Advanced Mathematics(Volume One)

Theory

$$\lim_{n\to\infty} x_n = a \leftrightarrow \forall\varepsilon > 0,\ \exists N \in \mathbb{N}^*,\ n > N,\ |x_n - a| < \varepsilon$$

1. Removable discontinuity:
$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$
2. Jump discontinuity:
$$\lim_{x\to 0} \begin{cases} x-1, & x<0 \\ 0, & x=0 \\ x+1, & x>0 \end{cases}$$
3. Infinite discontinuity:
$$\lim_{x\to\frac{\pi}{2}} \tan x$$
4. Oscillating discontinuity:
$$\lim_{x\to 0} \sin\frac{1}{x}$$

$$\lim_{\alpha} \frac{\beta}{\alpha} = 0 \leftrightarrow \beta = o(\alpha), \quad \sec x\cos x = 1, \quad \csc x \sin x = 1, \quad \tan^2 x + 1 = \sec^2 x$$

$$[f^{-1}(x)]' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\frac{\Delta x}{\Delta y}}; \quad \text{eg: } y' = (\arcsin x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad (\arctan x)' = \frac{1}{1+x^2}, \quad (\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

Rolle's Theorem: $f(x)$ is continuous on $[a,b]$, derivable on (a,b), and $f(a)=f(b)$.

$$\exists\,\xi \in (a,b), \quad f'(\xi) = 0$$

Lagrange's Theorem: $f(x)$ is continuous on $[a,b]$ and derivable on (a,b).

$$\exists\,\xi \in (a,b), \quad \frac{f(b)-f(a)}{b-a} = f'(\xi)$$

Cauchy's Theorem: $f(x)$ and $F(x)$ are continuous on $[a,b]$ and derivable on (a,b).

$$\exists\,\xi \in (a,b), \quad \frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(\xi)}{F'(\xi)}$$

Taylor's Theorem:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o((x-x_0)^n)$$

Maclaurin's Theorem:

$$f(x) = f(0) + f'(0)x + \ldots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n)$$

Concave function: $f''(x) > 0$:

$$f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$$

Convex function: $f''(x) < 0$:

$$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$$

Description of function graph: $f(x) = x^3 - x^2 - x + 1$

1. **Domain**: $\mathbb{R}$
   First derivative: $f'(x) = (3x+1)(x-1)$
   Second derivative: $f''(x) = 2(3x-1)$
2. **Zero points of** $f'(x)$: $f'\left(-\frac{1}{3}\right) = f'(1) = 0$, $f''\left(\frac{1}{3}\right) = 0$
3. **Table of variations**:

x	$(-\infty, -\frac{1}{3})$	$-\frac{1}{3}$	$(-\frac{1}{3}, \frac{1}{3})$	$\frac{1}{3}$	$(\frac{1}{3}, 1)$	1	$(1, +\infty)$
$f'(x)$	+	0	-	-	-	0	+
$f''(x)$	-	-	-	0	+	+	+
$y = f(x)$	$\nearrow$		$\searrow$	Inflection	$\searrow$		$\nearrow$

4. **Limits**: $\lim\limits_{x\to+\infty} y = +\infty$, $\lim\limits_{x\to-\infty} y = -\infty$
5. **Special points**: $f\left(-\frac{1}{3}\right) = \frac{32}{27}$, $f\left(\frac{1}{3}\right) = \frac{16}{27}$, $f(1) = 0$

Integrals

$$\int \tan x\,dx = -\ln|\cos x| + C, \quad \int \cot x\,dx = \ln|\sin x| + C$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C, \quad \int \csc x\,dx = \ln|\csc x - \cot x| + C$$
$$\int \sec^2 x\,dx = \tan x + C, \quad \int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C, \quad \int \csc x \cot x\,dx = -\csc x + C$$
$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C \quad (a>0), \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C \quad (a>0)$$
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C \quad (a>0), \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \quad (a>0)$$

Order: arc, log, power, exp, trig: $\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$
Trigonometric substitution: $u = \tan\frac{x}{2}$, $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$, $dx = \frac{2}{1+u^2}du$
Newton-Leibniz formula: $\int_a^b f'(x)dx = f(b) - f(a)$, $\left[\int_{\phi_1(x)}^{\phi_2(x)} f(t)dt\right]' = \phi_2'(x)\cdot f[\phi_2(x)] - \phi_1'(x)\cdot f[\phi_1(x)]$
Polar area1: $\rho = \rho(\theta)$, $\theta_1 = \alpha$, $\theta_2 = \beta$, $S = \frac{1}{2}\int_\alpha^\beta \rho^2(\theta)d\theta$
Polar area2: $\rho_1 = \rho_1(\theta)$, $\rho_2 = \rho_2(\theta)$, $\theta_1 = \alpha$, $\theta_2 = \beta$, $S = \frac{1}{2}\int_\alpha^\beta [\rho_2^2(\theta) - \rho_1^2(\theta)]d\theta$
Volume of revolution: $V_x = \pi\int_a^b f^2(x)dx$, $V_y = 2\pi\int_a^b xf(x)dx$, $L = \int_\alpha^\beta \sqrt{1 + f'^2(x)}dx$
Wallis formula: $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}, & n \geq 2 \text{ even} \\ \frac{(n-1)!!}{n!!}, & n \geq 3 \text{ odd} \end{cases}$

