Chapter 2
Solving Linear Programs

Companion slides of
Applied Mathematical Programming
by Bradley, Hax, and Magnanti
(Addison-Wesley, 1977)

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A systematic procedure for solving linear programs – the simplex method

- Proceeds by moving from one feasible solution to another, at each step improving the value of the objective function.
- Terminates after a finite number of such transitions.
- Two important characteristics of the simplex method:
  - The method is robust.
    - It solves any linear program;
    - it detects redundant constraints in the problem formulation;
    - it identifies instances when the objective value is unbounded over the feasible region; and
    - It solves problems with one or more optimal solutions.
    - The method is also self-initiating.
      - It uses itself either to generate an appropriate feasible solution, as required, to start the method, or to show that the problem has no feasible solution.
  - The simplex method provides much more than just optimal solutions.
    - it indicates how the optimal solution varies as a function of the problem data (cost coefficients, constraint coefficients, and righthand-side data).
    - information intimately related to a linear program called the dual to the given problem: **the simplex method automatically solves this dual problem along with the given problem**.

SIMPLEX METHOD — A PREVIEW

The canonical form

$$\text{Maximize } z = 0x_1 + 0x_2 - 3x_3 - \ x_4 + 20,$$

Objective function 1

subject to:

$$x_1 \qquad\qquad - 3x_3 + 3x_4 = 6, \qquad (1)$$

$$x_2 - 8x_3 + 4x_4 = 4, \qquad (2)$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

Any linear programming problem can be transformed so that it is in canonical form!

1. All decision variables are constrained to be nonnegative.
2. All constraints, except for the nonnegativity of decision variables, are stated as equalities.
3. The righthand-side coefficients are all nonnegative.
4. One decision variable is isolated in each constraint with a +1 coefficient (*x*1 in constraint (1) and *x*2 in constraint (2)). The variable isolated in a given constraint does not appear in any other constraint, and appears with a zero coefficient in the objective function.

Discussion

- Given any values for $x3$ and $x4$, the values of $x1$ and $x2$ are determined uniquely by the equalities.
  - In fact, setting $x3 = x4 = 0$ immediately gives a feasible solution with $x1 = 6$ and $x2 = 4$.
  - Solutions such as these will play a central role in the simplex method and are referred to as *basic feasible solutions*.
- In general, given a canonical form for any linear program, a basic feasible solution is given by setting the variable isolated in constraint j , called the j th *basic-variable*, equal to the righthand side of the j th constraint and by setting the remaining variables, called *nonbasic*, all to zero.
  Collectively the basic variables are termed a *basis*.

Discussion

- In the example above, the basic feasible solution
  $x1 = 6$, $x2 = 4$, $x3 = 0$, $x4 = 0$, is optimal.
  - For any other feasible solution, $x3$ and $x4$ must remain nonnegative.
  - Since their coefficients in the objective function are negative, if either $x3$ or $x4$ is positive, z will be less than 20.
  - Thus the maximum value for z is obtained when $x3 = x4 = 0$.

Optimality Criterion

- Suppose that, in a maximization problem, every nonbasic variable has a nonpositive coefficient in the objective function of a canonical form.
- Then the basic feasible solution given by the canonical form maximizes the objective function over the feasible region.

Unbounded Objective Value

$$\text{Maximize } z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20,$$

Objective function 2

subject to:

$$x_1 \qquad - 3x_3 + 3x_4 = 6, \qquad (1)$$

$$x_2 - 8x_3 + 4x_4 = 4, \qquad (2)$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

- Since $x3$ now has a positive coefficient in the objective function, it appears promising to increase the value of $x3$ as much as possible.
- Let us maintain $x4 = 0$, increase $x3$ to a value t to be determined, and update $x1$ and $x2$ to preserve feasibility.

