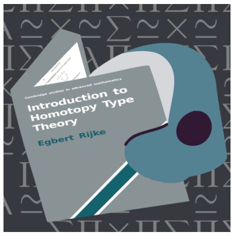

Worksheet 12 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 $(\star)$

For each of the following types, state how many unique[1] terms there are of that type, and list them. *Hint:* First figure out how many there are before defining them. Use Lemma 17.5.8 and think inductively!

(a)

$$\binom{\mathsf{Fin}\ 3}{\mathsf{Fin}\ 2}$$

There are three such elements:

$$(\textbf{Fin}\ 2, a),\ (\textbf{Fin}\ 2, b),\ (\textbf{Fin}\ 2, c) \quad : \quad \sum_{X:\mathcal{U}_{\textbf{Fin}\ 2}} X \hookrightarrow_{\textbf{d}} \textbf{Fin}\ 3$$

$$\begin{aligned} a(0) &\doteq 0 \\ a(1) &\doteq 1 \\ b(0) &\doteq 0 \\ b(1) &\doteq 2 \\ c(0) &\doteq 1 \\ c(1) &\doteq 2 \end{aligned}$$

Everything else is equal to one of these three, according to the characterization of identity types of binomial types we obtained.

(b)

$$\binom{\mathsf{Fin}\ n}{\mathbb{1}}$$

There are n such elements:

$$(\mathbb{1}, \textbf{const}_j) : \binom{\textbf{Fin}\ n}{\mathbb{1}} \qquad \text{for each } j : \mathsf{Fin}\ n$$

[1] Unique up to identity – assume univalence and function extensionality.

(c)

$$\binom{\mathsf{Fin}\ n}{\mathsf{Fin}\ n + \mathbb{1}}$$

This type is empty. There cannot be an embedding $\mathsf{Fin}(n) + \mathbb{1} \hookrightarrow \mathsf{Fin}\ n$.

2 $(\star\star)$

Consider a type A.

(a) We call a point $a : A$ *isolated* if the map $\mathsf{const}_a : \mathbb{1} \to A$ is a decidable embedding. Construct an equivalence

$$\binom{A}{\mathbb{1}} \simeq \sum_{a:A} \mathsf{is\text{-}isolated}(a).$$

Given (a, σ) on the right-hand side, we can construct the element

$$E(a, \sigma) := (\mathbb{1}, (\mathsf{const}_a, \sigma)) \colon \sum_{X:\mathcal{U}_{\mathbb{1}}} X \hookrightarrow_{\mathsf{d}} A.$$

This map is an equivalence: if (X, ψ) is any element of $\binom{A}{\mathbb{1}}$ then, since we know $\|X \simeq \mathsf{Fin}\ 1\|$, it must be that X is a singleton, i.e. $X = \mathbb{1}$. So, if ψ is a map $\mathbb{1} \to A$, then by function extensionality $\psi = \mathsf{const}_{\psi(\star)}$, so $(X, \psi) = E(a, \sigma)$ for some $a : A$. So the fibers of E are inhabited. We can readily check by function extensionality that E indeed has contractible fibers.

(b) Show that if A is a set, then $\binom{A}{\mathbb{1}} \simeq A$

This is a corollary of the previous part. This is because every point of a set is isolated: for any $a' : A$, the fiber

$$\mathsf{fib}_{\mathsf{const}_a}(a') \doteq \sum_{x:\mathbb{1}} \mathsf{const}_a(x) = a'$$

is equivalent to the identity type $a = a'$, which is a decidable proposition by the hypothesis that A is a set. So, since $\mathsf{is\text{-}isolated}$ is a proposition, we have

$$\binom{A}{\mathbb{1}} \simeq \sum_{a:A} \mathsf{is\text{-}isolated}(a) \simeq \sum_{a:A} \mathbb{1} \simeq A$$

(c) Construct an equivalence

$$\binom{A}{\mathbb{1}} \simeq \left(\sum_{X:\mathcal{U}} (X + \mathbb{1}) \simeq A\right)$$

conclude that the map $X \mapsto X + \mathbb{1}$ on a univalent universe $\mathcal{U}$ is 0-truncated.

(d) More generally, construct an equivalence

$$\binom{A}{B} \simeq \sum_{X:\mathcal{U}_B} \sum_{Y:\mathcal{U}} X + Y \simeq A$$

Given an $X : \mathcal{U}_B$ and a $Y : \mathcal{U}$ and an equivalence $e : X + Y \simeq A$, we construct an element of $\binom{A}{B}$ in the following way: X is, of course, the element of $\mathcal{U}_B$ we need, and then we obtain an embedding $\psi : X \hookrightarrow_{\mathsf{d}} A$ by composing the left injection map $X \hookrightarrow X + Y$ with the equivalence e. We can check that embeddings are preserved by composing with an equivalence, so this is an embedding. It is decidable because, given $a : A$, we can construct a term

$$\mathsf{fib}_\psi(a) + \neg\mathsf{fib}_\psi(a)$$

We do this by casing on $e^{-1}(a)$. If $e^{-1}(a)$ is $\mathsf{inl}(x)$ for some $x : X$, then, by construction, x is in $\mathsf{fib}_\psi(a)$. Otherwise, if $e^{-1}(a)$ is $\mathsf{inr}(y)$, then $e^{-1}(a)$ is not $\mathsf{inl}(x)$ for any $x : X$ and hence a is not $\psi(x)$ for any $x : X$, i.e. $\mathsf{fib}_\psi(a)$ is empty.

