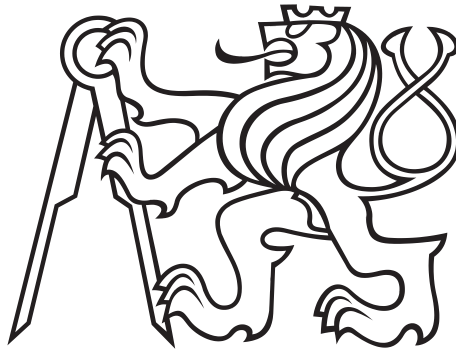


# RISK-AVERSE DISTRIBUTIONAL REINFORCEMENT LEARNING

A CVAR OPTIMIZATION APPROACH

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Silvestr Stanko: *Risk-Averse Distributional Reinforcement Learning*, a CVaR Optimization Approach, © February, 2018

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## ABSTRACT

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Short summary of the contents in English. . . a great guide by Kent Beck how to write good abstracts can be found here:

<https://plg.uwaterloo.ca/~migod/research/beckOOPSLA.html>

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## ABSTRAKT

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Český abstrakt

*We have seen that computer programming is an art,  
because it applies accumulated knowledge to the world,  
because it requires skill and ingenuity, and especially  
because it produces objects of beauty.*

— **knuth:1974** [**knuth:1974**]

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## ACKNOWLEDGMENTS

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Put your acknowledgments here.

Many thanks to everybody who already sent me a postcard!

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*Regarding LyX:* The LyX port was initially done by *Nicholas Mariette* in March 2009 and continued by *Ivo Pletikosić* in 2011. Thank you very much for your work and for the contributions to the original style.

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<sup>1</sup> Members of GuIT (Gruppo Italiano Utilizzatori di T<sub>E</sub>X e L<sup>A</sup>T<sub>E</sub>X)

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## INTRODUCTION

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A staple of an intelligent agent is the ability to reason and act over time in an environment, while working towards a desirable goal. This is the setting explored in reinforcement learning (RL), a branch of machine learning that focuses on dynamic decision making in unknown environments.

Recent advances in artificial intelligence (AI) are encouraging governments and corporations to deploy AI in high-stakes settings including driving cars autonomously, managing the power grid, trading on stock exchanges, and controlling autonomous weapon systems. As the industry steps away from specialized AI systems towards more general solutions, the demand for safe approaches to artificial intelligence increases.

In this thesis, we tackle one aspect of safe reinforcement learning, robustness, by considering the risk involved in acting in a non-deterministic, noisy environment.

### 1.1 MOTIVATION

Lately, there has been a surge of successes in machine learning research and applications, ranging from visual object detection [23] to machine translation [4]. Reinforcement learning has also been a part of this success, with excellent results regarding human-level control in computer games [28] or beating the best human players in the game of Go [37]. While these successes are certainly respectable and of great importance, reinforcement learning still has a long way to go before being applied on critical real-world decision-making tasks. This is partially caused by concerns of safety, as mistakes can be costly in the real world.

One of the problems encountered when training a reinforcement learning agent is sample efficiency, or the large amount of training time needed for the agent to successfully learn new and correct behaviors. The solution used by many [many] is to train the agent in simulation - it is indeed faster (as the simulation can run in parallel or faster than real-time), safer (we do not face any real danger in simulations) and cheaper than to train the agent in the real world. This approach then raises the question if an agent trained in simulation would perform well outside of the simulation.

Robustness, or distributional shift, is one of the identified issues of AI safety [1, 25] directly tied to the discrepancies between the environment the agent trains on and is tested on. Chow et al. [15] have shown that risk, a measure of uncertainty of the potential loss/reward, can be seen as equal to robustness, taking into account the differences during train- and test-time. This point is discussed in more detail in chapter 2.



