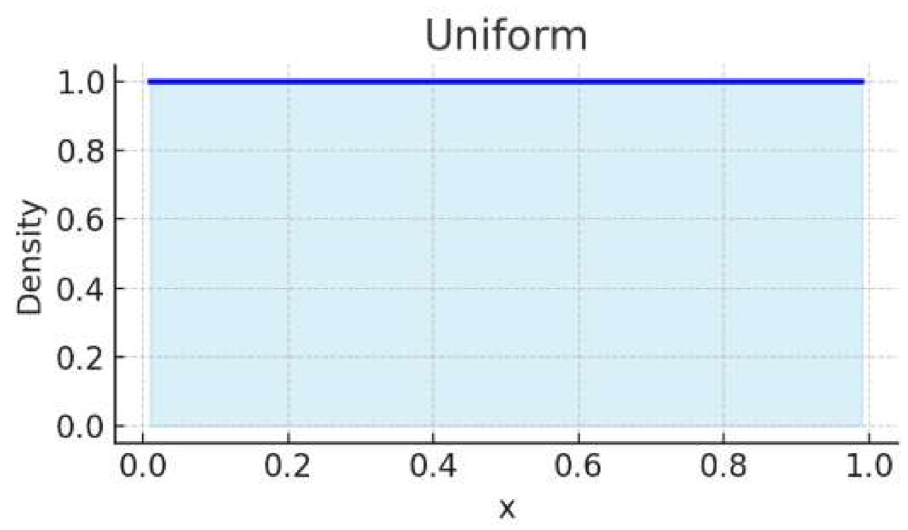
Probability distributions

domenica 7 settembre 2025 15:20

Uniform

$X \sim \mathcal{U}([a,b])$

$$\boxed{f(x) = \frac{1}{b-a}\,\mathbb{1}_{[a,b]}(x)}$$

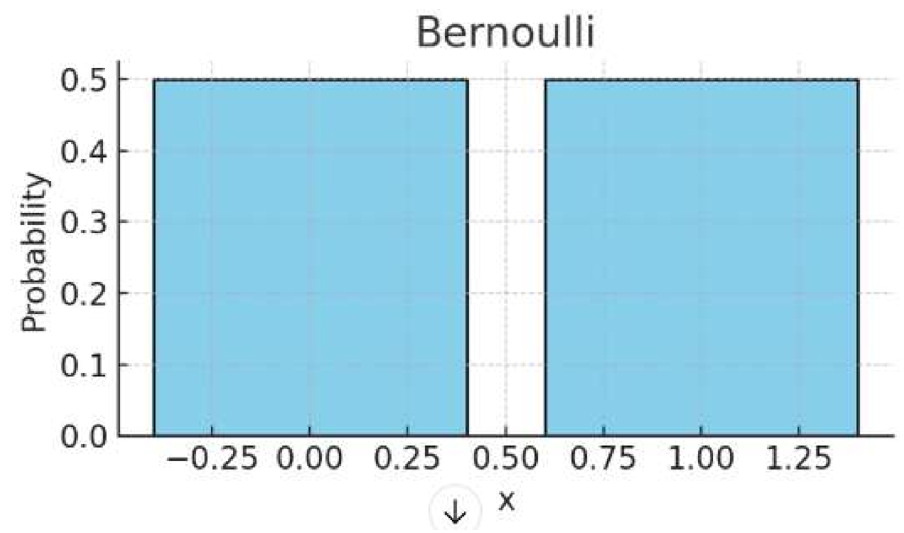


- $\mathbb{E}(X) = \dfrac{a+b}{2}$
- $\mathrm{Var}(X) = \dfrac{(b-a)^2}{12}$

Bernoulli

$X \sim Be(p)$

- $\mathbb{E}(X) = p$
- $\mathrm{Var}(X) = p(1-p)$



$$\mathbb{P}(X=0) = 1-p$$
$$\mathbb{P}(X=1) = p$$

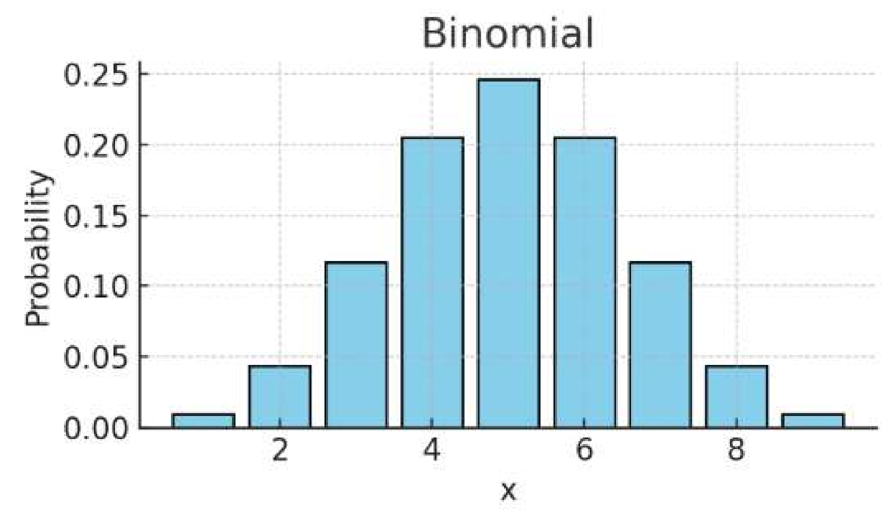
Binomial

⇝ represents k successes in n trials

