$ (P: internal force; e: deformation) and $U_M = \int_0^L \frac{M^2}{2EI} dx$

Castigliano's Thm :

$$\frac{\partial U}{\partial P_i} = \delta_i$$

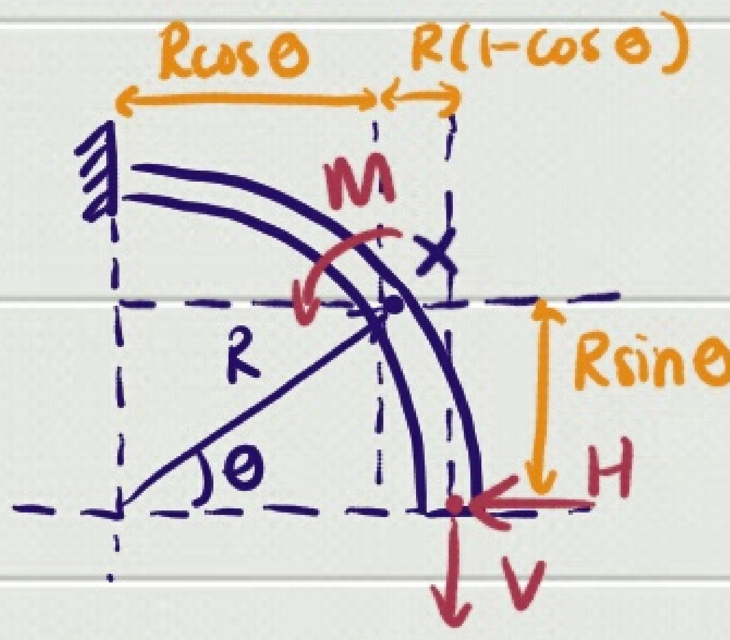
U: Strain energy; P_i: One of the applied forces; δ_i → Displacement in the direction of the force of its point of application.

The load can be a force (giving linear displacement) or a moment (giving angular displacement)

Castigliano's Thm (contd)

$$\delta_i = \int_0^L \frac{\partial}{\partial P_i}\left(\frac{M^2}{2EI}\right)dx = \int_0^L \frac{M}{EI}\cdot\frac{\partial M}{\partial P_i}\,dx$$

- Make use of dummy loads P_0.
- Curved beams :



Bending moment from H: $M_H = H\cdot R\sin\theta$

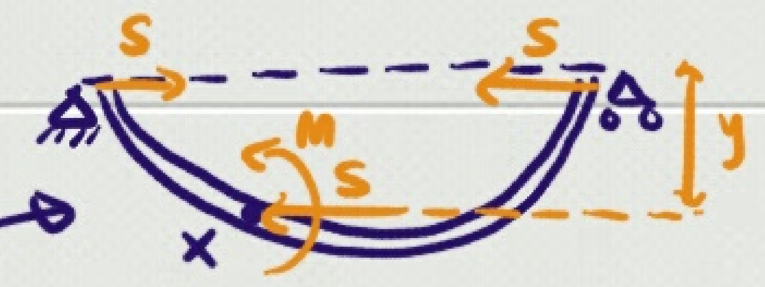
Bending moment from V: $M_V = V\cdot R(1-\cos\theta)$

$$\therefore M(\theta) = HR\sin\theta + VR(1-\cos\theta)$$

Beam Buckling —

Eule Buckling Theory :

$$\frac{d^2y}{dx^2} + \frac{S}{EI}y = 0$$



↳ General Solution: $y = A\sin\alpha x + B\cos\alpha x$

Sub in boundary conditions : $y(x) = A\sin\left(\frac{\pi}{L}x\right)$

$$\boxed{S_{cr} = \frac{\pi^2 EI}{Le^2}}$$

Pinned - Pinned	$Le = L$
Pinned - Fixed	$Le = 0.7L$
Free - Fixed	$Le = 2L$
Fixed - Fixed	$Le = 0.5L$

↳ Stresses

Normal force contribution : $\sigma_x' = -\frac{S}{A}$

Bending moment contribution : $\sigma_x'' = \frac{M}{I}h = \frac{Sh}{I}y$

Total Stress : $\sigma_x = \sigma_x' + \sigma_x'' = S\left(\frac{h}{I}y - \frac{1}{A}\right)$

