Supplementary Material for "A Boundary Value Problem Approach to Autonomous Rendezvous in Industrial Human-Drone Interaction"

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1. BACKGROUND ON INDIRECT METHOD

Let $[t_0, t_f]$ denote the finite time horizon, where $t_0 \in \mathbb{R}$ is the initial time, $t_f \in \mathbb{R}$ is the final time, and $t \in [t_0, t_f]$ denotes the continuous time variable. Let $\mathcal{U}$ denote the set of admissible control functions $u : [t_0, t_f] \to U \subseteq \mathbb{R}^m$, where U is a compact subset of the control space, and let $\mathcal{S}$ denote the set of admissible state trajectories $s : [t_0, t_f] \to \mathbb{R}^n$ with initial condition $s(t_0) = s_0$. The mapping $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ defines the system dynamics, the mapping $\xi : \mathbb{R}^n \to \mathbb{R}^q$ defines the terminal constraints at time t_f, the function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ defines the running cost, and the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ defines the terminal cost.

The optimal control problem, defined over the finite time horizon $[t_0, t_f]$, consists in determining $u^\star(\cdot) \in \mathcal{U}$ and the corresponding optimal state trajectory $s^\star(\cdot) \in \mathcal{S}$ that minimize the performance index

$$J[u(\cdot)] = \Phi(s(t_f)) + \int_{t_0}^{t_f} L(s(t), u(t), t)\, dt, \tag{1}$$

subject to the dynamical system

$$\dot{s}(t) = f(s(t), u(t), t), \qquad s(t_0) = s_0, \tag{2}$$

and the terminal equality constraints

$$\xi(s(t_f)) = 0. \tag{3}$$

According to the *Pontryagin Maximum Principle (PMP)* formulated in Kopp (1962), the Hamiltonian is introduced as

$$H(s(t), u(t), \lambda(t), t) = L(s(t), u(t), t) + \lambda^\top f(s(t), u(t), t), \tag{4}$$

where $\lambda(t) \in \mathbb{R}^n$ is the costate vector associated with the state dynamics. The necessary conditions of optimality consist of the state dynamics

$$\dot{s}(t) = \frac{\partial H}{\partial \lambda}(s(t), u(t), \lambda(t), t), \qquad s(t_0) = s_0, \tag{5}$$

the costate dynamics

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial s}(s(t), u(t), \lambda(t), t), \tag{6}$$

and the Hamiltonian minimization condition

$$u^\star(t) = \arg\min_{u(t)\in U} H(s(t), u(t), \lambda(t), t). \tag{7}$$

For fixed final time and terminal constraints, the terminal costate satisfies

$$\lambda(t_f) = \frac{\partial \Phi}{\partial s}(s(t_f)) + \left(\frac{\partial \xi}{\partial s}(s(t_f))\right)^\top \nu, \tag{8}$$

where $\nu \in \mathbb{R}^q$ is the multiplier vector associated with the terminal equality constraints. For free terminal state components, indexed by $i \in \mathcal{I} := \{1, \ldots, I\}$, the corresponding costate components satisfy

$$\lambda_i(t_f) = 0, \qquad \forall i \in \mathcal{I}. \tag{9}$$

When the final time t_f is free, the Hamiltonian satisfies the additional transversality condition

$$H(s(t_f), u(t_f), \lambda(t_f), t_f) = 0. \tag{10}$$

The coupled system composed of the state and costate equations, together with the Hamiltonian minimization and transversality conditions, defines a *two-point Boundary Value Problem (BVP)* (Avvakumov and Kiselev (2000)) in the unknown trajectories $s : [t_0, t_f] \to \mathbb{R}^n$, $\lambda : [t_0, t_f] \to \mathbb{R}^n$, and $u : [t_0, t_f] \to U$. The boundary specification is mixed in nature, with initial conditions imposed on the state vector at time t_0 and terminal conditions imposed on the costate vector and multipliers at time t_f. Hence, the BVP formulation constitutes the foundation of the *indirect method* (Wang (2009)), reformulating the optimal control problem as a differential system where the necessary conditions of the PMP are enforced through

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boundary conditions imposed at both the initial and final times.

