1 Problem

Find the distance from $P(3,-1,2)$ to the line $\begin{cases} x+y-z=-1 \\ 2x-y+z=4 \end{cases}$.

Solution

$$\vec{s}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix}=(0,-3,-3)\sim(0,1,1)$$

$$Q(1,-2,0), \overrightarrow{QP}=(2,1,2)$$

$$d=\frac{|\overrightarrow{QP}\times\vec{s}|}{|\vec{s}|}=\frac{|(-1,-2,2)|}{\sqrt{2}}=\boxed{\frac{3\sqrt{2}}{2}}$$

2 Problem

$|\vec{a}|=3$, $|\vec{b}|=4$, $|\vec{c}|=5$, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$
Find the value of $|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

Solution

$$\vec{c}=-\vec{a}-\vec{b}$$

$$\vec{b}\times\vec{c}=\vec{a}\times\vec{b}$$
$$\vec{c}\times\vec{a}=\vec{a}\times\vec{b}$$

$$\Rightarrow \quad \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=3(\vec{a}\times\vec{b})$$

$\sum\vec{a}\times\vec{b}=3(\vec{a}\times\vec{b})$ $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta=12$ ($\sin\theta=90^\circ$ from $\vec{a}\cdot\vec{b}=0$)

$$|3(\vec{a}\times\vec{b})|=3\times 12=\boxed{36}$$

3 Problem

Find the value of $L=\lim_{x\to 0}\frac{|\vec{a}+x\vec{b}|-|\vec{a}-x\vec{b}|}{x}$

Solution

$$\begin{aligned}
L&=\lim_{x\to 0}\frac{|\vec{a}+x\vec{b}|-|\vec{a}-x\vec{b}|}{x}\\
&=\lim_{x\to 0}\frac{(|\vec{a}+x\vec{b}|-|\vec{a}-x\vec{b}|)(|\vec{a}+x\vec{b}|+|\vec{a}-x\vec{b}|)}{x(|\vec{a}+x\vec{b}|+|\vec{a}-x\vec{b}|)}\\
&=\lim_{x\to 0}\frac{|\vec{a}+x\vec{b}|^2-|\vec{a}-x\vec{b}|^2}{x(|\vec{a}+x\vec{b}|+|\vec{a}-x\vec{b}|)}\\
&=\lim_{x\to 0}\frac{4x(\vec{a}\cdot\vec{b})}{x(2|\vec{a}|+O(x^2))}\\
&=\boxed{\frac{2(\vec{a}\cdot\vec{b})}{|\vec{a}|}}
\end{aligned}$$

4 Problem

Find the distance from point $P(1,2,-1)$ to the line $L : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-2}{3}$.

Solution

Given line L passes through point $Q(1,-1,2)$, $\vec{v} = (2,-1,3)$. $\overrightarrow{QP} = (0,3,-3)$.

Distance $d = \dfrac{\|\overrightarrow{QP} \times \vec{v}\|}{\|\vec{v}\|} = \dfrac{\|(-6,-6,-6)\|}{\sqrt{14}} = \dfrac{\sqrt{108}}{\sqrt{14}} = \boxed{\dfrac{3\sqrt{42}}{7}}$.

5 Problem

Find and classify the critical points of the function $f(x,y) = e^{-x}(x - y^3 + 3y)$.

Solution

Step 1: Find Critical Points

$$f_x = e^{-x}(-x + y^3 - 3y + 1)$$
$$f_y = e^{-x}(-3y^2 + 3)$$

Set $f_y = 0$:

$$-3y^2 + 3 = 0 \Rightarrow y = \pm 1$$

Critical points: $(-1,1)$ and $(3,-1)$.

Step 2: Second Derivative Test

$$f_{xx} = e^{-x}(x - y^3 + 3y - 2)$$
$$f_{xy} = e^{-x}(3y^2 - 3)$$
$$f_{yy} = e^{-x}(-6y)$$

At $(-1,1)$:

$$A = f_{xx}(-1,1) = e^1(-1 - 1 + 3 - 2) = -e$$
$$B = f_{xy}(-1,1) = e^1(3 - 3) = 0$$
$$C = f_{yy}(-1,1) = e^1(-6) = -6e$$
$$AC - B^2 = (-e)(-6e) - 0 = 6e^2 > 0 \text{ and } A < 0$$
$$\boxed{\Rightarrow \text{Local maximum at } (-1,1)}$$

At $(3,-1)$:

$$A = f_{xx}(3,-1) = e^{-3}(3 + 1 - 3 - 2) = -e^{-3}$$
$$B = f_{xy}(3,-1) = e^{-3}(3 - 3) = 0$$
$$C = f_{yy}(3,-1) = e^{-3}(6) = 6e^{-3}$$
$$AC - B^2 = (-e^{-3})(6e^{-3}) - 0 = -6e^{-6} < 0$$
$$\boxed{\Rightarrow \text{Saddle point at } (3,-1)}$$

6 Problem

Find points on $C: \begin{cases} x^2+y^2=2z^2 \\ x+y+3z=5 \end{cases}$ with extremal distances to the xOy plane.

Solution

Method 1: Lagrangian

$$\begin{aligned} \mathcal{L} &= z^2+\lambda(x^2+y^2-2z^2)+\mu(x+y+3z-5) \\ \mathcal{L}_x &= 2\lambda x+\mu=0 \\ \mathcal{L}_y &= 2\lambda y+\mu=0 \\ \mathcal{L}_z &= 2z(1-2\lambda)+3\mu=0 \end{aligned}$$

From $\mathcal{L}_x=\mathcal{L}_y$:

$$x=y$$

$$2x^2=2z^2 \Rightarrow x=\pm z$$

$$\begin{cases} x=z \Rightarrow 2z+3z=5 \Rightarrow (1,1,1) \\ x=-z \Rightarrow -2z+3z=5 \Rightarrow (-5,-5,5) \end{cases}$$

$$\boxed{\begin{cases} \text{Closest: } (1,1,1)\ (|z|=1) \\ \text{Farthest: } (-5,-5,5)\ (|z|=5) \end{cases}}$$

Method 2: Parametric Optimization

From the plane equation: $y=5-x-3z$. Substitute into the cone:

$$\begin{aligned} & x^2+(5-x-3z)^2=2z^2 \\ & 2x^2+2x(3z-5)+(25+9z^2-30z-2z^2)=0 \\ & 2x^2+(6z-10)x+(7z^2-30z+25)=0 \end{aligned}$$

For real solutions, discriminant $D \geq 0$:

$$\begin{aligned} & (6z-10)^2-8(7z^2-30z+25) \geq 0 \\ & -20z^2+120z-100 \geq 0 \\ & z^2-6z+5 \leq 0 \\ & (z-1)(z-5) \leq 0 \\ & \Rightarrow z \in [1,5] \end{aligned}$$

Extrema occur at endpoints and critical points:

$$\frac{d}{dz}\left(\frac{10-6z \pm \sqrt{-20z^2+120z-100}}{4}\right)=0$$
$$\Rightarrow z=1 \text{ or } 5$$

Corresponding points:

- $z=1 \Rightarrow x=y=1 \Rightarrow \boxed{(1,1,1)}$
- $z=5 \Rightarrow x=y=-5 \Rightarrow \boxed{(-5,-5,-5)}$

7 Problem

$A(1,3,4)$, $B(3,5,6)$, $C(2,5,8)$, $D(4,2,10)$, Find the value of: V_{ABCD}.

Solution

$$V = \frac{1}{6}\left|\det\begin{pmatrix}\overrightarrow{AB}\\ \overrightarrow{AC}\\ \overrightarrow{AD}\end{pmatrix}\right|, \quad \begin{aligned}\overrightarrow{AB} &= (2,2,2)\\ \overrightarrow{AC} &= (1,2,4)\\ \overrightarrow{AD} &= (3,-1,6)\end{aligned}$$

$$\begin{vmatrix} 2 & 2 & 2\\ 1 & 2 & 4\\ 3 & -1 & 6\end{vmatrix} = 2(16) - 2(-6) + 2(-7) = 30$$

$$V = \tfrac{1}{6} \times 30 = \boxed{5}$$

8 Problem

A plane passes through the z-axis and forms an angle of $\frac{\pi}{3}$ with the plane $2x + y - \sqrt{5}z = 0$. Find the equation of the plane.