Principal Stresses

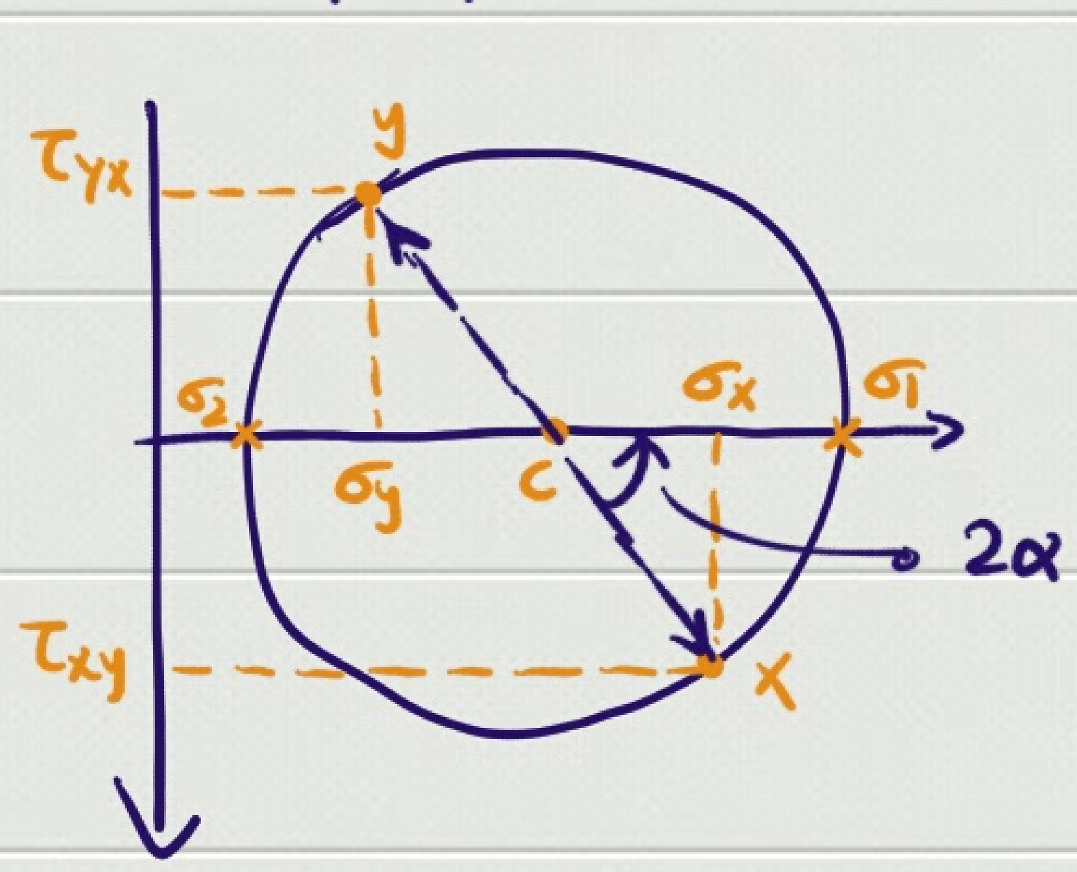
Stress tensor $\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$ 2D: $\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$

$$\sigma = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \tau = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

- Max and Min normal stresses are called principal stresses and occur at principal planes
- Tangential shear stresses are zero on principal planes

Mohr's Circle

Angles in mohr's circle are double angles and their direction is opposite to the physical one.



Centre $C \equiv \left(\frac{\sigma_x+\sigma_y}{2}, 0\right)$

Radius $R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$\sigma_1 = \bar{\sigma} + R$

$\sigma_2 = \bar{\sigma} - R$

$\alpha = \frac{1}{2}\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$

$\tau_{max} = R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2}$

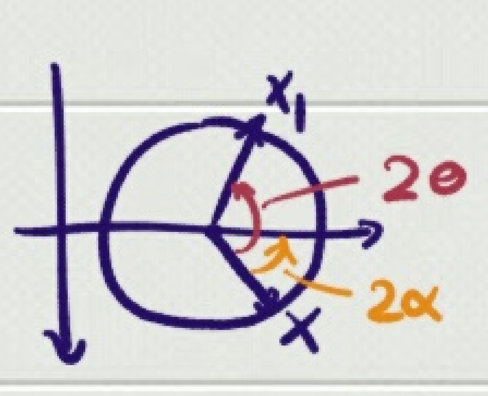
Always at $\theta = 45°$

Stress tensors for any plane at angle θ :