Discussion

$$\begin{aligned} x_1 \qquad\qquad - 3x_3 + 3x_4 &= 6, \\ x_2 - 8x_3 + 4x_4 &= 4, \\ z = 0x_1 + 0x_2 + 3x_3 - \ \ x_4 + 20, & \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 \qquad\qquad &= \ 6 + 3t, \\ x_2 \qquad &= \ 4 + 8t, \\ z &= 20 + 3t. \end{aligned}$$

- No matter how large *t* becomes, *x*1 and *x*2 remain nonnegative. In fact, as *t* approaches +∞, *z* approaches + ∞.
- In this case, the objective function is unbounded over the feasible region.

Unboundedness Criterion

- Suppose that, in a maximization problem, some nonbasic variable has a positive coefficient in the objective function of a canonical form.
- If that variable has negative or zero coefficients in all constraints, then the objective function is unbounded from above over the feasible region.

Improving a Nonoptimal Solution

$$\text{Maximize } z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20,$$

Objective function 3

subject to:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6, \\ x_2 - 8x_3 + 4x_4 &= 4, \end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

- As *x*4 increases, *z* increases.
- Maintaining *x*3 = 0, let us increase *x*4 to a value *t*, and update *x*1 and *x*2 to preserve feasibility.

Discussion

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6, \\ x_2 - 8x_3 + 4x_4 &= 4, \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 \qquad &= 6 - 3t, \\ x_2 &= 4 - 4t, \end{aligned}$$

$$z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20, \qquad\qquad z = 20 + t.$$

If $x1$ and $x2$ are to remain nonnegative, we require:

$$6 - 3t \geq 0, \qquad \text{that is, } t \leq \tfrac{6}{3} = 2$$

and

$$4 - 4t \geq 0, \qquad \text{that is, } t \leq \tfrac{4}{4} = 1.$$

Therefore, the largest value for t that maintains a feasible solution is $t = 1$.

When $t = 1$, the new solution becomes $x1 = 3$, $x2 = 0$, $x3 = 0$, $x4 = 1$, which has an associated value of $z = 21$ in the objective function.

Discussion

- Note that, in the new solution, $x4$ has a positive value and $x2$ has become zero.
- Since nonbasic variables have been given zero values before, it appears that $x4$ has replaced $x2$ as a basic variable.
- In fact, it is fairly simple to manipulate Eqs. (1) and (2) algebraically to produce a new canonical form, where $x1$ and $x4$ become the basic variables.

Discussion

- If *x*4 is to become a basic variable, it should appear with coefficient +1 in Eq. (2), and with zero coefficients in Eq. (1) and in the objective function.
- To obtain a +1 coefficient in Eq. (2), we divide that equation by 4.

$$
\begin{array}{lrcl}
(1) & x_1 \qquad\qquad - 3x_3 + 3x_4 = & 6, \\
(2) & x_2 - 8x_3 + 4x_4 = & 4,
\end{array}
\quad \rightarrow \quad
\begin{array}{rcl}
x_1 \qquad\qquad - 3x_3 + 3x_4 = & 6, \\
\frac{1}{4}x_2 - 2x_3 + x_4 = & 1.
\end{array}
$$

Discussion

- To eliminate *x*4 from the first constraint, we may multiply Eq. (2') by 3 and subtract it from constraint (1).

(1) $x_1 \qquad - 3x_3 + 3x_4 = 6,$

(2') $\frac{1}{4}x_2 - 2x_3 + x_4 = 1.$

$$\rightarrow$$

$$x_1 - \tfrac{3}{4}x_2 + 3x_3 = 3,$$
$$\tfrac{1}{4}x_2 - 2x_3 + x_4 = 1.$$

- We may rearrange the objective function and write it as:

$$(-z) \qquad - 3x_3 + x_4 = -20$$

and use the same technique to eliminate *x*4; that is, multiply (20) by –1 and add to Eq. (1):

$$(-z) - \tfrac{1}{4}x_2 - x_3 = -21.$$

The new global system becomes

$$\text{Maximize } z = 0x_1 - \tfrac{1}{4}x_2 - \quad x_3 + 0x_4 + 21,$$

subject to:

$$\begin{aligned} x_1 - \tfrac{3}{4}x_2 + 3x_3 \qquad\quad &= 3, \\ \tfrac{1}{4}x_2 - 2x_3 + \quad x_4 &= 1, \end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