Solution

$$ax + by = 0$$

$$\vec{n_1} = (2, 1, -\sqrt{5}), \vec{n_2} = (a, b, 0)$$

$$\cos\frac{\pi}{3} = \frac{|\vec{n_1}\cdot\vec{n_2}|}{\|\vec{n_1}\|\|\vec{n_2}\|} = \frac{|2a+b|}{\sqrt{10}\cdot\sqrt{a^2+b^2}} = \frac{1}{2}$$

$$2|2a+b| = \sqrt{10(a^2+b^2)}$$

$$4(4a^2+4ab+b^2) = 10(a^2+b^2) \Rightarrow 6a^2 + 16ab - 6b^2 = 0$$

$$3\left(\frac{a}{b}\right)^2 + 8\left(\frac{a}{b}\right) - 3 = 0$$

Let $k = \frac{a}{b}$:

$$3k^2 + 8k - 3 = 0 \Rightarrow k = -3 \text{ or } \frac{1}{3}$$

$$\frac{a}{b} = -3 \Rightarrow x + 3y = 0$$

$$\frac{a}{b} = \frac{1}{3} \Rightarrow -3x + y = 0$$

The plane equations are $\boxed{x + 3y = 0}$ or $\boxed{-3x + y = 0}$.

9 Problem

Prove that the lines $L_1 : \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $L_2 : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-2}{1}$ are skew lines, find their common perpendicular, and calculate the distance between them.

Solution

Part 1: Prove Skew Lines

Direction vectors and points:

- For L_1: $\vec{v}_1 = (1, 2, 3)$, passing through $P_1(0, 0, 0)$
- For L_2: $\vec{v}_2 = (1, 1, 1)$, passing through $P_2(1, -1, 2)$

Verify skew condition:

$$\boxed{(\overrightarrow{P_1P_2} \times \vec{v}_1) \cdot \vec{v}_2 = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 5 \neq 0}$$

Part 2: Common Perpendicular

Direction vector of perpendicular:

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = (-1, 2, -1)$$

Plane Π_1 containing L_1 parallel to $\vec{n}$:

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = 0 \Rightarrow 4x + y - 2z = 0$$

Plane Π_2 containing L_2 parallel to $\vec{n}$:

$$\begin{vmatrix} x-1 & y+1 & z-2 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{vmatrix} = 0 \Rightarrow x - z + 1 = 0$$

Common perpendicular line:

$$\begin{cases} 4x + y - 2z = 0 \\ x - z + 1 = 0 \end{cases}$$

Parametric form: $\boxed{\frac{x}{1} = \frac{y+2}{2} = \frac{z-1}{1}}$

Part 3: Distance Calculation

Distance between skew lines:

$$d = \frac{|(\overrightarrow{P_1P_2} \times \vec{v}_1) \cdot \vec{v}_2|}{|\vec{v}_1 \times \vec{v}_2|} = \frac{5}{\sqrt{6}} = \boxed{\frac{5\sqrt{6}}{6}}$$

10 Problem

$x^2 - 6xy + 10y^2 - 2yz - z^2 + 18 = 0$

Find the extreme points and extreme values of $z = z(x, y)$

Solution

$$2x - 6y - 2y\frac{\partial z}{\partial x} - 2z\frac{\partial z}{\partial x} = 0$$
$$(2y + 2z)\frac{\partial z}{\partial x} = 2x - 6y$$
$$\frac{\partial z}{\partial x} = \frac{x - 3y}{y + z}$$
$$-6x + 20y - 2z - 2y\frac{\partial z}{\partial y} - 2z\frac{\partial z}{\partial y} = 0$$
$$(2y + 2z)\frac{\partial z}{\partial y} = -6x + 20y - 2z$$
$$\frac{\partial z}{\partial y} = \frac{-3x + 10y - z}{y + z}$$

Set $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$:

$$\frac{x - 3y}{y + z} = 0 \Rightarrow x = 3y$$
$$\frac{-3x + 10y - z}{y + z} = 0 \Rightarrow -3x + 10y - z = 0$$
$$-9y + 10y - z = 0 \Rightarrow y - z = 0 \Rightarrow z = y$$
$$(9, 3, 3), (-9, -3, -3)$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{(1)(y + z) - (x - 3y)\frac{\partial z}{\partial x}}{(y + z)^2} = \frac{1}{y + z}$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(-3)(y + z) - (x - 3y)(1 + \frac{\partial z}{\partial y})}{(y + z)^2} = \frac{-3}{y + z}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{(10 - \frac{\partial z}{\partial y})(y + z) - (-3x + 10y - z)(1 + \frac{\partial z}{\partial y})}{(y + z)^2} = \frac{10}{y + z}$$

For $(9, 3, 3)$:

$$A = \frac{\partial^2 z}{\partial x^2} = \frac{1}{6}, \quad B = \frac{\partial^2 z}{\partial x \partial y} = \frac{-3}{6} = -\frac{1}{2}, \quad C = \frac{\partial^2 z}{\partial y^2} = \frac{10}{6} = \frac{5}{3}$$
$$AC - B^2 = \left(\frac{1}{6}\right)\left(\frac{5}{3}\right) - \left(-\frac{1}{2}\right)^2 = \frac{5}{18} - \frac{1}{4} = \frac{10}{36} - \frac{9}{36} = \frac{1}{36} > 0$$

Since $A > 0$, this is a local minimum. For $(-9, -3, -3)$:

$$A = \frac{1}{-6} = -\frac{1}{6}, \quad B = \frac{-3}{-6} = \frac{1}{2}, \quad C = \frac{10}{-6} = -\frac{5}{3}$$
$$AC - B^2 = \left(-\frac{1}{6}\right)\left(-\frac{5}{3}\right) - \left(\frac{1}{2}\right)^2 = \frac{5}{18} - \frac{1}{4} = \frac{1}{36} > 0$$

Since $A < 0$, this is a local maximum.

- A local minimum at $\boxed{(9, 3)}$ with value $\boxed{z = 3}$
- A local maximum at $\boxed{(-9, -3)}$ with value $\boxed{z = -3}$

11 Problem

$$\begin{cases} u = x^2 + y^2 + z^2, \\ z = x^2 + y^2, \\ x + y + z = 4. \end{cases}$$

Find the minimum and maximum values of u.

Solution

Method 1: Substitution

Substitute the second equation into the third equation:

$$x^2 + x + y^2 + y = 4$$

Complete the squares:

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{9}{2}$$

This represents a circle in the xy-plane with radius $\frac{3\sqrt{2}}{2}$. Now express u in terms of x and y:

$$u = x^2 + y^2 + (x^2 + y^2)^2$$

Let $r = x^2 + y^2$:

$$u = r + r^2$$

The maximum and minimum occur at the extreme values of r. From the circle equation:

$$r_{\max} = \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)^2 = 8$$

$$r_{\min} = \left(\frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)^2 = 2$$

Thus:

$$\boxed{u_{\min} = 6}$$

$$\boxed{u_{\max} = 72}$$

Method 2: Lagrange Multipliers

Substitute the second equation into the third:

$$x + y + x^2 + y^2 = 4$$

Define the Lagrangian:

$$\mathcal{L} = x^2 + y^2 + z^2 + \lambda_1(z - x^2 - y^2) + \lambda_2(x + y + z - 4)$$

Take partial derivatives and set them to zero:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 2x - 2\lambda_1 x + \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= 2y - 2\lambda_1 y + \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial z} &= 2z + \lambda_1 + \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= z - x^2 - y^2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_2} &= x + y + z - 4 = 0
\end{aligned}$$