$\sigma_{x_1} = \bar{\sigma} + R\cos 2(\theta - \alpha)$

$\sigma_{y_1} = \bar{\sigma} - R\cos 2(\theta - \alpha)$

$\tau_{x_1y_1} = -R\sin 2(\theta - \alpha)$



Stress Strain Relations

Poisson's Ratio $\nu = -\frac{\text{transverse strain}}{\text{longitudinal strain}} = -\frac{\varepsilon_t}{\varepsilon_l}$

Uniaxial: $\sigma_1 = E\varepsilon_1$

Biaxial: $\sigma_1 = \frac{E}{1-\nu^2}\left[\varepsilon_1 + \nu\varepsilon_2\right]$

$\sigma_2 = \frac{E}{1-\nu^2}\left[\varepsilon_2 + \nu\varepsilon_1\right]$

Triaxial: $\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_1 + \nu(\varepsilon_2 + \varepsilon_3)\right]$

$\sigma_2 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_2 + \nu(\varepsilon_3 + \varepsilon_1)\right]$

$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_3 + \nu(\varepsilon_1 + \varepsilon_2)\right]$

Failure Criteria

- Brittle (breaking of bonds) → Brittle materials
- Plastic (spreading of dislocations) → Ductile materials

Max and min principal stress theory (Galileo-Rankine): Failure occurs if max principal stress equals tensile strength or if min principal stress equals compressive strength

i.e if $\sigma > \sigma_{ft}$ or $\sigma < \sigma_{fc}$

Max normal strain theory (St Venant-Grashof): safe if at all points:

$\varepsilon_1 < \varepsilon_f$ & $\varepsilon_2 < \varepsilon_f$ & $\varepsilon_3 < \varepsilon_f$

max principal strain

strain corresponding to tensile strength

Max Shear Stress theory (Tresca): safe if at all points: $\tau_{max} < \tau_y$ — max shear stress produced in a tension test specimen of same material

→ $\tau_y = \frac{1}{2}\sigma_y$

$(\tau_{max})_3 = \frac{|\sigma_1-\sigma_2|}{2}$; $(\tau_{max})_2 = \frac{|\sigma_3-\sigma_1|}{2}$; $(\tau_{max})_3 = \frac{|\sigma_2-\sigma_3|}{2}$

Therefore: $|\sigma_1-\sigma_2|$; $|\sigma_2-\sigma_3|$; $|\sigma_3-\sigma_1| < \sigma_y$

Max strain energy Theory (Beltrami): $\frac{dW}{dV} < \frac{dW_y}{d}$ — Strain Energy Density (area under stress-strain graph)

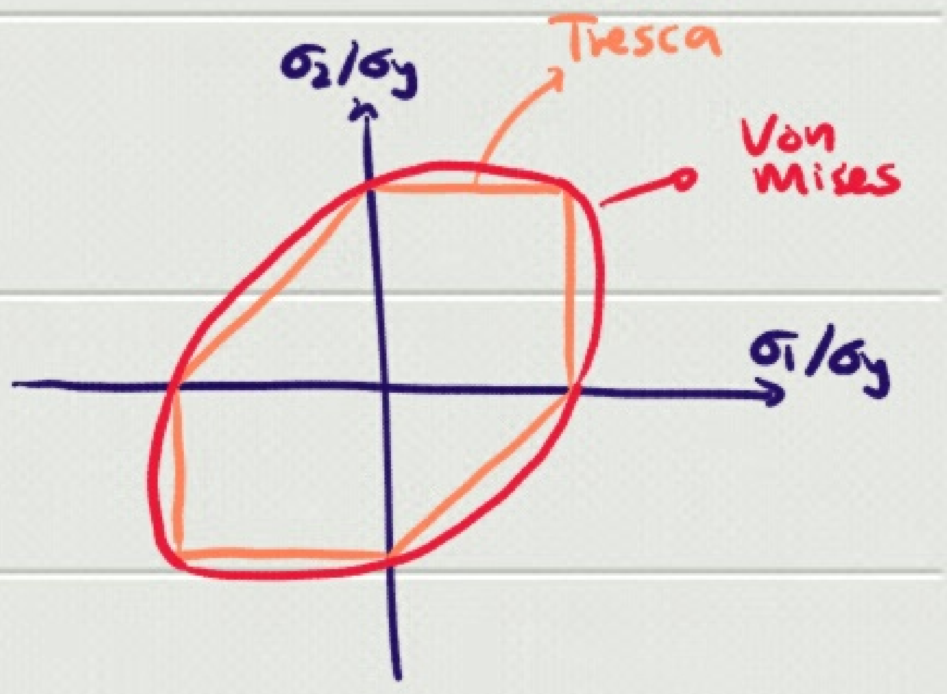
$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) < \sigma_y^2$

$\frac{dW}{dV} = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$

Distortion energy theory (Von Mises): $\frac{dW_{DY}}{dV} < \frac{dW_{DY}}{dV}$ — Distortion Strain Energy (strain energy associated with a change of shape.)

$$\boxed{\sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}} < \sigma_y}$$

→ Best for ductile materials. But tresca is more conservative.



