1 Elastic Collision Derivation

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \tag{1}$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \tag{2}$$

$$v_{a1} + v_{b1} = v_{a2} + v_{b2} \tag{3}$$

$$\boxed{v_1' = \frac{(m_1 - m_2) v_1 + 2 m_2 v_2}{m_1 + m_2}} \tag{4}$$

$$\boxed{v_2' = \frac{(m_2 - m_1) v_2 + 2 m_1 v_1}{m_1 + m_2}} \tag{5}$$

2 Kinematic Equations for Rigid Body Rotation

$$\alpha = \frac{d\omega}{dt} \implies \int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha \, dt \tag{1}$$

$$\omega = \omega_0 + \alpha t \iff \boxed{v = v_0 + at} \tag{2}$$

$$\frac{d\theta}{dt} = \omega_0 + \alpha t \implies \int_{\theta_0}^{\theta} d\theta = \int_0^t (\omega_0 + \alpha t) \, dt \tag{3}$$

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2} \alpha t^2 \iff \boxed{s = v_0 t + \frac{1}{2} a t^2} \tag{4}$$

$$\alpha = \omega \frac{d\omega}{d\theta} \implies \int_{\omega_0}^{\omega} \omega \, d\omega = \int_{\theta_0}^{\theta} \alpha \, d\theta \tag{5}$$

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0) \iff \boxed{v^2 - v_0^2 = 2as} \tag{6}$$

3 Particle Motion VS Rigid Body Rotation

Quantity	**Particle Motion**	**Quantity**	**Rigid Body Rotation**
Velocity	$\vec{v} = \frac{d\vec{r}}{dt}$	Angular velocity	$\vec{\omega} = \frac{d\vec{\theta}}{dt}$
Acceleration	$\vec{a} = \frac{d\vec{v}}{dt}$	Angular acceleration	$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$
Force	$\vec{F} = m\vec{a}$	Torque	$\vec{M} = J\vec{\alpha}$
Inertia	Mass m	Inertia	Moment of inertia $J = \int r^2 dm$
Momentum	$\vec{p} = m\vec{v}$	Angular momentum	$\vec{L} = J\vec{\omega}$
Law	$\vec{F} = \frac{d\vec{p}}{dt}$	Law	$\vec{M} = \frac{d\vec{L}}{dt}$
Impulse	$\int \vec{F} dt = \Delta\vec{p}$	Angular impulse	$\int \vec{M} dt = \Delta\vec{L}$
Energy	$T = \frac{1}{2} m v^2$	Rotational energy	$T = \frac{1}{2} J \omega^2$
Work	$W = \int \vec{F} \cdot d\vec{r}$	Rotational work	$W = \int \vec{M} \cdot d\vec{\theta}$
Work-energy	$W = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_2^2$	Work-energy	$W = \frac{1}{2} J \omega_1^2 - \frac{1}{2} J \omega_2^2$

4 Problem

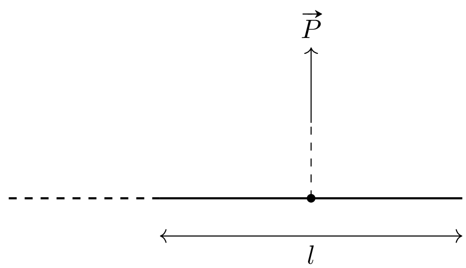
Find the moment of inertia for the following physical models using the advanced mathematical definition of moment of inertia:

1. Thin rod of length l, rotation axis through center perpendicular to rod.
2. Cylinder of radius R, rotation axis along geometric axis.
3. Thin ring of radius R, rotation axis along geometric axis.
4. Sphere of radius R, rotation axis along any diameter.
5. Cylindrical shell with outer radius R_2, inner radius R_1, rotation axis along geometric axis.
6. Thin rod of length l, rotation axis through one end perpendicular to rod.