From the first two equations:

$$2x(1 - \lambda_1) = -\lambda_2$$

$$2y(1 - \lambda_1) = -\lambda_2$$

Thus:

$$x = y \quad \text{or} \quad \lambda_1 = 1$$

<u>Case 1: $x = y$</u> Substitute $y = x$ into the constraints:

$$z = 2x^2$$

$$2x + 2x^2 = 4$$

$$x^2 + x - 2 = 0$$

Solutions:

$$x = 1 \Rightarrow y = 1, z = 2$$

$$x = -2 \Rightarrow y = -2, z = 8$$

Calculate u for these points:

$$u(1, 1, 2) = 1 + 1 + 4 = 6$$

$$u(-2, -2, 8) = 4 + 4 + 64 = 72$$

<u>Case 2: $\lambda_1 = 1$</u> From the third equation:

$$2z + 1 + \lambda_2 = 0$$

From the first equation with $\lambda_1 = 1$:

$$\lambda_2 = 0$$

$$z = -\frac{1}{2}$$

But from $z = x^2 + y^2$, $z \geq 0$, so no solution in this case. Therefore, the extreme values are:

$$u_{\text{min}} = 6 \quad \text{at} \quad (1, 1, 2)$$

$$u_{\text{max}} = 72 \quad \text{at} \quad (-2, -2, 8)$$

Both methods yield the same results:

$$\boxed{u_{\text{min}} = 6}$$

$$\boxed{u_{\text{max}} = 72}$$

12 Problem

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Find the maximum value of the product xyz.

Solution

Method 1: Trigonometric Substitution

We can parameterize the variables using spherical coordinates: Let:

$$x = a \sin\alpha \cos\beta$$

$$y = b \sin\alpha \sin\beta$$

$$z = c \cos\alpha$$

where $0 \leq \alpha \leq \pi$ and $0 \leq \beta \leq 2\pi$. The product becomes:

$$xyz = (a \sin\alpha \cos\beta)(b \sin\alpha \sin\beta)(c \cos\alpha) = abc \sin^2\alpha \cos\alpha \sin\beta \cos\beta$$

Using the double-angle identity:

$$\sin\beta \cos\beta = \frac{1}{2} \sin 2\beta$$

This reaches its maximum when $\sin 2\beta = 1$, $\beta = \frac{\pi}{4}$. Now we need to maximize:

$$f(\alpha) = \sin^2\alpha \cos\alpha$$

Take the derivative:

$$f'(\alpha) = 2\sin\alpha \cos^2\alpha - \sin^3\alpha = \sin\alpha(2\cos^2\alpha - \sin^2\alpha)$$

Set $f'(\alpha) = 0$:

$$\sin\alpha = 0 \quad \text{or} \quad 2\cos^2\alpha - \sin^2\alpha = 0$$

The non-trivial solution is:

$$2\cos^2\alpha = \sin^2\alpha$$

$$\tan^2\alpha = 2$$

$$\alpha = \arctan\sqrt{2}$$

Substituting back:

$$\sin\alpha = \sqrt{\frac{2}{3}}, \quad \cos\alpha = \sqrt{\frac{1}{3}}$$

$$f(\alpha) = \left(\frac{2}{3}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$$

Thus the maximum product is:

$$\boxed{\frac{abc}{3\sqrt{3}}}$$

Method 2: Lagrange Multipliers

We want to maximize $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. The Lagrange condition is:

$$\nabla f = \lambda \nabla g$$

Which gives:

$$(yz, xz, xy) = \lambda \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right)$$

This leads to the system:

$$yz = \frac{2\lambda x}{a^2} \tag{1}$$

$$xz = \frac{2\lambda y}{b^2} \tag{2}$$

$$xy = \frac{2\lambda z}{c^2} \tag{3}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{4}$$

Multiply equation (1) by x, equation (2) by y, and equation (3) by z:

$$xyz = \frac{2\lambda x^2}{a^2} = \frac{2\lambda y^2}{b^2} = \frac{2\lambda z^2}{c^2}$$

This implies:

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Let $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = k$. From the constraint (4):

$$3k = 1 \Rightarrow k = \frac{1}{3}$$

Thus:

$$x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

The maximum product is:

$$xyz = \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = \boxed{\frac{abc}{3\sqrt{3}}}$$

13 Problem

Continuity, Partial Derivatives, Differentiability, Continuity of Partial Derivatives

$$z = f(x,y) = \begin{cases} 0, & x^2+y^2=0 \\ (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & x^2+y^2 \neq 0 \end{cases}$$

Solution

(i) Continuity at (0,0)

To check continuity at $(0,0)$, we examine:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Since $0 \leq |(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)| \leq x^2+y^2$ and $x^2+y^2 \to 0$, by the squeeze theorem:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

$\boxed{\checkmark}$

(ii) Partial Derivatives at (0,0)

Using the definition:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{|h|}\right) = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} k\sin\left(\frac{1}{|k|}\right) = 0$$

$\boxed{\checkmark}$

(iii) Differentiability at (0,0)

Check:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-f_x(0,0)h-f_y(0,0)k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right) = 0$$

$\boxed{\checkmark}$

(iv) Continuity of Partial Derivatives

For $(x,y) \neq (0,0)$:

$$f_x(x,y) = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Along $y=0$ as $x \to 0$:

$$f_x(x,0) = 2x\sin\left(\frac{1}{|x|}\right) - \operatorname{sgn}(x)\cos\left(\frac{1}{|x|}\right)$$

$\boxed{\times}$

14 Problem

Calculate the double integral: $\iint_D \left(x^2 - |x| \arctan y + x^3 e^y\right) dxdy$, $D = \{(x, y) \mid x^2 + y^2 < 1\}$

Solution

Symmetry Analysis

First, we simplify the integral using symmetry properties:

- The term $x^3 e^y$ is odd in x over the symmetric domain D, so its integral vanishes:

$$\iint_D x^3 e^y dxdy = 0$$

- The term $-|x| \arctan y$ is odd in y over the symmetric domain D, so its integral also vanishes:

$$\iint_D -|x| \arctan y dxdy = 0$$

Thus, the original integral simplifies to:

$$\iint_D x^2 dxdy$$

Method 1: Polar Coordinates with Wallis' Formula

Convert to polar coordinates where $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\theta$, and $r \in [0, 1]$, $\theta \in [0, 2\pi]$:

$$\begin{aligned}
\iint_D x^2 dxdy &= \int_0^{2\pi} \int_0^1 (r\cos\theta)^2 rdrd\theta \\
&= \int_0^{2\pi} \cos^2\theta d\theta \int_0^1 r^3 dr \\
&= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} \cdot \left[\frac{r^4}{4}\right]_0^1 \\
&= \boxed{\frac{\pi}{4}}
\end{aligned}$$

Method 2: Using Rotation Symmetry

By the rotation symmetry of the unit disk, we have:

$$\iint_D x^2 dxdy = \iint_D y^2 dxdy$$

Therefore:

$$\begin{aligned}
2\iint_D x^2 dxdy &= \iint_D x^2 dxdy + \iint_D y^2 dxdy \\
&= \iint_D (x^2 + y^2) dxdy \\
&= \int_0^{2\pi} \int_0^1 r^2 \cdot rdrd\theta \\
&= 2\pi \int_0^1 r^3 dr \\
&= \frac{\pi}{2}
\end{aligned}$$

Thus:

$$\boxed{\iint_D \left(x^2 - |x| \arctan y + x^3 e^y\right) dxdy = \frac{\pi}{4}}$$

15 Problem

A thin plate occupies the closed region D bounded by the parabola $y = x^2$ and the line $y = x$. The surface density at point (x, y) is $\mu(x, y) = x^2 y$. Find the centroid of the thin plate.