Mohr's Criterion

→ Failure occurs when the Mohr's circle becomes tangent to the failure envelope

→ Coulomb-Mohr criterion for brittle materials

Thin Walled Cylinders.

$$\boxed{\text{Hoop Stress } \sigma_\theta = \frac{pR}{t}} \qquad \boxed{\text{Longitudinal Stress } \sigma_z = \frac{pR}{2t} = \frac{\sigma_\theta}{2}}$$

σ_r neglible compared to σ_θ

Thick Walled Cylinders.

↳ Radial Variation of hoop stress becomes considerable and radial stress no longer negligible

Compatibility Relations: $\varepsilon_r = \frac{du}{dr}$; $\varepsilon_\theta = \frac{u}{r}$ (u → radial displacement)

– Stress Strain Relationship in radial coords:

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r)$$

Combines into Euler's DE : $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$

With general solution : $u = Ar + \frac{B}{r}$

Some steps later : $\boxed{\sigma_r = C - \frac{D}{r^2}}$ $\boxed{\sigma_\theta = C + \frac{D}{r^2}}$ →

Boundary Conditions for Lame's Eqns:
Sub @ $r = r_i$, $\sigma_r = -p_i$
and @ $r = r_o$, $\sigma_r = -p_o$

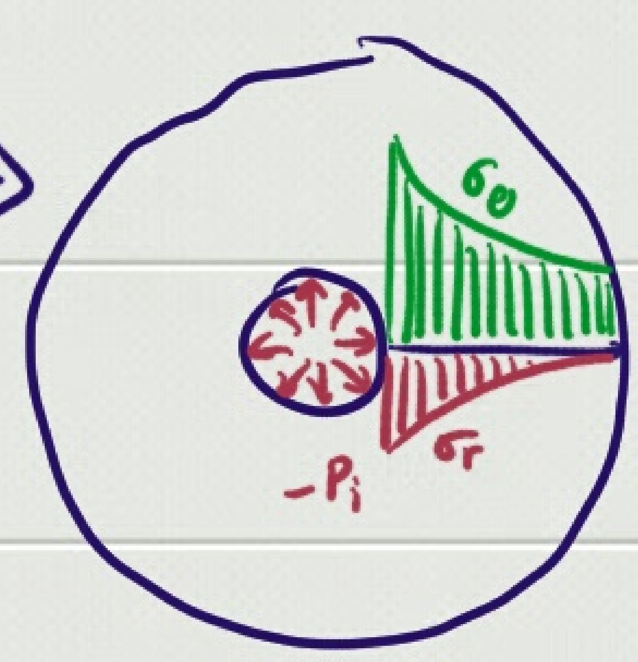
Lame's Equations: $\sigma_r = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} - \frac{(p_i - p_o) r_i^2 r_o^2}{(r_o^2 - r_i^2) r^2}$ and $\sigma_\theta = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} + \frac{(p_i - p_o) r_i^2 r_o^2}{(r_o^2 - r_i^2) r^2}$