This procedure for generating a new basic variable is called *pivoting*

- Now the problem is in canonical form with $x1$ and $x4$ as basic variables, and z has increased from 20 to 21.
- Consequently, we are in a position to reapply the arguments of this section, beginning with this improved solution.
- However, in this case, the new canonical form satisfies the optimality criterion since all nonbasic variables have nonpositive coefficients in the objective function, and thus the basic feasible solution $x1 = 3$, $x2 = 0$, $x3 = 0$, $x4 = 1$, is optimal.

Improvement Criterion

- Suppose that, in a maximization problem, some nonbasic variable has a positive coefficient in the objective function of a canonical form.
- If that variable has a positive coefficient in some constraint, then a new basic feasible solution may be obtained by pivoting.

Discussion

- Recall that we chose the constraint to pivot in (and consequently the variable to drop from the basis) by determining *which basic variable* first goes to zero as we increase the nonbasic variable $x4$.
- The constraint is selected by taking the ratio of the righthand-side coefficients to the coefficients of $x4$ in the constraints, i.e., by performing the *ratio test*:

$$\min\left\{\frac{6}{3}, \frac{4}{4}\right\}$$

Discussion

- Note, however, that if the coefficient of $x4$ in the second constraint were −4 instead of +4, the values for $x1$ and $x2$ would be given by:

$$\begin{aligned} x_1 \qquad\qquad - 3x_3 + 3x_4 &= 6, \\ x_2 - 8x_3 - 4x_4 &= 4, \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 \qquad &= 6 - 3t, \\ x_2 &= 4 + 4t, \end{aligned}$$

  so that as $x4 = t$ increases from 0, $x2$ never becomes zero. In this case, we would increase $x4$ to $t = 6/3 = 2$.
- This observation applies in general for any number of constraints, so that we need never compute ratios for nonpositive coefficients of the variable that is coming into the basis.

Ratio and Pivoting Criterion

- When improving a given canonical form by introducing variable xs into the basis, pivot in a constraint that gives the minimum ratio of righthand-side coefficient to corresponding xs coefficient.
- Compute these ratios only for constraints that have a positive coefficient for xs .

REDUCTION TO CANONICAL FORM – PART 1

Reduction to Canonical Form

- To this point we have been solving linear programs posed in canonical form with
  - (1) nonnegative variables,
  - (2) equality constraints,
  - (3) nonnegative righthand-side coefficients, and
  - (4) one basic variable isolated in each constraint.
- We will now show how to transform any linear program to this canonical form.

Inequality constraints

$$40x_1 + 10x_2 + 6x_3 \leq 55.0,$$
$$40x_1 + 10x_2 + 6x_3 \geq 32.5.$$

Introduce two new nonnegative variables:

- *x*5 measures the amount that the consuption of resource falls *short* of the maximum available, and is called a *slack variable*;
- *x*6 is the amount of product in *excess* of the minimum requirement and is called a *surplus variable*.

$$\begin{aligned} 40x_1 + 10x_2 + 6x_3 \quad + x_5 \quad &= 55.0, \\ 40x_1 + 10x_2 + 6x_3 \quad - x_6 &= 32.5. \end{aligned}$$

SIMPLEX METHOD
—A SIMPLE EXAMPLE

A simple example

The owner of a shop producing automobile trailers wishes to determine the best mix for his three products: flat-bed trailers, economy trailers, and luxury trailers. His shop is limited to working 24 days/month on metalworking and 60 days/month on woodworking for these products. The following table indicates production data for the trailers.

