1 Derivative

1.1

Given:

$$f(x) = \frac{1}{\ln(1+x)} - \frac{1}{x}, \quad x \in [0,1], \quad \lambda = \max_{x\in[0,1]} f(x), \quad \mu = \min_{x\in[0,1]} f(x)$$

As $x \to 0^+$:

$$\ln(1+x) = x - \frac{x^2}{2} + O(x^3)$$
$$\frac{1}{\ln(1+x)} = \frac{1}{x} \cdot \frac{1}{1 - \frac{x}{2} + O(x^2)} = \frac{1}{x}\left(1 + \frac{x}{2} + O(x^2)\right) = \frac{1}{x} + \frac{1}{2} + O(x)$$
$$f(x) = \left(\frac{1}{x} + \frac{1}{2}\right) - \frac{1}{x} + O(x) = \frac{1}{2} + O(x) \to \frac{1}{2}$$

At $x = 1$:

$$f(1) = \frac{1}{\ln 2} - 1$$

The derivative is:

$$f'(x) = -\frac{1}{(1+x)\ln^2(1+x)} + \frac{1}{x^2}$$

Setting $f'(x) = 0$ gives:

$$\frac{1}{x^2} = \frac{1}{(1+x)\ln^2(1+x)} \Rightarrow (1+x)\ln^2(1+x) = x^2$$

Numerical analysis shows $f'(x) < 0$ on $[0,1]$, so $f(x)$ is strictly decreasing. Therefore:

$$\lambda = \lim_{x\to 0^+} f(x) = \frac{1}{2}, \quad \mu = f(1) = \frac{1}{\ln 2} - 1$$

1.2

Given $f(x)$ has continuous second-order derivatives on $[a,b]$. Prove:

$$\lim_{n\to\infty} n^2 \left[\int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{2k-1}{2n}(b-a)\right)\right] = \frac{(b-a)^2}{24}\left[f'(b) - f'(a)\right]$$

Proof: Consider the error for one subinterval $[x_{k-1}, x_k]$ where $x_k = a + \frac{k}{n}(b-a)$ and midpoint $c_k = a + \frac{2k-1}{2n}(b-a)$. By Taylor expansion around c_k:

$$f(x) = f(c_k) + f'(c_k)(x - c_k) + \frac{f''(\xi_k)}{2}(x - c_k)^2$$
$$\int_{x_{k-1}}^{x_k} f(x)dx = f(c_k)h + \frac{f''(\eta_k)}{24}h^3$$

where $h = \frac{b-a}{n}$.

Summing over all subintervals:

$$\int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f(c_k) = \sum_{k=1}^{n} \frac{f''(\eta_k)}{24}h^3 = \frac{(b-a)^3}{24n^2} \cdot \frac{1}{n}\sum_{k=1}^{n} f''(\eta_k)$$

As $n \to \infty$, $\frac{1}{n}\sum_{k=1}^{n} f''(\eta_k) \to \frac{1}{b-a}\int_a^b f''(x)dx = \frac{f'(b)-f'(a)}{b-a}$. Therefore:

$$\lim_{n\to\infty} n^2 \left[\int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f(c_k)\right] = \frac{(b-a)^2}{24}[f'(b) - f'(a)]$$

2 Limits

2.1

Compute:

$$I = \lim_{x \to 0} \frac{\ln(\cos x)}{x^2}$$

Using Taylor expansion:

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2} + O(x^4) \\
\ln(\cos x) &= \ln\left(1 - \frac{x^2}{2} + O(x^4)\right) = -\frac{x^2}{2} + O(x^4) \\
I &= \lim_{x \to 0} \frac{-\frac{x^2}{2} + O(x^4)}{x^2} = -\frac{1}{2}
\end{aligned}$$

2.2

Compute:

$$I = \lim_{x \to 0} \frac{\ln(\cos x + x \sin 2x)}{e^{x^2} - \sqrt[3]{1 - x^2}}$$

Expand numerator:

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2} + O(x^4) \\
x \sin 2x &= 2x^2 - \frac{4}{3}x^4 + O(x^6) \\
\cos x + x \sin 2x &= 1 + \frac{3}{2}x^2 + O(x^4) \\
\ln(\cos x + x \sin 2x) &= \frac{3}{2}x^2 + O(x^4)
\end{aligned}$$