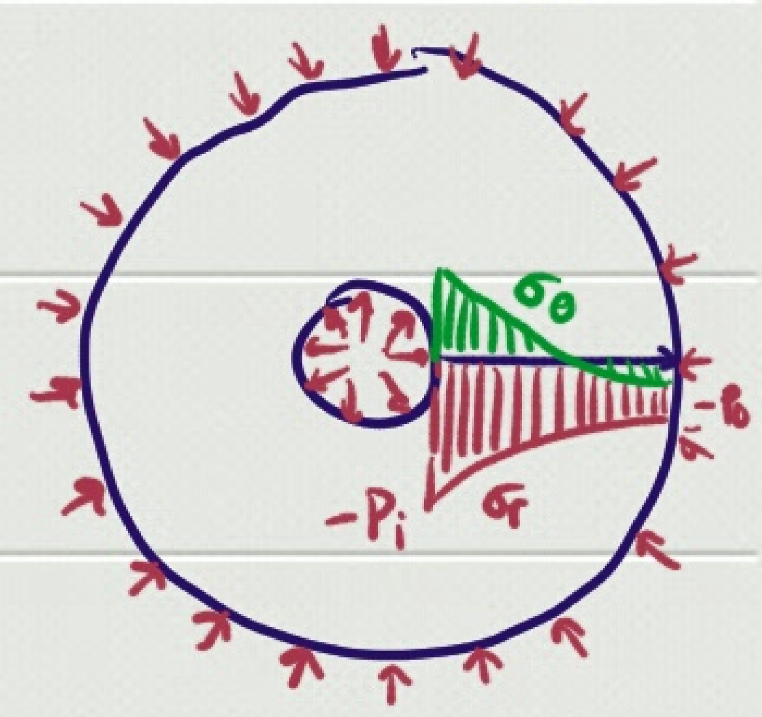
Stress variation : (p_i only)

$\sigma_r = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$ $\sigma_\theta = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$ →

$\sigma_r(r_i) = -p_i$ $\sigma_\theta(r_i) = p_i \frac{r_i^2 + r_o^2}{r_o^2 - r_i^2}$

$\sigma_r(r_o) = 0$ $\sigma_\theta(r_o) = 2\frac{p_i r_i^2}{r_o^2 - r_i^2}$



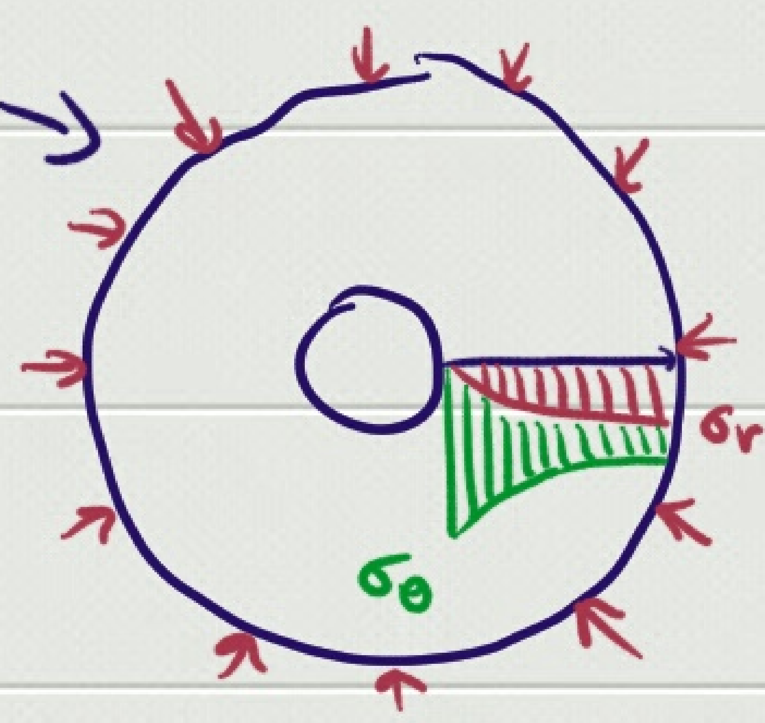


Stress Variation : (p_o only)

$\sigma_r = \frac{-p_o r_o^2}{r_o^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right)$ $\sigma_\theta = \frac{-p_o r_o^2}{r_o^2 - r_i^2}\left(1 + \frac{r_i^2}{r^2}\right)$ →

$\sigma_r(r_i) = 0$ $\sigma_\theta(r_i) = -\frac{2p_o r_o^2}{r_o^2 - r_i^2}$

$\sigma_r(r_o) = -p_o$ $\sigma_\theta(r_o) = -p_o\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}$