	Usage per unit of trailer			*Resources availabilities*
	Flat-bed	*Economy*	*Luxury*	
Metalworking days	$\frac{1}{2}$	2	1	24
Woodworking days	1	2	4	60
Contribution ($\$ \times 100$)	6	14	13	

LP Model

Let the decision variables of the problem be:

$x_1 =$ Number of flat-bed trailers produced per month,

$x_2 =$ Number of economy trailers produced per month,

$x_3 =$ Number of luxury trailers produced per month.

$$\text{Maximize } z = 6x_1 + 14x_2 + 13x_3,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}x_1 + 2x_2 + x_3 &\le 24, \\ x_1 + 2x_2 + 4x_3 &\le 60, \end{aligned}$$

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Canonical form

$$\text{Maximize } z = 6x_1 + 14x_2 + 13x_3,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}x_1 + 2x_2 + x_3 + x_4 &= 24, \\ x_1 + 2x_2 + 4x_3 + x_5 &= 60, \\ x_j \geq 0 \qquad (j = 1, 2, \ldots, 5). \end{aligned}$$

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_4	24	$\frac{1}{2}$	2	1	1	
x_5	60	1	2	4		1
$(-z)$	0	+6	+14	+13		

$$(-z) + 6x_1 + 14x_2 + 13x_3 = 0.$$

Iterations

Tableau 1

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
x_4	24	$\frac{1}{2}$	2	1	1		$\boxed{1}$	24/2
x_5	60	1	2	4		1	$\boxed{2}$	60/2
$(-z)$	0	$+6$	$+14$	$+13$			$\boxed{3}$	

Tableau 2

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
x_2	12	$\frac{1}{4}$	1	$\frac{1}{2}$	$\frac{1}{2}$		$\boxed{4} = \frac{1}{2}\boxed{1}$	12/(1/2)
x_5	36	$\frac{1}{2}$		3	-1	1	$\boxed{5} = \boxed{2} - 2\boxed{4}$	36/3
$(-z)$	-168	$+\frac{5}{2}$		$+6$	-7		$\boxed{6} = \boxed{3} - 14\boxed{4}$	

Iterations

Tableau 3

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
← x_2	6	$\frac{1}{6}$	1		$\frac{2}{3}$	$-\frac{1}{6}$	$\boxed{7} = \boxed{4} - \frac{1}{2}\boxed{8}$	$6/(1/6)$
x_3	12	$\frac{1}{6}$		1	$-\frac{1}{3}$	$\frac{1}{3}$	$\boxed{8} = \frac{1}{3}\boxed{5}$	$12/(1/6)$
$(-z)$	-240	$+\frac{3}{2}$			-5	-2	$\boxed{9} = \boxed{6} - 6\boxed{8}$	
		↑						

Tableau 4

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations
x_1	36	1	6		4	-1	$\boxed{10} = 6\boxed{7}$
x_3	6		-1	1	-1	$\frac{1}{2}$	$\boxed{11} = \boxed{8} - \frac{1}{6}\boxed{10}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$	$\boxed{12} = \boxed{9} - \frac{3}{2}\boxed{10}$

Minimization problems

- Enters the basis the nonbasic variable that has a negative coefficient in the objective function of a canonical form.
- The solution is optimal when every nonbasic variable has a nonnegative coefficient in the objective function of a canonical form.

Formal procedure

These steps apply to either the phase I or phase II problem.

Simplex Algorithm (Maximization Form)

STEP (0) The problem is initially in canonical form and all $\overline{b}_i \geq 0$.

STEP (1) If $\overline{c}_j \leq 0$ for $j = 1, 2, \ldots, n$, then *stop*; we are optimal. If we continue then there exists some $\overline{c}_j > 0$.

STEP (2) Choose the column to pivot in (i.e., the variable to introduce into the basis) by:

$$\overline{c}_s = \max_j \{\overline{c}_j \,|\overline{c}_j > 0\}.^*$$

If $\overline{a}_{is} \leq 0$ for $i = 1, 2, \ldots, m$, then *stop*; the primal problem is unbounded. If we continue, then $\overline{a}_{is} > 0$ for some $i = 1, 2, \ldots, m$.

