# Mathematical Programming

Optimization problem:

$$
\begin{aligned}
&\min \text{ or } \max\ u = f(x), \quad x \in \Omega \\
&\text{subject to:} \\
&h_i(x) = 0, \quad i = 1, 2, \ldots, m \\
&g_j(x) \leq 0, \quad j = 1, 2, \ldots, p
\end{aligned}
$$

# Statistical Regression

Regression equation:

$$
\begin{cases} y = \beta_0 + \beta_1 x + \varepsilon \\ \mathbb{E}[\varepsilon] = 0, \quad \mathrm{Var}(\varepsilon) = \sigma^2 \end{cases}
$$

Parameter estimates:

$$
\begin{cases} \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \\ \hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{cases}
$$

Confidence intervals:

$$
\begin{cases} \beta_0 : \left[\hat{\beta}_0 - t_{1-\alpha/2}(n-2)\hat{\sigma}_e \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{L_{xx}}}, \hat{\beta}_0 + t_{1-\alpha/2}(n-2)\hat{\sigma}_e \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{L_{xx}}}\right] \\ \beta_1 : \left[\hat{\beta}_1 - \frac{t_{1-\alpha/2}(n-2)\hat{\sigma}_e}{\sqrt{L_{xx}}}, \hat{\beta}_1 + \frac{t_{1-\alpha/2}(n-2)\hat{\sigma}_e}{\sqrt{L_{xx}}}\right] \end{cases}
$$

Sum of squares:

$$
\begin{aligned}
\mathrm{TSS} &= \sum y_i^2 = \sum (Y_i - \bar{Y})^2 \\
\mathrm{ESS} &= \sum \hat{y}_i^2 = \sum (\hat{Y}_i - \bar{Y})^2 \\
\mathrm{RSS} &= \sum e_i^2 = \sum (Y_i - \hat{Y}_i)^2
\end{aligned}
$$

Goodness of fit:

$$
R^2 = \frac{\mathrm{ESS}}{\mathrm{TSS}} = 1 - \frac{\mathrm{RSS}}{\mathrm{TSS}}
$$

F-test:

$$
F = \frac{U}{Q_e/(n-2)} \sim F(1, n-2), \quad \begin{cases} U = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 \\ Q_e = \sum_{i=1}^n (y_i - \hat{y}_i)^2 \end{cases}
$$

Decision rule:

$$
\begin{cases} F > F_{1-\alpha}(1, n-2) : \text{Reject } H_0 \\ F \leq F_{1-\alpha}(1, n-2) : \text{Accept } H_0 \end{cases}
$$

Prediction interval:

$$
\left(\hat{y} - \hat{\sigma}_e \sqrt{1 + \sum_{i=0}^k \sum_{j=0}^k c_{i,j} x_i x_j}\, t_{1-\alpha/2}(n-k-1), \hat{y} + \hat{\sigma}_e \sqrt{1 + \sum_{i=0}^k \sum_{j=0}^k c_{i,j} x_i x_j}\, t_{1-\alpha/2}(n-k-1)\right)
$$

# Principal Component Analysis

Covariance:

$$
\mathrm{cov}(x, y) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})
$$

Correlation coefficient:

$$
\mathrm{corr}(x_j, x_k) = \frac{\sum_{i=1}^n (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k)}{\sqrt{\sum_{i=1}^n (x_{ij} - \bar{x}_j)^2 \sum_{i=1}^n (x_{ik} - \bar{x}_k)^2}} = \frac{\mathrm{cov}(x_j, x_k)}{\sqrt{\mathrm{var}(x_j)\mathrm{var}(x_k)}}
$$

Data matrix:

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{bmatrix}$$

Standardization methods:

$$\begin{cases} x_{ij}^* = \frac{x_{ij} - \bar{x}_j}{\sigma_j}, & i = 1, \ldots, n; j = 1, \ldots, m \\ x_{ij}^* = \frac{x_{ij} - \min(x_j)}{\max(x_j) - \min(x_j)}, & i = 1, \ldots, n; j = 1, \ldots, m \end{cases}$$

Variance contribution rate:

$$e_k = \frac{\mathrm{var}(F_k)}{\sum_{i=1}^{p} \mathrm{var}(F_i)} = \frac{\lambda_k}{\sum_{i=1}^{p} \lambda_i}$$

## Differential Equations

Numerical integration (Trapezoidal rule):

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} f(t, y(t))\, dt \approx \frac{x_{i+1} - x_i}{2} \left[ f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1})) \right]$$

Numerical solution:

$$\begin{cases} y_{i+1} = y_i + \frac{h}{2} \left[ f(x_i, y_i) + f(x_{i+1}, y_{i+1}) \right] \\ y_0 = y(x_0) \end{cases}$$

## Time Series

Autocovariance:

$$\gamma_X(t, s) = \mathbb{E}[(X_t - \mu_X(t))(X_s - \mu_X(s))], \quad t, s \in T$$

Autocorrelation:

$$\rho_X(t, s) = \frac{\gamma_X(t, s)}{\sqrt{D_X(t) D_X(s)}}, \quad t, s \in T$$

Sample autocovariance:

$$\hat{\gamma}(k) = \frac{1}{n} \sum_{i=1}^{n-k} (x_i - \bar{x})(x_{i+k} - \bar{x})$$

Sample autocorrelation:

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} = \frac{\sum_{i=1}^{n-k} (x_i - \bar{x})(x_{i+k} - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

MA(q) model autocorrelation:

$$\rho_k = \begin{cases} 1, & k = 0 \\ \frac{-\theta_k + \sum_{i=1}^{q-k} \theta_i \theta_{k+i}}{1 + \sum_{i=1}^{q} \theta_i^2}, & 1 \leq k \leq q \\ 0, & k > q \end{cases}$$

Model selection criteria:

$$\mathrm{FPE}_p = \frac{n + p}{n - p} \hat{\sigma}^2$$

$$\mathrm{AIC} = \ln(\hat{\sigma}^2) + \frac{2}{n}(p + q + 1)$$

where $\hat{\sigma}^2$ is the variance estimate, $n$ is sample size, and $p, q$ are model orders.