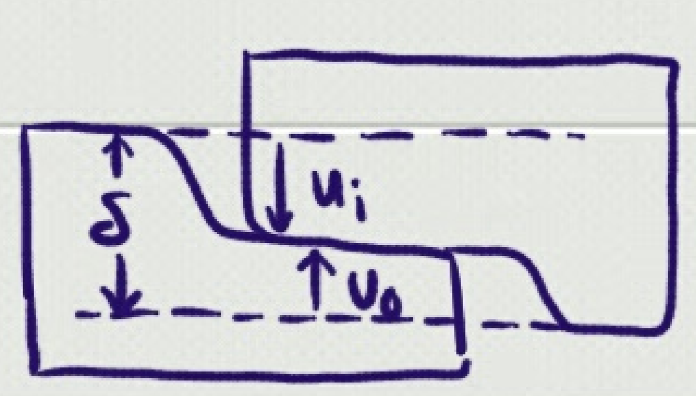
Longitudinal Stress

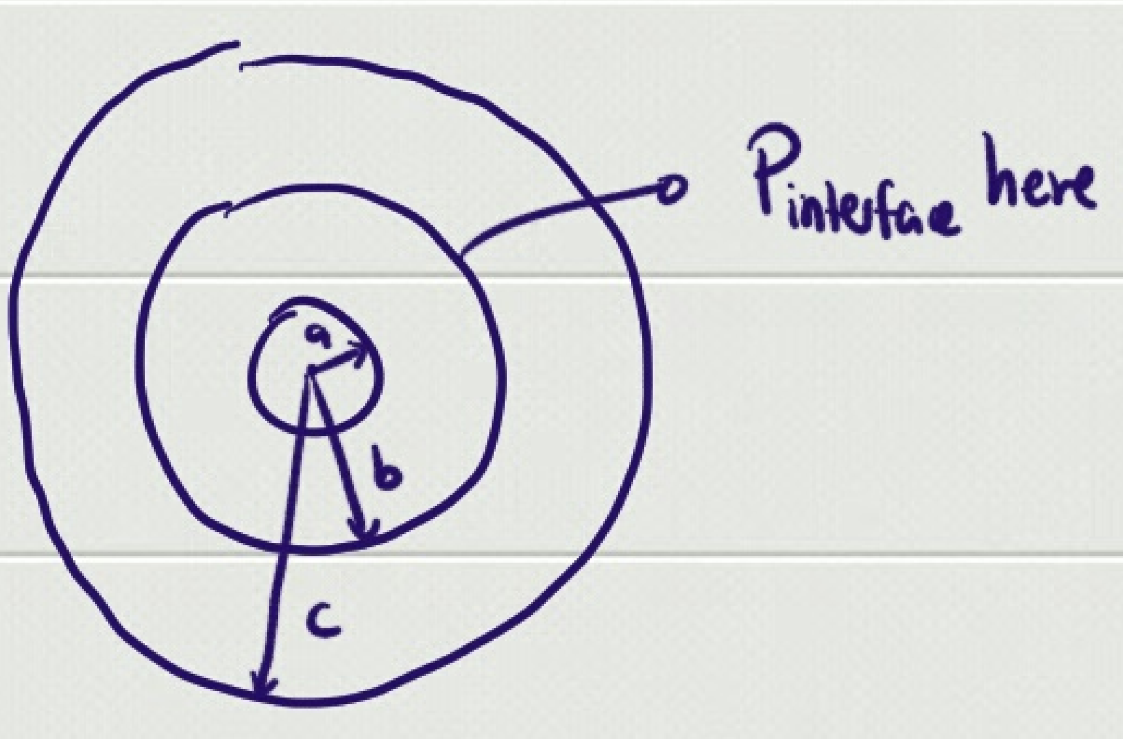
↳ End Pistons : $\sigma_z = 0$

End Caps : $(p_o - p_i)\frac{r_i^2}{r_o^2 - r_i^2}$

Compound Cylinders

Interference $\delta = u_o - u_i$





Stress state at interface:

hoop stress of inner cylinder at b

Poisson's ratio of inner cylinder

radial stress of inner cylinder at b

E of inner cylinder

$$u_i = \frac{b}{E_i}\left[\sigma_{\theta,i}(b) - \nu_i \sigma_{r,i}(b)\right]$$ ← Sub Lamé & boundary conds

$$\left.\begin{aligned}\sigma_{r,i}(a) &= 0 = A - \frac{B}{a^2}\\ \sigma_{r,i}(b) &= -P_{int} = A - \frac{B}{b^2}\end{aligned}\right\}$$ Find eq^{n}s for A and B in terms of P_{int}, a and b and sub into $\sigma_{r,i}(b)$ and $\sigma_{\theta,i}(b)$

$$u_i = -P_{int}\frac{b}{E_i}\left(\frac{b^2+a^2}{b^2-a^2} - \nu_i\right)$$

Do the same for outer cylinder

$$u_o = \frac{b}{E_o}\left[\sigma_{\theta,o}(b) - \nu_o\sigma_{r,o}(b)\right]$$ ← Sub Lamé & BC:

$$\sigma_{r,o}(b) = -P_{int} = A - \frac{B}{b^2}$$
$$\sigma_{r,o}(c) = 0 = A - \frac{B}{c^2}$$

$$u_o = P_{int}\frac{b}{E_o}\left(\frac{c^2+b^2}{c^2-b^2} + \nu_o\right)$$

Combining $\delta = u_o - u_i$:

$$\delta = P_{int}\frac{b}{E}\left[\frac{2b^2(c^2-a^2)}{(c^2-b^2)(b^2-a^2)}\right]$$ (For single material $E_i = E_o = E$ and $\nu_i = \nu_o = \nu$)