STEP (3) Choose row r to pivot in (i.e., the variable to drop from the basis) by the ratio test:

$$\frac{\overline{b}_r}{\overline{a}_{rs}} = \min_i \left\{ \frac{\overline{b}_i}{\overline{a}_{is}} \,\middle|\, \overline{a}_{is} > 0 \right\}.$$

STEP (4) Replace the basic variable in row r with variable s and re-establish the canonical form (i.e., pivot on the coefficient $\overline{a}_{rs}$).

STEP (5) Go to step (1).

STEP (4) Pivoting

$$
\begin{array}{ccccc|ccccc|c}
x_1 & \cdots & x_r & \cdots & x_m & x_{m+1} & \cdots & x_s & \cdots & x_n & \\
\hline
1 & & & & & \bar{a}_{1,\,m+1} & \cdots & \bar{a}_{1s} & \cdots & \bar{a}_{1n} & \bar{b}_1 \\
& \ddots & & & & \vdots & & & & \vdots & \vdots \\
& & 1 & & & \bar{a}_{r,\,m+1} & \cdots & \boxed{\bar{a}_{rs}} & \cdots & \bar{a}_{rn} & \bar{b}_r \\
& & & \ddots & & \vdots & & & & \vdots & \vdots \\
& & & & 1 & \bar{a}_{m,\,m+1} & \cdots & \bar{a}_{ms} & \cdots & \bar{a}_{mn} & \bar{b}_m \\
\hline
& & & & & \bar{c}_{m+1} & \cdots & \bar{c}_s & \cdots & \bar{c}_n & -\bar{z}_0
\end{array}
$$

$\downarrow$ Normalization

$$
\begin{array}{ccccc|ccccc|c}
1 & & & & & \bar{a}_{1,\,m+1} & \cdots & \bar{a}_{1s} & \cdots & \bar{a}_{1n} & \bar{b}_1 \\
& \ddots & & & & \vdots & & & & \vdots & \vdots \\
& & \left(\dfrac{1}{\bar{a}_{rs}}\right) & & & \left(\dfrac{\bar{a}_{r,\,m+1}}{\bar{a}_{rs}}\right) & \cdots & 1 & \cdots & \left(\dfrac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \left(\dfrac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
& & & \ddots & & \vdots & & & & \vdots & \vdots \\
& & & & 1 & \bar{a}_{m,\,m+1} & \cdots & \bar{a}_{ms} & \cdots & \bar{a}_{mn} & \bar{b}_m \\
\hline
& & & & & \bar{c}_{m+1} & \cdots & \bar{c}_s & \cdots & \bar{c}_n & -\bar{z}_0
\end{array}
$$

STEP (4) Pivoting

1					$\bar{a}_{1,m+1}$	$\cdots$	$\bar{a}_{1s}$	$\cdots$	$\bar{a}_{1n}$	$\bar{b}_1$
	$\ddots$				$\vdots$				$\vdots$	$\vdots$
		$\left(\frac{1}{\bar{a}_{rs}}\right)$			$\left(\frac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right)$	$\cdots$	1	$\cdots$	$\left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right)$	$\left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right)$
			$\ddots$		$\vdots$				$\vdots$	$\vdots$
				1	$\bar{a}_{m,m+1}$	$\cdots$	$\bar{a}_{ms}$	$\cdots$	$\bar{a}_{mn}$	$\bar{b}_m$
					$\bar{c}_{m+1}$	$\cdots$	$\bar{c}_s$	$\cdots$	$\bar{c}_n$	$-\bar{z}_0$

