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Module Design for Pilots Training with Full Flight Simulator

Abstract. This study addresses a practical problem in the design of training modules for pilots with Full Flight Simulators (FFS), using operations research techniques. FFSs are highly precise tools that accurately replicate real-world flight scenarios, making them indispensable for pilot training and evaluation. Given their high cost and limited availability, airlines must strategically design modules for pilots' periodic recurrent assessments. To enhance aviation safety, we formulate a combinatorial optimization objective function to select a set of modules that minimizes a pilot's lowest score, enabling early detection of potential weaknesses. Our research underscores the importance of accounting for correlations in pilots' performance across modules. To this end, we develop a suite of optimization methods and provide corresponding managerial insights. Extensive experimental evaluations demonstrate that our approach significantly outperforms traditional methods reliant on intuition and experience.

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Key words: module design, Full Flight Simulators, simulation optimization

1. Background

The Full Flight Simulator (FFS) is a highly sophisticated flight training device designed to replicate the cockpit environment and operational dynamics of a specific aircraft model with exceptional fidelity (ICA 2015). It is capable of simulating the entire spectrum of flight operations, from takeoff and cruise to landing, encompassing both routine procedures and complex scenarios such as engine failures, hydraulic system malfunctions, severe weather conditions, and even rare events like bird strikes or runway contamination (Adv 2024). The simulation integrates visual, auditory, and tactile feedback, alongside precise instrument readouts and control responses, to provide an immersive experience that closely mirrors actual flight conditions. Certified to the highest standards, such as Level D by aviation authorities like the Federal Aviation Administration (FAA) or the European Union Aviation Safety Agency (EASA), the FFS achieves unparalleled precision, enabling zero flight time training where pilots can qualify without operating an actual aircraft (Federal Aviation Administration 1995, European Union Aviation Safety Agency 2020). However, this extraordinary accuracy comes at a significant cost, with a single FFS unit priced in the tens of millions of dollars, rendering its manufacture and maintenance a substantial financial commitment (CAE 2023). Most airlines typically own only a limited number of FFS units outright or instead lease them from manufacturers such as CAE or L3Harris and dedicated flight training centers (IBA 2023).

The primary application of the FFS lies in pilot training and qualification maintenance, with a particular emphasis on mandatory recurrent training to renew licenses. Pursuant to regulations set by the International Civil Aviation Organization (ICAO) and national aviation authorities, pilots are required to undergo recurrent training every six to twelve months, typically conducted in an FFS, to validate their proficiency (Fli 2024).During qualification assessments, pilots are evaluated through a series of subjects designed to test their ability to manage diverse operational challenges. These subjects are carefully crafted based on real-world flight data and historical incident analyses(National Transportation Safety Board 2012). Common subjects include "Engine Failure on Takeoff," where pilots must stabilize the aircraft and execute a safe return to the airport following an engine malfunction immediately after liftoff; "Low Visibility Approach," which tests precision navigation and landing in simulated adverse weather; and "Emergency Descent and Decompression," requiring rapid altitude reduction and oxygen management in response to cabin pressure loss (Pil 2023). These assessments, which may span several hours, evaluate technical proficiency, decision-making speed, and crew coordination. Successful completion of these subjects results in license renewal, while deficiencies necessitate further training, ensuring consistent competency across the pilot workforce (European Union Aviation Safety Agency 2020).

Given the high cost of FFS and the critical need for regular pilot assessments, airlines must carefully balance budgetary constraints with the demand for efficient and effective evaluations. However, the current practice of selecting assessment subjects is often random or reliant on airlines' experiential judgment, which can fail to identify specific deficiencies or redundantly evaluate areas of established proficiency. This work aims to develop dedicated optimization approaches to enhance the selection of assessment subjects, maximizing the effectiveness of the evaluation process.

Suppose there is a pool of M different training modules from which m are selected in one training session due to cost restriction. Ideally, these m subjects should collectively cover a diverse range of skills to efficiently identify a pilot's weaknesses. Unlike conventional approaches, our strategy incorporates the correlations and variability in pilot performance across these modules, driven by individual differences in skills such as hand-eye coordination, spatial awareness, and stress resilience. For instance, pilots with strong coordination skills often excel in subjects like

"Precision Approach" and "Instrument Flying," as both require similar competencies, including accurate interpretation of flight instruments, stable aircraft control, and anticipation of spatial positioning (Damos 2003). As a result, high performance in one module frequently predicts success in the other due to shared skill requirements. Conversely, pilots with superior stress management capabilities may outperform in complex emergency scenarios such as "Multi-Engine Failure", but struggle in subjects like "Crosswind Landing" if their coordination is less developed (Szczepanska et al. 2025). These correlations arise from overlapping skill sets and selecting multiple highly correlated subjects may lead to redundant testing.Accounting for these correlations is therefore critical in optimizing the selection of assessment modules to enhance training efficiency and safety (Duruaku et al. 2024).

We illustrate this with a simple example, where we need to select two subjects out of three for a pilot's assessment. From an aviation safety perspective, we aim to select the two subjects that best identify the subject with the lowest score, and thus the assessment metric is the minimum score of the two selected subjects. From an optimization perspective, the following objective function is considered:

$$\min_{x_1,x_2} \mathbb{E}[\min\{Y(x_1),...,Y(x_2)\}], \tag{1}$$

where x_1, x_2 are two selected subjects from the subjects pool $\{A, B, C\}$ and $Y(x_i)$ is the score on x_i. The scores of each pilot on these three subjects follow a discrete distribution with three scenarios (shown in Table 1), each with probability 1/3. From the table, we can observe that the correlation between scores for subjects A and B is significantly higher than that between A and C or B and C, due to the differing skill requirements for the subjects, as previously discussed. If correlations are ignored, the first two subjects might be chosen (due to their lower mean scores). However, calculations reveal that the latter two subjects should be selected. This is precisely because their lower correlation allows them to more independently yield lower scores, increasing the likelihood of

Table 1 Discrete distribution of the scores

Scenario	A score	B score	C score
1	9	9.5	7.5
2	6	5	8
3	10	9.5	10
Expectation	8.33	8	8.5

identifying the pilot's weakest performance. Therefore, we need to design an optimization method that fully accounts for the correlations between scores of different subjects.

Addressing this problem presents multiple challenges. First, in practical applications, the expected values of these distributions are likely unknown. While we can infer correlations between scores based on the skill requirements of different subjects, estimating the mean scores of pilots across these subjects is considerably more difficult. Second, even when the parameters of the score distributions are fully known, computing the expected minimum value of several correlated distributions is inherently complex, particularly when selecting an optimal subset of mm modules from a larger set to minimize this value. This combinatorial optimization problem is computationally challenging and generally does not admit an exact solution, necessitating the use of approximation algorithms. Hence, an approach that simultaneously learns from data of previous test results and optimizes the selection process is necessary.

To address the first challenge of estimating score means, we develop a combinatorial multi-armed bandit (CMAB) algorithm based on a Bayesian framework that assigns a prior distribution to the score means and continuously updates it through iterative testing. This learning framework, however, critically relies on an α-approximation oracle to select near-optimal module combinations at each iteration based on the current estimates, which directly confronts the second computational challenge. To tackle the computational complexity challenge, we exploit the submodular structure of our objective function. This key property allows us to develop efficient greedy-style algorithms that serve as effective α-approximation oracles for module selection. By combining these two components—the Bayesian learning framework and α-approximation oracles for combinatorial selection—we create an integrated solution that simultaneously learns from evaluation data and optimizes the selection process.

This paper addresses a problem of significant practical relevance, representing an innovative effort to apply optimization and operations research algorithms to real-world challenges. Previously, the design of pilot training modules has relied heavily on intuition and experience. Beyond its practical significance, the optimization problem explored here also carries substantial theoretical value, presenting multiple challenges as outlined earlier. Moreover, while related to existing research domains, it exhibits notable distinctions. Our work primarily aligns with two categories of optimization studies. The first is simulation-based optimization (Fu 2015), which employs simulation models to support decision-making in real-world systems (FFS is a highly precise simulation tool). Certain simulation optimization studies account for correlations in experimental results when selecting multiple design alternatives (Xie et al. 2016). These correlations primarily arise from the use of common random numbers in computer simulations. However, no prior work in this field has targeted the same objective function as ours, as most focus on evaluating a single design's performance without an internal minimization operation. The second category is combinatorial multi-armed bandit research (Chen et al. 2013a), where some studies select multiple alternatives and use the best as the evaluation metric, akin to our objective function (1). Yet, to the best of our knowledge, none have addressed the complex reward structure in (1) while simultaneously accounting for correlations among alternatives. Although correlations in CMAB settings have been studied in prior works, these typically focus on simpler reward functions that depend only on marginal statistics rather than the full joint distribution of the selected arms.

The remainder of the paper is organized as follows. Section 2 reviews the related works. Section3 formulates the objective function. Section 4 presents our Bayesian Combinatorial Upper Confidence Bound (BCUCB) framework and its regret bound. Section 5 develops α-approximation oracles to tackle the computational challenge within the BCUCB framework. Section 6 reports numerical experiments. Finally, Section 7 concludes the paper.

2. Related works

We review relevant works in three aspects: existing approaches for pilot training modules design, simulation-based optimization, and combinatorial MAB.