Solution

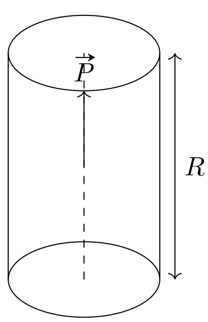
1. Thin rod (center axis):

$$J = \int_{-l/2}^{l/2} r^2 \lambda \, dr = \lambda \left[\frac{r^3}{3} \right]_{-l/2}^{l/2} = \frac{\lambda l^3}{12} = \boxed{\frac{ml^2}{12}} \tag{1}$$



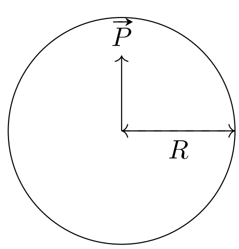
2. Solid cylinder:

$$J = \int_0^R r^2 \sigma 2\pi r \, dr = 2\pi\sigma \left[\frac{r^4}{4} \right]_0^R = \frac{\pi\sigma R^4}{2} = \boxed{\frac{mR^2}{2}} \tag{2}$$



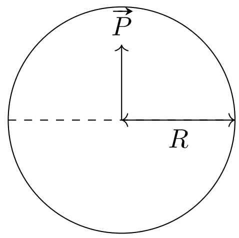
3. Thin ring:

$$J = \int_0^{2\pi} R^2 \lambda R \, d\theta = 2\pi R^3 \lambda = \boxed{mR^2} \tag{3}$$



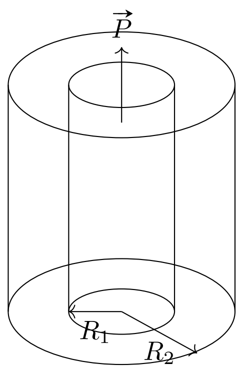
4. Solid sphere:

$$J = \int_0^R r^2 \rho 4\pi r^2 \, dr = 4\pi\rho \left[\frac{r^5}{5}\right]_0^R = \frac{4\pi\rho R^5}{5} = \boxed{\frac{2mR^2}{5}} \tag{4}$$



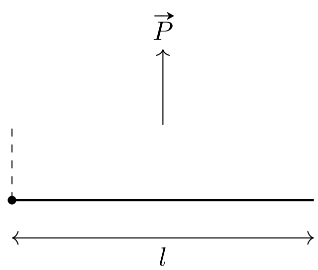
5. Cylindrical shell:

$$J = \int_{R_1}^{R_2} r^2 \sigma 2\pi r \, dr = 2\pi\sigma \left(\frac{R_2^4}{4} - \frac{R_1^4}{4}\right) = \boxed{\frac{m}{2}(R_1^2 + R_2^2)} \tag{5}$$



6. Thin rod (end axis):

$$J = \int_0^l r^2 \lambda \, dr = \lambda \left[\frac{r^3}{3}\right]_0^l = \frac{\lambda l^3}{3} = \boxed{\frac{ml^2}{3}} \tag{6}$$



5 Problem

Prove that planetary orbits in the solar system are elliptical under Newtonian gravity.

Solution

We begin with Newton's law of gravitation and his second law of motion:

$$\vec{F} = -\frac{GMm}{r^2}\vec{e_r} \quad \text{(Gravitational force)} \tag{1}$$
$$\vec{F} = m\vec{a} \quad \text{(Newton's second law)} \tag{2}$$

In polar coordinates (r, θ), the acceleration has two components:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e_\theta} \tag{3}$$

Equating the components gives two equations of motion:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} \quad \text{(Radial equation)} \tag{4}$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \text{(Angular equation)} \tag{5}$$

The angular equation can be rewritten as:

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{6}$$

This implies conservation of angular momentum:

$$L = r^2\dot{\theta} = \text{constant} \tag{7}$$

To solve the radial equation, we make a substitution $u = 1/r$ and change variables using the chain rule:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{L}{r^2} = -L\frac{du}{d\theta} \tag{8}$$
$$\ddot{r} = -L\frac{d^2u}{d\theta^2}\dot{\theta} = -L^2u^2\frac{d^2u}{d\theta^2} \tag{9}$$