$X \sim Bi(n,p) \rightsquigarrow X = X_1 + X_2 + \dots + X_n$ s.t. $X_i \sim Be(p)$ $\forall i = 1:n$ iiD

$$\boxed{\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

- $\mathbb{E}(X) = np$
- $\mathrm{Var}(X) = np(1-p)$



NOTES:

$$Bi\left(n, \frac{\lambda}{n}\right) \xrightarrow[n \to \infty]{} P(\lambda)$$

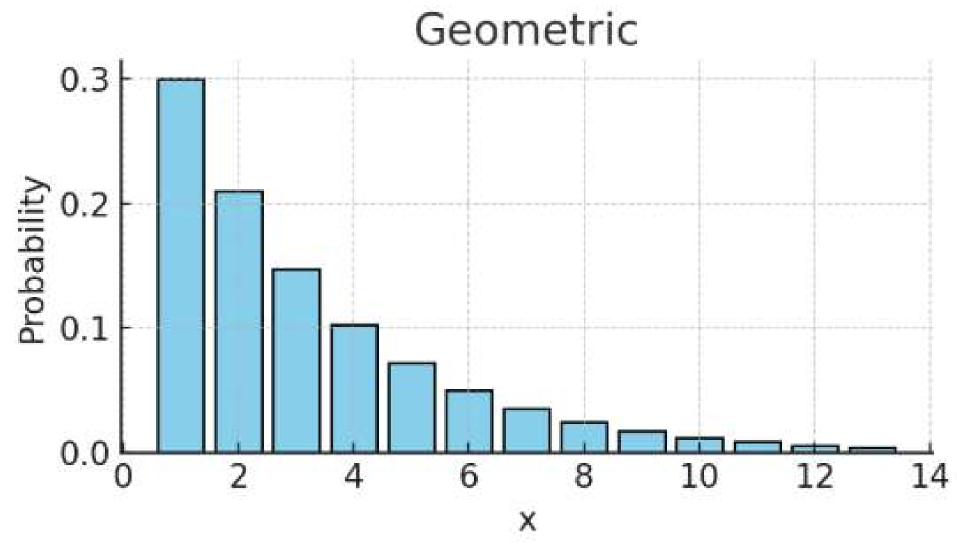
Geometric

$\rightsquigarrow$ represents the first success on the k-th trial

$X \sim Ge(p)$

$$\boxed{\mathbb{P}(X=k) = p(1-p)^{k-1}}$$

- $\mathbb{E}(X) = \frac{1}{p}$
- $Var(X) = \frac{(1-p)}{p^2}$



NOTES:

It has no memory:

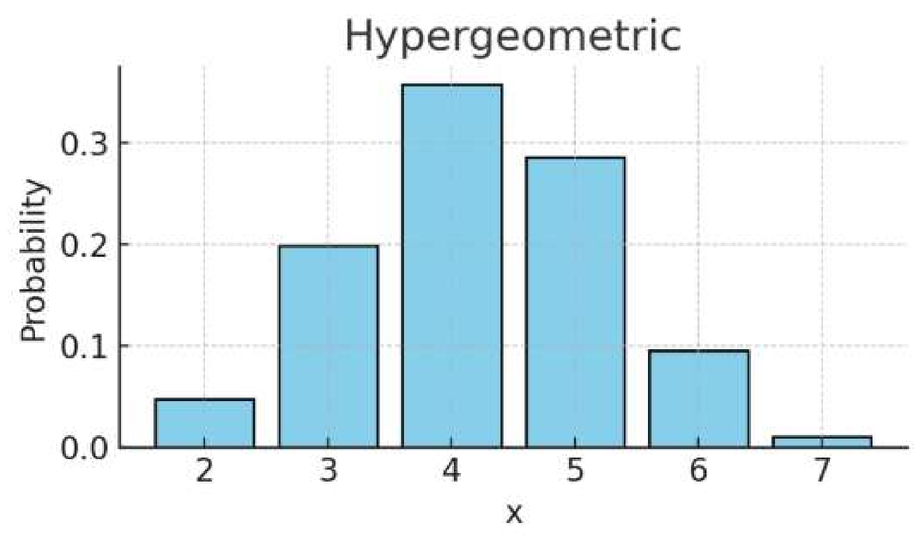
$$\mathbb{P}(X = m+n \mid X > n) = \mathbb{P}(X = m)$$

Hypergeometric

$\rightsquigarrow$ represents the probability of drawing, from a set of n balls of which h are white and n–h are red, exactly k white balls when r balls are drawn in total (without replacement)

$X \sim H(n, h, r)$

$$\boxed{\mathbb{P}(X=k) = \frac{\binom{h}{k}\binom{n-h}{r-k}}{\binom{n}{r}}}$$



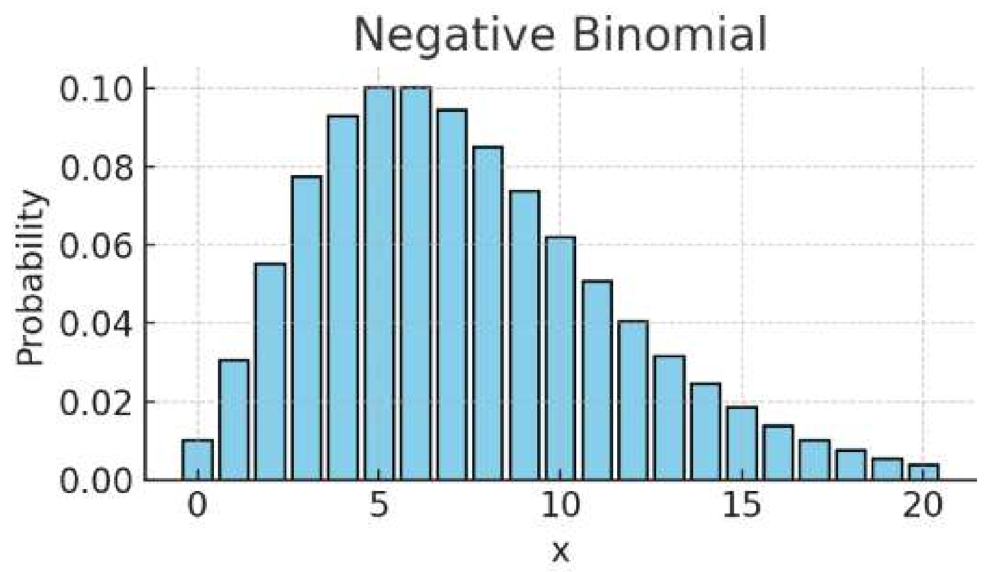
- $\mathbb{E}(X) = \frac{rh}{n}$
- $Var(X) = \frac{r(n-r)h(n-h)}{n^2(n-1)}$

Pascal $\rightsquigarrow$ T_n counts the number of failures before the n-th success

$T_n \sim NB(p,n) \rightsquigarrow T_n := \min\{t : X_1 + \dots + X_{t+n} = n\}$ s.t. $X_i \sim Be(p)$ $\forall i = 1:n$ IID

$$\boxed{\mathbb{P}(T_n = k) = \binom{k+n-1}{k} p^n (1-p)^k}$$

$$\boxed{\bullet\ \mathbb{E}(T_n) = \frac{n}{p}} \qquad \boxed{\bullet\ Var(T_n) = n\,\frac{1-p}{p^2}}$$