Expand denominator:

$$\begin{aligned}
e^{x^2} &= 1 + x^2 + O(x^4) \\
\sqrt[3]{1 - x^2} &= 1 - \frac{1}{3}x^2 + O(x^4) \\
e^{x^2} - \sqrt[3]{1 - x^2} &= \frac{4}{3}x^2 + O(x^4)
\end{aligned}$$

Therefore:

$$I = \lim_{x \to 0} \frac{\frac{3}{2}x^2 + O(x^4)}{\frac{4}{3}x^2 + O(x^4)} = \frac{3/2}{4/3} = \frac{9}{8}$$

3 Integrals

3.1

$$I = \int \sec x dx$$

Multiply numerator and denominator by $\sec x + \tan x$:

$$I = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Let $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x)dx$:

$$I = \int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C$$

3.2

$$I = \int \frac{dx}{x^2 + a^2}$$

Let $x = a \tan t$, $dx = a \sec^2 t dt$:

$$I = \int \frac{a \sec^2 t}{a^2(\tan^2 t + 1)} dt = \frac{1}{a} \int dt = \frac{1}{a} t + C = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

3.3

$$I = \int \frac{dx}{\sqrt{x^2 + a^2}}$$

Let $x = a \tan t$, $dx = a \sec^2 t dt$, $\sqrt{x^2 + a^2} = a \sec t$:

$$I = \int \frac{a \sec^2 t}{a \sec t} dt = \int \sec t dt = \ln |\sec t + \tan t| + C$$

Substitute back: $\sec t = \frac{\sqrt{x^2+a^2}}{a}$, $\tan t = \frac{x}{a}$:

$$I = \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 + a^2}}{a} \right| + C = \ln \left| x + \sqrt{x^2 + a^2} \right| + C'$$

4 Multiple Integrals

4.1

$$D = \{(x, y) \mid x^2 + y^2 \leq \pi\}, \quad I = \iint_D \left(\sin(x^2)\cos(y^2) + x\sqrt{x^2 + y^2}\right) dxdy$$

The second term $x\sqrt{x^2 + y^2}$ is odd in x, so its integral over the symmetric domain D is zero. Thus:

$$I = \iint_D \sin(x^2)\cos(y^2)dxdy$$

Using the identity:

$$\sin(x^2)\cos(y^2) = \frac{1}{2}[\sin(x^2 + y^2) + \sin(x^2 - y^2)]$$

The $\sin(x^2 - y^2)$ term integrates to zero due to symmetry. Use polar coordinates:

$$\begin{aligned} x &= r\cos\theta, \quad y = r\sin\theta \\ I &= \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{\pi}} \sin(r^2) \cdot rdrd\theta \\ &= \pi\int_0^{\sqrt{\pi}} \sin(r^2)rdr \end{aligned}$$

Let $u = r^2$, $du = 2rdr$:

$$I = \pi\int_0^{\pi} \sin u \cdot \frac{du}{2} = \frac{\pi}{2}[-\cos u]_0^{\pi} = \frac{\pi}{2}(1 - (-1)) = \pi$$

4.2

$$I = \iint_S (x^2 - x)dydz + (y^2 - y)dzdx + (z^2 - z)dxdy$$

where S is the upper side of the upper hemisphere $x^2 + y^2 + z^2 = R^2$ $(z \geq 0)$.

Use spherical coordinates:

$$\begin{aligned} &x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta \\ &0 \leq r \leq R, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq 2\pi \end{aligned}$$

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} (2r\cos\theta - 3)r^2\sin\theta drd\theta d\phi \\ &= 2\pi\int_0^{\pi/2}\int_0^{R} (2r^3\cos\theta - 3r^2)\sin\theta drd\theta \\ &= 2\pi\int_0^{\pi/2} \left(\frac{R^4}{2}\cos\theta - R^3\right)\sin\theta d\theta \\ &= 2\pi\left[-\frac{R^4}{4}\cos^2\theta + R^3\cos\theta\right]_0^{\pi/2} \\ &= 2\pi\left(\frac{R^4}{4} - R^3\right) = \frac{\pi R^4}{2} - 2\pi R^3 \end{aligned}$$