Substituting into the radial equation:

$$-L^2u^2\frac{d^2u}{d\theta^2} - \frac{L^2u^3}{u^2} = -GMu^2 \tag{10}$$
$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{L^2} \tag{11}$$

This is an inhomogeneous second-order differential equation. The general solution is:

$$u(\theta) = \frac{GM}{L^2} + A\cos(\theta - \theta_0) \tag{12}$$

Converting back to r and defining $e = AL^2/GM$:

$$\boxed{r(\theta) = \frac{L^2/GM}{1 + e\cos(\theta - \theta_0)}} \tag{13}$$

This is exactly the polar equation of a conic section with eccentricity e.
For planets, $0 \leq e < 1$, which corresponds to elliptical orbits.

6 Problem

Maxwell's Equations in Integral Form:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{1}$$

$$\oint_S \vec{B} \cdot d\vec{S} = 0 \tag{2}$$

$$\oint_L \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} \tag{3}$$

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{S} \tag{4}$$

7 Problem

Physical Quantity	Formula	Unit Symbol	SI Base Units
Current	$I = \frac{dQ}{dt}$	A	A
Voltage	$V = IR$	V	kg·m^2·s^{-3}·A^{-1}
Electric Charge	$Q = CV$	C	A·s
Capacitance	$C = \frac{Q}{V}$	F	kg^{-1}·m^{-2}·s^4·A^2
Resistance	$R = \frac{V}{I}$	Ω	kg·m^2·s^{-3}·A^{-2}
Conductance	$G = \frac{1}{R}$	S	kg^{-1}·m^{-2}·s^3·A^2
Inductance	$V = L\frac{dI}{dt}$	H	kg·m^2·s^{-2}·A^{-2}
Electric Field Strength	$\vec{E} = \frac{\vec{F}}{q}$	V/m	kg·m·s^{-3}·A^{-1}
Magnetic Flux	$\Phi_B = \int \vec{B} \cdot d\vec{A}$	Wb	kg·m^2·s^{-2}·A^{-1}
Magnetic Flux Density	$\vec{B} = \mu_0 \vec{H} + \vec{M}$	T	kg·s^{-2}·A^{-1}
Electric Displacement Field	$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$	C/m^2	A·s·m^{-2}
Magnetic Field Strength	$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$	A/m	A·m^{-1}
Permittivity	$\epsilon = \epsilon_0 \epsilon_r$	F/m	kg^{-1}·m^{-3}·s^4·A^2
Permeability	$\mu = \mu_0 \mu_r$	H/m	kg·m·s^{-2}·A^{-2}
Power	$P = VI$	W	kg·m^2·s^{-3}
Energy	$W = \int P\, dt$	J	kg·m^2·s^{-2}
Polarization Density	$\vec{P} = \epsilon_0 \chi_e \vec{E}$	C/m^2	A·s·m^{-2}
Magnetization	$\vec{M} = \chi_m \vec{H}$	A/m	A·m^{-1}

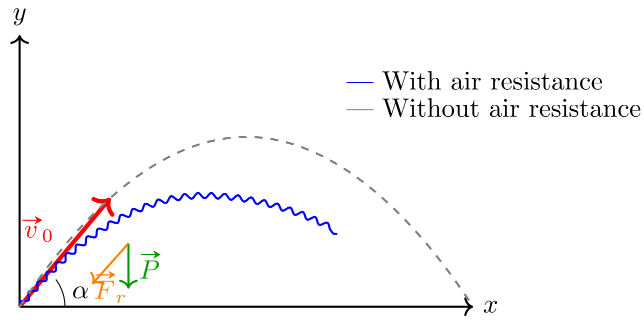
8 Problem

Charge Distribution	Electric Field
Point charge with charge q	$E = \dfrac{q}{4\pi\epsilon_0 r^2}$
Uniformly charged spherical shell with radius R and charge q	$E = \begin{cases} 0, & \text{inside the sphere} \\ \frac{q}{4\pi\epsilon_0 r^2}, & \text{outside the sphere} \end{cases}$
Uniformly charged solid sphere with radius R and charge q	$E = \begin{cases} \frac{qr}{4\pi\epsilon_0 R^3}, & \text{inside the sphere} \\ \frac{q}{4\pi\epsilon_0 r^2}, & \text{outside the sphere} \end{cases}$
Infinitely long uniformly charged line with linear charge density λ	$E = \dfrac{\lambda}{2\pi\epsilon_0 r}$
Infinitely large uniformly charged plane with surface charge density σ	$E = \dfrac{\sigma}{2\epsilon_0}$