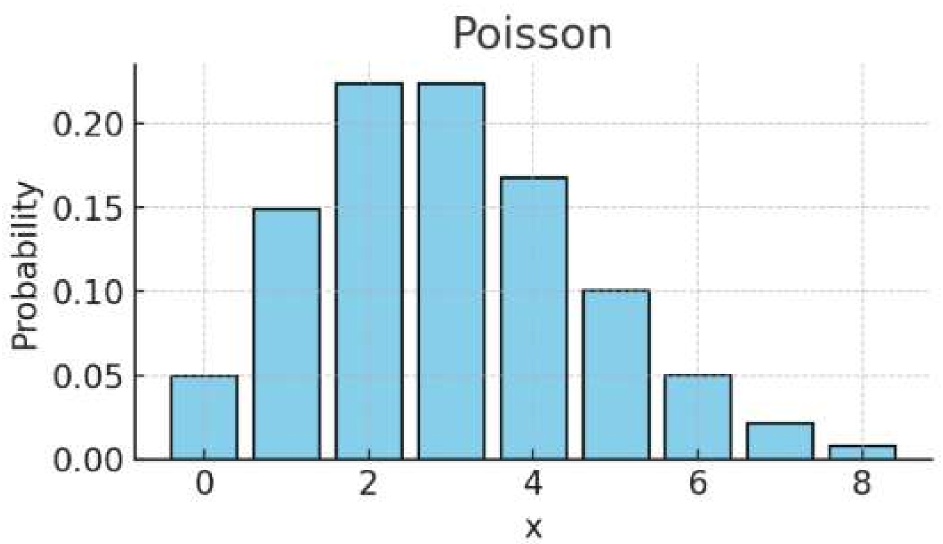
NOTES:

$$NB\left(\frac{r-\lambda}{r}, r\right) \xrightarrow[r\to\infty]{} P(\lambda)$$

Poisson $\rightsquigarrow$ expresses the probability of the number of events occurring successively and independently within a given time interval, given that on average λ events occur

$X \sim P(\lambda)$

$$\boxed{\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}}$$



$$\boxed{\bullet\ \mathbb{E}(X) = \lambda}$$

$$\boxed{\bullet\ Var(X) = \lambda}$$

NOTES:

$$Y = \sum_{i=1}^{n} Y_i \quad \text{s.t. } Y_i \sim P(\lambda_i)\ \forall i = 1:n \Rightarrow Y \sim P\left(\sum_{i=1}^{n} \lambda_i\right)$$