2. STATE–COSTATE DYNAMICS

According to the PMP, the necessary conditions of optimality for a continuous–time system with Hamiltonian $H^{\mathrm{d}}\big(s^{\mathrm{d}}(t), u^{\mathrm{d}}(t), \lambda^{\mathrm{d}}(t), t\big)$ consist of the canonical equations

$$\dot{s}^{\mathrm{d}}(t) = \frac{\partial H^{\mathrm{d}}}{\partial \lambda^{\mathrm{d}}}\big(s^{\mathrm{d}}(t), u^{\mathrm{d}}(t), \lambda^{\mathrm{d}}(t), t\big), \qquad s^{\mathrm{d}}(t_0) = \mathrm{s}_0^{\mathrm{d}}, \tag{11}$$

$$\dot{\lambda}^{\mathrm{d}}(t) = -\frac{\partial H^{\mathrm{d}}}{\partial s^{\mathrm{d}}}\big(s^{\mathrm{d}}(t), u^{\mathrm{d}}(t), \lambda^{\mathrm{d}}(t), t\big), \tag{12}$$

together with the Hamiltonian minimization condition

$$u^{\mathrm{d},\star}(t) = \arg\min_{u^{\mathrm{d}}(t)\in\mathcal{U}} H^{\mathrm{d}}\big(s^{\mathrm{d}}(t), u^{\mathrm{d}}(t), \lambda^{\mathrm{d}}(t), t\big). \tag{13}$$

By substituting the explicit dynamics and cost function of the drone, the extended Hamiltonian reads

$$\begin{aligned} H^{\mathrm{d}}\big(s^{\mathrm{d}}(t), u^{\mathrm{d}}(t), t\big) &= L^{\mathrm{d}}\big(s^{\mathrm{d}}(t), u^{\mathrm{d}}(t), t\big) + \lambda_p^{\mathrm{d}}(t)^\top v^{\mathrm{d}}(t) \\ &\quad + \lambda_v^{\mathrm{d}}(t)^\top \left(-\frac{\mathrm{k}^{\mathrm{d}}}{\mathrm{m}^{\mathrm{d}}}\, v^{\mathrm{d}}(t) + \frac{1}{\mathrm{m}^{\mathrm{d}}}\, u^{\mathrm{d}}(t)\right), \end{aligned} \tag{14}$$

where the running cost is

$$\begin{aligned} L^{\mathrm{d}}\big(s^{\mathrm{d}}(t), u^{\mathrm{d}}(t), t\big) &= \|u^{\mathrm{d}}(t)\|_{\mathrm{R}}^2 + \|v^{\mathrm{d}}(t)\|_{\mathrm{Q}}^2 \\ &\quad + \big(v^{\mathrm{d}}(t) - v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t)\big)^\top \mathrm{S}\big(v^{\mathrm{d}}(t) - v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t)\big). \end{aligned} \tag{15}$$

The canonical state equations follow from the partial derivative of the Hamiltonian with respect to the costate variables:

$$\frac{\partial H^{\mathrm{d}}}{\partial \lambda^{\mathrm{d}}(t)} = \begin{pmatrix} \dfrac{\partial H^{\mathrm{d}}}{\partial \lambda_p^{\mathrm{d}}(t)} \\ \dfrac{\partial H^{\mathrm{d}}}{\partial \lambda_v^{\mathrm{d}}(t)} \end{pmatrix} \Longrightarrow \begin{cases} \dot{p}^{\mathrm{d}}(t) = v^{\mathrm{d}}(t), \\ \dot{v}^{\mathrm{d}}(t) = -\dfrac{\mathrm{k}^{\mathrm{d}}}{\mathrm{m}^{\mathrm{d}}}\, v^{\mathrm{d}}(t) + \dfrac{1}{\mathrm{m}^{\mathrm{d}}}\, u^{\mathrm{d}}(t). \end{cases} \tag{16}$$

The Hamiltonian minimization condition gives

$$\frac{\partial H^{\mathrm{d}}}{\partial u^{\mathrm{d}}(t)} = 0 \quad \Rightarrow \quad u^{\mathrm{d},\star}(t) = \mathrm{proj}_{\mathcal{U}}\big(-\tfrac{1}{2\mathrm{m}^{\mathrm{d}}}\,\mathrm{R}^{-1}\lambda_v^{\mathrm{d}}(t)\big), \tag{17}$$

yielding the explicit state dynamics

$$\dot{p}^{\mathrm{d}}(t) = v^{\mathrm{d}}(t), \qquad \dot{v}^{\mathrm{d}}(t) = -\frac{\mathrm{k}^{\mathrm{d}}}{\mathrm{m}^{\mathrm{d}}}\, v^{\mathrm{d}}(t) + u^{\mathrm{d},\star}(t). \tag{18}$$