9 Problem

The air resistance is given by $\vec{F}_r = -k\vec{v}$.
Find the trajectory equation for a projectile with mass m, initial velocity $\vec{v}_0$, and projection angle α.

Solution



Horizontal Motion

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = -kv_x \tag{1}$$

$$\frac{\mathrm{d}v_x}{v_x} = -\frac{k}{m}\mathrm{d}t \tag{2}$$

$$\ln v_x = -\frac{k}{m}t + C \tag{3}$$

$$v_x = v_{0x}e^{-kt/m} = v_0\cos\alpha \cdot e^{-kt/m} \tag{4}$$

$$x = \int_0^t v_x \mathrm{d}t = \frac{mv_0\cos\alpha}{k}\left(1 - e^{-kt/m}\right) \tag{5}$$

Vertical Motion

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -mg - kv_y \tag{6}$$

$$\frac{\mathrm{d}v_y}{v_y + mg/k} = -\frac{k}{m}\mathrm{d}t \tag{7}$$

$$\ln\left(v_y + \frac{mg}{k}\right) = -\frac{k}{m}t + C \tag{8}$$

$$v_y = \left(v_0\sin\alpha + \frac{mg}{k}\right)e^{-kt/m} - \frac{mg}{k} \tag{9}$$

$$y = \int_0^t v_y \mathrm{d}t = \frac{m}{k}\left(v_0\sin\alpha + \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right) - \frac{mg}{k}t \tag{10}$$

Eliminating Time Parameter

$$e^{-kt/m} = 1 - \frac{kx}{mv_0\cos\alpha} \tag{11}$$

$$t = -\frac{m}{k}\ln\left(1 - \frac{kx}{mv_0\cos\alpha}\right) \tag{12}$$

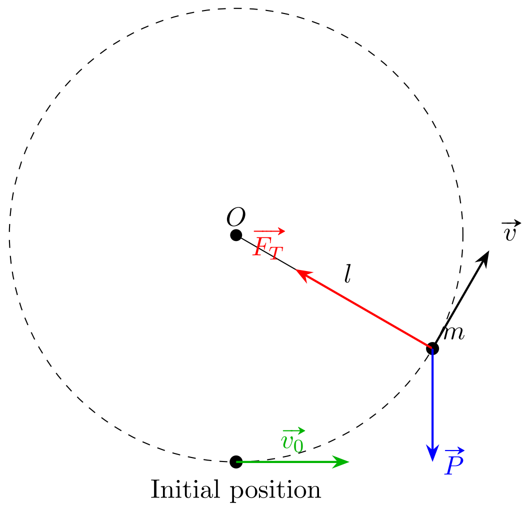
Final Result

$$\boxed{y = \left(\tan\alpha + \frac{mg}{kv_0\cos\alpha}\right)x + \frac{m^2g}{k^2}\ln\left(1 - \frac{kx}{mv_0\cos\alpha}\right)} \tag{13}$$

10 Problem

Given: Light rope of length l, mass m at one end, fixed at point O.
Initial position: lowest point with initial horizontal velocity $\vec{v_0}$.
Calculate both the instantaneous speed of the pendulum bob and the corresponding cord tension at any point during its circular motion.