Gamma

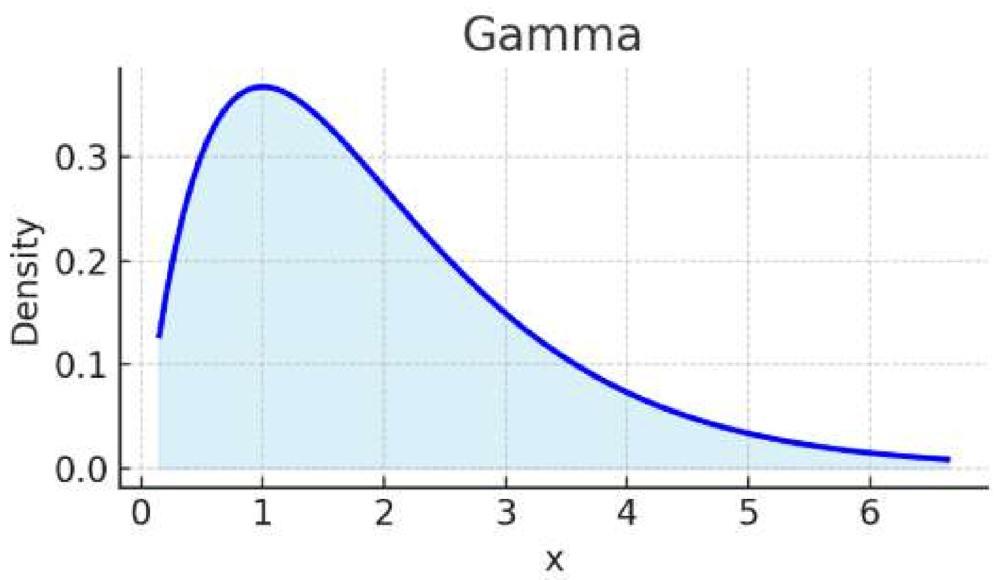
$X \sim \Gamma(k, \theta)$

$$\boxed{f(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}}}$$

where $\Gamma(k) = \int_0^{+\infty} t^{k-1} e^{-t} \, dt$

Euler Gamma function

- $\mathbb{E}(X) = k\theta$
- $Var(X) = k\theta^2$



NOTES:

1) $\Gamma(1, \theta) = \mathcal{E}(1/\theta)$

2) $\Gamma\left(\frac{n}{2}, \frac{1}{2}\right) = \chi^2(n)$

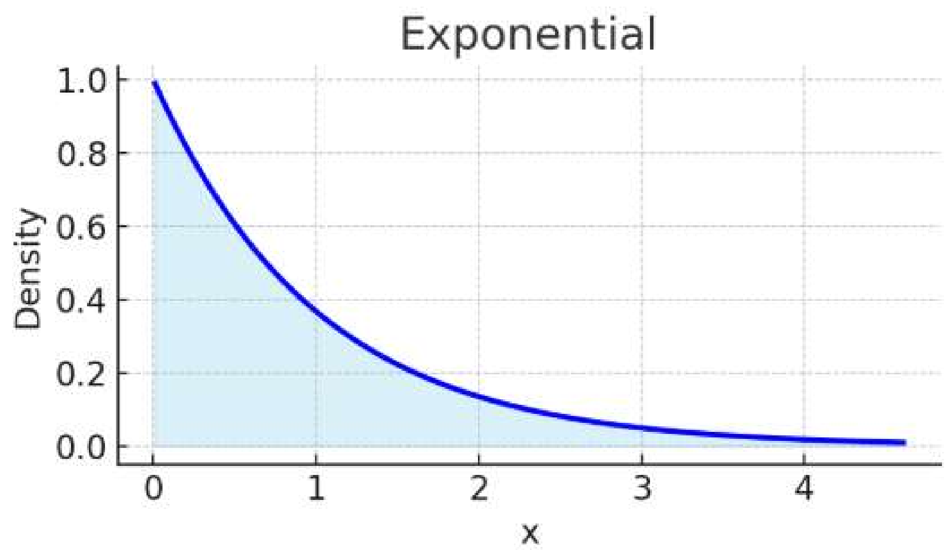
3) $X \sim \Gamma(k_1, \theta)$ $\quad Y \sim \Gamma(k_2, \theta)$ $\quad Z = \frac{X}{X+Y} \Rightarrow Z \sim Beta(k_1, k_2)$

Exponential

→ describes the lifetime of a phenomenon that does not age (i.e., it is memoryless)

$X \sim \mathcal{E}(\lambda)$

$$\boxed{f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}}$$



- $\mathbb{E}(X) = \frac{1}{\lambda}$
- $Var(X) = \frac{1}{\lambda^2}$

NOTES:

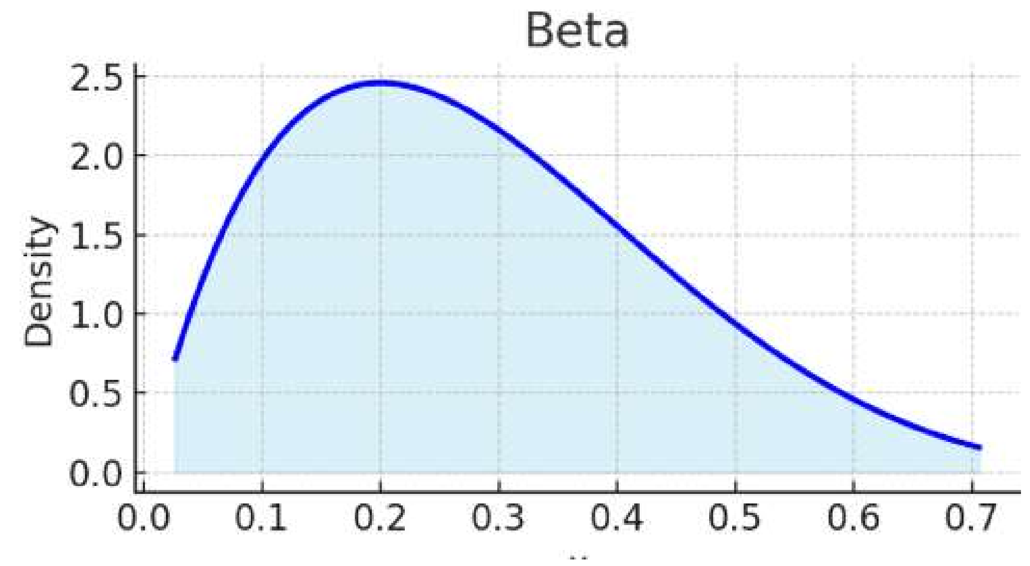
$\mathcal{E}\left(\frac{1}{2}\right) = \chi^2(2)$

Beta $\rightsquigarrow$ this distribution arises naturally in Bayesian inference, because it governs the probability p of a Bernoulli process after observing α – 1 successes and β – 1 failures, when p is a priori uniformly distributed between 0 and 1

$X \sim Beta(\alpha, \beta)$

$$\boxed{f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}} \qquad \text{where} \quad B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$\boxed{\bullet\ \mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}} \qquad \boxed{\bullet\ Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$



NOTES :

1) $X \sim Beta(\alpha, \beta) \Rightarrow 1 - X \sim Beta(\beta, \alpha)$

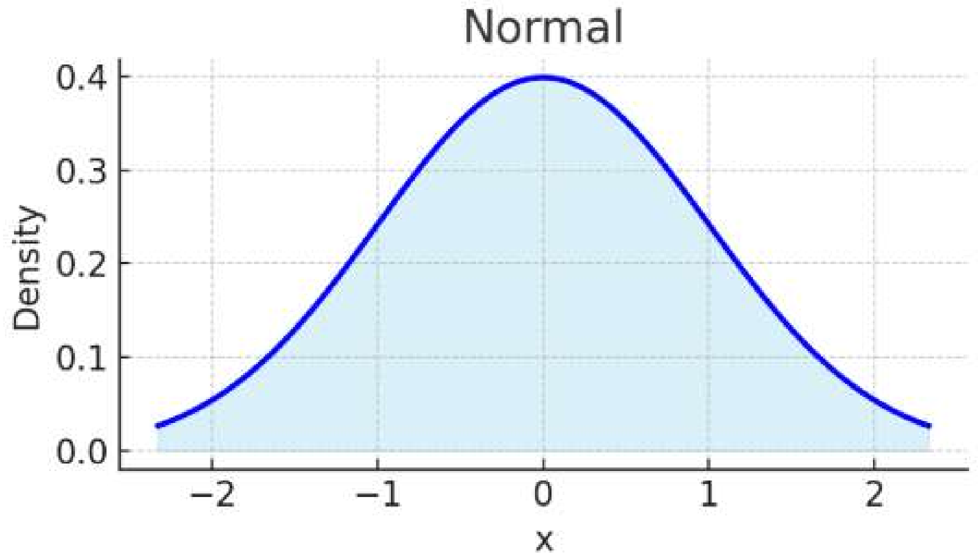
2) $Beta(1,1) = \mathcal{U}([0,1])$

3) $Y = a + (b-a)X$ s.t. $X \sim Beta(\alpha, \beta) \Rightarrow Y \sim Beta(\alpha, \beta)$ defined on the interval $[a, b]$

Normal

$$X \sim N(\mu, \sigma^2) \quad \rightsquigarrow \quad Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$



- $\boxed{\mathbb{E}(X) = \lambda}$
- $\boxed{Var(X) = \sigma^2}$

NOTES:

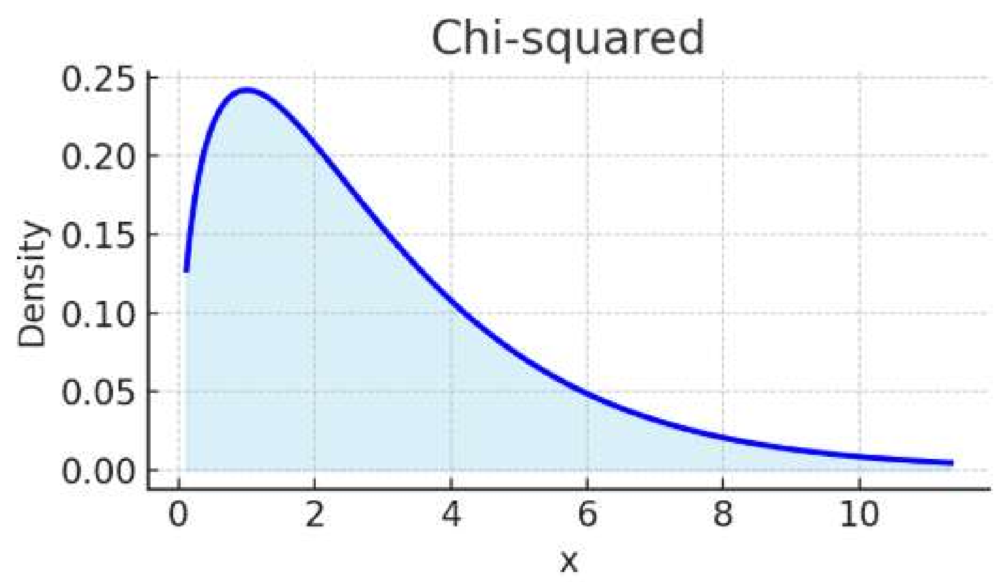
1) $Y = \sum_{i=1}^{n} \alpha_i X_i$ s.t. $X_i \sim N(\mu_i, \sigma_i^2)$ $\forall i = 1:n \Rightarrow Y \sim N\left(\sum_{i=1}^{n} \alpha_i \mu_i, \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2\right)$

2) $P(\lambda) \sim N(\lambda, \lambda)$ for λ really huge

χ^2 - squared

$$X \sim \chi^2(k) \quad \leadsto \quad X = \sum_{i=1}^{k} X_i^2 \quad \text{s.t. } X_i \sim N(0,1) \quad \forall i = 1:k \quad \text{IID}$$

$$\boxed{f(x) = \frac{1}{2^{k/2}\,\Gamma(k/2)}\, x^{\frac{k}{2}-1}\, e^{-x/2} \quad \text{for } x > 0}$$



$$\text{where} \quad \Gamma(k/2) = \begin{cases} \sqrt{\pi}\,\dfrac{(k-2)!!}{2^{(k-1)/2}} & k \text{ odd} \\ \left(\dfrac{k}{2}-1\right)! & k \text{ even} \end{cases}$$

- $\boxed{\mathbb{E}(X) = k}$
- $\boxed{Var(X) = 2k}$

NOTES:

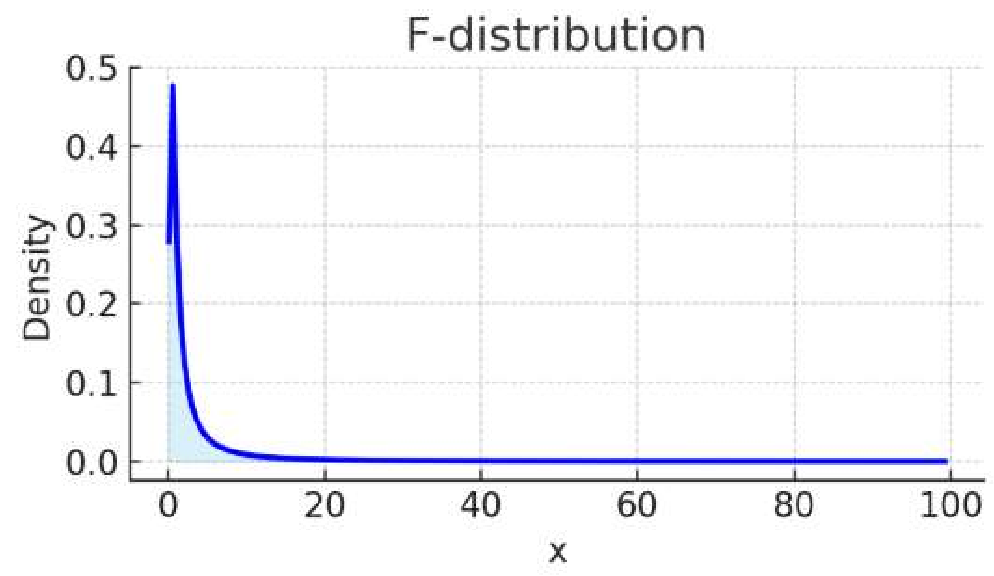
1) $\chi^2(k) \xrightarrow[k \to \infty]{} N(0,1)$