Combinatorial MAB Our module selection problem naturally fits into the combinatorial multi-armed bandit (CMAB) framework with semi-bandit feedback, where an agent sequentially selects a subset of base arms (super arm) and observes outcomes from all selected arms (Chen et al. 2013b). The CMAB literature predominantly makes two simplifying assumptions: either base arms are *mutually independent* (Zhang and Combes 2025, Combes et al. 2015, Cuvelier et al. 2021), or the reward function is *linear* in base arm outcomes or depends only on *marginal means* (Demirel and Tekin 2021, Kveton et al. 2015). In contrast, our problem requires accounting for the full joint distribution's influence on the non-linear minimum-based reward function. Our work is most closely related to (Chen et al. 2016), who also consider the expected maximum of selected arms as the reward function but do not incorporate correlations between base arms. (Wang et al. 2023) also examines a similar maximum reward function but under a more restrictive full-bandit feedback setting with discrete outcomes, whereas we consider continuous distributions with semi-bandit feedback. In an orthogonal direction, (Degenne and Perchet 2016) considers correlated base arms with a known covariance matrix under a sub-Gaussian assumption, which aligns with our setting in terms of correlation structure, but focuses on linear reward functions, which fundamentally differ from our minimum-based reward structure that requires complete joint distribution information.

3. problem formulation

Our goal is to efficiently identify pilots' deficiencies through rational module design, with these deficiencies measured by their scores during the module evaluation process. Suppose there are M different training modules from which m are selected in one training session due to the time and cost restriction. Then, we select m modules to maximally expose a pilot's deficiencies, such that the minimum score across these m modules is as low as possible. The optimization problem can be formulated as follows:

$$\min_{x_1,...,x_m} \mathbb{E}_\xi[\min\{Y(x_1,\xi),...,Y(x_m,\xi)\}], \tag{2}$$

where $x_1,...,x_m$ represent the m modules in the training session, taken from the module set $\mathcal{X} = \{x_1,...,x_M\}$, and $Y(x_1,\xi)$ is the score the pilot obtain in module x_i. It is evaluated through simulation at FFS. In this model, we assume the score vector in the population $\{Y(x_1),...,Y(x_M)\}$ follows a multivariate normal distribution. For each specific person, his score is a sample from this distribution. In equation (2), the random vector ξ is used to illustrate the randomness in this multivariate normal distribution. The inner minimization is to find the minimum score one pilot obtain across the given m training modules. We take the minimum value as we need to find the weakness of the pilots considering flight safety. The outer minimization is the find the best set of training sessions in detecting the pilot inadequacy.

4. A Combinatorial MAB Algorithm for Unknown Score Means

4.1. Model Formulation

We assume that the score has Gaussian noise: $Y(x) = Z(x) + \mathcal{N}(0,\sigma(x))$ and the joint distribution for $Y(x)_{x\in\{\mathcal{X}\}}$ is a multivariate normal distribution with mean vector $\theta = [Z(x_1),...,Z(x_M)]^T$ and covariance matrix Λ. We use a Bayesian approach to model θ and assume a normal prior:

$$\theta \sim \mathcal{N}(\mu_0,\Sigma_0).$$

As with Xie et al. (2016), we assume Λ, μ_0, Σ_0 are known.

The ith entry of a vector v is denoted as $v(i)$ and the (i,j)th entry of a matrix M is denoted as $M(i,j)$. For an ordered collection of m alternatives $\mathbf{x} = (x_1,...,x_m)$ with element $x_i \in \{1,...,M\}$ for each i, we use $v(\mathbf{x})$ to denote the subvector of v with the ith entry equal to $v(x_i)$. We further denote by $M(\mathbf{x},\mathbf{x}')$ the m-by-m submatrix of M with the (i,j)th entry equal to $M(x_i,x_j')$.

We consider a situation where in each iteration t, one pilot will attend m different modules $\mathbf{x}_t = (x_{t,1},...,x_{t,m})^T$ and we obtain his score vector $\mathbf{y}_t = (Y(x_{t,1}),...,Y(x_{t,m}))^T$. The conditional distribution of $\mathbf{y}_t$ is:

$$\mathbf{y}_t|\theta,\mathbf{x}_t \sim \mathcal{N}(\theta(\mathbf{x}_t),\Lambda(\mathbf{x}_t,\mathbf{x}_t)).$$

Let $\mathbb{X}_t = (\mathbf{x}_1^T,...,\mathbf{x}_t^T)$ denote the concatenation of the design points of the previous t iteration and similarly $\mathbb{Y}_t = (\mathbf{y}_1^T,...,\mathbf{y}_t^T)^T$. Then, the posterior distribution for θ is:

$$\theta_n|\mathbb{X}_t,\mathbb{Y}_t \sim \mathcal{N}(\mu_t,\Sigma_t),$$

where for any vector $\mathbf{x} = (x_1,x_2,...,x_m)$,

$$\mu_t(\mathbf{x}) = \mu_0(\mathbf{x}) + \Sigma_0(\mathbf{x},\mathbb{X}_\mathbf{t})(\boldsymbol{\Sigma}_\mathbf{0}(\mathbb{X}_\mathbf{n},\mathbb{X}_\mathbf{t}) + \boldsymbol{\Gamma}_\mathbf{t})^{-\mathbf{1}}(\mathbb{Y}_\mathbf{t} - \mu_\mathbf{0}(\mathbb{X}_\mathbf{t})),$$

$$\Sigma_n(\mathbf{x},\mathbf{x}) = \Sigma_0(\mathbf{x},\mathbf{x}) - \Sigma_0(\mathbf{x},\mathbb{X}_\mathbf{t})(\boldsymbol{\Sigma}_\mathbf{0}(\mathbb{X}_\mathbf{t},\mathbb{X}_\mathbf{t}) + \boldsymbol{\Gamma}_\mathbf{t})^{-\mathbf{1}}\boldsymbol{\Sigma}_\mathbf{0}(\mathbb{X}_\mathbf{t},\mathbf{x}),$$

where Γ_t can be calculated from the observed samples since we know the covariance matrix Λ.

4.2. The BCUCB Algorithm

To address the challenge of unknown score means, we propose a Bayesian Combinatorial Upper Confidence Bound (BCUCB) algorithm. This algorithm iteratively maintains a bayesian posterior distribution over the mean scores $\boldsymbol{\theta}$ and leverages an α-approximation oracle for module selection.

The algorithm operates in two phases: an initial exploration phase that ensures each module is evaluated at least once, followed by the main optimization phase. At each iteration t, the algorithm constructs a lower confidence bound (LCB) estimate $\tilde{\boldsymbol{\mu}}_t = \boldsymbol{\mu}_{t-1} - \boldsymbol{\beta}_t \circ \boldsymbol{\sigma}_{t-1}$ based on the current posterior distribution. This LCB vector is then passed to an α-approximation oracle O that selects a module set S_t aiming to minimize the expected minimum score. The specific design and implementation of this oracle will be detailed in the next section. After evaluating the selected modules on a new pilot, the posterior distribution is updated with the observed scores. The complete procedure is summarized in Algorithm 1.

Algorithm 1 Bayesian Combinatorial Upper Confidence Bound (BCUCB)

```
Require: Module set $\mathcal{X}$; Covariance matrix $\boldsymbol{\Lambda}$; Prior parameters $\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0$; Oracle $O$; Iterations $N_0, N$
 1: Initialize $\boldsymbol{\mu}_0$, $\boldsymbol{\Sigma}_0$ and count vector $\mathbf{T}_0 \leftarrow \mathbf{0}$
 2: for $i = 1$ to $N_0$ do                                                    ▷ Initial exploration phase
 3:     Play a super arm $S_i$ that contains module $i$
 4:     Observe $\mathbf{y}_i$,Update $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{T}_i)$,Calculate $R(S_i)$          ▷ Same as steps 11, 12, and 13
 5: end for
 6: for $t = N_0 + 1$ to $N$ do                                                ▷ Main optimization phase
 7:     $\boldsymbol{\sigma}_{t-1} \leftarrow \sqrt{\mathrm{diag}(\boldsymbol{\Sigma}_{t-1})}$                        ▷ Compute posterior standard deviations
 8:     $\boldsymbol{\beta}_t[x] \leftarrow \sqrt{6 \ln t / \mathbf{T}_{t-1}[x]}$ for each $x \in \mathcal{X}$  ▷ $\mathbf{T}_{t-1}[x]$ represents the number of times module
    $x$ was observed before round $t-1$
 9:     $\tilde{\boldsymbol{\mu}}_t \leftarrow \boldsymbol{\mu}_{t-1} - \boldsymbol{\beta}_t \circ \boldsymbol{\sigma}_{t-1}$                    ▷ Construct LCB-based mean estimate
10:     $S_t \leftarrow O(\tilde{\boldsymbol{\mu}}, \boldsymbol{\Lambda})$                                  ▷ Query $\alpha$-approximation oracle
11:     Observe score vector $\mathbf{y}_t$ for modules $S_t$ on a new pilot.
12:     Update $(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$ with $(\mathbf{y}_t, S_t)$; Update $\mathbf{T}_t[x] \leftarrow \mathbf{T}_t[x] + 1$ for each $x \in S_t$
13:     Calculate $R(S_t) = \mathbb{E}[\min\{Y(x) : x \in S_t\}]$, track minimal $R(S_t)$ and corresponding $S_t$
14: end for
15: return $S^*$ with minimal $R(S^*)$
```

4.3. Regret bounds

We establish both distribution-dependent and distribution-independent regret bounds for Algorithm 1, proving $O(\log T)$ and $O(\sqrt{T \log T})$ rates respectively.

For any approximation parameter $0 < \alpha < 1$, we define a super arm S as *bad* if $R(S) > \alpha R(S^*)$, where S^* denotes the true optimal super arm. The suboptimality gap for each super arm S is defined as:

$$\Delta_S = \max\{R(S) - \alpha \cdot R(S^*), 0\}.$$

Let $\mathcal{F}_B = \{S \in \mathcal{F} \mid \Delta_S > 0\}$ denote the set of all bad super arms, and let $E_B \subseteq \mathcal{X}$ be the set of modules contained in at least one bad super arm. For each module $i \in E_B$, we define its minimal suboptimality gap as:

$$\Delta_{i,\min} = \min\{\Delta_S \mid S \in \mathcal{F}_B, i \in S\}.$$

Let σ^2 represent the maximum variance among all modules (the largest diagonal element of covariance matrix $\mathbf{\Lambda}$). Recall that $m = |S|$ is the cardinality of the selected super arm, and $y_{\max}$ is the maximum possible score for any module.