Solution



Using differential work-energy principle:

$$\mathrm{d}W = \mathrm{d}T + \mathrm{d}U \tag{1}$$

$$\vec{P} \cdot \mathrm{d}\vec{r} = \mathrm{d}\left(\frac{1}{2}m\vec{v} \cdot \vec{v}\right) + \mathrm{d}(mgy) \tag{2}$$

$$-mg\,\mathrm{d}y = m\vec{v} \cdot \mathrm{d}\vec{v} + mg\,\mathrm{d}y \tag{3}$$

$$\int_{v_0}^{v} v\,\mathrm{d}v = -g\int_{0}^{l(1-\cos\theta)} \mathrm{d}y \tag{4}$$

$$\frac{1}{2}(v^2 - v_0^2) = -gl(1-\cos\theta) \tag{5}$$

$$v^2 = v_0^2 - 2gl(1-\cos\theta) \tag{6}$$

Radial dynamics using polar coordinates:

$$\sum F_r = -F_T + P_r = m\ddot{r} - mr\dot{\theta}^2 \tag{7}$$

$$-F_T + mg\cos\theta = -ml\dot{\theta}^2 \quad \text{(since } r = l \text{ constant)} \tag{8}$$

$$F_T = m\left(g\cos\theta + l\dot{\theta}^2\right) \tag{9}$$

$$\text{Using } v = l\dot{\theta} \text{ and energy result:} \tag{10}$$

$$F_T = m\left(g\cos\theta + \frac{v_0^2}{l} - 2g(1-\cos\theta)\right) \tag{11}$$

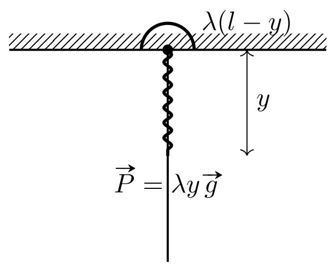
$$\boxed{v = \sqrt{v_0^2 + 2gl(\cos\theta - 1)}} \tag{12}$$

$$\boxed{F_T = m\left(\frac{v_0^2}{l} - 2g + 3g\cos\theta\right)} \tag{13}$$

11 Problem

A flexible chain of length l and linear mass density λ lies on a table with a small hole.
One end of the chain hangs slightly down through the hole, while the rest is piled around it.
Due to a disturbance, the chain starts to fall under its own weight.
Neglecting friction, find the relationship between the velocity v and the falling distance y.

Solution



$$\frac{d}{dt}(\lambda y v) = \lambda y g \implies \lambda v \frac{dy}{dt} + \lambda y \frac{dv}{dt} = \lambda y g \tag{1}$$

$$v^2 + yv\frac{dv}{dy} = yg \tag{2}$$

$$\text{Let } v^2 = u \Rightarrow u + \frac{y}{2}\frac{du}{dy} = yg \tag{3}$$

$$\frac{d}{dy}(uy^2) = 2gy^3 \implies uy^2 = \frac{gy^4}{2} \tag{4}$$

$$\boxed{v = \left(\frac{2gy}{3}\right)^{1/2}} \tag{5}$$

Method 1: Direct Integration

$$\int_0^t \lambda y g dt = \lambda y v$$

$$g\int_0^y y dy = \int_0^v y v dy$$

$$\frac{1}{2}gy^2 = \frac{1}{3}yv^2$$

$$\boxed{v = \left(\frac{2gy}{3}\right)^{1/2}} \tag{6}$$

Method 2: Energy Conservation

$$\lambda y g dy = \frac{1}{2}\lambda d(yv^2)$$

$$g y dy = \frac{1}{2}d(yv^2)$$

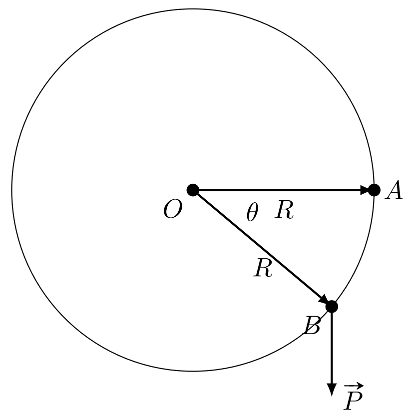
$$\frac{1}{2}gy^2 = \frac{1}{2}yv^2$$

$$\boxed{v = \left(\frac{2gy}{3}\right)^{1/2}} \tag{7}$$

12 Problem

A smooth ring with radius R is placed in a vertical plane.
A small ball with mass m slides on the ring and can move freely without friction.
The ball starts from rest at point A.
Find the angular velocity ω of the ball about O when it reaches point B.