$\downarrow$ Elimination of x_s

1		$-\left(\frac{\bar{a}_{1s}}{\bar{a}_{rs}}\right)$			$\bar{a}_{1,m+1} - \bar{a}_{1s}\left(\frac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right)$	$\cdots$	0	$\cdots$	$\bar{a}_{1n} - \bar{a}_{1s}\left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right)$	$\bar{b}_1 - \bar{a}_{1s}\left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right)$
	$\ddots$				$\vdots$				$\vdots$	$\vdots$
		$\left(\frac{1}{\bar{a}_{rs}}\right)$			$\left(\frac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right)$	$\cdots$	1	$\cdots$	$\left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right)$	$\frac{\bar{b}_r}{\bar{a}_{rs}}$
			$\ddots$		$\vdots$				$\vdots$	$\vdots$
		$-\left(\frac{\bar{a}_{ms}}{a_{rs}}\right)$		1	$\bar{a}_{m,m+1} - \bar{a}_{ms}\left(\frac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right)$	$\cdots$	0	$\cdots$	$\bar{a}_{mn} - \bar{a}_{ms}\left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right)$	$\bar{b}_m - \bar{a}_{ms}\left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right)$
		$-\left(\frac{\bar{c}_s}{\bar{a}_{rs}}\right)$			$\bar{c}_{m+1} - \bar{c}_s\left(\frac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right)$	$\cdots$	0	$\cdots$	$\bar{c}_n - \bar{c}_s\left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right)$	$-\bar{z}_0 - \bar{c}_s\left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right)$

REDUCTION TO CANONICAL FORM – PART 2

Reduction to Canonical Form

- To this point we have been solving linear programs posed in canonical form with
  - (1) nonnegative variables,
  - (2) equality constraints,
  - (3) nonnegative righthand-side coefficients, and
  - (4) one basic variable isolated in each constraint.
- We will now show how to transform any linear program to this canonical form.

Free variables

$$\underset{\begin{pmatrix}\text{Production}\\ \text{in period } t\end{pmatrix}}{x_t} + \underset{\begin{pmatrix}\text{Inventory}\\ \text{from period } (t-1)\end{pmatrix}}{I_{t-1}} = \underset{\begin{pmatrix}\text{Demand in}\\ \text{period } t\end{pmatrix}}{d_t} + \underset{\begin{pmatrix}\text{Inventory at}\\ \text{end of period } t\end{pmatrix}}{I_t}.$$

- Production x_t must be nonnegative.
- However, inventory I_t may be positive or negative, indicating either that there is a surplus of goods to be stored or that there is a shortage of goods and some must be produced later.
- To formulate models with free variables, we introduce two nonnegative variables I_t^+ and I_t^-, and write, as a substitute for I_t everywhere in the model:

$$I_t = I_t^+ - I_t^-$$

Artificial variables

- To obtain a canonical form, we must make sure that, in each constraint, one basic variable can be isolated with a +1 coefficient.
- Some constraints already will have this form (for example, slack variables have always this property).
- When there are no “volunteers” to be basic variables we have to resort to *artificial variables*.

Artificial variables

new nonnegative
basic variable

- Add a new (completely fictitious) variable to any equation that requires one:

$$40x_1 + 10x_2 + 6x_3 - x_6 = 32.5 \rightarrow 40x_1 + 10x_2 + 6x_3 - x_6 + x_7 = 32.5,$$

- Any solution with *x*7 = 0 is feasible for the original problem, but those with *x*7 > 0 are not feasible.
- Consequently, we should attempt to drive the artificial variable to zero.

Driving an artificial variable to zero – the *big M method*

- In a minimization problem, this can be accomplished by attaching a high unit cost M (>0) to x_7 in the objective function
  - For maximization, add the penalty $-M\,x_7$ to the objective function).
- For M sufficiently large, x_7 will be zero in the final linear programming solution, so that the solution satisfies the original problem constraint without the artificial variable.
- If $x_7 > 0$ in the final tableau, then there is no solution to the original problem where the artificial variables have been removed; that is, we have shown that the problem is infeasible.

artificial variables $\neq$ slack variables

Driving an artificial variable to zero – the *phase I–phase II procedure*

- Phase I determines a canonical form for the problem by solving a linear program related to the original problem formulation.
  - New objective function minimizing the sum of the artificial variables.
- The second phase starts with this canonical form to solve the original problem.