THEOREM 1. *The α-approximation regret of Algorithm* **??** *over T rounds satisfies the following upper bounds:*

1. ***Distribution-dependent bound:***

$$\sigma^2 m \sum_{i \in E_B} \frac{2136}{\Delta_{i,\min}} \ln T + \left(\frac{\pi^2}{6} + 1\right) \alpha y_{max} M.$$

2. ***Distribution-independent bound:***

$$93\sigma\sqrt{MmT\ln T} + \left(\frac{\pi^2}{6} + 1\right) \alpha y_{max} M.$$

5. Design of α-Approximation Oracles

This section details the design of the α-approximation oracle O used in the BCUCB algorithm. The oracle's purpose is to efficiently select a set of m modules that (approximately) minimizes the expected minimum score given current parameter estimates. Our approach leverages the submodular structure of the objective function to develop oracles with theoretical guarantees.

5.1. Submodularity and Greedy Approximation

We first recall the definition of submodularity. For a finite set $\mathcal{V}$ (the set of all training modules $\mathcal{X}$ in our context), a set function $F : 2^{\mathcal{V}} \to \mathbb{R}$ is **submodular** if it satisfies the diminishing returns property: for any subsets $S \subseteq T \subseteq \mathcal{V}$ and any element $x \in \mathcal{V} \setminus T$,

$$F(S \cup \{x\}) - F(S) \geq F(T \cup \{x\}) - F(T).$$

This means the marginal gain of adding a new element x to a set S is at least as large as the marginal gain of adding the same element to a superset T of S.

In our problem, we define the set function for a module subset $S \subseteq \mathcal{X}$ as $R(S) = \mathbb{E}[\min\{Y(x) : x \in S\}]$. Our goal is to minimize $R(S)$. To frame this for a greedy algorithm that maximizes a utility, we define a utility function $U(S) = y_{\max} - R(S)$. Here $y_{\max}$ is the maximum score a pilot could achieve on any module. Then we give the following theorem:

THEOREM 2. *Given that the module score vector $\mathbf{Y} = (Y(x_1,\xi),\ldots,Y(x_m,\xi))$ follows a multivariate normal distribution $\mathcal{N}(\boldsymbol{\mu},\mathbf{\Sigma})$. For $S \subseteq \mathcal{X}$, the set function $U(S) = y_{max} - R(S) = y_{max} - \mathbb{E}[\min\{Y(x) : x \in S\}]$ is:*

1. ***Monotone Increasing****: For any $S \subseteq T \subseteq \mathcal{X}$, $U(S) \leq U(T)$.*

2. ***Submodular****: For any $S \subseteq T \subseteq \mathcal{X}$ and any module $x \in \mathcal{X} \setminus T$, $U(S \cup \{x\}) - U(S) \geq U(T \cup \{x\}) - U(T)$.*

Theorem 2 establishes that $U(S)$ is a monotone increasing, submodular function. Therefore, the problem of selecting a set S of size m to maximize $U(S)$ is an instance of maximizing a monotone submodular function under a cardinality constraint.

For this class of problems, the classical greedy algorithm provides a strong approximation guarantee. The algorithm starts with an empty set $S_0 = \emptyset$. At each step $k = 1, ..., m$, it adds the module x that provides the highest marginal gain:

$$x_k = \underset{x \in \mathcal{X} \setminus S_{k-1}}{\arg\max} \left[U(S_{k-1} \cup \{x\}) - U(S_{k-1}) \right], \tag{3}$$

and then updates $S_k = S_{k-1} \cup \{x_k\}$.

Theorem 3 **(Nemhauser et al. (1978))**. *Let S^* be the optimal set of size m that maximizes $U(S)$, and let S_m^{greedy} be the set of size m obtained by the greedy algorithm. If U is a monotone, non-negative, submodular function, then the greedy algorithm achieves the following performance guarantee:*

$$U(S_m^{greedy}) \geq \left(1 - \frac{1}{e}\right) U(S^*) \approx 0.632 \cdot U(S^*).$$

This theoretical guarantee allows us to use the greedy algorithm as an α-approximation oracle with $\alpha = (1 - 1/e)$. However, a direct implementation requires computing the expected minimum value, which remains computationally challenging. While Sample Average Approximation (SAA) can be employed, it requires extensive Monte Carlo simulations and becomes prohibitive for large-scale problems.

5.2. Efficient Heuristic Oracles

To address the computational challenge, we derive inspiration from the analytical solution for the bivariate case:

Lemma 1 **(Clark (1961))**. *For two Gaussian random variables $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ with correlation ρ, the expectation $\mathbb{E}[\min\{X_1, X_2\}]$ has closed-form solution:*

$$\mathbb{E}[\min\{X_1, X_2\}] = \mu_1 \Phi(\frac{\mu_2 - \mu_1}{\theta}) + \mu_2 \Phi(\frac{\mu_1 - \mu_2}{\theta}) - \theta \phi(\frac{\mu_2 - \mu_1}{\theta})$$

where $\theta = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$, and ϕ, Φ are PDF and CDF of standard normal distribution.

This analytical insight motivates the development of more efficient heuristic oracles that approximate the marginal gain without expensive numerical integration.

5.2.1. Improvement-based Oracle As established in Section 4.1.1, the expected value of the minimum of two module scores, $\mathbb{E}[\min\{Y(x_1),Y(x_2)\}]$, admits an exact closed-form solution under the bivariate normal distribution assumption. Leveraging this analytical tractability, we define a pairwise measure of deficiency enhancement for any two modules x_1 and x_2. Specifically, the **improvement** (Ipr) of incorporating module x_2 given a baseline module x_1 is defined as:

$$\mathrm{Ipr}(x_2|x_1) = \mathbb{E}[\min\{Y(x_1),Y(x_2)\}] - \mathbb{E}[Y(x_1)].$$

This quantity captures the expected reduction in the minimum score by adding x_2 to an assessment already containing x_1. It follows directly from the definition that $\mathrm{Ipr}(x_2|x_1) = 0$ when x_1 and x_2 are identical.

Next, consider an existing set of selected modules $S = \{x_2, x_3, x_4\}$. We can extend the pairwise improvement concept by defining the marginal improvement of adding a new module x_1 to the set S as:

$$\mathrm{Ipr}(x_1|S) = \mathbb{E}\left[\min\left\{Y(x) : x \in S \cup \{x_1\}\right\}\right] - \mathbb{E}\left[\min\left\{Y(x) : x \in S\right\}\right].$$

This metric quantifies the expected reduction in the overall minimum score resulting from the inclusion of x_1 into the current assessment portfolio S.

Now let's consider an extreme case, assuming x_1 and x_2 are exactly the same (i.e. $\mathrm{Ipr}(x_1|x_2) = 0$), regardless of the value of $\mathrm{Ipr}(x_1|x_3)$ and $\mathrm{Ipr}(x_1|x_4)$, the value of $\mathrm{Ipr}(x_1|S)$ is always 0. This inspires us to speculate that the value of the improvement of module x_1 over set S is dominated by pairwise minima: $\min\limits_{x \in S} \mathrm{Ipr}(x_1|x)$. So we can define the contribution of module x_1 to set S:

$$\mathrm{Ctr}(x_1|S) = \min_{x \in S} \mathrm{Ipr}(x_1|x)$$

Here, the contribution measure $\mathrm{Ctr}(x_1|S)$ represents quantifies the reduction potential in the minimum score when adding module x to set S.

Based on this we can get our improvement-based approach as follows:

Algorithm 1 likewise employs a greedy strategy for problem resolution, yet it achieves significantly higher computational efficiency compared to the SAA-based greedy algorithm, rendering it suitable for large-scale applications. However, this gain in efficiency comes at the cost of solution quality for more complex problem instances, where the SAA-based algorithm frequently yields superior optimization results. To harness the complementary strengths of both methodologies—the computational efficiency of the heuristic and the solution quality of the SAA-based approach—we propose a novel hybrid greedy algorithm. This mixed strategy, which synergistically combines both approaches, is detailed in the following section.

Algorithm 2 Improvement-based Oracle

```
Input: module set $\mathcal{X}$, number of selected modules $m$.
Initialize $S = \{x_1, x_2\}$, where $x_1 = \min_{x\in\mathcal{X}} \mathbb{E}[Y(x)]$, $x_2 = \max_{x\in\mathcal{X}} \mathrm{Ipr}(x_2|x_1)$.
for $i = 3$ to $m$ do
    Choose $x_i = \arg\max_{x\in\mathcal{X}/S} \mathrm{Ctr}(x|S)$
    Update $S = S \cup x_i$
end for
return $S$
```

5.2.2. Mixed Oracle The workflow for Mixed approach is described below(**Algorithm 2**). In Step 4, the algorithm identifies the q modules that offer the highest contribution to the current selected modules S. This step functions as a preliminary screening phase, designed to efficiently prune the candidate pool and mitigate the computational burden associated with evaluating all M modules. The parameter q is user-defined; a larger value for q yields a more comprehensive search, leading to a solution that more closely approximates the one obtained by the full SAA-based greedy algorithm. Our numerical experiments presented in Section 6 shows that a value of $q = 10$ can be sufficient for achieving near-optimal performance in most practical instances, effectively balancing computational efficiency with solution quality.