2) $X \sim \chi^2(m) \quad Y \sim \chi^2(n) \quad Z = X + Y \Rightarrow Z \sim \chi^2(m+n)$

Fisher

$$F \sim F(m,n) \rightsquigarrow F := \frac{X/m}{Y/n} \quad \text{s.t. } X \sim \chi^2(m) \quad Y \sim \chi^2(n)$$

$$\boxed{f(x) = \frac{1}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \frac{1}{x} \left(\frac{m^m n^n x^m}{(mx+n)^{m+n}} \right)^{1/2}}$$



$$\boxed{\bullet\ \mathbb{E}(F) = \frac{n}{n-2} \quad \text{for } n > 2}$$

$$\boxed{\bullet\ Var(F) = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)} \quad \text{for } n > 4}$$

NOTES:

1) $B = \dfrac{mF}{mF+n} \sim Beta\left(\frac{m}{2}, \frac{n}{2}\right)$

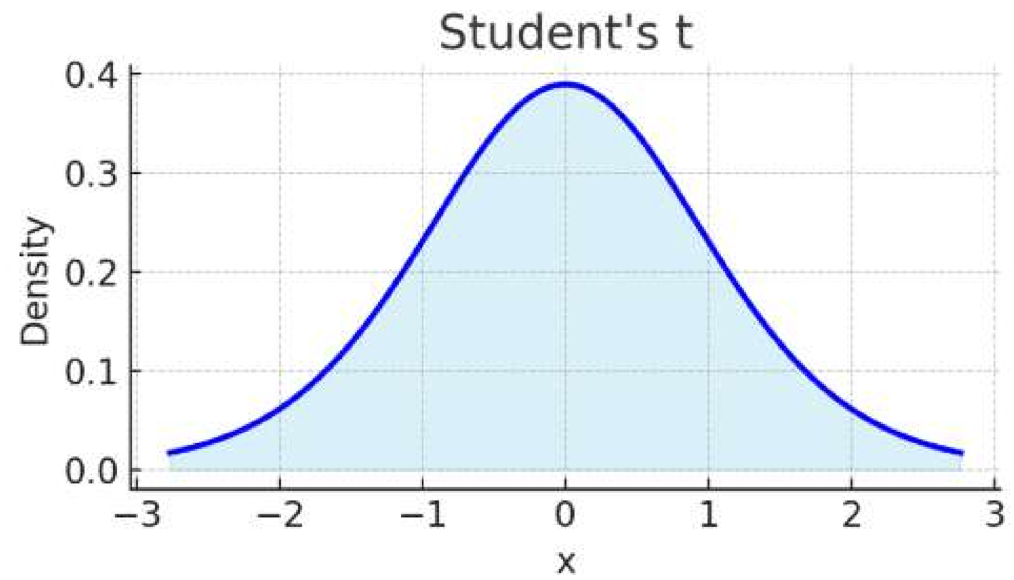
2) $T \sim t(\nu) \Rightarrow F = T^2 \sim F(1,\nu)$

t - student

$$X \sim t(\nu) \implies X = \frac{Z}{\sqrt{K/n}} \quad \text{s.t. } Z \sim N(0,1) \quad K \sim \chi^2(n)$$

with n degree of freedom

$$\boxed{f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{(n+1)}{2}}}$$



- $\boxed{\mathbb{E}(X) = 0}$

- $\boxed{Var(X) = \frac{n}{n-2} \quad \text{for } n > 2}$

NOTES:

$$\mathbb{E}_k(X) = \begin{cases} 0 & k \text{ odd} \\ \dfrac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right) n^{k/2}}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} & k \text{ even} \end{cases}$$