Solution



Method 1: Energy Conservation

$$\Delta h = R\sin\theta \tag{1}$$

$$\frac{1}{2}mv^2 = mg\Delta h \tag{2}$$

$$v^2 = 2gR\sin\theta \tag{3}$$

$$v = R\omega \tag{4}$$

$$\boxed{\omega = \sqrt{\frac{2g\sin\theta}{R}}} \tag{5}$$

Method 2: Calculus Approach

$$\vec{P} = m\vec{g} \tag{6}$$

$$\text{Tangential component: } P_\tau = -mg\sin\phi \quad \text{(angle } \phi \text{ measured from horizontal axis)} \tag{7}$$

$$\tau = P_\tau R = -mgR\sin\phi \tag{8}$$

$$I\alpha = \tau \Rightarrow mR^2\frac{d\omega}{dt} = -mgR\sin\phi \tag{9}$$

$$\frac{d\omega}{dt} = -\frac{g}{R}\sin\phi \tag{10}$$

$$\omega\frac{d\omega}{d\phi} = -\frac{g}{R}\sin\phi \quad \text{(chain rule)} \tag{11}$$

$$\int_0^\omega \omega\, d\omega = -\frac{g}{R}\int_0^\theta \sin\phi\, d\phi \tag{12}$$

$$\frac{\omega^2}{2} = \frac{g}{R}(\cos\theta - 1) \tag{13}$$

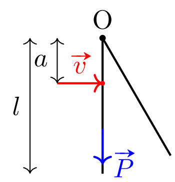
$$\text{From geometry: } \cos\theta = 1 - \sin\theta \tag{14}$$

$$\boxed{\omega = \sqrt{\frac{2g\sin\theta}{R}}} \tag{15}$$

13 Problem

A uniform rod of length l and mass m' can rotate freely about pivot O.
A bullet of mass m with initial velocity $\vec{v}$ hits the rod at distance a from O, causing the rod to swing to a maximum angle of 30°.
Find the initial speed v of the bullet.

Solution



$$mva = \left(\frac{1}{3}m'l^2 + ma^2\right)\omega \tag{1}$$

$$\frac{1}{2}\left(\frac{1}{3}m'l^2 + ma^2\right)\omega^2 = \left(\frac{m'l}{2} + ma\right)g(1 - \cos 30°) \tag{2}$$

$$\boxed{v = \sqrt{(2-\sqrt{3})(m'l + 2ma)(m'l^2 + 3ma^2)g/6}/ma} \tag{3}$$

14 Problem

Two long parallel straight wires with radius R, the distance between their centers is d ($d \gg R$).
Find the capacitance per unit length of the wires.

Solution

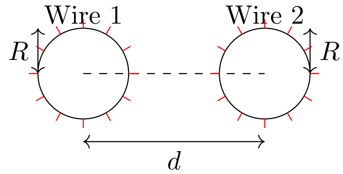


Figure 1: Cross-section of two parallel wires with charge distributions

For a single wire with charge λ per unit length, the potential at distance r is:

$$V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_0}{r}\right) \tag{1}$$

For two wires with opposite charges $\pm\lambda$, the potential difference is:

$$U = V_+ - V_- = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{r_0}{R}\right) - \ln\left(\frac{r_0}{d-R}\right)\right] \tag{2}$$