Algorithm 3 Mixed Oracle

```
Input: module set $\mathcal{X}$, number of selected modules $m$, size of candidate set $q = 10$.
Initialize $S = \{x_1, x_2\}$, where $x_1 = \min_{x\in\mathcal{X}} \mathbb{E}[Y(x)]$, $x_2 = \max_{x\in\mathcal{X}} \mathrm{Ipr}(x_2|x_1)$.
for $i = 3$ to $m$ do
    Choose candidate set $C \in \mathcal{X}$, $s.t$ $C = \arg\max_{|C|=q} \sum_{x\in C} \mathrm{Ctr}(x|S)$
    Choose $x_i = \arg\max_{x\in C} [U(S \cup \{x\}) - U(S)]$ by SAA
    Update $S = S \cup x_i$
end for
return $S$
```

These oracles provide efficient alternatives to direct SAA-based greedy optimization while maintaining strong empirical performance, as demonstrated in our numerical experiments.

6. Numerical Experiments

We conduct extensive numerical experiments to evaluate the performance of the proposed heuristic algorithms under the known-means scenario and GP-UCB algorithm under the unknown-means scenario. Our experiments are designed to systematically assess the impact of problem scale and correlation structure on algorithmic performance.

6.1. Experimental Setup

6.1.1. Problem Scales We examine four distinct problem scales to represent varying levels of computational complexity:

- **Medium Scale I:** Selecting 8 from 50 modules ($m = 8, M = 50$)
- **Medium Scale II:** Selecting 16 from 50 modules ($m = 16, M = 50$)
- **Large Scale I:** Selecting 10 from 200 modules ($m = 10, M = 200$)
- **Large Scale II:** Selecting 20 from 200 modules ($m = 20, M = 200$)

6.1.2. Correlation Structures For each problem scale, we investigate two fundamental correlation structures:

1. Grouped Structure: Modules are partitioned into groups with distinctive correlation patterns.

- For $M = 50$: 5 groups of 10 modules each
- For $M = 200$: 8 groups of 25 modules each
- Within-group correlation: $\rho_{\text{in}} \approx 0.8$ (highly correlated)
- Between-group correlation: $\rho_{\text{out}} \approx 0$ (weakly correlated)

This structure is further divided into two scenarios:

- **Worst Case:** Modules with the smallest mean scores are concentrated in one group. Additionally, we introduce a positive correlation between module means and noise levels: modules with smaller means exhibit lower assessment variance, while those with larger means have higher variance.

- **Best Case:** Mean scores are distributed evenly across all groups, and noise levels are independent of mean scores.

2. Random Structure: All pairwise correlations are randomly generated with non-negative values, representing an unstructured scenario without inherent module clustering. Noise levels are generated independently of mean scores.

6.1.3. Data Generation Procedure The ground-truth mean score vector $\boldsymbol{\mu}_{\text{true}} \in \mathbb{R}^M$ is generated by sampling from a multivariate normal distribution $\mathcal{N}(\mathbf{70}, \mathbf{\Sigma}_0)$, where $\mathbf{70}$ is a vector of baseline scores. The covariance matrix $\mathbf{\Lambda}$ is then constructed based on the specified correlation structure:

- For grouped structures, $\mathbf{\Lambda}$ is constructed such that modules within the same group exhibit high correlations ($\rho \approx 0.8$), while modules from different groups show negligible correlations ($\rho \approx 0$)
- For the **Worst Case** scenario, the diagonal elements of $\mathbf{\Lambda}$ (variances) are set to be positively correlated with the mean scores: $\mathbf{\Lambda}_{ii} \propto \boldsymbol{\mu}_{\text{true},i}$, which creates a more challenging environment
- For random structures, $\mathbf{\Lambda}$ is randomly generated with non-negative entries

Table 2 Summary of Experimental Configurations

Scale	**Structure**	**Configuration**
50/8	Grouped	5 groups; $\rho_{\text{in}} \approx 0.8$; $\rho_{\text{out}} \approx 0$; Worst/Best Case
50/16	Grouped	5 groups; $\rho_{\text{in}} \approx 0.8$; $\rho_{\text{out}} \approx 0$; Worst/Best Case
200/10	Grouped	8 groups; $\rho_{\text{in}} \approx 0.8$; $\rho_{\text{out}} \approx 0$; Worst/Best Case
200/20	Grouped	8 groups; $\rho_{\text{in}} \approx 0.8$; $\rho_{\text{out}} \approx 0$; Worst/Best Case
50/8	Random	Random non-negative correlations
50/16	Random	Random non-negative correlations
200/10	Random	Random non-negative correlations
200/20	Random	Random non-negative correlations

To comprehensively evaluate the performance of our proposed algorithms (Improvement, Mixed, and SAA), we compare them against two intuitive and straightforward benchmark policies:

1. **Smallest Mean (SM)**: This naive baseline selects the m modules with the smallest mean scores, entirely ignoring the covariance structure (variances and correlations) among them.
2. **Clustering-based Selection (CLS)**: This policy first clusters the M modules into m groups based on their correlation matrix using spectral clustering, aiming to group highly correlated modules together. It then selects the module with the smallest mean score from each cluster. This approach seeks diversity by explicitly minimizing correlation within the selected set.

subsectionExperiments with known Means

The experimental results across all 12 scenarios (3 structures × 4 scales) are succinctly presented in Table 3. The best objective value(expected minimum score $\mathbb{E}[\min\{Y(x_1),...,Y(x_m)\}]$) achieved in each scenario is highlighted in **bold**. We also show the running time of the algorithms.

Table 3 Experimental Results: Comparison of Objective Value and Computation Time

Scale	Case	SM		CLS		Improvement		Mixed		SAA	
		Obj.	Time(s)	Obj.	Time(s)	Obj.	Time(s)	Obj.	Time(s)	Obj.	Time(s)
50/8	Best	56.77	0.0003	57.07	0.0008	56.76	0.10	**56.52**	3.01	56.53	13.47
	Worst	60.72	0.0002	58.59	0.0011	57.81	0.11	57.75	2.97	**57.61**	13.15
	Random	54.28	0.0001	54.41	0.0007	53.00	0.10	52.68	2.90	**52.67**	12.62
50/16	Best	55.59	0.0002	55.62	0.0005	55.63	0.37	55.41	7.17	**55.36**	27.23
	Worst	57.90	0.0001	56.72	0.0027	56.64	0.38	56.59	7.03	**56.10**	27.44
	Random	52.53	0.0002	53.17	0.0005	52.48	0.36	**52.21**	7.16	52.22	27.22
200/10	Best	48.77	0.0001	48.87	0.0021	48.77	0.69	**48.76**	4.48	48.80	72.13
	Worst	52.28	0.0001	53.18	0.0017	51.45	0.68	51.22	4.50	**51.12**	73.72
	Random	47.59	0.0002	47.09	0.0017	46.19	0.71	45.88	4.57	**45.84**	72.85
200/20	Best	**47.87**	0.0001	48.33	0.0019	47.97	2.86	47.91	11.83	47.94	167.31
	Worst	51.90	0.0001	51.76	0.0020	50.38	2.81	50.12	12.07	**49.97**	172.30
	Random	44.31	0.0001	45.18	0.0029	44.23	2.83	44.08	11.69	**43.98**	169.09

The comprehensive evaluation reveals that the SAA-based greedy algorithm achieves the best results in most cases, while the Mixed algorithm produces nearly identical results with minimal optimality gaps, demonstrating their superior performance. In contrast, both SM and CLS exhibit significant performance degradation compared to SAA in the challenging Worst Case and Random Case configurations, although the Improvement algorithm shows relatively better performance than these benchmarks while still falling short of Mixed and SAA. For the Best Case conditions, all algorithms except CLS yield comparable results. Overall, the proposed algorithms (Improvement, Mixed, and SAA) consistently outperform the benchmark approaches, confirming the value of systematic optimization; however, it is important to note that SAA's computational time increases substantially with problem scale, particularly as the total number of modules M grows, while the performance differences between all algorithms diminish as the selection size m increases. These results collectively indicate that while SAA provides optimal performance, Mixed offers an attractive balance between solution quality and computational efficiency for practical applications.

6.2. Experiments with Unknown Means

In practical applications, the true means of module scores are typically unknown and must be estimated through sequential testing. This subsection evaluates the performance of the proposed Bayesian learning framework under this more realistic and challenging setting, using the same experimental configuration described in Section 6.1.

6.2.1. Learning Framework and Algorithms We employ a Gaussian Process Upper Confidence Bound (GP-UCB) approach to address the unknown means scenario. The algorithmic framework, outlined in Algorithm **??**, maintains Gaussian posterior distributions for the mean scores of all modules. The algorithm proceeds in two phases: an efficient initialization phase that ensures each module is observed at least once using the minimal number of super arms, followed by a main optimization phase that balances exploration and exploitation.

Three algorithms are compared within this framework:

1. **GP-UCB (Proposed)**: Our proposed algorithm employs a Lower Confidence Bound (LCB) strategy for module selection. At each iteration t, it constructs pessimistic mean estimates $\tilde{\boldsymbol{\mu}}_t = \boldsymbol{\mu}_{t-1} - \boldsymbol{\beta}_t \circ \boldsymbol{\sigma}_{t-1}$, where $\boldsymbol{\beta}_t[x] = \sqrt{6 \ln t / \mathbf{T}_{t-1}[x]}$ controls the exploration-exploitation trade-off. The oracle O then selects the super arm S_t that minimizes the expected minimum score based on these conservative estimates.

2. **GP-SM**: This benchmark algorithm follows the same Bayesian learning framework but employs a greedy selection strategy. At each iteration, it simply selects the m modules with the smallest posterior mean estimates $\boldsymbol{\mu}_{t-1}$, ignoring uncertainty information and correlation structure. This approach corresponds to the SM algorithm in the known-means scenario.

