Lab 1 – Mathematical Proof

The problem is stated as follows:

> *Grab a stick. Pick a random point on the stick and break it in two. Take the longer piece. Now pick a random point on the longer piece and break it in two, to make three pieces altogether. What is the probability you can form a triangle?*

We claim that the probability is $\log(4) - 1 \approx 0.386$, and proceed to verify this claim.

Note that the criteria for creating a valid triangle from 3 sides is dependent on the fact that any two side lengths add up to a value greater than the third side length. This is usually stated as the *triangle inequality*: $||x + y|| \leq ||x|| + ||y||$, $\forall x, y \in V$, where V is some vector space on which a norm is defined (including the familiar $\mathbb{R}^2$, the vector space within which this problem resides).

Proof

The length of the stick has no bearing on the probability that it will form a triangle if broken as described, so we take a length of 1 for convenience. Pick some point p on this stick, breaking it into the proportion $p : 1-p$. Furthermore, assume $p < 1 - p$, and thus $p < 1/2$.

Now we wish to find the range of values within which we can break the "$1 - p$" stick such that we form a valid triangle (in terms of p). Once we have found a range, we can calculate the probability that our second choice of break point will lie in that range (again, in terms of p).

Note firstly that wherever we now decide to break the stick, we cannot produce a side of length greater than $1/2$, otherwise we could not possibly form a triangle (by the triangle inequality above). Thus the greatest distance we can possibly mark off on the stick (from the point p) is $1/2$. We're only considering these distances starting from the point p (otherwise we would not be choosing points on the "$1 - p$" side), so we add p to give us the absolute point from the start of the stick, which gives us $p + 1/2$ (this will be necessary later).

Now we need to find the minimum possible distance we can mark off from the point p, and in turn the absolute distance from the start of the whole stick. Assume we have already chosen such a point. Then the easiest way to calculate this is to consider the following expression:

$$p + x + y = 1$$

... where p, x, and y are the lengths of the sides we generate by picking such a point. If we incorporate our max bound into this, we should be able to solve for the minimum bound, since they are essentially duals to one another: when one side is at its maximum length, the other is necessarily at its minimum length.

$$p + \frac{1}{2} + x = 1 \quad \Rightarrow \quad x = \frac{1}{2} - p$$

Thus, the minimum length of this 'y' stick is $1/2 - p$, which when taken from our starting point p gives us the point $p + (1/2 - p) = 1/2$.

So we now have the range of values $(1/2, 1/2 + p)$ within which we can make a valid triangle, which clearly has length p. If we take this distance as a fraction of the length of the total range of values we can choose from (which has length $1 - p$), then we will get the probability that we choose a point on the "$1 - p$" stick that forms a triangle:

$$\frac{p}{1-p} = \text{probability of forming a triangle given } p < \frac{1}{2}$$

Getting the probability of forming a triangle for any given p is fine, but we need to find the probability over *all* possible p. Thus, since this is a continuous probability distribution, we need only integrate over all valid values of p, which lie between 0 and $1/2$, as we assumed at the beginning.

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{p}{1-p} dp \\
&= \int_0^{\frac{1}{2}} \frac{1}{1-p} - 1 \, dp \\
&= [-\log(1-p) - p]_0^{\frac{1}{2}} \\
&= -\log\left(\frac{1}{2}\right) - \frac{1}{2} \\
&= \log(2) - \frac{1}{2}
\end{aligned}$$

Now recall that we only considered half the problem (i.e. where $p < 1/2$), but since the other half will have the same probability as the first half, we can simply multiply our answer by 2 to get our final answer, and we are done.

$$2\log(2) - 1 = \log(4) - 1 \approx 0.386 \qquad \square$$