Double Pendulum equations of motion and numerical results

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Abstract

To be abstracted :)

Contents

1 Introduction

To be introduced :)

I'll be modelling a pendulum composed of two rigid rods as opposed to point masses hanged on ropes.

2 Equations of motion

Starting with the equations as derived on paper, will fill with explanation after!

2.1 The Lagrangian

We'll define each rod $i = \{1, 2\}$ as having length (l_i), mass (m_i) and moment of inertia about its center of mass (I_{ci}). Additionally we will have the gravitational acceleration norm g as a constant ($g \cong 9.8[\mathrm{m/s^2}]$). Normal vectors for the two "arms" of the pendulum are

$$\begin{aligned} \hat{n}_1 &= (\sin\theta, -\cos\theta)^T && (1) \\ \hat{n}_2 &= (\sin\varphi, -\cos\varphi)^T. && (2) \end{aligned}$$

Positions of the centre of mass for the two rods are

$$\begin{aligned} \vec{r}_{c1} &= \frac{l_1}{2}\hat{n}_1 && (3) \\ \vec{r}_{c2} &= l_1\hat{n}_1 + \frac{l_2}{2}\hat{n}_2. && (4) \end{aligned}$$

The Lagrangian is taken as the sum of kinetic terms minus the sum of potential terms, which gives

$$L = \frac{1}{2}m_1|\dot{\vec{r}}_{c1}|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_{c2}|^2 + \frac{1}{2}I_{c1}|\vec{\omega}_{c1}|^2 + \frac{1}{2}I_{c2}|\vec{\omega}_{c2}|^2 - m_1 g y_{c1} - m_2 g y_{c2}, \tag{5}$$

where $\dot{\vec{\omega}}_{ci}$ are the angular rates around the centres of mass for the two rods. Note that constant terms have been neglected (since a Lagrandian is unique up to linear transformation with constants). It is easy to see that $|\vec{\omega}_{c1}| = \dot{\theta}$ and $|\vec{\omega}_{c2}| = \dot{\varphi}$. Also, given a homogeneous rod of mass m and length l, the moment of inertia around its centre of mass is

$$I_c = \frac{1}{12}ml^2. \tag{6}$$

Using (1)–(4) in (5), one gets the full form

$$\begin{aligned} L = \; & \left(\tfrac{1}{6}m_1 l_1^2 + \tfrac{1}{2}m_2 l_2^2\right)\dot{\theta}^2 + \tfrac{1}{6}m_2 l_2^2\dot{\varphi}^2 + \tfrac{1}{2}m_2 l_1 l_2\cos(\theta-\varphi)\dot{\theta}\dot{\varphi} && (7) \\ & +\tfrac{1}{2}m_1 g l_1\cos\theta + m_2 g\left(l_1\cos\theta + \tfrac{l_2}{2}\cos\varphi\right). \end{aligned}$$

A lot of the clutter comes from constants, so I've chosen to simplify to

$$L = a_\theta\dot{\theta}^2 + a_\varphi\dot{\varphi}^2 + a_{\mathrm{mix}}\cos(\theta-\varphi)\dot{\theta}\dot{\varphi} + b_\theta\cos\theta + b_\varphi\cos\varphi, \tag{8}$$

where I've defined the constants

$$\begin{aligned} a_\theta &= \frac{1}{6}m_1 l_1^2 + \frac{1}{2}m_2 l_2^2 && (9) \\ a_\varphi &= \frac{1}{6}m_2 l_2^2 && (10) \\ a_{\mathrm{mix}} &= \frac{1}{2}m_2 l_1 l_2 && (11) \\ b_\theta &= l_1 g\left(\frac{m_1}{2} + m_2\right) && (12) \\ b_\varphi &= \frac{1}{2}l_2 g m_2. && (13) \end{aligned}$$

At this point the equations of motion can be readily derived but for numerical reasons I'll switch to Hamiltonian formalism.

2.2 The Hamiltonian

3 Conclussions

To be concluded :)

References

[1] H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*.