3. **GP-CLS**: This algorithm incorporates correlation information through clustering. It first partitions the M modules into m groups using spectral clustering based on the correlation matrix $\boldsymbol{\Lambda}$, then selects the module with the smallest posterior mean from each cluster. This strategy mirrors the CLS approach in the known-means setting while maintaining the Bayesian learning framework.

6.2.2. Experimental Configuration The experimental setup mirrors that of the known-means scenario in terms of problem scales and correlation structures (Section 6.1). All algorithms share the same Gaussian prior initialization $\mathcal{N}(\mathbf{70}, \boldsymbol{\Sigma}_0)$ and update their posterior distributions identically upon receiving new observations. Each algorithm executes for a total of $N = 150$ iterations, with the initial phase using the minimal number of super arms required to cover all modules ($N_0 = \lceil M/m \rceil$). All results are averaged over **30** independent runs with different random seeds.

6.2.3. Performance Metrics We employ three complementary metrics:

1. **Cumulative Regret**: We define the optimal reward R^* as the expected minimum score obtained by the SAA-based greedy algorithm with access to the true means. The cumulative regret is then calculated as $\sum_{t=1}^{N}(R^* - R(S_t))$, where $R(S_t)$ is the actual reward obtained by the selected super arm S_t at iteration t. Additionally, we analyze the *instant regret* $R^* - R(S_t)$ at each iteration to examine the transient behavior and convergence properties.

2. **Simple Regret (obs_based)**: Simple Regret quantifies the gap between the currently believed best solution and the true optimal solution. For obs_based definition, we maintain a running estimate of the best super arm encountered thus far. Specifically, at each iteration t, we compute the expected minimum score for all previously tested super arms using their empirical means and select the minimal value as the current best estimate: $\tilde{R}_t = \min_{k=1,\ldots,t} \mathbb{E}[\min\{Y(x) : x \in S_k\}]$. The simple regret is then defined as $R^* - \tilde{R}_t$.

3. **Simple Regret (global_based)**: This definition employs a more optimistic approach by treating the current posterior means as ground truth. At each iteration t, we apply the SAA-based greedy algorithm using the current posterior mean estimates $\boldsymbol{\mu}_{t-1}$ to identify the theoretically optimal super arm S_t^* under these beliefs. The simple regret is then computed as $R^* - \mathbb{E}[\min\{Y(x) : x \in S_t^*\}]$, where the expectation is calculated using the true (but unknown) means.

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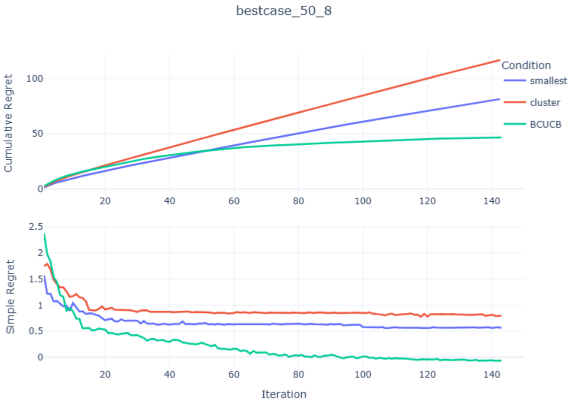

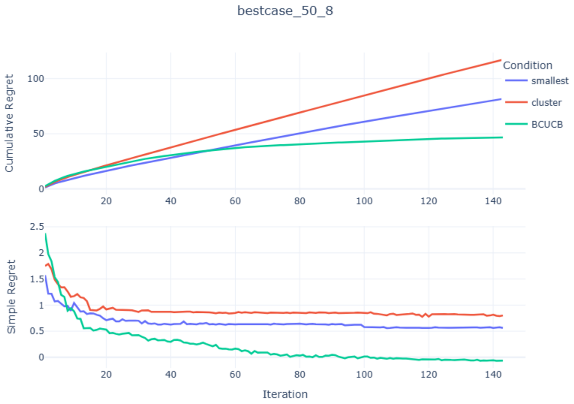

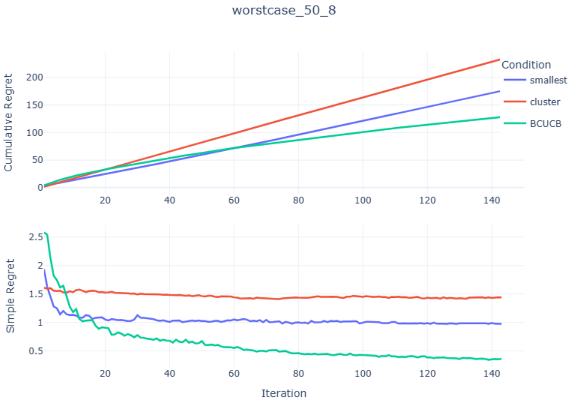

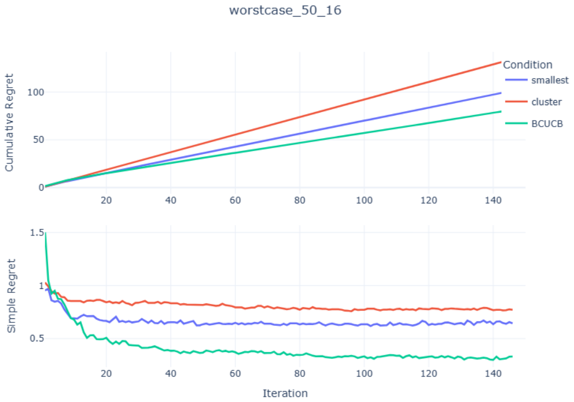

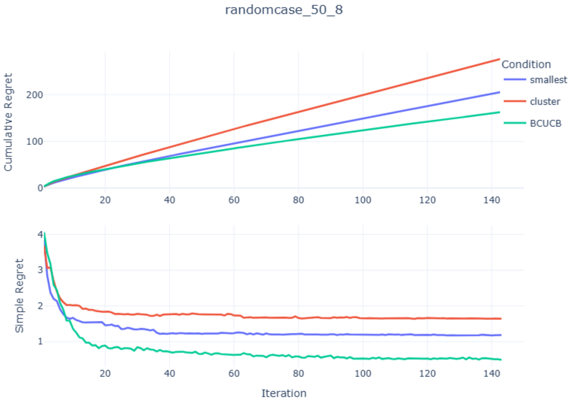

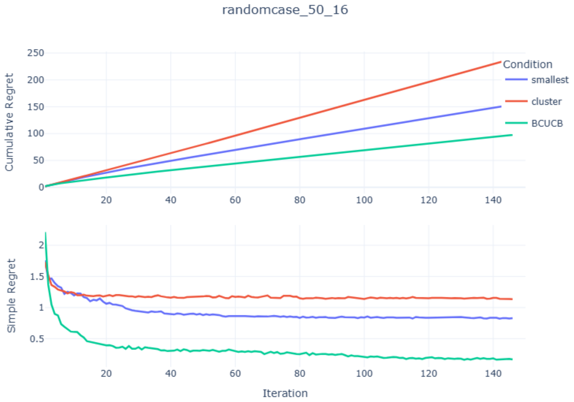

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Online Appendix

EC.1. Proof of Theorem 2

Proof. Monotone Increasing is quite obvious. Here we show that $U(\cdot)$ is submodular. i.e., for any $S \subseteq T \subseteq \mathcal{X}$ and $x \in \mathcal{X} \setminus T$:

$$U(S \cup \{x\}) - U(S) \geq U(T \cup \{x\}) - U(T).$$

which is equivalent to:

$$R(S) - R(S \cup \{x\}) \geq R(T) - R(T \cup \{x\}).$$

Let $Y_S = \min_{i \in S} Y(i, \xi)$, $Y_T = \min_{i \in T} Y(i, \xi)$, $Y_x = Y(x, \xi)$. Then $Y_T \leq Y_S$. We have:

$$R(S) - R(S \cup \{x\}) = \mathbb{E}\left[Y_S - \min\{Y_S, Y_x\}\right],$$

$$R(T) - R(T \cup \{x\}) = \mathbb{E}\left[Y_T - \min\{Y_T, Y_x\}\right].$$

Consider the function $g(y) = y - \min\{y, x\}$. For fixed x:

- If $y \leq x$, then $g(y) = 0$;
- If $y > x$, then $g(y) = y - x$.

Thus, $g(y)$ is non-decreasing in y. Since $Y_A \geq Y_B$, we have:

$$Y_S - \min\{Y_S, Y_x\} \geq Y_T - \min\{Y_T, Y_x\}.$$

Taking expectation:

$$\mathbb{E}\left[Y_S - \min\{Y_S, Y_x\}\right] \geq \mathbb{E}\left[Y_T - \min\{Y_T, Y_x\}\right],$$

which implies:

$$U(S \cup \{x\}) - U(S) \geq U(T \cup \{x\}) - U(T).$$

Therefore, $U(\cdot)$ is submodular. ■

EC.1.1. Proof of Theorem 1

LEMMA EC.1. *For any $\epsilon > 0$ and any $n = T_{i,t-1} \in \mathbb{Z}_+$, we have :*

$$\Pr[|\mu_{t-1}(i) - \mu(i)| \geq \epsilon_i] \leq \exp(-\frac{n\epsilon_i^2}{2\sigma^2(i)})$$

Proof. Fix $t \geq 1$ and $i \in [m]$. Conditioned on $(\mathbf{y}_1, ..., \mathbf{y}_{t-1}), \{S_1, ..., S_{t-1}\}$ are deterministic, and $\mu(i) \sim N(\mu_{t-1}(i), \sigma^2_{t-1}(i))$. Now, if $r \sim N(0,1)$,then:

$$\begin{aligned}\Pr\{r > c\} &= e^{-c^2/2}(2\pi)^{-1/2}\int e^{-(r-c)^2/2-c(r-c)}dr \\ &\leq e^{-c^2/2}\Pr\{r > 0\} = (1/2)e^{-c^2/2}\end{aligned}$$

for $c > 0$, since $e^{-c(r-c)} \leq 1$ for $r \geq c$. Therefore, $\Pr\{|\mu_{t-1}(i) - \mu(i)| > \lambda_i \sigma_{t-1}(i)\} \leq e^{-\lambda_i/2}$