Solution

Solve $x^2 = x$ to get $(0, 0)$ and $(1, 1)$.

$$m = \iint_D x^2 y \, dA = \int_0^1 \int_{x^2}^{x} x^2 y \, dy dx = \frac{1}{2} \int_0^1 (x^4 - x^6) \, dx = \frac{1}{35}$$

$$M_y = \iint_D x \cdot x^2 y \, dA = \frac{1}{2} \int_0^1 (x^5 - x^7) \, dx = \frac{1}{48}$$

$$M_x = \iint_D y \cdot x^2 y \, dA = \frac{1}{3} \int_0^1 (x^5 - x^8) \, dx = \frac{1}{54}$$

$$\bar{x} = \frac{M_y}{m} = \frac{35}{48}, \quad \bar{y} = \frac{M_x}{m} = \frac{35}{54}$$

Thus the centroid is at $\boxed{\left(\frac{35}{48}, \frac{35}{54}\right)}$.

16 Problem

$D = \{(x, y) \mid 0 \le y \le \sin x, \ 0 \le x \le \pi\}$. Compute:$\iint_D (x^2 + y^2) \, d\sigma$.

Solution

$$\iint_D (x^2 + y^2) \, d\sigma = \int_0^\pi \left(\int_0^{\sin x} (x^2 + y^2) \, dy \right) dx$$

$$\int_0^{\sin x} (x^2 + y^2) \, dy = \left[x^2 y + \frac{y^3}{3} \right]_0^{\sin x} = x^2 \sin x + \frac{\sin^3 x}{3}$$

$$\int_0^\pi \left(x^2 \sin x + \frac{\sin^3 x}{3} \right) dx = \int_0^\pi x^2 \sin x \, dx + \frac{1}{3} \int_0^\pi \sin^3 x \, dx$$

$$\begin{aligned} \int_0^\pi x^2 \sin x \, dx &= \left[-x^2 \cos x \right]_0^\pi + \int_0^\pi 2x \cos x \, dx \\ &= \pi^2 + 2 \left([x \sin x]_0^\pi - \int_0^\pi \sin x \, dx \right) \\ &= \pi^2 - 4 \end{aligned}$$

$$\begin{aligned} \frac{1}{3} \int_0^\pi \sin^3 x \, dx &= \frac{1}{3} \int_0^\pi \frac{3 \sin x - \sin 3x}{4} \, dx \\ &= \frac{1}{12} \left[-3 \cos x + \frac{\cos 3x}{3} \right]_0^\pi \\ &= \frac{4}{9} \end{aligned}$$

$$\boxed{\iint_D (x^2 + y^2) \, d\sigma = \pi^2 - \frac{32}{9}}$$

17 Problem

$D = \{(x,y) | -a \le x \le a, -b \le y \le b\}$. Compute:$\iint_D e^{\max\{b^2x^2, a^2y^2\}}\, d\sigma$.

Solution

Divide D into two regions:

1. $D_1 = \{(x,y) \in D | b|x| \ge a|y|\}$
2. $D_2 = \{(x,y) \in D | b|x| < a|y|\}$

In D_1: $\max\{b^2x^2, a^2y^2\} = b^2x^2$

$$\iint_{D_1} e^{b^2x^2}\, d\sigma = 4\int_0^a \int_0^{\frac{bx}{a}} e^{b^2x^2}\, dy\, dx = \frac{4b}{a}\int_0^a xe^{b^2x^2}\, dx$$

Let $u = b^2x^2$, then $du = 2b^2x\, dx$:

$$= \frac{2}{ab}\int_0^{a^2b^2} e^u\, du = \frac{2}{ab}(e^{a^2b^2} - 1)$$

Similarly for D_2:

$$\iint_{D_2} e^{a^2y^2}\, d\sigma = \frac{2}{ab}(e^{a^2b^2} - 1)$$

Adding both results:

$$\iint_D e^{\max\{b^2x^2, a^2y^2\}}\, d\sigma = \boxed{\frac{4}{ab}(e^{a^2b^2} - 1)}$$

18 Problem

Let $D = \{(x,y) | x^2 + y^2 \le r^2\}$. Compute:$\lim_{r\to 0} \frac{\iint_D e^{x^2-y^2}\cos(x+y)\, d\sigma}{\pi r^2}$

Solution

Using Taylor expansion near $(0,0)$:

$$e^{x^2-y^2} \approx 1 + (x^2 - y^2)$$
$$\cos(x+y) \approx 1 - \frac{(x+y)^2}{2}$$

The integrand becomes approximately $1 + O(x^2 + y^2)$. Thus:

$$\iint_D e^{x^2-y^2}\cos(x+y)\, d\sigma \approx \text{Area}(D) = \pi r^2$$

Therefore:

$$\boxed{\lim_{r\to 0} \frac{\iint_D e^{x^2-y^2}\cos(x+y)\, d\sigma}{\pi r^2} = 1}$$

19 Problem

Compute:$I = \int_0^1 \int_0^{1-x} \int_{x+y}^1 \frac{\sin z}{z}\, dz\, dy\, dx$

Solution

Method 1: Projection on yOz Plane

Change integration order:

$$\begin{aligned}
I &= \int_0^1 \int_0^z \int_0^{z-y} \frac{\sin z}{z}\, dx\, dy\, dz \\
&= \int_0^1 \frac{\sin z}{z} \left(\int_0^z (z-y)\, dy \right) dz \\
&= \int_0^1 \frac{\sin z}{z} \cdot \frac{z^2}{2}\, dz \\
&= \frac{1}{2} \int_0^1 z \sin z\, dz \\
&= \frac{1}{2} \Big[-z\cos z + \sin z \Big]_0^1 \\
&= \boxed{\frac{1}{2}(\sin 1 - \cos 1)}
\end{aligned}$$

Method 2: Sequential Reduction

Original integral with changed approach:

$$\begin{aligned}
I &= \int_0^1 (1-x) \left(\int_x^1 \frac{\sin z}{z} dz \right) dx \\
&= \int_0^1 \sin x \left(1 - \frac{x}{2} \right) dx \\
&= \int_0^1 \sin x\, dx - \frac{1}{2} \int_0^1 x \sin x\, dx \\
&= \Big[-\cos x \Big]_0^1 - \frac{1}{2} \Big[-x\cos x + \sin x \Big]_0^1 \\
&= (1 - \cos 1) - \frac{1}{2}(\sin 1 - \cos 1) \\
&= \boxed{\frac{1}{2}(\sin 1 - \cos 1)}
\end{aligned}$$

20 Problem

Prove that: $\int_0^1 dx \int_x^1 dy \int_x^y f(x)f(y)f(z)\,dz = \frac{1}{6}\left(\int_0^1 f(t)\,dt\right)^3$

Solution

Let $u = F(x) = \int_0^x f(t)\,dt$.

$$\begin{aligned}
&\int_0^1 f(x)dx \int_x^1 f(y)dy \int_x^y f(z)dz \\
=&\int_0^1 f(x)dx \int_x^1 f(y)[F(y) - F(x)]dy \\
=&\int_0^1 f(x)\left[\frac{1}{2}F(y)^2\Big|_x^1 - F(x)(F(1) - F(x))\right]dx \\
=&\int_0^1 f(x)\left[\frac{1}{2}F(1)^2 - \frac{1}{2}F(x)^2 - F(x)F(1) + F(x)^2\right]dx \\
=&\frac{1}{2}\int_0^1 f(x)[F(1)^2 - 2F(x)F(1) + F(x)^2]dx \\
=&\frac{1}{2}\int_0^{F(1)} [F(1)^2 - 2uF(1) + u^2]du \\
=&\frac{1}{2}\left[F(1)^2u - F(1)u^2 + \frac{1}{3}u^3\right]_0^{F(1)} \\
=&\frac{1}{2}\left(F(1)^3 - F(1)^3 + \frac{1}{3}F(1)^3\right) \\
=&\frac{1}{6}F(1)^3 \\
=&\boxed{\frac{1}{6}\left(\int_0^1 f(t)dt\right)^3}
\end{aligned}$$

21 Problem

Let the surface Σ be the finite part of $z = \frac{1}{2}(x^2 + y^2)$ cut by the plane $z = 2$. Evaluate the surface integral $\iint_\Sigma z\, dS$.