The costate dynamics is obtained from the negative partial derivative of the Hamiltonian with respect to the state variables:

$$\dot{\lambda}^{\mathrm{d}}(t) = -\begin{pmatrix} \dfrac{\partial H^{\mathrm{d}}}{\partial p^{\mathrm{d}}(t)} \\ \dfrac{\partial H^{\mathrm{d}}}{\partial v^{\mathrm{d}}(t)} \end{pmatrix}. \tag{19}$$

Since $H^{\mathrm{d}}(\cdot)$ depends on the drone position $p^{\mathrm{d}}(t)$ only through the safety term $v_{\mathrm{safe}}(t) = \tilde{v}_{\mathrm{safe}}(p^{\mathrm{d}}(t), p^{\mathrm{h}}(t))$, its partial derivative with respect to position can be expressed as

$$\frac{\partial H^{\mathrm{d}}}{\partial p^{\mathrm{d}}(t)} = \frac{\partial L}{\partial v_{\mathrm{safe}}(t)}\,\frac{\partial v_{\mathrm{safe}}(t)}{\partial p^{\mathrm{d}}(t)}, \qquad \frac{\partial v_{\mathrm{safe}}(t)}{\partial p^{\mathrm{d}}(t)} = \tilde{v}'_{\mathrm{safe}}(d(t))\,\hat{r}(t), \tag{20}$$

where the unit relative position vector is defined as $\hat{r}(t) = \frac{p^{\mathrm{d}}(t) - \mathrm{p}^{\mathrm{h}}(t)}{\|p^{\mathrm{d}}(t) - \mathrm{p}^{\mathrm{h}}(t)\|}$, and points from the human operator toward the drone and $\tilde{v}'_{\mathrm{safe}}(d(t))$ is equal to

$$\tilde{v}'_{\mathrm{safe}}(d(t)) = \begin{cases} 0, & d(t) \le \mathrm{d}_{\mathrm{stop}}, \\ \dfrac{\mathrm{v}_{\mathrm{cruise}}}{\mathrm{d}_{\mathrm{cruise}} - \mathrm{d}_{\mathrm{stop}}}, & \mathrm{d}_{\mathrm{stop}} < d(t) < \mathrm{d}_{\mathrm{cruise}}, \\ 0, & d(t) \ge \mathrm{d}_{\mathrm{cruise}}. \end{cases} \tag{21}$$

Let $e(t) = v^{\mathrm{d}}(t) - v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t)$, where $\hat{v}^{\mathrm{d}}(t) = v^{\mathrm{d}}(t)/\|v^{\mathrm{d}}(t)\|$ represents the instantaneous heading direction of the drone. The derivative of the running cost L^{d} with respect to $v_{\mathrm{safe}}(t)$ is then

$$\frac{\partial L}{\partial v_{\mathrm{safe}}(t)} = -\,2\,\hat{v}^{\mathrm{d}}(t)^\top \mathrm{S}\big(v^{\mathrm{d}}(t) - v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t)\big), \tag{22}$$

which, when combined with the chain rule above, gives

$$\frac{\partial H^{\mathrm{d}}}{\partial p^{\mathrm{d}}(t)} = -\,2\,\tilde{v}'_{\mathrm{safe}}(d(t))\,\hat{r}(t)\,\hat{v}^{\mathrm{d}}(t)^\top \mathrm{S}\big(v^{\mathrm{d}}(t) - v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t)\big). \tag{23}$$

Thus, the position costate dynamics is defined as

$$\dot{\lambda}_p^{\mathrm{d}}(t) = 2\,\tilde{v}'_{\mathrm{safe}}(d(t))\,\hat{r}(t)\,\hat{v}^{\mathrm{d}}(t)^\top \mathrm{S}\big(v^{\mathrm{d}}(t) - v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t)\big). \tag{24}$$