Differential Equations

Separable variable: $\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)dx$
Homogeneous equation: $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$ Let $u = \frac{y}{x} \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx} \Rightarrow \int \frac{du}{g(u)-u} = \int \frac{dx}{x}$
First order linear differential equation: $\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C\right]$
Bernoulli equation: $y' + p(x)y = q(x)y^\alpha$ Let $z = y^{1-\alpha} \Rightarrow z' + (1-\alpha)p(x)z = (1-\alpha)q(x) \quad (\alpha \neq 0, 1)$
Second order constant coefficient: $y'' + py' + qy = 0$
Characteristic equation: $\mu^2 + p\mu + q = 0$

- Distinct real roots $\mu_1 \neq \mu_2$: $y = C_1e^{\mu_1 x} + C_2e^{\mu_2 x}$
- Repeated real root $\mu_1 = \mu_2$: $y = (C_1 + C_2x)e^{\mu_1 x}$
- Complex roots $\mu_{1,2} = \alpha \pm \beta i$: $y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$

Advanced Mathematics(Volume Two)

Directional Derivative

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{f(x_0 + t\alpha, y_0 + t\beta, z_0 + t\gamma) - f(x_0, y_0, z_0)}{t} \tag{1}$$

Definition of Gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \tag{2}$$

Polar Coordinates

$$\iint\limits_{D} f(x, y) \, dA = \iint\limits_{D'} f(r \cos \theta, r \sin \theta) \, r \, dr d\theta \tag{3}$$

Spherical Coordinates

$$\iiint\limits_{\Omega} f(x, y, z) \, dV = \iiint\limits_{\Omega'} f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) \, r^2 \sin \phi \, dr d\phi d\theta \tag{4}$$

First Type of Line Integral

$$\int_L f(x, y, z) \, ds = \int_{\alpha}^{\beta} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2} \, dt \tag{5}$$

Second Type of Line Integral

$$\int_L \vec{F} \cdot d\vec{r} = \int_L P dx + Q dy + R dz = \int_{\alpha}^{\beta} \left[P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right] dt \tag{6}$$

First Type of Surface Integral

$$\iint_{\Sigma} f(x, y, z) \, dS = \iint_{D_{xy}} f(x, y, z(x, y)) \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} \, dx dy \tag{7}$$

Second Type of Surface Integral

$$\iint_{\Sigma} \vec{F} \cdot d\vec{S} = \iint_{\Sigma} P dy dz + Q dz dx + R dx dy = \iint_{D_{xy}} \left[P \left(-\frac{\partial z}{\partial x} \right) + Q \left(-\frac{\partial z}{\partial y} \right) + R \right] dx dy \tag{8}$$

Ratio Test / Root Test

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \sqrt[n]{|a_n|} \tag{9}$$

Leibniz Test for Alternating Series

$$\text{If } \sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ with } a_n > 0 \text{ satisfies: } a_{n+1} \leq a_n \text{ and } \lim_{n \to \infty} a_n = 0, \text{ then the series converges.} \tag{10}$$

Power Series Expansion of Common Functions

$$B_{2n} = (-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n) \tag{1}$$

$$E_{2n} = (-1)^{n}\frac{2^{2n+2}(2n)!}{\pi^{2n+1}}\beta(2n+1) \tag{2}$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots, \quad x \in (-\infty, +\infty) \tag{3}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots, \quad x \in (-\infty, +\infty) \tag{4}$$

$$\tan x = \sum_{n=1}^{\infty}\frac{B_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots, \quad |x| < \frac{\pi}{2} \tag{5}$$

$$\cot x = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} B_{2n}}{(2n)!}x^{2n-1} = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \cdots, \quad 0 < |x| < \pi \tag{6}$$

$$\sec x = \sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}x^{2n} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots, \quad |x| < \frac{\pi}{2} \tag{7}$$

$$\csc x = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}2(2^{2n-1}-1)B_{2n}}{(2n)!}x^{2n-1} = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \cdots, \quad 0 < |x| < \pi \tag{8}$$

$$\arcsin x = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots, \quad |x| \le 1 \tag{9}$$

$$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1}, \quad |x| \le 1 \tag{10}$$

$$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad |x| \le 1 \tag{11}$$

$$\cot^{-1} x = \frac{\pi}{2} - \arctan x = \frac{\pi}{2} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}, \quad |x| \le 1 \tag{12}$$

$$\sec^{-1} x = \arccos\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}x^{-2n-1}, \quad |x| \ge 1 \tag{13}$$

$$\csc^{-1} x = \arcsin\left(\frac{1}{x}\right) = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}x^{-2n-1}, \quad |x| \ge 1 \tag{14}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots, \quad x \in (-\infty, +\infty) \tag{15}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 + \cdots, \quad x \in (-1, 1) \tag{16}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots, \quad x \in (-1, 1) \tag{17}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad x \in (-1, 1] \tag{18}$$

$$-\ln(1-x) = \sum_{n=1}^{\infty}\frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \quad x \in [-1, 1) \tag{19}$$

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots, \quad x \in (-1, 1) \tag{20}$$

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n, \quad x \in (-1, 1) \tag{21}$$