

1 Streams

In Python, we can use iterators to represent infinite sequences (for example, the generator for all natural numbers). However, Scheme does not support iterators. Let's see what happens when we try to use a Scheme list to represent an infinite sequence of natural numbers:

```
scm> (define (naturals n)
      (cons n (naturals (+ n 1))))
naturals
scm> (naturals 0)
Error: maximum recursion depth exceeded
```

Because `cons` is a regular procedure and both its operands must be evaluated before the pair is constructed, we cannot create an infinite sequence of integers using a Scheme list. Instead, our Scheme interpreter supports *streams*, which are *lazy* Scheme lists. The first element is represented explicitly, but the rest of the stream's elements are computed only when needed. Computing a value only when it's needed is also known as *lazy evaluation*.

```
scm> (define (naturals n) (define nat (naturals 0))
      (cons-stream n (naturals (+ n 1))))
naturals
scm> (define nat (naturals 0))
nat
scm> (car nat)
0
scm> (cdr nat)
#[promise (not forced)]
scm> (car (cdr-stream nat))
1
scm> (car (cdr-stream (cdr-stream nat)))
2
```

We use the special form `cons-stream` to create a stream:

```
(cons-stream <operand1> <operand2>)
```

`cons-stream` is a special form because the second operand is not evaluated when evaluating the expression. To evaluate this expression, Scheme does the following:

1. Evaluate the first operand.
2. Construct a promise containing the second operand.
3. Return a pair containing the value of the first operand and the promise.

2 Streams

To actually get the rest of the stream, we must call `cdr-stream` on it to force the promise to be evaluated. Note that this argument is only evaluated once and is then stored in the promise; subsequent calls to `cdr-stream` returns the value without recomputing it. This allows us to efficiently work with infinite streams like the `naturals` example above. We can see this in action by using a non-pure function to compute the rest of the stream:

```
scm> (define (compute-rest n)
...>   (print 'evaluating!)
...>   (cons-stream n nil))
compute-rest
scm> (define s (cons-stream 0 (compute-rest 1)))
s
scm> (car (cdr-stream s))
evaluating!
1
scm> (car (cdr-stream s))
1
```

Here, the expression `compute-rest 1` is only evaluated the first time `cons-stream` is called, so the symbol `evaluating!` is only printed the first time.

When displaying a stream, the first element of the stream and the promise are displayed separated by a dot (this indicates that they are part of the same pair, with the promise as the `cdr`). If the value in the promise has not been evaluated by calling `cdr-stream`, we consider it to be not forced. Otherwise, we consider it forced.

```
scm> (define s (cons-stream 1 nil))
s
scm> s
(1 . #[promise (not forced)])
scm> (cdr-stream s) ; nil
()
scm> s
(1 . #[promise (forced)])
```

Streams are very similar to Scheme lists in that they are also recursive structures. Just like the `cdr` of a Scheme list is either another Scheme list or `nil`, the `cdr-stream` of a stream is either a stream or `nil`. The difference is that whereas both arguments to `cons` are evaluated upon calling `cons`, the second argument to `cons-stream` isn't evaluated until the first time that `cdr-stream` is called.

Here's a summary of what we just went over:

- `nil` is the empty stream
- `cons-stream` constructs a stream containing the value of the first operand and a promise to evaluate the second operand
- `car` returns the first element of the stream
- `cdr-stream` computes and returns the rest of stream

Questions

1.1 What would Scheme display?

As you work through these problems, remember that streams have two important components:

- Lazy evaluation – so the remainder of the stream isn't computed until explicitly requested.
- Memoization – so anything we compute won't be recomputed.

The examples here stretch these concepts to the limit. In most practical use cases, you may find you rarely need to redefine functions that compute the remainder of the stream.

```
scm> (define (has-even? s)
      (cond ((null? s) #f)
            ((even? (car s)) #t)
            (else (has-even? (cdr-stream s)))))

has-even?
scm> (define (f x) (* 3 x))
f
scm> (define nums (cons-stream 1 (cons-stream (f 3) (cons-stream (f 5) nil))))
nums

scm> nums
(1 . #[promise (not forced)])
scm> (cdr nums)
(9 15) #[promise (not forced)]
scm> (cdr-stream nums)
(9 . #[promise (not forced)])
scm> nums
(1 . #[promise (forced)])
scm> (define (f x) (* 2 x))
f
scm> (cdr-stream nums)
(9 . #[promise (not forced)])
scm> (cdr-stream (cdr-stream nums))
(10 . #[promise (not forced)])
scm> (has-even? nums)
#t
```

← **cdr just returns the second element of the pair**

← **cdr-stream will not evaluate the second part, even its value is clear**

- 1.2 Using streams can be tricky! Compare the following two implementations of `filter-stream`, the first is a correct implementation whereas the second is wrong in some way. What's wrong with the second implementation?

; Correct

```
(define (filter-stream f s)
  (cond
    ((null? s) nil)
    ((f (car s)) (cons-stream (car s) (filter-stream f (cdr-stream s))))
    (else (filter-stream f (cdr-stream s)))))
```

; Incorrect

```
(define (filter-stream f s)
  (if (null? s) nil
      (let ((rest (filter-stream f (cdr-stream s))))
        (if (f (car s))
            (cons-stream (car s) rest)
            rest))))
```

I guess if the `s` is an infinite list, then we will not get result.