The *phase I–phase II procedure*

$$\text{Maximize } z = -3x_1 + 3x_2 + 2x_3 - 2x_4 - x_5 + 4x_6,$$

subject to:

Slack variables added

$$\begin{array}{rcl}
x_1 - x_2 + x_3 - x_4 - 4x_5 + 2x_6 - x_7 + x_9 \phantom{+ x_{10} + x_{11}} & = & 4, \\
-3x_1 + 3x_2 + x_3 - x_4 - 2x_5 + x_8 \phantom{+ x_9 + x_{10} + x_{11}} & = & 6, \\
-x_3 + x_4 + x_6 + x_{10} \phantom{+ x_{11}} & = & 1, \\
x_1 - x_2 + x_3 - x_4 - x_5 \phantom{+ x_6 - x_7 + x_8 + x_9 + x_{10}} + x_{11} & = & 0,
\end{array}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, 11).$$

Artificial variables added

- Any feasible solution to the augmented system with all artificial variables equal to zero provides a feasible solution to the original problem.
- Since the artificial variables *x*9, *x*10, and *x*11 are all nonnegative, they are all zero only when their sum *x*9 + *x*10 + *x*11 is zero.
- Consequently, the artificial variables can be eliminated by ignoring the original objective function for the time being and minimizing *x*9 + *x*10 + *x*11 (i.e., minimizing the sum of all artificial variables).

The *phase I–phase II procedure*

- If the minimum sum is 0, then the artificial variables are all zero and a feasible, but not necessarily optimal, solution to the original problem has been obtained.
- If the minimum is greater than zero, then every solution to the augmented system has $x9 + x10 + x11 > 0$, so that *some* artificial variable is still positive. In this case, the original problem has no feasible solution.

Phase 1 model

$$\text{Maximize} \quad w = \quad - x_9 - x_{10} - x_{11} \quad \Leftrightarrow$$

$$w = 2x_1 - 2x_2 + x_3 - x_4 - 5x_5 + 3x_6 - x_7 + 0x_9 + 0x_{10} + 0x_{11} - 5.$$

subject to:

$$\begin{aligned}
x_1 - x_2 + x_3 - x_4 - 4x_5 + 2x_6 - x_7 \quad + x_9 \qquad &= 4, \\
-3x_1 + 3x_2 + x_3 - x_4 - 2x_5 \qquad + x_8 \qquad &= 6, \\
- x_3 + x_4 \quad + x_6 \qquad + x_{10} \quad &= 1, \\
x_1 - x_2 + x_3 - x_4 - x_5 \qquad + x_{11} &= 0,
\end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, 11).$$

- The artificial variables have zero coefficients in the phase I objective: they are basic variables.
- Note that the initial coefficients for the nonartificial variable *xj* in the w equation is the sum of the coefficients of *xj* from the equations with an artificial variable.

Iterations

Initial tableau

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	Equation identification and transformations	Ratio test
										Artificial variables				
x_9	4	1	−1	1	−1	−4	2	−1		1			$\boxed{1}$	4/2
x_8	6	−3	3	1	−1	−2	0	0	1				$\boxed{2}$	
← x_{10}	1	0	0	−1	1	0	1	0			1		$\boxed{3}$	1/1
x_{11}	0	1	−1	1	−1	−1	0	0				1	$\boxed{4}$	
$(-z)$	0	−3	3	2	−2	−1	4	0					$\boxed{5}$	
$(-w)$	5	2	−2	1	−1	−5	3	−1					$\boxed{6}$	
							↑							

