

Double Pendulum equations of motion and numerical results

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Abstract

To be abstracted :)

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1 Introduction

To be introduced :)

I'll be modelling a pendulum composed of two rigid rods as opposed to point masses hanged on ropes.

2 Equations of motion

Starting with the equations as derived on paper, will fill with explanation after!

2.1 The Lagrangian

We'll define each rod $i = \{1, 2\}$ as having length (l_i), mass (m_i) and moment of inertia about its center of mass (I_{ci}). Additionally we will have the gravitational acceleration norm g as a constant ($g \cong 9.8[\text{m/s}^2]$). Normal vectors for the two "arms" of the pendulum are

$$\hat{n}_1 = (\sin \theta, -\cos \theta)^T \quad (1)$$

$$\hat{n}_2 = (\sin \varphi, -\cos \varphi)^T. \quad (2)$$

Positions of the centre of mass for the two rods are

$$\vec{r}_{c1} = \frac{l_1}{2} \hat{n}_1 \quad (3)$$

$$\vec{r}_{c2} = l_1 \hat{n}_1 + \frac{l_2}{2} \hat{n}_2. \quad (4)$$

The Lagrangian is taken as the sum of kinetic terms minus the sum of potential terms, which gives

$$L = \frac{1}{2} m_1 |\dot{\vec{r}}_{c1}|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_{c2}|^2 + \frac{1}{2} I_{c1} |\dot{\vec{\omega}}_{c1}|^2 + \frac{1}{2} I_{c2} |\dot{\vec{\omega}}_{c2}|^2 - m_1 g y_{c1} - m_2 g y_{c2}, \quad (5)$$

where $\dot{\vec{\omega}}_{ci}$ are the angular rates around the centres of mass for the two rods. Note that constant terms have been neglected (since a Lagrangian is unique up to linear transformation with constants). It is easy to see that $|\dot{\vec{\omega}}_{c1}| = \dot{\theta}$ and $|\dot{\vec{\omega}}_{c2}| = \dot{\varphi}$. Also, given a homogeneous rod of mass m and length l , the moment of inertia around its centre of mass is

$$I_c = \frac{1}{12} m l^2. \quad (6)$$

Using (1)–(4) in (5), one gets the full form

$$L = \left(\frac{1}{6} m_1 l_1^2 + \frac{1}{2} m_2 l_2^2 \right) \dot{\theta}^2 + \frac{1}{6} m_2 l_2^2 \dot{\varphi}^2 + \frac{1}{2} m_2 l_1 l_2 \cos(\theta - \varphi) \dot{\theta} \dot{\varphi} + \frac{1}{2} m_1 g l_1 \cos \theta + m_2 g \left(l_1 \cos \theta + \frac{l_2}{2} \cos \varphi \right). \quad (7)$$

A lot of the clutter comes from constants, so I've chosen to simplify to

$$L = a_\theta \dot{\theta}^2 + a_\varphi \dot{\varphi}^2 + a_{\text{mix}} \cos(\theta - \varphi) \dot{\theta} \dot{\varphi} + b_\theta \cos \theta + b_\varphi \cos \varphi, \quad (8)$$

where I've defined the constants

$$a_\theta = \frac{1}{6} m_1 l_1^2 + \frac{1}{2} m_2 l_2^2 \quad (9)$$

$$a_\varphi = \frac{1}{6} m_2 l_2^2 \quad (10)$$

$$a_{\text{mix}} = \frac{1}{2} m_2 l_1 l_2 \quad (11)$$

$$b_\theta = l_1 g \left(\frac{m_1}{2} + m_2 \right) \quad (12)$$

$$b_\varphi = \frac{1}{2} l_2 g m_2. \quad (13)$$

At this point the equations of motion can be readily derived.

2.2 The Equations of Motion

Lagrange's equations from a minimum action principle give dynamical equations

$$\frac{d}{dt} \left(\frac{dL}{d\dot{q}^i} \right) = \frac{dL}{dq^i}, \quad (14)$$

where q^i are the general positions (configuration space parameters) and \dot{q}^i are their derivatives (general velocities). The combination of all of these $\{q^1, q^2, \dots, q^N, \dot{q}^1, \dot{q}^2, \dots, \dot{q}^N\}$ is the full state of the system (phase space). In our case $N = 2$ and the state is $\vec{s} = (\theta, \varphi, \dot{\theta}, \dot{\varphi})^T$. I will further denote $\dot{\theta}$ and $\dot{\varphi}$.

Computing the equations (14) for the Lagrangian (8) yields the equations of motion

$$2a_\theta \ddot{\theta} + a_{\text{mix}} \cos(\varphi - \theta) \ddot{\varphi} = -b_\theta \sin \theta + a_{\text{mix}} \sin(\varphi - \theta) \dot{\varphi}^2 \quad (15)$$

$$2a_\varphi \ddot{\varphi} + a_{\text{mix}} \cos(\varphi - \theta) \ddot{\theta} = -b_\varphi \sin \varphi - a_{\text{mix}} \sin(\varphi - \theta) \dot{\theta}^2 \quad (16)$$

To be continued :)

3 Conclusions

To be concluded :)

References

- [1] H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*.