Solution

The surface is $z = \frac{1}{2}(x^2 + y^2)$ with $z \leq 2$. In polar coordinates:

$$z = \frac{1}{2}r^2, \quad r \leq 2$$

The surface element is:

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy = \sqrt{1 + x^2 + y^2}\, dxdy = \sqrt{1 + r^2}\, rdrd\theta$$

The integral becomes:

$$\iint_\Sigma z\, dS = \int_0^{2\pi} \int_0^2 \frac{1}{2}r^2 \sqrt{1 + r^2}\, rdrd\theta$$

Let $u = 1 + r^2$, $du = 2rdr$:

$$= \frac{\pi}{2} \int_1^5 (u - 1)\sqrt{u}\, du = \frac{\pi}{2} \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^5 = \boxed{\frac{2\pi(25\sqrt{5} + 1)}{15}}$$

22 Problem

Let Σ be the outer surface of the sphere $x^2 + y^2 + z^2 = 9$. Evaluate the surface integral $\iint_\Sigma z\, dxdy$.

Solution

Method 1: Divergence Theorem

$$\iint_\Sigma z\, dxdy = \iiint_V \left(\frac{\partial z}{\partial z}\right) dV = \iiint_V 1\, dV = \frac{4}{3}\pi(3)^3 = \boxed{36\pi}$$

Method 2: Projection Method

$$\begin{gathered} x = \beta \cos \alpha \\ y = \beta \sin \alpha \\ dxdy = \beta\, d\beta d\alpha \\ \alpha \in [0, 2\pi], \beta \in [0, 3] \end{gathered}$$

$$\begin{aligned} \iint_\Sigma z\, dxdy &= \iint_{\text{upper}} z\, dxdy + \iint_{\text{lower}} z\, dxdy \\ &= \iint_D \sqrt{9 - x^2 - y^2}\, dxdy + \iint_D (-\sqrt{9 - x^2 - y^2})\, dxdy \\ &= 2\iint_D \sqrt{9 - x^2 - y^2}\, dxdy \\ &= 2\int_0^{2\pi} \int_0^3 \sqrt{9 - \beta^2}\, \beta\, d\beta d\alpha \\ &= 2\int_0^{2\pi} d\alpha \int_0^3 \beta(9 - \beta^2)^{1/2}\, d\beta \\ &= 2 \cdot 2\pi \cdot \left[-\frac{1}{3}(9 - \beta^2)^{3/2}\right]_0^3 = \boxed{36\pi} \end{aligned}$$

23 Problem

Compute the line integral $I = \oint_D xy\,dx + z^2\,dy + zx\,dz$, where D is the intersection curve of $z = \sqrt{x^2+y^2}$ and $x^2+y^2 = 2ax$ $(a>0)$, oriented counterclockwise when viewed from the positive z-axis.

Solution

By Stokes' theorem:

$$I = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

where $\vec{F} = (xy, z^2, zx)$.
Compute the curl:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & z^2 & zx \end{vmatrix} = (-\vec{z}, -\vec{z}, -\vec{x})$$

The surface S is the cone $z = \sqrt{x^2+y^2}$ within the cylinder $x^2+y^2 = 2ax$. Using cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r$$
$$d\vec{S} = (-z_x, -z_y, 1)\,dxdy = \left(-\frac{x}{z}, -\frac{y}{z}, 1\right) dxdy$$

The dot product:

$$(\nabla \times \vec{F}) \cdot d\vec{S} = (-z)\left(-\frac{x}{z}\right) + (-z)\left(-\frac{y}{z}\right) + (-x)(1) = y$$

Thus:

$$I = \iint_S y\,dxdy$$

The projection on xy-plane is the circle $x^2+y^2 \le 2ax$:

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r\sin\theta \cdot r\,drd\theta \\ &= \int_{-\pi/2}^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta \\ &= \frac{8a^3}{3} \int_{-\pi/2}^{\pi/2} \sin\theta\cos^3\theta\,d\theta \\ &= \boxed{\pi a^3} \end{aligned}$$

24 Problem

Let Σ be the lower side of the surface $z = x^2 + y^2$ where $0 \le z \le a^2$.
Evaluate the surface integral: $\iint_\Sigma (y - x^2 + z^2)dydz + (x - z^2 + y^2)dzdx + (z - y^2 + x^2)dxdy$

Solution

Using Gauss's divergence theorem, we convert the surface integral to a volume integral:

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz$$

where $P = y - x^2 + z^2$, $Q = x - z^2 + y^2$, $R = z - y^2 + x^2$. Compute the divergence:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = -2x + 2y + 1$$

By symmetry, the $-2x + 2y$ terms integrate to zero over the circular region. Thus:

$$\iiint_V 1\, dxdydz = \text{Volume} = \int_0^{a^2} \pi z\, dz = \boxed{\frac{\pi}{2} a^4}$$

25 Problem

Let Σ be the oriented surface $z = x^2 + y^2$ $(0 \le z \le 1)$, with its normal vector forming an acute angle with the positive z-axis. Evaluate the surface integral: $\iint_\Sigma (2x + z)\, dy\, dz + z\, dx\, dy$

Solution

Project Σ onto the xy-plane ($D : x^2 + y^2 \le 1$). The normal vector is $(-2x, -2y, 1)$, and since the z-component is positive, it satisfies the acute angle condition. Convert the integral:

$$\iint_\Sigma = \iint_D \left[-(2x + z)\frac{\partial z}{\partial x} - z\frac{\partial z}{\partial y} + z \right] dx\, dy$$

Substitute $z = x^2 + y^2$ and simplify:

$$\iint_D \left[-(2x + x^2 + y^2)(2x) + (x^2 + y^2) \right] dx\, dy = \iint_D (-4x^2 - 2x^3 - 2xy^2 + x^2 + y^2) dx\, dy$$

Using polar coordinates:

$$\int_0^{2\pi} \int_0^1 (-3r^2 \cos^2\theta - 2r^3 \cos^3\theta - 2r^3 \cos\theta \sin^2\theta + r^2 \sin^2\theta) r\, dr\, d\theta$$

Simplify and evaluate:

$$\int_0^{2\pi} \int_0^1 (-3r^3 \cos^2\theta + r^3 \sin^2\theta) dr\, d\theta = \boxed{-\frac{\pi}{2}}$$

26 Problem

Given $f(0) = 0$, $f'(0) = 1$, and the equation $[xy(x+y) - f(x)y]dx + [f'(x) + x^2y]dy = 0$ is an exact differential equation. Find its general solution.

Solution

For exactness, require $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ where:

$$P = xy(x+y) - f(x)y$$
$$Q = f'(x) + x^2y$$

Compute partial derivatives:

$$\frac{\partial P}{\partial y} = x^2 + 2xy - f(x)$$
$$\frac{\partial Q}{\partial x} = f''(x) + 2xy$$

Set them equal:

$$x^2 + 2xy - f(x) = f''(x) + 2xy$$
$$f''(x) + f(x) = x^2$$

Solve the ODE:

$$f(x) = A\cos x + B\sin x + x^2 - 2$$
$$\text{Using } f(0) = 0 \Rightarrow A = 2$$
$$f'(0) = 1 \Rightarrow B = 1$$
$$\Rightarrow f(x) = 2\cos x + \sin x + x^2 - 2$$

Now integrate the exact equation:

$$\frac{\partial F}{\partial x} = xy^2 + (2\cos x + \sin x - 2)y$$
$$\frac{\partial F}{\partial y} = -\sin x + 2\cos x + x^2y + x^2 - 2$$

Integrate to find $F(x, y)$:

$$F(x,y) = \frac{x^2y^2}{2} + 2y\sin x - y\cos x - 2xy + g(y) = \boxed{2y\sin x - y\cos x + \frac{x^2y^2}{2} - 2xy + C}$$

27 Problem

Given $f(0) = \frac{1}{2}$ and the integral $\int_L [e^x + f(x)]y\,dx - f(x)\,dy$ is path-independent. Find the value from $(0,0)$ to $(1,1)$.