Tableau 2

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	Equation identification and transformations	Ratio test
x_9	2	1	−1	3	−3	−4		−1		1	−2		$\boxed{7} = \boxed{1} - 2\boxed{3}$	2/3
x_8	6	−3	3	1	−1	−2		0	1		0		$\boxed{8} = \boxed{2}$	6/1
x_6	1	0	0	−1	1	0	1	0			1		$\boxed{9} = \boxed{3}$	
← x_{11}	0	1	−1	1	−1	−1		0			0	1	$\boxed{10} = \boxed{4}$	0/1
$(-z)$	−4	−3	3	6	−6	−1		0			−4		$\boxed{11} = \boxed{5} - 4\boxed{3}$	
$(-w)$	2	2	−2	4	−4	−5		−1			−3		$\boxed{12} = \boxed{6} - 3\boxed{3}$	
				↑										

Iterations

Tableau 3

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}		
x_9	2	-2	2		0	-1		-1		1	-2	-3	$\boxed{13} = \boxed{7} - 3\boxed{10}$	2/2
x_8	6	-4	4		0	-1		0	1		0	-1	$\boxed{14} = \boxed{8} - \boxed{10}$	6/4
x_6	1	1	-1		0	-1	1	0			1	1	$\boxed{15} = \boxed{9} + \boxed{10}$	
x_3	0	1	-1	1	-1	-1		0			0	1	$\boxed{16} = \boxed{10}$	
$(-z)$	-4	-9	9		0	5		0			-4	-6	$\boxed{17} = \boxed{11} - 6\boxed{10}$	
$(-w)$	2	-2	2		0	-1		-1			-3	-4	$\boxed{18} = \boxed{12} - 4\boxed{10}$	

Tableau 4

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}		
x_2	1	-1	1		0	$-\frac{1}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$	-1	$-\frac{3}{2}$	$\boxed{19} = \frac{1}{2}\boxed{13}$	
x_8	2	0			0	1		2	1	-2	4	5	$\boxed{20} = \boxed{14} - 4\boxed{19}$	2/1
x_6	2	0			0	$-\frac{3}{2}$	1	$-\frac{1}{2}$		$\frac{1}{2}$	0	$-\frac{1}{2}$	$\boxed{21} = \boxed{15} + \boxed{19}$	
x_3	1	0		1	-1	$-\frac{3}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$	-1	$-\frac{1}{2}$	$\boxed{22} = \boxed{16} + \boxed{19}$	
$(-z)$	-13	0			0	$\frac{19}{2}$		$\frac{9}{2}$		$-\frac{9}{2}$	5	$\frac{15}{2}$	$\boxed{23} = \boxed{17} - 9\boxed{19}$	
$(-w)$	0	0			0	0		0		-1	-1	-1	$\boxed{24} = \boxed{18} - 2\boxed{19}$	

End of phase I. All artificial variables are nonbasic, so proceed with phase II, dropping the w-equation and maintaining $x_9 = x_{10} = x_{11} = 0$ (i.e., never introduce an artificial variable into the basis).

Iterations

Final tableau

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
x_2	2	-1	1		0			$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	1
x_5	2	0			0	1		2	1	-2	4	5
x_6	5	0			0		1	$\frac{5}{2}$	$\frac{3}{2}$	$-\frac{5}{2}$	6	7
x_3	4	0		1	-1			$\frac{5}{2}$	$\frac{3}{2}$	$-\frac{5}{2}$	5	7
$(-z)$	-32	0			0			$-\frac{29}{2}$	$-\frac{19}{2}$	$\frac{29}{2}$	-33	-40

$\boxed{25} = \boxed{19} + \frac{1}{2}\boxed{20}$

$\boxed{26} = \boxed{20}$

$\boxed{27} = \boxed{21} + \frac{3}{2}\boxed{20}$

$\boxed{28} = \boxed{22} + \frac{3}{2}\boxed{20}$

$\boxed{29} = \boxed{23} - \frac{19}{2}\boxed{20}$

End of phase II. Substituting for the original variables, the optimal solution is $y_1 = -2$, $y_2 = 4$, $y_3 = 2$, $y_4 = 5$, max $z = 32$.