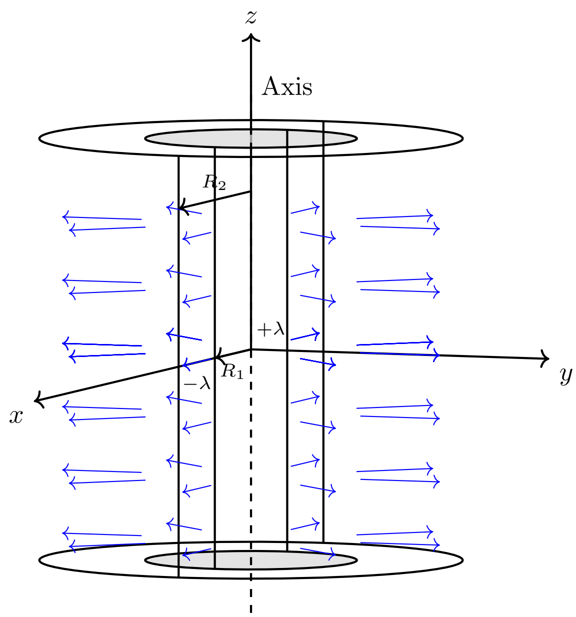
$$U = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{d-R}{R}\right) \approx \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{d}{R}\right) \quad \text{(since } d \gg R\text{)} \tag{3}$$

$$\boxed{C_l = \frac{\lambda}{U} \approx \frac{\pi\epsilon_0}{\ln(d/R)}} \tag{4}$$

15 Problem

A long straight cylindrical conductor with radius R_1 and a coaxial thin cylindrical conductor shell with radius R_2 are separated by air.
The line charge densities are $+\lambda$ and $-\lambda$ respectively. Given the breakdown field strength of air E_b, find the maximum R_1 that maximizes the stored energy without causing air breakdown.

Solution



$$\vec{E}(r) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r} \quad (R_1 < r < R_2) \tag{1}$$

$$\text{At } r = R_1 : \quad E_{\max} = \frac{\lambda}{2\pi\epsilon_0 R_1} \leq E_b \tag{2}$$

$$\lambda_{\max} = 2\pi\epsilon_0 R_1 E_b \tag{3}$$

$$V = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{r} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{R_2}{R_1}\right) \tag{4}$$

$$C = \frac{\lambda}{V} = \frac{2\pi\epsilon_0}{\ln(R_2/R_1)} \tag{5}$$

$$U = \frac{1}{2}CV^2 = \frac{\lambda^2}{4\pi\epsilon_0} \ln\left(\frac{R_2}{R_1}\right) \tag{6}$$

$$\text{Substitute } \lambda_{\max} : \quad U(R_1) = \pi\epsilon_0 E_b^2 R_1^2 \ln\left(\frac{R_2}{R_1}\right) \tag{7}$$

$$\frac{dU}{dR_1} = 2\pi\epsilon_0 E_b^2 R_1 \ln\left(\frac{R_2}{R_1}\right) - \pi\epsilon_0 E_b^2 R_1 = 0 \tag{8}$$

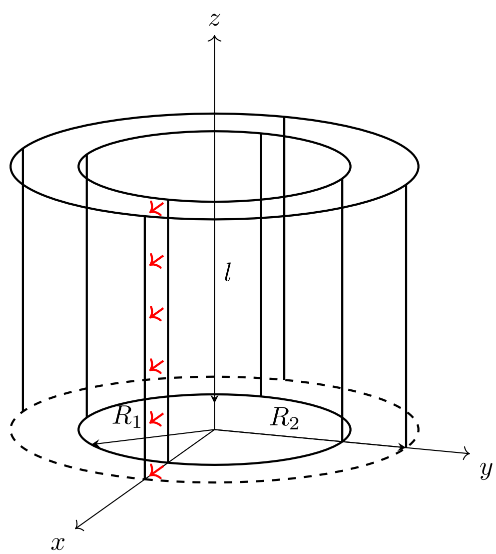
$$\ln\left(\frac{R_2}{R_1}\right) = \frac{1}{2} \Rightarrow R_1 = \frac{R_2}{\sqrt{e}} \tag{9}$$

$$\text{With breakdown condition:} \quad R_1^{\max} = \boxed{\frac{E_b R_2}{2\sqrt{e}}} \tag{10}$$

16 Problem

A metal cylindrical shell with inner radius R_1, outer radius R_2, length l, and resistivity ρ. If the inner edge has higher potential than the outer edge, when the potential difference is U, what is the radial current in the cylinder?