Solution

Since the integral is path-independent, we have:

$$\frac{\partial}{\partial y}[e^x + f(x)]y = \frac{\partial}{\partial x}[-f(x)] \Rightarrow e^x + f(x) = -f'(x)$$

Solve the ODE $f'(x) + f(x) = -e^x$ with $f(0) = \frac{1}{2}$:

$$f(x) = -\frac{1}{2}e^x + \frac{1}{e^x}$$

Choose path $(0,0) \to (1,0) \to (1,1)$:

$$\int_0^1 0\,dx + \int_0^1 -f(1)\,dy = -f(1) = \boxed{\frac{1}{2}e - \frac{1}{e}}$$

28 Problem

For all smooth oriented closed surfaces Σ in the half-space $x > 0$, the surface integral satisfies: $\iint\limits_{\Sigma} xf(x)dydz - xyf(x)dzdx - e^{2x}zdxdy = 0$, Given $\lim\limits_{x\to 0^+} f(x) = 1$, find $f(x)$.

Solution

By Gauss's divergence theorem, for any closed surface Σ:

$$\iiint\limits_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz = 0$$

where $P = xf(x)$, $Q = -xyf(x)$, $R = -e^{2x}z$. This gives:

$$\frac{\partial}{\partial x}(xf(x)) + \frac{\partial}{\partial y}(-xyf(x)) + \frac{\partial}{\partial z}(-e^{2x}z) = 0$$

Simplifying:

$$f(x) + xf'(x) - xf(x) - e^{2x} = 0$$
$$f'(x) + \left(\frac{1}{x} - 1\right)f(x) = \frac{e^{2x}}{x}$$

Using the standard solution formula $y' + p(x)y = q(x)$:

$$\text{Integrating factor: } \mu(x) = e^{\int p(x)dx} = e^{\int\left(\frac{1}{x}-1\right)dx} = e^{\ln x - x} = xe^{-x}$$

$$\begin{aligned}\text{Solution: } f(x) &= \frac{1}{\mu(x)}\left(\int \mu(x)q(x)dx + C\right) \\ &= \frac{e^x}{x}\left(\int xe^{-x}\cdot\frac{e^{2x}}{x}dx + C\right) \\ &= \frac{e^x}{x}\left(\int e^x dx + C\right) \\ &= \frac{e^x}{x}(e^x + C)\end{aligned}$$

Applying the initial condition $\lim\limits_{x\to 0^+} f(x) = 1$: $\lim_{x\to 0^+} \frac{e^x(e^x+C)}{x} = 1 \Rightarrow C = -1$

Thus the solution is:

$$f(x) = \boxed{\frac{e^x(e^x - 1)}{x}}$$

29 Problem

Compute the surface integral $I = \iint_\Sigma x^2 dS$,
where Σ is the part of the cylinder $x^2 + y^2 = a^2$ between $z = 0$ and $z = h$ ($h > 0$).

Solution

Method 1: Using Symmetry

$$\begin{aligned}
I &= \iint_\Sigma x^2 dS \\
&= \frac{1}{2} \iint_\Sigma (x^2 + y^2) dS \quad \text{(by symmetry)} \\
&= \frac{a^2}{2} \iint_\Sigma dS \\
&= \frac{a^2}{2} \times 2\pi a h \\
&= \boxed{\pi a^3 h}
\end{aligned}$$

Method 2: Projection onto yOz-plane

$$\begin{aligned}
I &= \iint_\Sigma x^2 dS \\
&= \int_0^{2\pi} \int_0^h a^2 \cos^2\theta \cdot \frac{a}{|\sin\theta|} dz d\theta \quad \text{(parameterizing } x = a\cos\theta) \\
&= a^3 h \int_0^{2\pi} \frac{\cos^2\theta}{|\sin\theta|} d\theta \\
&= 4a^3 h \int_0^{\pi/2} \cot\theta \cos\theta d\theta \quad \text{(by symmetry)} \\
&= \boxed{\pi a^3 h}
\end{aligned}$$

Method 3: Cylindrical Coordinates Parameterization

$$\begin{aligned}
I &= \iint_\Sigma x^2 dS \\
&= \int_0^h \int_0^{2\pi} (a\cos\theta)^2 \cdot a\, d\theta dz \quad (dS = a d\theta dz \text{ on cylinder}) \\
&= a^3 \int_0^h dz \int_0^{2\pi} \cos^2\theta\, d\theta \\
&= a^3 h \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta \\
&= \frac{a^3 h}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} \\
&= \frac{a^3 h}{2} \cdot 2\pi \\
&= \boxed{\pi a^3 h}
\end{aligned}$$

30 Problem

Let Σ be the part of $z = x^2 + y^2$ below $z = 1$. Compute the surface integral $I = \iint_\Sigma |xyz|\, dS$.

Solution

Using polar coordinates:

$$\begin{aligned}
z &= x^2 + y^2 = r^2 \\
dS &= \sqrt{1+4r^2}\, r\, dr\, d\theta \\
I &= \int_0^{2\pi} \int_0^1 |r^3 \cos\theta \sin\theta| \cdot r^2 \sqrt{1+4r^2}\, dr\, d\theta \\
&= \left(\int_0^{2\pi} |\cos\theta \sin\theta|\, d\theta \right) \left(\int_0^1 r^5 \sqrt{1+4r^2}\, dr \right) \\
&= \frac{1}{4} \int_0^1 u^2 \sqrt{1+4u}\, du \quad (u = r^2) \\
&= \boxed{\frac{1}{4}\left(\frac{25\sqrt{5}}{21} - \frac{1}{105} \right)}
\end{aligned}$$

31 Problem

Prove that $(yze^{xyz} + 2x)dx + (zxe^{xyz} + 3y^2)dy + (xye^{xyz} + 4z^3)dz$ is an exact differential, and find its potential function.

Solution

$$\begin{aligned}
\frac{\partial P}{\partial y} &= e^{xyz}(1 + xyz) = \frac{\partial Q}{\partial x} \\
\frac{\partial P}{\partial z} &= e^{xyz}(1 + xyz) = \frac{\partial R}{\partial x} \\
\frac{\partial Q}{\partial z} &= e^{xyz}(1 + xyz) = \frac{\partial R}{\partial y}
\end{aligned}$$

Find the potential function U:

$$\begin{aligned}
U &= \int P\, dx \\
&= \int (yze^{xyz} + 2x)\, dx \\
&= e^{xyz} + x^2 + f(y,z) \\
\frac{\partial U}{\partial y} &= zxe^{xyz} + f_y = Q \\
&\Rightarrow f_y = 3y^2 \\
&\Rightarrow f(y,z) = y^3 + g(z) \\
\frac{\partial U}{\partial z} &= xye^{xyz} + g'(z) = R \\
&\Rightarrow g'(z) = 4z^3 \\
&\Rightarrow g(z) = z^4 + C
\end{aligned}$$

Thus, the potential function is:

$$\boxed{x^2 + y^3 + z^4 + e^{xyz} + C}$$

32 Problem

Find the limit: $\lim_{t\to 0} \frac{1}{\pi t^4} \iiint_{x^2+y^2+z^2\le t^2} f\left(\sqrt{x^2+y^2+z^2}\right)\, dv$

Solution

Using spherical coordinates ($r = \sqrt{x^2+y^2+z^2}$):

$$\text{Integral} = 4\pi \int_0^t f(r) r^2 \, dr$$

For $f(0) \neq 0$, the limit becomes ∞. For $f(0) = 0$:

$$\text{Limit} = \lim_{t\to 0} \frac{4\pi \int_0^t f(r) r^2 \, dr}{\pi t^4} = \lim_{t\to 0} \frac{f(t)t^2}{t^3} = f'(0)$$

Final answer: $\boxed{\begin{cases} f'(0), & f(0) = 0 \\ \infty, & f(0) \neq 0 \end{cases}}$

33 Problem

Evaluate:$\lim_{n\to\infty} \sum_{i=1}^n \sum_{j=1}^n \frac{n}{(n+i)(n^2+j^2)}$

Solution

We can rewrite the expression as a double Riemann sum:

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^n \sum_{j=1}^n \frac{n}{(n+i)(n^2+j^2)} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1+\frac{i}{n}} \cdot \frac{1}{n} \sum_{j=1}^n \frac{1}{1+\left(\frac{j}{n}\right)^2} \quad \text{(Factoring the expression)} \\
&= \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1+\frac{i}{n}}\right) \left(\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \frac{1}{1+\left(\frac{j}{n}\right)^2}\right) \quad \text{(Separating the limits)} \\
&= \left(\int_0^1 \frac{dx}{1+x}\right) \left(\int_0^1 \frac{dy}{1+y^2}\right) \quad \text{(Recognizing Riemann sums)} \\
&= [\ln(1+x)]_0^1 \cdot [\arctan y]_0^1 \quad \text{(Evaluating integrals)} \\
&= (\ln 2 - \ln 1) \cdot \left(\frac{\pi}{4} - 0\right) \\
&= \boxed{\frac{\pi \ln 2}{4}}
\end{aligned}$$

34 Problem

Given $r = \sqrt{x^2 + y^2 + z^2}$, Σ is the outer surface of the sphere $x^2 + y^2 + z^2 = a^2$.
Evaluate: $\iint_\Sigma \frac{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}{r^3}$

Solution

Method 1: Gauss's Divergence Theorem

Let $\vec{F} = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$. The divergence is:

$$\nabla \cdot \vec{F} = \frac{3}{r^3} - \frac{3(x^2 + y^2 + z^2)}{r^5} = 0 \quad (r \neq 0)$$

Since $\vec{F}$ is singular at the origin, we consider a small sphere Σ_ϵ of radius ϵ around the origin. By the divergence theorem:

$$\iint_\Sigma \vec{F} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{F}) dV + \iint_{\Sigma_\epsilon} \vec{F} \cdot d\vec{S} = \boxed{4\pi}$$

Method 2: Symmetry and Rotation

By symmetry, the three terms contribute equally:

$$\iint_\Sigma \frac{x\,dy\,dz}{r^3} = \iint_\Sigma \frac{y\,dz\,dx}{r^3} = \iint_\Sigma \frac{z\,dx\,dy}{r^3}$$

On Σ, $r = a$, so we evaluate one component:

$$\iint_\Sigma \frac{z\,dx\,dy}{a^3} = \frac{1}{a^3} \iint_D 2a\,dx\,dy = \frac{2}{a^2} \cdot \pi a^2 = 2\pi$$

where D is the projection. Total integral is $3 \times \frac{4\pi}{3}$: $\boxed{4\pi}$

Method 3: Surface Area Integral

Parameterize using spherical coordinates (with α and β):

$$\vec{r}(\alpha, \beta) = a(\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha)$$

The normal vector is $\vec{n} = a^2 \sin\alpha(\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha)$. Then:

$$\vec{F} \cdot \vec{n} = \frac{a^3 \sin\alpha}{a^3} = \sin\alpha$$

Integrating over $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq 2\pi$:

$$\int_0^{2\pi} \int_0^\pi \sin\alpha\, d\alpha d\beta = \boxed{4\pi}$$

35 Problem

Compute the triple integral $\iiint_\Omega z\,dx\,dy\,dz$,
where Ω is the region bounded by the cone $z = (h/R)\sqrt{x^2+y^2}$ and the plane $z = h$ ($R > 0$, $h > 0$).

Solution

Method 1: Eliminating z

The region Ω can be described as:

$$\frac{h}{R}\sqrt{x^2+y^2} \le z \le h$$

Projection on xy-plane is $x^2+y^2 \le R^2$.

$$\begin{aligned}
\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{(h/R)\sqrt{x^2+y^2}}^{h} z\,dz\,dy\,dx &= \frac{1}{2}\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\left[h^2-\frac{h^2}{R^2}(x^2+y^2)\right]dy\,dx \\
&= \frac{h^2}{2R^2}\int_0^{2\pi}\int_0^R (R^2-r^2)r\,dr\,d\theta \\
&= \frac{h^2}{2R^2}\cdot 2\pi\cdot\frac{R^4}{4} = \boxed{\frac{1}{4}\pi R^2h^2}
\end{aligned}$$

Method 2: Cross-section at height z

At height z, the cross-section is a disk with radius $r = (R/h)z$.

$$\begin{aligned}
\int_0^h z\left[\iint_{x^2+y^2\le(Rz/h)^2} dx\,dy\right]dz &= \int_0^h z\left[\pi\left(\frac{Rz}{h}\right)^2\right]dz \\
&= \frac{\pi R^2}{h^2}\int_0^h z^3\,dz = \boxed{\frac{1}{4}\pi R^2h^2}
\end{aligned}$$

Method 3: Spherical Coordinates

$$\alpha\in[0,2\pi],\quad \beta\in\left[0,\arctan\left(\frac{R}{h}\right)\right],\quad \rho\in\left[0,\frac{h}{\cos\beta}\right],$$

$$\begin{aligned}
\iiint_\Omega z\,dx\,dy\,dz &= \int_0^{2\pi}\int_0^{\arctan(R/h)}\int_0^{h\sec\beta}(\rho\cos\beta)\cdot\rho^2\sin\beta\,d\rho\,d\beta\,d\alpha \\
&= 2\pi\int_0^{\arctan(R/h)}\cos\beta\sin\beta\left(\int_0^{h\sec\beta}\rho^3\,d\rho\right)d\beta \\
&= 2\pi\int_0^{\arctan(R/h)}\cos\beta\sin\beta\left(\frac{h^4\sec^4\beta}{4}\right)d\beta \\
&= \frac{\pi h^4}{2}\int_0^{\arctan(R/h)}\tan\beta\sec^2\beta\,d\beta \\
&= \frac{\pi h^4}{2}\left[\frac{\tan^2\beta}{2}\right]_0^{\arctan(R/h)} \\
&= \frac{\pi h^4}{4}\left(\frac{R^2}{h^2}\right) \\
&= \boxed{\frac{1}{4}\pi R^2h^2}
\end{aligned}$$

36 Problem

Find the sum:$S = \sum_{n=0}^{\infty} \frac{(-1)^n(n^2-n+1)}{2^n}$

Solution

Decompose the general term:

$$\frac{(-1)^n(n^2-n+1)}{2^n} = (-1)^n\frac{n^2}{2^n} - (-1)^n\frac{n}{2^n} + \left(-\frac{1}{2}\right)^n$$

Compute each series separately:

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{n}{2^n} = \frac{-\frac{1}{2}}{(1+\frac{1}{2})^2} = -\frac{2}{9}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{n^2}{2^n} = \frac{-\frac{1}{2}(1-\frac{1}{2})}{(1+\frac{1}{2})^3} = -\frac{2}{27}$$

$$S = -\frac{2}{27} - \left(-\frac{2}{9}\right) + \frac{2}{3} = \boxed{\frac{22}{27}}$$

37 Problem

Expansion of $f(x) = \cos x$ as a Sine Series on $[0, \pi]$

Solution

We perform an odd extension of $f(x)$ to $[-\pi, \pi]$:

$$f_{\text{odd}}(x) = \begin{cases} \cos x, & 0 \le x \le \pi \\ -\cos(-x), & -\pi \le x < 0 \end{cases}$$

The sine series coefficients are:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(nx)\, dx$$

Using the identity $\cos x \sin(nx) = \frac{1}{2}[\sin(n+1)x + \sin(n-1)x]$:

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi}\sin(n+1)x\,dx + \int_0^{\pi}\sin(n-1)x\,dx\right]$$

Evaluating the integrals:

$$b_n = \frac{1}{\pi}\left[\frac{1-(-1)^{n+1}}{n+1} + \frac{1-(-1)^{n-1}}{n-1}\right]$$

For $n = 1$:

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin(2x)\,dx = 0$$

For $n \ge 2$:

$$b_n = \frac{2n[1+(-1)^n]}{\pi(n^2-1)}$$

Only even n terms remain ($n = 2, 4, 6, \ldots$). The final expansion is:

$$\boxed{\cos x = \sum_{k=1}^{\infty}\frac{4k}{\pi(4k^2-1)}\sin(2kx)}$$

38 Problem

If the series $\sum_{n=1}^{\infty} a_n$ converges,
provide counterexamples showing that the following series may not converge:

(i) $\displaystyle\sum_{n=1}^{\infty} |a_n|$

(ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} a_n a_{n+1}$

Solution

1. For $\boxed{a_n = \dfrac{(-1)^n}{n}}$: $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.
2. For $\boxed{a_n = \dfrac{(-1)^n}{n}}$: $\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.
3. For $\boxed{a_n = \dfrac{(-1)^n}{\sqrt{n}}}$: $\sum_{n=1}^{\infty} a_n a_{n+1} = -\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n(n+1)}}$ diverges since $\dfrac{1}{\sqrt{n(n+1)}} \geq \dfrac{1}{2n}$.