Assume that arm i has been observed n times, yielding n observations. Further assume that these n observations are our only observational data. Using GP modeling, we obtain:

$$\sigma^2_n(i) = k(i,i) - \mathbf{k}_n^\top(i)(\mathbf{K}_n + \sigma^2(i)\mathbf{I}_n)^{-1}\mathbf{k}_n(i)$$

where:

- $\mathbf{k}_n(i) = [k(i,i), \dots, k(i,i)]^\top \in \mathbb{R}^n$
- $\mathbf{K}_n = \sigma_0^2 \mathbf{1}_n \mathbf{1}_n^\top$ $(\sigma_0^2 = k(i,i))$

For $\mathbf{A} = \sigma^2(i)\mathbf{I}_n$, $\mathbf{u} = \sigma_0^2\mathbf{1}_n$, $\mathbf{v} = \mathbf{1}_n$, by **Sherman-Morrison** formula:

$$\begin{aligned}(\mathbf{K}_n + \sigma^2(i)\mathbf{I}_n)^{-1} &= \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\top\mathbf{A}^{-1}}{1 + \mathbf{v}^\top\mathbf{A}^{-1}\mathbf{u}} \\ &= \frac{1}{\sigma^2(i)}\mathbf{I}_n - \frac{\sigma_0^2/\sigma(i)^4 \cdot \mathbf{1}_n\mathbf{1}_n^\top}{1 + n\sigma_0^2/\sigma^2(i)}\end{aligned}$$

So we have:

$$\mathbf{k}_n^\top(i)(\mathbf{K}_n + \sigma^2(i)\mathbf{I}_n)^{-1}\mathbf{k}_n(i) = \frac{\sigma_0^4 n}{\sigma^2(i) + n\sigma_0^2}$$

Substitute the results into the posterior variance formula:

$$\begin{aligned}\sigma^2_n(i) &= \sigma_0^2 - \frac{\sigma_0^4 n}{\sigma^2(i) + n\sigma_0^2} = \frac{\sigma_0^2\sigma^2(i)}{\sigma^2(i) + n\sigma_0^2} \\ &= \frac{n\sigma_0^2}{\sigma^2(i) + n\sigma_0^2} \cdot \frac{\sigma^2(i)}{n} \leq \frac{\sigma^2(i)}{n}\end{aligned}$$

Considering that adding other observation points will not increase the uncertainty of arm i, so we have $\sigma_{t-1}(i) \leq \sigma_n(i) \leq \frac{\sigma(i)}{\sqrt{n}}$. Therefore:

$$\Pr\left\{|\mu_{t-1}(i) - \mu(i)| > \lambda_i \frac{\sigma(i)}{\sqrt{n}}\right\} \leq \Pr\{|\mu_{t-1}(i) - \mu(i)| > \lambda_i\sigma_{t-1}(i)\} \leq e^{-\lambda_i/2}$$

Denote $\epsilon_i = \lambda_i \frac{\sigma(i)}{\sqrt{n}}$, then we have $\Pr[|\mu_{t-1}(i) - \mu(i)| \geq \epsilon_i] \leq \exp(-\frac{n\epsilon_i^2}{2\sigma^2(i)})$ ■

Lemma EC.2. *If for any $i \in [m]$ we have $\mu_1(i) \geq \mu_2(i)$ (Abbreviated as $\boldsymbol{\mu}_1 \geq \boldsymbol{\mu}_2$), then for any super arm $S \in \mathcal{F}$,we have:*

$$r_{P_1}(S) \geq r_{P_2}(S)$$

Lemma EC.3. *If for any $i \in [m]$ we have $\mu_1(i) - \mu_2(i) \leq \Lambda_i (\Lambda_i > 0)$ ($\boldsymbol{\mu}_1 \geq \boldsymbol{\mu}_2$). For any super arm $S \in \mathcal{F}$, let $\Lambda_S = \max_{i \in S} \Lambda_i$, then we have:*

$$r_{P_1}(S) - r_{P_2}(S) \leq \Lambda_S$$

Proof. For any super arm $S = \{i_1, ..., i_n\}$, denote the corresponding mean vectors in distributions P_1 and P_2 as $\boldsymbol{\mu}_{1,S} = [\mu_1(i_1), ..., \mu_1(i_n)]^T$, $\boldsymbol{\mu}_{2,S} = [\mu_1(i_2), ..., \mu_2(i_n)]^T$. Note that P_1 and P_2 are both multivariate normal distributions, and have same covariance matrices Σ_S.

Now, let $\mathbf{z} = [z_1, ..., z_n]^T \sim N(0, \Sigma_S)$. We have:

$$r_{P_1}(S) = \int \cdots \int_{\mathbf{z}} \min\{\mu_1(i_1) + z_1, \mu_1(i_2) + z_2, \ldots, \mu_1(i_n) + z_n\}\, dz_1 \ldots dz_k$$

$$r_{P_2}(S) = \int \cdots \int_{\mathbf{z}} \min\{\mu_2(i_1) + z_1, \mu_2(i_2) + z_2, \ldots, \mu_2(i_n) + z_n\}\, dz_1 \ldots dz_k$$

Denote $f(\boldsymbol{\mu}_{1,S} + \mathbf{z}) = \min\{\mu_1(i_1) + z_1, \mu_1(i_2) + z_2, \ldots, \mu_1(i_n) + z_n\}$ and $f(\boldsymbol{\mu}_{2,S} + \mathbf{z}) = \min\{\mu_2(i_1) + z_1, \mu_2(i_2) + z_2, \ldots, \mu_2(i_n) + z_n\}$. We want to show that $f(\mu_{1,S} + \mathbf{z}) - f(\mu_{2,S} + \mathbf{z}) \leq \Lambda_S$.

If there exists $i, j \in S = \{i_1, ..., i_n\}$ s.t. $\mu_1(i) + z_i = f(\mu_{1,S} + \mathbf{z})$, $\mu_2(j) + z_j = f(\mu_{2,S} + \mathbf{z})$, $f(\mu_{1,S} + \mathbf{z}) - f(\mu_{2,S} + \mathbf{z}) > \Lambda_S$. Since $\mu_1(i) + z_i = f(\mu_{1,S} + \mathbf{z}) = \min\{\mu_1(i_1) + z_1, \mu_1(i_2) + z_2, \ldots, \mu_1(i_n) + z_n\}$, we have: $\mu_1(j) + z_j > \mu_1(i) + z_i$.

Thus $\mu_1(j) - \mu_2(j) = \mu_1(j) + z_j - (\mu_2(j) + z_j) > \mu_1(i) + z_i - (\mu_2(j) + z_j) > \Lambda_S$, which is contradict with $\boldsymbol{\mu}_1 \geq \boldsymbol{\mu}_2$.

Therefore

$$\begin{aligned} r_{P_1}(S) - r_{P_2}(S) &= \int \cdots \int_{\mathbf{z}} f(\boldsymbol{\mu}_{1,S} + \mathbf{z}) - f(\boldsymbol{\mu}_{2,S} + \mathbf{z})\, dz_1 \ldots dz_k \\ &\leq \int \cdots \int_{\mathbf{z}} \Lambda_S\, dz_1 \ldots dz_k = \Lambda_S \end{aligned}$$

■

Lemma EC.4. *Define an event in each round $t(m+1) \leq t \leq T$:*

$$\mathcal{H}_t = \left\{ 0 < \Delta_{S_t} \leq 2 \max_{i \in S_t} \sigma(i) \sqrt{\frac{6 \ln(t)}{T_{i,t-1}}} \right\}$$

Then the α-approximation regret in T rounds is at most

$$\mathbb{E}[\sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t}] + (\frac{\pi^2}{6} + 1)\alpha M m.$$

Proof. For each super arm S, we define:

$$\Delta_S = \max\{r_P(S) - \alpha \cdot r_P(S^*), 0\}$$

From $m+1 \leq t \leq T$, Define an event:

$$\varepsilon_t = \left\{ \text{there exists } i \in [m] \text{ such that } |\hat{\mu}^i_{T_{i,t-1}} - \mu^i| \geq \sigma(i)\sqrt{\frac{6\ln(t)}{T_{i,t-1}}} \right\}$$

We bound the α-approximation regret as:

$$\begin{aligned} \mathrm{Reg}_{P,\alpha}(T) &= \sum_{t=1}^{T} \mathbb{E}\big[\alpha r_P(S^*) - r_P(S_t)\big] \\ &\leq \sum_{t=1}^{T} \mathbb{E}[\Delta_{S_t}] \\ &= \mathbb{E}[\sum_{t=1}^{m} \Delta_{S_t}] + \mathbb{E}[\sum_{t=m+1}^{T} \mathbb{I}\{\varepsilon_t\}\Delta_{S_t}] \\ &+ \mathbb{E}[\sum_{t=m+1}^{T} \mathbb{I}\{\neg\varepsilon_t\}\Delta_{S_t}] \end{aligned} \tag{EC.1}$$

(a) the first term

Let $M = 100$, The first term can be trivially bounded as:

$$\mathbb{E}[\sum_{t=1}^{m} \Delta_{S_t}] \leq \sum_{t=1}^{m} \alpha \cdot r_P(S^*) \leq m \cdot \alpha M \tag{EC.2}$$