Rotating Discs

Another Euler's DE: $$\frac{d^2u}{dr^2} + \frac{1}{r}\cdot\frac{du}{dr} - \frac{u}{r^2} = \frac{1-\nu^2}{E}\rho\omega^2 r$$ due to rotation

General solution: $$u = Ar + \frac{B}{r} - \frac{1-\nu^2}{E}\frac{\rho\omega^2 r^3}{8};$$ } Sub u and $\frac{du}{dr}$ into constitutive laws for σ_r and σ_θ

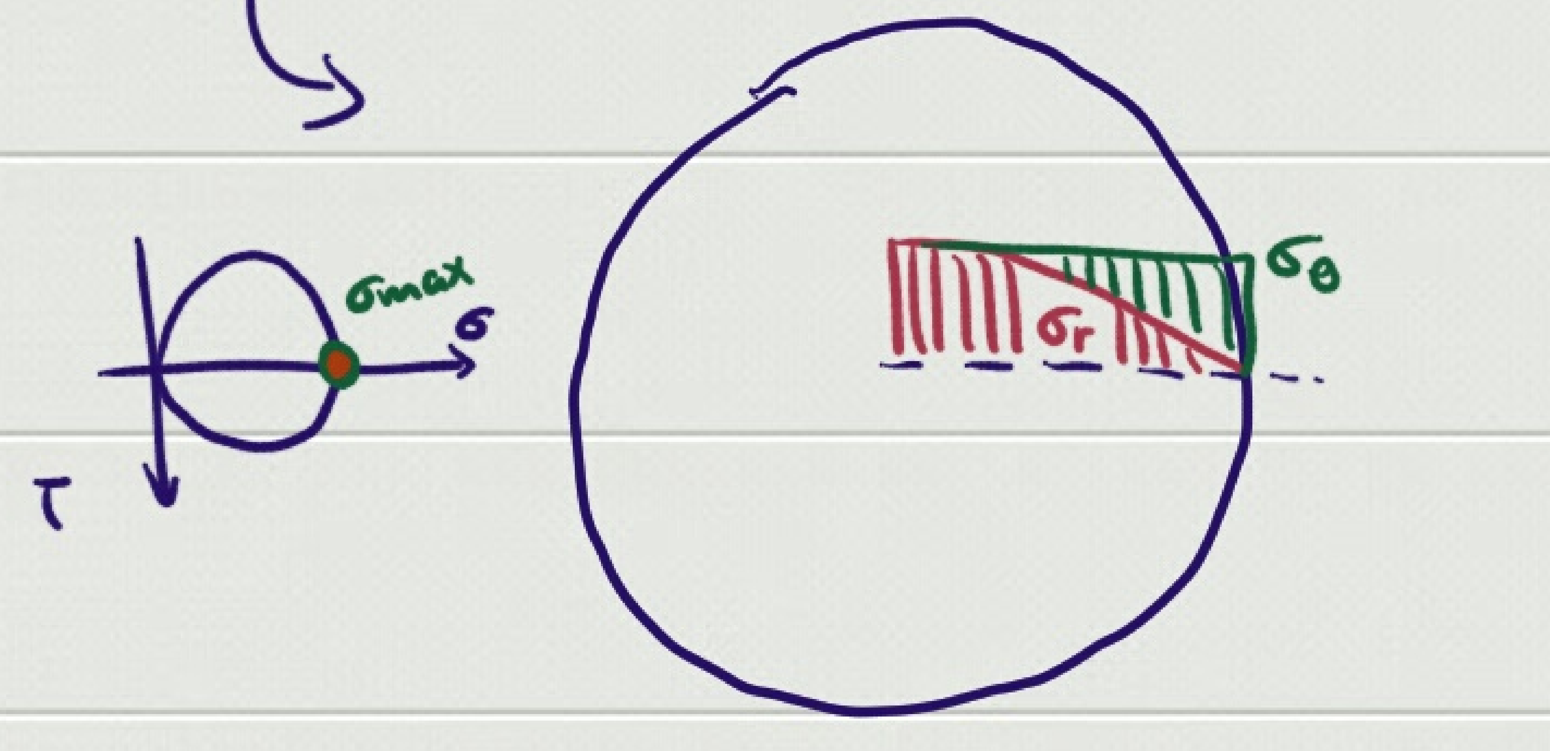
Hence: $$\boxed{\begin{aligned}\sigma_r &= C - \frac{D}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2\\ \sigma_\theta &= C + \frac{D}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2\end{aligned}}$$