Solution



Consider a thin cylindrical shell at radius r with thickness dr:

$$dR = \rho \frac{dr}{A} = \rho \frac{dr}{2\pi r l} \tag{1}$$

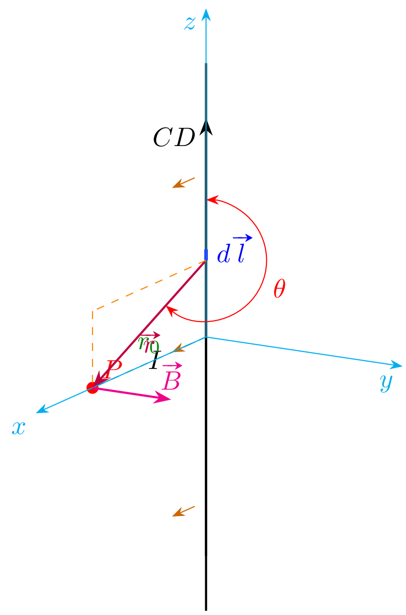
$$R = \int_{R_1}^{R_2} dR = \frac{\rho}{2\pi l} \int_{R_1}^{R_2} \frac{dr}{r} = \frac{\rho}{2\pi l} \ln\left(\frac{R_2}{R_1}\right) \tag{2}$$

$$\boxed{I = \frac{U}{R} = \frac{U}{\frac{\rho}{2\pi l} \ln\left(\frac{R_2}{R_1}\right)} = \frac{2\pi l U}{\rho \ln(R_2/R_1)}} \tag{3}$$

17 Problem

The magnetic field of a current-carrying straight wire.
In vacuum, there is a long straight wire CD carrying current I.
Find the magnetic induction $\vec{B}$ at an arbitrary point P near the wire.
The perpendicular distance between point P and the wire is r_0.

Solution



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \vec{r}}{r^3} \tag{1}$$

$$|\vec{dB}| = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2} \tag{2}$$

$$\text{From geometry: } r = \frac{r_0}{\sin\theta}, \quad l = -r_0\cot\theta \tag{3}$$

$$dl = \frac{r_0}{\sin^2\theta}d\theta \tag{4}$$

$$\text{Substituting into the integral:} \tag{5}$$

$$B = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{dl\sin\theta}{r^2} = \frac{\mu_0 I}{4\pi}\int_0^{\pi}\frac{(r_0 d\theta/\sin^2\theta)\sin\theta}{(r_0^2/\sin^2\theta)} \tag{6}$$

$$B = \frac{\mu_0 I}{4\pi r_0}\int_0^{\pi}\sin\theta d\theta \tag{7}$$

$$B = \frac{\mu_0 I}{4\pi r_0}[-\cos\theta]_0^{\pi} = \frac{\mu_0 I}{4\pi r_0}(1-(-1)) \tag{8}$$

$$\boxed{B = \frac{\mu_0 I}{2\pi r_0}} \tag{9}$$

18 Problem

Electromagnetic catapult principle: Two parallel cylindrical conductor rails with length L, radius R, and separation d ($L \gg d$).
A rod-shaped metal projectile of mass m connects the rails, forming a circuit with external power supply carrying current I.
(1) Find the magnetic force on the projectile.
(2) If the projectile starts from the middle of the rails with acceleration distance $L/2$, find its exit velocity.

Solution

(1) The magnetic field at distance r from a single wire:

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$

Force on the projectile:

$$\vec{F} = I \int_R^{R+d} \vec{dl} \times \vec{B}$$

$$F = \frac{\mu_0 I^2}{2\pi} \ln\left(\frac{R+d}{R}\right) \tag{1}$$

(2) Using work-energy theorem:

$$\frac{1}{2} m v^2 = F \cdot \frac{L}{2}$$

$$\boxed{v = \sqrt{\frac{\mu_0 I^2 L}{2\pi m} \ln\left(\frac{R+d}{R}\right)}}$$