(b) the second term

By Lemma 1 we know that for any $i \in [m], t \geq m+1$, denote $c_i = \sigma(i)\sqrt{\frac{6\ln(t)}{T_{i,t-1}}}$, we have:

$$\Pr[|\hat{\mu}_{t-1}(i) - \mu(i)| \geq c_i] \leq \exp(-\frac{nc_i^2}{2\sigma^2(i)}) = t^{-3}$$

Therefore

$$\begin{aligned} \mathbb{E}[\sum_{t=m+1}^{T} \mathbb{I}\{\varepsilon_t\}] &\leq \sum_{t=m+1}^{T} \sum_{i=1}^{m} \sum_{l=1}^{t-1} t^{-3} \\ &\leq m \sum_{t=m+1}^{T} t^{-2} \\ &\leq \frac{\pi^2}{6} m \end{aligned} \tag{EC.3}$$

and then the second term in (6) can be bounded as

$$\mathbb{E}[\sum_{t=m+1}^{T} \mathbb{I}\{\varepsilon_t\}\Delta_{S_t}] \leq \frac{\pi^2}{6} m \cdot (\alpha \cdot r_P(S^*)) \leq \frac{\pi^2}{6} \alpha M m \tag{EC.4}$$

(c) the third term

We fix $t > m$ and first asuume $\neg\varepsilon_t$ happens. For each $i \in [m]$, since $\neg\varepsilon_t$ happens, we have:

$$|\hat{\mu}_{T_{i,t-1}}(i) - \mu(i)| < c_i \quad \forall i \in [m] \tag{EC.5}$$

Recall that in round t, the input to the oracle is $\underline{P} = \mathcal{N}(\underline{\boldsymbol{\mu}}, \Sigma)$, where the mean vector of $\underline{P}$ is:

$$\underline{\mu}(i) = \hat{\mu}(i) - c_i \quad \forall i \in [m] \tag{EC.6}$$

From (10) and (11) we konw that $\mu(i) > \underline{\mu}(i) > \mu(i) - 2c_i$ for all $i \in [m]$. Thus, from Lemma 2 we have:

$$r_{\underline{P}}(S) \leq r_P(S) \quad \forall S \in \mathcal{F}. \tag{EC.7}$$

and from Lemma 3 we have:

$$r_{\underline{P}}(S) \geq r_P(S) - 2\Lambda_S \quad \forall S \in \mathcal{F}. \tag{EC.8}$$

where $\Lambda = \max_i c_i$

Also, from the fact that the algorithm chooses S_t in the t-th round, we have:

$$r_{\underline{P}}(S_t) \leq \alpha \cdot \min_{S \in \mathcal{F}} r_{\underline{P}}(S) \leq \alpha \cdot r_{\underline{P}}(S^*). \tag{EC.9}$$

From (12),(13) and (14) we have:

$$r_P(S) - 2\Lambda_S \leq r_{\underline{P}}(S_t) \leq \alpha \cdot r_{\underline{P}}(S^*) \leq \alpha \cdot r_P(S^*) \tag{EC.10}$$

which implies:

$$\Delta_{S_t} \leq 2\Lambda_S$$

Therefore, when $\neg\varepsilon_t$ happens, we always have $\Delta_{S_t} \leq 2 \max_{i \in S_t} c_i$.

This implies:

$$\{\neg\varepsilon_t, \Delta_{S_t} > 0\} \implies \{0 < \Delta_{S_t} \leq 2 \max_{i \in S_t} c_i\} = \mathcal{H}_t.$$

Hence, the third term in (6) can be bounded as:

$$\begin{aligned}\mathbb{E}[\sum_{t=m+1}^{T}\mathbb{I}\{\neg\varepsilon_t\}\Delta_{S_t}] &= \mathbb{E}[\sum_{t=m+1}^{T}\mathbb{I}\{\neg\varepsilon_t.\Delta_{S_t}>0\}\Delta_{S_t}] \\ &\leq \mathbb{E}[\sum_{t=m+1}^{T}\mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t}]\end{aligned} \tag{EC.11}$$

Finally, by combining (6),(7),(9),(16) we have:

$$\mathrm{Reg}_{P,\alpha}(T) \leq \mathbb{E}[\sum_{t=m+1}^{T}\mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t}] + (\frac{\pi^2}{6}+1)\alpha Mm$$

■

Now it remains to bound $\mathbb{E}[\sum_{t=m+1}^{T}\mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t}]$.

Define two decreasing sequences of positive contants:

$$1=\beta_0>\beta_1>\beta_2>\dots$$

$$\alpha_1>\alpha_2>\dots$$

such that $\lim_{k\to\infty}\alpha_k=\lim_{k\to\infty}\beta_k=0$. We choose $\{\alpha_k\}$ and $\{\beta_k\}$ as in Theorem 4 of (Kveton et al. 2014), which satisfy:

$$\sqrt{6}\sum_{k=1}^{\infty}\frac{\beta_{k-1}-\beta_k}{\sqrt{\alpha_k}}\leq 1 \tag{EC.12}$$

and

$$\sum_{k=1}^{\infty}\frac{\alpha_k}{\beta_k}<267. \tag{EC.13}$$

For $t\in\{m+1,...,T\}$ and $k\in\mathbb{Z}_+$, let

$$m_{k,t}=\begin{cases}\alpha_k\left(\frac{2\sigma K}{\Delta_{S_t}}\right)^2\cdot\ln(T) & \Delta_{S_t}>0,\\ +\infty & \Delta_{S_t}=0,\end{cases}$$

and

$$A_{k,t}=\{i\in S_t|T_{i,t-1}\leq m_{k,t}\}.$$

Then we define an event:

$$\mathcal{G}_{k,t}=\{|A_{k,t}|\geq\beta_k K\},$$

which means "in the $t-th$ round, at least $\beta_k K$ arms in S_t had been observed at most $m_{k,t}$ times."

LEMMA EC.5. *In the $t-th$ round, at least $\beta_k K$ arms in S_t had been observed at most $m_{k,t}$ times.*

Proof. Assume that $\mathcal{H}_t$ happens and that none of $\mathcal{G}_{1,t}, \mathcal{G}_{2,t}, \ldots$ happens. Then $|A_{k,t}| < \beta_k K$ for all $k \in \mathbb{Z}_+$.

Let $A_{0,t} = S_t$ and $\bar{A}_{k,t} = S_t \setminus A_{k,t}$ for $k \in \mathbb{Z}_+ \cup \{0\}$. It is easy to see $\bar{A}_{k-1,t} \subseteq \bar{A}_{k,t}$ for all $k \in \mathbb{Z}_+$. Note that $\lim_{k\to\infty} m_{k,t} = 0$. Thus there exists $N \in \mathbb{Z}_+$ such that $\bar{A}_{k,t} = S_t$ for all $k \geq N$, and then we have $S_t = \bigcup_{k=1}^{\infty} \left(\bar{A}_{k,t} \setminus \bar{A}_{k-1,t}\right)$. Finally, note that for all $i \in \bar{A}_{k,t}$, we have $T_{i,t-1} > m_{k,t}$. Therefore

$$\begin{aligned}
\sum_{i\in S_t} \frac{1}{\sqrt{T_{i,t-1}}} &= \sum_{k=1}^{\infty} \sum_{i\in \bar{A}_{k,t}\setminus \bar{A}_{k-1,t}} \frac{1}{\sqrt{T_{i,t-1}}} \leq \sum_{k=1}^{\infty} \sum_{i\in \bar{A}_{k,t}\setminus \bar{A}_{k-1,t}} \frac{1}{\sqrt{m_{k,t}}} \\
&= \sum_{k=1}^{\infty} \frac{\left|\bar{A}_{k,t} \setminus \bar{A}_{k-1,t}\right|}{\sqrt{m_{k,t}}} = \sum_{k=1}^{\infty} \frac{\left|A_{k-1,t} \setminus A_{k,t}\right|}{\sqrt{m_{k,t}}} = \sum_{k=1}^{\infty} \frac{\left|A_{k-1,t}\right| - \left|A_{k,t}\right|}{\sqrt{m_{k,t}}} \\
&= \frac{|S_t|}{\sqrt{m_{1,t}}} + \sum_{k=1}^{\infty} |A_{k,t}| \left(\frac{1}{\sqrt{m_{k+1,t}}} - \frac{1}{\sqrt{m_{k,t}}}\right) \\
&< \frac{K}{\sqrt{m_{1,t}}} + \sum_{k=1}^{\infty} \beta_k K \left(\frac{1}{\sqrt{m_{k+1,t}}} - \frac{1}{\sqrt{m_{k,t}}}\right) \\
&= \sum_{k=1}^{\infty} \frac{(\beta_{k-1} - \beta_k) K}{\sqrt{m_{k,t}}}.
\end{aligned}$$

Note that we asuume $\mathcal{H}_t$ happens. Denote $\sigma = \max_{i\in S_t} \sigma(i)$, then we have:

$$\begin{aligned}
\Delta_{S_t} &\leq 2\Lambda_S = 2 \max_{i\in S_t} \sigma(i) \sqrt{\frac{6\ln(t)}{T_{i,t-1}}} \\
&\leq 2\sigma \cdot \sqrt{6\ln(T)} \cdot \max_{i\in S_t} \frac{1}{\sqrt{T_{i,t-1}}} \\
&\leq 2\sigma \cdot \sqrt{6\ln(T)} \cdot \sum_{i\in S_t} \frac{1}{\sqrt{T_{i,t-1}}} \\
&< 2\sigma \cdot \sqrt{6\ln(T)} \cdot \sum_{k=1}^{\infty} \frac{(\beta_{k-1} - \beta_k) K}{\sqrt{m_{k,t}}} \\
&= \sqrt{6} \sum_{k=1}^{\infty} \frac{\beta_{k-1} - \beta_k}{\sqrt{\alpha_k}} \cdot \Delta_{S_t} \leq \Delta_{S_t},
\end{aligned}$$

We reach a contradiction here. The proof of lemma 5 is completed. ■

By Lemma 5 we have

$$\sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{H}_t\} \Delta_{S_t} \leq \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{G}_{k,t}, \Delta_{S_t} > 0\} \Delta_{S_t}.$$

For $i \in [m], k \in \mathbb{Z}_+, t \in \{m+1, ..., T\}$, define an event $Gi, k, t = Gk, tiSt, Ti, t1mk, t. \mathcal{G}_{i,k,t} = \mathcal{G}_{k,t} \wedge \{i \in S_t, T_{i,t-1} \le m_{k,t}\}$.