For the velocity component, the Hamiltonian depends explicitly and implicitly on $v^{\mathrm{d}}(t)$ through the running cost L^{d} and the dynamic term. The partial derivative is

$$\frac{\partial H^{\mathrm{d}}}{\partial v^{\mathrm{d}}(t)} = \frac{\partial L^{\mathrm{d}}}{\partial v^{\mathrm{d}}(t)} + \lambda_p^{\mathrm{d}}(t) - \frac{\mathrm{k}^{\mathrm{d}}}{\mathrm{m}^{\mathrm{d}}}\,\lambda_v^{\mathrm{d}}(t), \tag{25}$$

which leads to the costate dynamics

$$\dot{\lambda}_v^{\mathrm{d}}(t) = \frac{\mathrm{k}^{\mathrm{d}}}{\mathrm{m}^{\mathrm{d}}}\,\lambda_v^{\mathrm{d}}(t) - \lambda_p^{\mathrm{d}}(t) - \frac{\partial L^{\mathrm{d}}}{\partial v^{\mathrm{d}}(t)}. \tag{26}$$

Considering

$$L^{\mathrm{d}} = \|u^{\mathrm{d}}(t)\|_{\mathrm{R}}^2 + \|v^{\mathrm{d}}(t)\|_{\mathrm{Q}}^2 + e(t)^\top \mathrm{S} e(t), \tag{27}$$

the derivative of L^{d} with respect to $v^{\mathrm{d}}(t)$ reads

$$\frac{\partial L^{\mathrm{d}}}{\partial v^{\mathrm{d}}(t)} = 2\mathrm{Q}\,v^{\mathrm{d}}(t) + 2\mathrm{S}\,e(t)\,\frac{\partial e(t)}{\partial v^{\mathrm{d}}(t)}. \tag{28}$$

Since $e(t)$ depends on $v^{\mathrm{d}}(t)$ both directly and through $\hat{v}^{\mathrm{d}}(t)$, its derivative is

$$\begin{aligned} \frac{\partial e(t)}{\partial v^{\mathrm{d}}(t)} &= \mathrm{I} - v_{\mathrm{safe}}(t)\,\frac{\partial \hat{v}^{\mathrm{d}}(t)}{\partial v^{\mathrm{d}}(t)}, \\ \frac{\partial \hat{v}^{\mathrm{d}}(t)}{\partial v^{\mathrm{d}}(t)} &= \frac{1}{\|v^{\mathrm{d}}(t)\|}\big(\mathrm{I} - \hat{v}^{\mathrm{d}}(t)\,\hat{v}^{\mathrm{d}}(t)^\top\big). \end{aligned} \tag{29}$$

Assuming that $v^{\mathrm{d}}(t)$ varies slowly so that its direction changes negligibly, the operator $(\mathrm{I} - \hat{v}^{\mathrm{d}}\hat{v}^{\mathrm{d}\top})$ acts as a projection onto the plane orthogonal to $\hat{v}^{\mathrm{d}}(t)$. Hence,

$$(\mathrm{I} - \hat{v}^{\mathrm{d}}\hat{v}^{\mathrm{d}\top})\,\big(v^{\mathrm{d}}(t) - v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t)\big) \simeq 0, \tag{30}$$

which implies that only the component of $v^{\mathrm{d}}(t)$ along its heading direction is relevant in the derivative. Under this assumption, (30) simplifies to the approximate form

$$\frac{\partial L^{\mathrm{d}}}{\partial v^{\mathrm{d}}(t)} \simeq 2(\mathrm{Q} + \mathrm{S})\,v^{\mathrm{d}}(t) - 2\mathrm{S}\,v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t). \tag{31}$$

Since $v_{\mathrm{safe}}(t)$ depends only on the drone position and not explicitly on the velocity, no additional coupling term appears in the velocity costate dynamics. Substituting the expression of $\frac{\partial L^{\mathrm{d}}}{\partial v^{\mathrm{d}}(t)}$ into the canonical costate equation yields

$$\dot{\lambda}_v^{\mathrm{d}}(t) = \frac{\mathrm{k}^{\mathrm{d}}}{\mathrm{m}^{\mathrm{d}}}\,\lambda_v^{\mathrm{d}}(t) - \lambda_p^{\mathrm{d}}(t) - 2(\mathrm{Q}+\mathrm{S})\,v^{\mathrm{d}}(t) + 2\,\mathrm{S}\,v_{\mathrm{safe}}(t)\,\hat{v}^{\mathrm{d}}(t). \tag{32}$$

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