- 1.3 Write a function `map-stream`, which takes a function `f` and a stream `s`. It returns a new stream which has all the elements from `s`, but with `f` applied to each one.

```
(define (map-stream f s)
```

```
(define (map-stream f s)
  (if (null? s) '()
      (cons-stream (f (car s)) (map-stream f (cdr-stream s)))))
```

```
scm> (define evens (map-stream (lambda (x) (* x 2)) nat))
```

```
evens
```

```
scm> (car (cdr-stream evens))
```

```
2
```

- 1.4 Write a function `slice` which takes in a stream `s`, a `start`, and an `end`. It should return a Scheme list that contains the elements of `s` between index `start` and `end`, not including `end`. If the stream ends before `end`, you can return `nil`.

```
(define (slice s start end)
```

```
(define (slice s start end)
  (if (null? s)
      '()
      (cond
        ((> start 0)
         (slice (cdr-stream s) (- start 1) (- end 1)))
        ((<= end 0)
         '())
        (else
         (cons (car s) (slice (cdr-stream s) start (- end 1)))))))
```

```
scm> (define nat (naturals 0)) ; See naturals procedure defined earlier
```

```
nat
```

```
scm> (slice nat 4 12)
```

```
(4 5 6 7 8 9 10 11)
```

- 1.5 Since streams only evaluate the next element when they are needed, we can combine infinite streams together for interesting results! Use it to define a few of our favorite sequences. We've defined the function `combine-with` for you below, as well as an example of how to use it to define the stream of even numbers.

```
(define (combine-with f xs ys)
  (if (or (null? xs) (null? ys))
      nil
      (cons-stream
        (f (car xs) (car ys))
        (combine-with f (cdr-stream xs) (cdr-stream ys)))))

scm> (define evens (combine-with + (naturals 0) (naturals 0)))
evens
scm> (slice evens 0 10)
(0 2 4 6 8 10 12 14 16 18)
```

For these questions, you may use the `naturals` stream in addition to `combine-with`.

- i. (define factorials

```
(define factorials
  (cons-stream 1 (combine-with * factorials (naturals 1)))))
```

```
scm> (slice factorials 0 10)
(1 1 2 6 24 120 720 5040 40320 362880)
```

- ii. (define fibs

```
(define fibs
  (cons-stream 0 (cons-stream 1 (combine-with + fibs (cdr-stream fibs)))))
```

```
scm> (slice fibs 0 10)
(0 1 1 2 3 5 8 13 21 34)
```

- iii. (Extra for practice) Write `exp`, which returns a stream where the n th term represents the degree- n polynomial expansion for e^x , which is $\sum_{i=0}^n x^i/i!$.

You may use `factorials` in addition to `combine-with` and `naturals` in your solution.

(define (exp x)

```
(define (exp x)
  (cons-stream 1 (combine-with + (exp x) (cdr-stream (exp-helper2 x)))))
```

```
scm> (slice (exp 2) 0 5)
(1 3 5 6.333333333333333 7)
```

- 1.6 We can even represent the sequence of all prime numbers as an infinite stream! Define a function `sieve`, which takes in a stream of increasing numbers and returns a stream containing only those numbers which are not multiples of an earlier number in the stream. We can define `primes` by sifting all natural numbers starting at 2. Look online for the **Sieve of Eratosthenes** if you need some inspiration.

Hint: You might find using `filter-stream` as defined earlier helpful.

`(define (sieve s)`

```
(define (sieve s)
  (if (null? s)
      '()
      (cons-stream (car s)
                    (sieve (filter-stream
                           (lambda (x) (not (= 0 (remainder x (car s))))
                           (cdr-stream s)))))))
```

`(define primes`

`(sieve (naturals 2)))`

`scm> (slice primes 0 10)`

`(2 3 5 7 11 13 17 19 23 29)`

2 Extra Questions

- 2.1 Write a macro that takes an expression and returns a parameter-less lambda procedure with the expression as its body

```
(define-macro (make-lambda expr)
```

```
(define-macro (make-lambda expr)
  (list 'lambda () expr))
```

```
scm> (make-lambda (print 'hi))
(lambda () (print (quote hi)))
scm> (make-lambda (/ 1 0))
(lambda () (/ 1 0))
scm> (define print-3 (make-lambda (print 3)))
print-3
scm> (print-3)
3
```

- 2.2 Using the `make-lambda` macro you defined Question 1, define `make-stream`, a macro which returns a pair of elements, where the second element is not evaluated until `cdr-stream` is called on it. Also define the procedure `cdr-stream`, which takes in a stream returned by `make-stream` and returns the result of evaluating the second element in the stream pair.

Unlike the streams we've seen in lecture and earlier in discussion, if you repeatedly call `cdr-stream` on a stream returned by `make-stream`, you may evaluate an expression multiple times.

```
(define-macro (make-stream first second)
```

```
(define-macro (make-stream first second)
  (list 'cons (list (list 'make-lambda first)) (list 'make-lambda second)))

(define (cdr-stream stream)
  ((cdr stream)))
```

```
scm> (define a (make-stream (print 1) (make-stream (print 2) nil)))
1
a
scm> (define b (cdr-stream a))
2
b
scm> (cdr-stream b)
()
```