Then by the definitions of $\mathcal{G}_{k,t}$ and $\mathcal{G}_{i,k,t}$ we have

$$\mathbb{I}\{\mathcal{G}_{k,t}, \Delta_{S_t} > 0\} \le \frac{1}{\beta_k K} \sum_{i \in E_{\mathrm{B}}} \mathbb{I}\{\mathcal{G}_{i,k,t}, \Delta_{S_t} > 0\}.$$

Therefore $Tt = m + 1IHtStiEBk = 1Tt = m + 1IGi, k, t, St > 0StkK. \sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t} \le \sum_{i \in E_{\mathrm{B}}} \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{G}_{i,k,t}, \Delta_{S_t} > 0\} \frac{\Delta_{S_t}}{\beta_k K}$.

For each arm $i \in E_B$, suppose i is contained in N_i bad super arms $S^{\mathrm{B}}_{i,1}, S^{\mathrm{B}}_{i,2}, \ldots, S^{\mathrm{B}}_{i,N_i}$. Let $\Delta_{i,l} = \Delta_{S^{\mathrm{B}}_{i,l}} (l \in [N_i])$. Without loss of generality, we assume $\Delta_{i,1} \ge \Delta_{i,2} \ge \ldots \ge \Delta_{i,N_i}$. Note that $\Delta_{i,N_i} = \Delta_{i,\min}$. For convenience, we also define $\Delta_{i,0} = +\infty$, i.e., $\alpha_k \left(\frac{2\sigma K}{\Delta_{i,0}}\right)^2 = 0$. Then we have

$$\begin{aligned}
&\sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t} \\
&\le \sum_{i \in E_{\mathrm{B}}} \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \sum_{l=1}^{N_i} \mathbb{I}\{\mathcal{G}_{i,k,t}, S_t = S^{\mathrm{B}}_{i,l}\} \frac{\Delta_{S_t}}{\beta_k K} \\
&\le \sum_{i \in E_{\mathrm{B}}} \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \sum_{l=1}^{N_i} \mathbb{I}\{T_{i,t-1} \le m_{k,t}, S_t = S^{\mathrm{B}}_{i,l}\} \frac{\Delta_{i,l}}{\beta_k K} \\
&= \sum_{i \in E_{\mathrm{B}}} \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \sum_{l=1}^{N_i} \mathbb{I}\left\{T_{i,t-1} \le \alpha_k \left(\frac{2\sigma K}{\Delta_{i,l}}\right)^2 \ln T, S_t = S^{\mathrm{B}}_{i,l}\right\} \frac{\Delta_{i,l}}{\beta_k K} \\
&= \sum_{i \in E_{\mathrm{B}}} \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \sum_{l=1}^{N_i} \sum_{j=1}^{l} \mathbb{I}\left\{\alpha_k \left(\frac{2\sigma K}{\Delta_{i,j-1}}\right)^2 \ln T < T_{i,t-1} \le \alpha_k \left(\frac{2\sigma K}{\Delta_{i,j}}\right)^2 \ln T, S_t = S^{\mathrm{B}}_{i,l}\right\} \frac{\Delta_{i,l}}{\beta_k K} \\
&\le \sum_{i \in E_{\mathrm{B}}} \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \sum_{l=1}^{N_i} \sum_{j=1}^{l} \mathbb{I}\left\{\alpha_k \left(\frac{2\sigma K}{\Delta_{i,j-1}}\right)^2 \ln T < T_{i,t-1} \le \alpha_k \left(\frac{2\sigma K}{\Delta_{i,j}}\right)^2 \ln T, S_t = S^{\mathrm{B}}_{i,l}\right\} \frac{\Delta_{i,j}}{\beta_k K} \\
&\le \sum_{i \in E_{\mathrm{B}}} \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \sum_{l=1}^{N_i} \sum_{j=1}^{N_i} \mathbb{I}\left\{\alpha_k \left(\frac{2\sigma K}{\Delta_{i,j-1}}\right)^2 \ln T < T_{i,t-1} \le \alpha_k \left(\frac{2\sigma K}{\Delta_{i,j}}\right)^2 \ln T, S_t = S^{\mathrm{B}}_{i,l}\right\} \frac{\Delta_{i,j}}{\beta_k K} \\
&\le \sum_{i \in E_{\mathrm{B}}} \sum_{k=1}^{\infty} \sum_{t=m+1}^{T} \sum_{j=1}^{N_i} \mathbb{I}\left\{\alpha_k \left(\frac{2\sigma K}{\Delta_{i,j-1}}\right)^2 \ln T < T_{i,t-1} \le \alpha_k \left(\frac{2\sigma K}{\Delta_{i,j}}\right)^2 \ln T\right\} \frac{\Delta_{i,j}}{\beta_k K}
\end{aligned}$$

$$\leq \sum_{i\in E_{\mathrm{B}}}\sum_{k=1}^{\infty}\sum_{j=1}^{N_i}\left(\alpha_k\left(\frac{2\sigma K}{\Delta_{i,j}}\right)^2\ln T-\alpha_k\left(\frac{2\sigma K}{\Delta_{i,j-1}}\right)^2\ln T\right)\frac{\Delta_{i,j}}{\beta_k K}$$
$$=4\sigma^2 K\left(\sum_{k=1}^{\infty}\frac{\alpha_k}{\beta_k}\right)\ln T\cdot\sum_{i\in E_{\mathrm{B}}}\sum_{j=1}^{N_i}\left(\frac{1}{\Delta_{i,j}^2}-\frac{1}{\Delta_{i,j-1}^2}\right)\Delta_{i,j}$$
$$\leq 1068\sigma^2 K\ln T\cdot\sum_{i\in E_{\mathrm{B}}}\sum_{j=1}^{N_i}\left(\frac{1}{\Delta_{i,j}^2}-\frac{1}{\Delta_{i,j-1}^2}\right)\Delta_{i,j},$$

where the last inequality is due to (18).

Finally, for each $i\in E_B$ we have

$$\begin{aligned}\sum_{j=1}^{N_i}\left(\frac{1}{\Delta_{i,j}^2}-\frac{1}{\Delta_{i,j-1}^2}\right)\Delta_{i,j}&=\frac{1}{\Delta_{i,N_i}}+\sum_{j=1}^{N_i-1}\frac{1}{\Delta_{i,j}^2}(\Delta_{i,j}-\Delta_{i,j+1})\\&\leq\frac{1}{\Delta_{i,N_i}}+\int_{\Delta_{i,N_i}}^{\Delta_{i,1}}\frac{1}{x^2}\mathrm{d}x\\&=\frac{2}{\Delta_{i,N_i}}-\frac{1}{\Delta_{i,1}}\\&<\frac{2}{\Delta_{i,\min}}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{t=m+1}^{T}\mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t}&\leq 1068\sigma^2 K\ln T\cdot\sum_{i\in E_{\mathrm{B}}}\frac{2}{\Delta_{i,\min}}\\&=\sigma^2 K\sum_{i\in E_{\mathrm{B}}}\frac{2136}{\Delta_{i,\min}}\ln T.\end{aligned}\tag{EC.14}$$

Combining (19) with Lemma 4, the distribution-dependent regret bound in Theorem 1 is proved.

To prove the distribution-independent bound, we decompose $\sum_{t=m+1}^{T}\mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t}$ into two parts:

$$\begin{aligned}\sum_{t=m+1}^{T}\mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t}&=\sum_{t=m+1}^{T}\mathbb{I}\{\mathcal{H}_t,\Delta_{S_t}\leq\epsilon\}\Delta_{S_t}\\&\quad+\sum_{t=m+1}^{T}\mathbb{I}\{\mathcal{H}_t,\Delta_{S_t}>\epsilon\}\Delta_{S_t}\\&\leq\epsilon T+\sum_{t=m+1}^{T}1\{\mathcal{H}_t,\Delta_{S_t}>\epsilon\}\Delta_{S_t},\end{aligned}\tag{EC.15}$$

where $\epsilon>0$ is a constant to be determined. The second term can be bounded in the same way as in the proof of the distribution-dependent regret bound, except that we only consider the case

$\Delta_{S_t} > \epsilon$. Thus we can replace (19) by

$$\begin{aligned} &\sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{H}_t, \Delta_{S_t} > \epsilon\}\Delta_{S_t} \\ &\leq \sigma^2 K \sum_{i \in E_{\mathrm{B}}, \Delta_{i,\min} > \epsilon} \frac{2136}{\Delta_{i,\min}} \ln T \\ &\leq \sigma^2 K m \frac{2136}{\epsilon} \ln T. \end{aligned} \tag{EC.16}$$

It follows that

$$\sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t} \leq \epsilon T + \sigma^2 K m \frac{2136}{\epsilon} \ln T.$$

Finally, letting $\epsilon = \sqrt{\frac{2136\sigma^2 K m \ln T}{T}}$, we get

$$\sum_{t=m+1}^{T} \mathbb{I}\{\mathcal{H}_t\}\Delta_{S_t} \leq 2\sqrt{2136\sigma^2 K m T \ln T} < 93\sigma\sqrt{mKT \ln T}.$$

Combining this with Lemma 4, we conclude the proof of the distribution-independent regret bound in Theorem 1. ■