39 Problem

Evaluate the limit:$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{3^k} \left(1 + \frac{1}{k}\right)^{k^2}$

Solution

We have:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{3^k} \left(1 + \frac{1}{k}\right)^{k^2} \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{e}{3}\right)^k = \lim_{n\to\infty} \frac{C}{n} = \boxed{0}$$

where we used $\left(1 + \frac{1}{k}\right)^{k^2} \leq e^k$ and $C = \sum_{k=1}^{\infty} \left(\frac{e}{3}\right)^k < \infty$ since $\frac{e}{3} < 1$.

40 Problem

Given the function $f(x) = \dfrac{2x^2}{1+x^2}$, find the value of the 6th derivative at zero: $f^{(6)}(0)$

Solution

First, expand $f(x)$ as a power series:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Thus,

$$f(x) = 2x^2 \sum_{n=0}^{\infty} (-1)^n x^{2n} = 2 \sum_{n=0}^{\infty} (-1)^n x^{2n+2}$$

The coefficient of x^6 is $2(-1)^2 = 2$. Therefore:

$$\frac{f^{(6)}(0)}{6!} = 2 \implies f^{(6)}(0) = \boxed{2 \cdot 6!}$$

41 Problem

Find the sum function of the power series: $\sum_{n=2}^{\infty} \frac{x^n}{n^2-1}$

Solution

$$\begin{aligned}
S(x) &= \sum_{n=2}^{\infty} \frac{x^n}{n^2-1} \\
&= \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{x^n}{n-1} - \frac{x^n}{n+1}\right) \quad \text{(Partial fractions)} \\
&= \frac{x}{2}\sum_{k=1}^{\infty}\frac{x^k}{k} - \frac{1}{2x}\sum_{m=3}^{\infty}\frac{x^m}{m} \quad \text{(Index shift)} \\
&= -\frac{x}{2}\ln(1-x) + \frac{1}{2x}\left(\ln(1-x) + x + \frac{x^2}{2}\right) \quad \text{(Series for } \ln(1-x)\text{)} \\
&= \frac{x+2}{4} + \frac{\ln(1-x)}{2x}(1-x^2) \quad \text{(Simplified form)}
\end{aligned}$$

$$\boxed{\sum_{n=2}^{\infty}\frac{x^n}{n^2-1} = \begin{cases} \frac{x+2}{4} + \frac{1-x^2}{2x}\ln(1-x) & \text{otherwise} \\ 0, & x=0 \end{cases}}$$

42 Problem

Expand the function $f(x) = \frac{1}{4}\ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\arctan x - x$ into a power series of x:

Solution

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}, \quad \text{for } |x|<1$$

$$\ln(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}, \quad \text{for } |x|<1$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \sum_{k=1}^{\infty}\left[(-1)^{k+1}+1\right]\frac{x^k}{k} = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

$$\arctan x = \sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{2m+1}, \quad \text{for } |x|\le 1$$

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} + \frac{1}{2}\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{2m+1} - x$$

Notice that the $-x$ term cancels with the $n=0$ and $m=0$ terms from the series.

$$f(x) = \frac{1}{2}\sum_{k=1}^{\infty}\left[1+(-1)^k\right]\frac{x^{2k+1}}{2k+1}$$

The term $1+(-1)^k$ is non-zero only when k is even. Let $k=2n$:

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty}2\cdot\frac{x^{4n+1}}{4n+1} = \boxed{\sum_{n=1}^{\infty}\frac{x^{4n+1}}{4n+1}}$$

43 Problem

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$$

Solution

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n}{n+2} = 1 \quad \Rightarrow \quad R = 1 \tag{1}$$

$$x = 1 : \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converges} \tag{2}$$

$$x = -1 : \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \text{ converges} \tag{3}$$

$$\text{Thus } x \in [-1, 1] \tag{4}$$

$$S(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = x \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} \tag{5}$$

$$\text{Let } f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} \tag{6}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n+1} = \frac{1}{x^2} \sum_{n=2}^{\infty} \frac{x^n}{n} = \frac{1}{x^2} \left(-\ln(1-x) - x\right) \tag{7}$$

$$f(x) = \int \frac{-\ln(1-x) - x}{x^2} dx = (1-x)\ln(1-x) + x \tag{8}$$

$$\Rightarrow S(x) = x[(1-x)\ln(1-x) + x] \quad \text{for } x \in [-1, 1) \tag{9}$$

$$\text{At } x = 1 : \lim_{x\to 1^-} S(x) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1 \tag{10}$$

$$\boxed{S(x) = \begin{cases} x + x(1-x)\ln(1-x), & x \in [-1, 1) \\ 1, & x = 1 \end{cases}} \tag{11}$$

44 Problem

Prove convergence of $\sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx$.

Solution

Method 1: Alternating Series Test

Let $I_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx$. Since $\frac{1}{(n+1)\pi} \leq \frac{1}{x} \leq \frac{1}{n\pi}$ on $[n\pi, (n+1)\pi]$, we have:

- For even n, $\sin x \leq 0 \Rightarrow I_n \leq 0$
- For odd n, $\sin x \geq 0 \Rightarrow I_n \geq 0$

The series alternates in sign. We estimate $|I_n| \leq \frac{1}{n\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| dx = \frac{2}{n\pi}$.
Since $\frac{2}{n\pi}$ decreases to 0, by the Alternating Series Test.

$$\boxed{\sum I_n \text{ converges}}$$

Method 2: Absolute Convergence via Comparison

We show $\sum |I_n|$ converges. Note that:

$$|I_n| \leq \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx \leq \frac{1}{n\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| dx = \frac{2}{n\pi}$$

Since $\sum_{n=1}^{\infty} \frac{2}{n\pi} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, this approach fails. Instead, consider:

$$|I_n| \leq \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx \leq \frac{1}{n\pi} \int_{0}^{\pi} |\sin u| du = \frac{2}{n\pi}$$

While this gives the same bound, we can improve the estimate by integration by parts:

$$I_n = -\frac{\cos x}{x}\Big|_{n\pi}^{(n+1)\pi} - \int_{n\pi}^{(n+1)\pi} \frac{\cos x}{x^2} dx$$

The boundary terms telescope and the remaining integral is absolutely convergent since $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$.

$$\boxed{\sum I_n \text{ converges}}$$

Method 3: Taylor Expansion

Let $I_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx$. Substitute $x = n\pi + t$:

$$I_n = (-1)^n \int_0^{\pi} \frac{\sin t}{n\pi + t} dt$$

Expand $\sin t$ and $(n\pi + t)^{-1}$:

$$\frac{\sin t}{n\pi + t} = \frac{t - \frac{t^3}{6} + \cdots}{n\pi} \left(1 - \frac{t}{n\pi} + \frac{t^2}{(n\pi)^2} - \cdots\right)$$

$$I_n \approx (-1)^n \left[\frac{\pi}{2n} - \frac{\pi}{3n^2} + O\left(\frac{1}{n^3}\right)\right]$$

The series $\sum I_n$ converges as:

- $\sum (-1)^n \frac{\pi}{2n}$ converges (alternating series)
- $\sum \frac{1}{n^2}$ and higher order terms converge absolutely

$$\boxed{\sum I_n \text{ converges}}$$