3 $(\star\star)$

Given a type A, the type of **unordered pairs** in A is defined to be

$$\mathsf{unordered\text{-}pairs}(A) := \sum_{X:BS_2} X \to A$$

(a) Construct an embedding

$$\binom{A}{\mathsf{bool}} \hookrightarrow \mathsf{unordered\text{-}pairs}(A)$$

Why does $\mathsf{unordered\text{-}pairs}(A)$ have, in general, more elements than $\binom{A}{\mathsf{bool}}$? Which elements of $\mathsf{unordered\text{-}pairs}(A)$ are *not* in the image of this embedding?

Observe that $\mathcal{U}_{\mathsf{bool}}$, the type of types $X : \mathcal{U}$ such that $\|X \simeq \mathsf{bool}\|$ is the same thing as BS_2. So we can say

$$\mathsf{unordered\text{-}pairs}(A) = \sum_{X:\mathcal{U}_{\mathsf{bool}}} X \to A.$$

Since $\binom{A}{\mathsf{bool}}$ is defined as

$$\sum_{X:\mathcal{U}_{\mathsf{bool}}} X \hookrightarrow_{\mathsf{d}} A$$

we can define a function $\binom{A}{\mathsf{bool}} \to \mathsf{unordered\text{-}pairs}(A)$ by sending each (X, ψ) to itself ("forgetting" that ψ is a decidable embedding). This is an embedding, by function extensionality.
The unordered pairs which are in $\binom{A}{\mathsf{bool}}$ are the ones whose two components are *distinct*. However, $\mathsf{unordered\text{-}pairs}(A)$ additionally contains unordered pairs whose components are the same. For example,

$$(\mathsf{bool}, \mathsf{const}_7) : \mathsf{unordered\text{-}pairs}(\mathbb{N})$$

because bool is a 2-element type and $\mathsf{const}_7 : \mathsf{bool} \to \mathbb{N}$. This encodes the unordered pair whose two components are both 7. But const_7 is not an embedding – the fiber of 7 is bool itself, which is not a proposition – so this is not an element of $\binom{\mathbb{N}}{\mathsf{bool}}$.

(b) The type of homotopy commutative binary operations from A to B is defined as

$$\mathsf{unordered\text{-}pairs}(A) \to B.$$

Show that if B is a set, then this type is equivalent to the type

$$\sum\nolimits_{m:A\to A\to B} \prod\nolimits_{x,y:A} m(x,y) = m(y,x).$$

(c) Show that the type of undirected graphs in $\mathcal{U}$ with at most one edge between any two vertices

$$\sum_{V:\mathcal{U}} (\mathsf{unordered\text{-}pairs}(V) \to \mathsf{Prop})$$

is equivalent to the type

$$\sum\nolimits_{V:\mathcal{U}} \sum\nolimits_{E:V\to V\to \mathsf{Prop}} \prod\nolimits_{x,y:V} E(x,y) \to E(y,x).$$

4 $(\star\star)$

Consider the following claim.

$$\binom{\mathsf{Fin}(n)}{\mathsf{bool}} \simeq \sum_{k:\mathsf{Fin}(n)} \mathsf{Fin}(k) \qquad (*)$$

Prove (*) by induction on n, using the equivalences from Lemma 17.5.8 and the identities proved above. You should not need to unfold the definition of $\binom{A}{B}$ or explicitly construct any decidable embeddings.

Start with $n \doteq 0$. The Lemma tells us that

$$\binom{\emptyset}{B + \mathbb{1}} \simeq \emptyset$$

so, since $\mathsf{bool} = \mathbb{1} + \mathbb{1}$, the left-hand side is equivalent to $\emptyset$. Since $\mathsf{Fin}(0) \doteq \emptyset$, the right hand side is also empty.
Now suppose

$$\binom{\mathsf{Fin}(n)}{\mathsf{bool}} \simeq \sum_{k:\mathsf{Fin}(n)} \mathsf{Fin}(k)$$

for some $n : \mathbb{N}$. Again, using the Lemma and basic identities of finite sets, we have

$$\binom{\mathsf{Fin}(\mathsf{suc}\ n)}{\mathsf{bool}} = \binom{\mathsf{Fin}(n) + \mathbb{1}}{\mathbb{1} + \mathbb{1}} \simeq \binom{\mathsf{Fin}(n)}{\mathbb{1}} + \binom{\mathsf{Fin}(n)}{\mathsf{bool}}$$

By Question 2b above and the fact that $\mathsf{Fin}(n)$ is a set, we know $\binom{\mathsf{Fin}\ n}{\mathbb{1}} \simeq \mathsf{Fin}\ n$. Applying the inductive hypothesis, we now have

$$\binom{\mathsf{Fin}(\mathsf{suc}\ n)}{\mathsf{bool}} \simeq (\mathsf{Fin}\ n) + \sum_{k:\mathsf{Fin}(n)} \mathsf{Fin}(k)$$

The right-hand side can easily be seen to be equivalent to $\sum_{k:\mathsf{Fin}(\mathsf{suc}\ n)} \mathsf{Fin}(k)$, as desired.

5 $(\star\star)$

Given a finite type A, show that the following are equivalent:

(i) The type of all decidable equivalence relations on A

(ii) The type of all surjective maps $A \to X$ into a finite type X

(iii) The type of finite types X equipped with a family $Y : X \to \mathsf{Fin}$ of finite types, such that each $Y(x)$ is inhabited and equipped with an equivalence

$$e : \left(\sum_{x:X} Y(x)\right) \simeq A$$

1. The type of all decidable partitions of A, i.e. the type of all $P : (A \to \mathsf{dProp}) \to \mathsf{dProp}$ such that each Q in P is inhabited, and such that for each $x : A$ the type of $Q : A \to \mathsf{dProp}$ such that $Q(x)$ holds and $P(Q)$ holds is contractible.