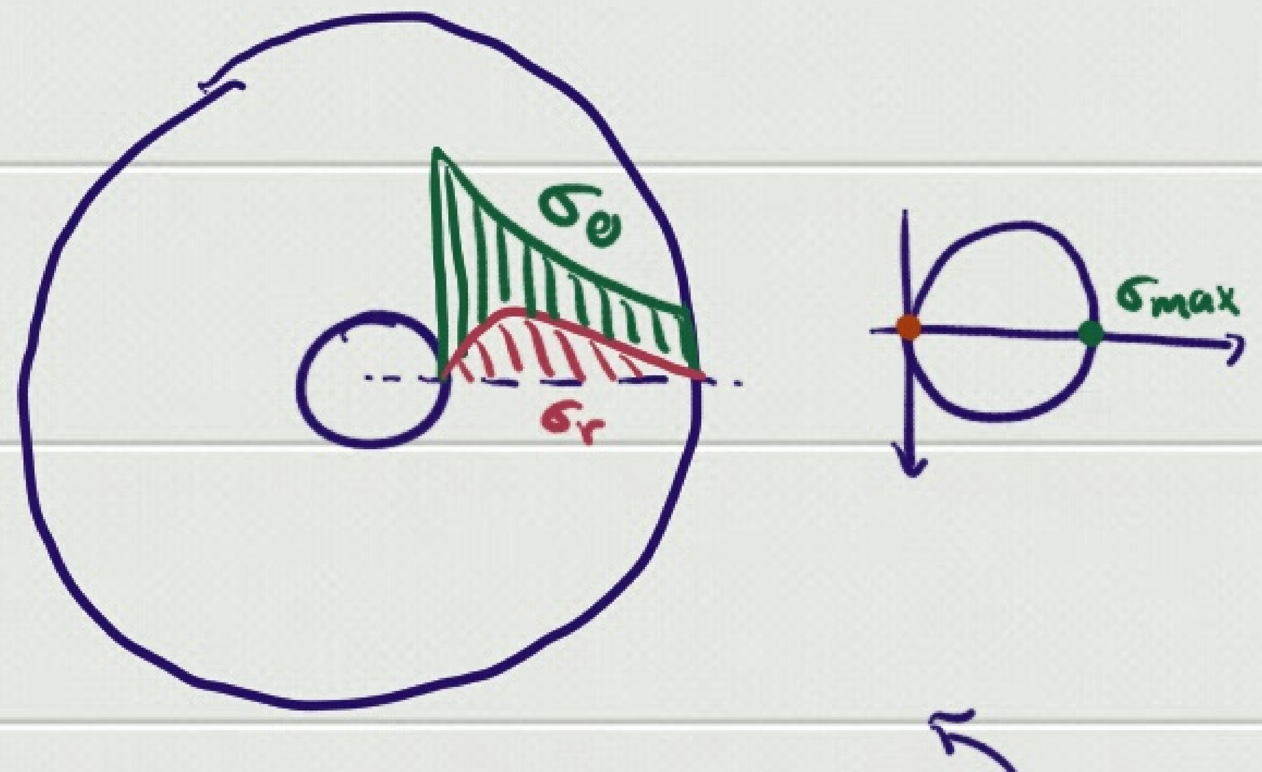
Continuous Disc:

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(r_o^2 - r^2)$$
$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 r_o^2 - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

— max stress at centre of disc ($r=0$)

$$\sigma_{max} = \frac{3+\nu}{8}\rho\omega^2 r_o^2$$





Disc with Central Hole:

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(r_i^2 + r_o^2 - \frac{r_i^2 r_o^2}{r^2} - r^2\right)$$
$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right)$$

— max stress at inner radius r_i

$$\sigma_{max} = \frac{\rho\omega^2}{4}\left[(3+\nu)r_o^2 + (1-\nu)r_i^2\right]$$