While the term risk is a general one, we will focus on a concrete notion of risk - a particular risk metric called Conditional Value-at-Risk (CVaR). Due to its favorable computational properties, CVaR has been recognized as the industry standard for measuring risk in finance, as in 2014 the Basel Committee on Banking Supervision changed its guidelines for banks to replace VaR (a previously used metric) with CVaR for assessing market risk [16]. The metric also satisfies the recently proposed axioms of risk in robotics [26].

Another motivational point, aside from robustness, might be one of general decision-making. Commonly encountered in finance, decision makers face the problem of maximizing profits while keeping the risks to a minimum. The solutions to problems encountered in this thesis can therefore be seen as ones of general time-dependent risk-averse decision making.

The aim of this thesis is to consider reinforcement learning agents that maximize Conditional Value-at-Risk instead of the usual expected value, hereby learning a robust, risk-averse policy. The word *distributional* in the title emphasizes that our approach takes inspirations from, or directly uses, the recent advances in distributional reinforcement learning [6, 17].

## 1.2 THESIS OUTLINE

## 1.3 CONTRIBUTIONS

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## PRELIMINARIES

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The goal of this chapter is to provide a formal background on the covered material, together with a unified notation (which differs quite a lot from publication to publication). After we establish some basic theoretical foundations in Section 2.1, we remind the reader of the basic notions of reinforcement learning in Section 2.2 and of the recently explored and useful distributional reinforcement learning in Section 2.3. We follow up with the basics of risk together with the crucial CVaR measure in Section 2.4.

The interested reader is welcome to explore the books and publications referenced throughout this chapter and in Section 2.6. An informed reader may choose to skip to Section 2.5 where we formalize the problems tackled in this thesis.

### 2.1 PROBABILITY ESSENTIALS

**unclear:** define random variable and expectation?

### 2.2 REINFORCEMENT LEARNING

The idea that we learn by interacting with our environment is probably the first to occur to us when we think about the nature of learning. Reinforcement learning [39] is a sub-field of machine learning that deals with time-dependent decision making in an unknown environment. The learner (often called agent) is not told which actions to take, but instead must discover which actions yield the most reward by trying them out. In the most interesting and challenging cases, actions may affect not only the immediate reward but also subsequent situations and rewards. These two characteristics, trial-and-error search and delayed reward are the most important distinguishing features of reinforcement learning.

The general interaction between the agent and an environment can be seen in Figure ???. In each time-step  $t$ , the agent receives an observation  $x_t$  and a reward  $r_t$  and picks an action  $a_t$  and the process repeats. Below we formalize all the necessary notions of states, actions and rewards as a Markov Decision Process.

#### 2.2.1 Markov Decision Processes

Markov Decision Process (MDP, Bellman [8]) is a classical formalization of sequential decision making, where actions influence not just immediate rewards, but also subsequent situations, or states, and through those future rewards. They are a math-

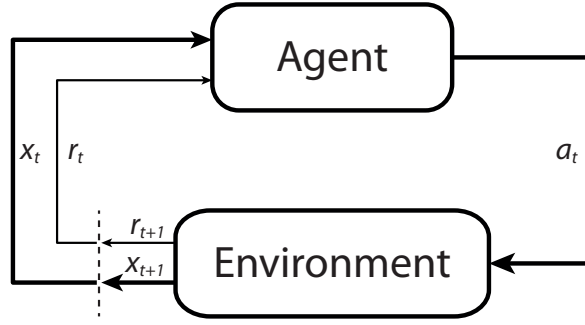


Figure 2.1: The Reinforcement learning cycle

ematically idealized form of the reinforcement learning problem for which precise theoretical statements can be made.

The word Markov points to the fact that we assume that the state transitions of an MDP satisfy the Markov property [???]. This means that the conditional probability distribution of future states of the process depends only upon the present state and not the whole history of events that preceded it.

**Definition 1.** *MDP is a 5-tuple  $\mathcal{M} = (\mathcal{X}, \mathcal{A}, R, P, \gamma)$ , where*

*$\mathcal{X}$  is the finite state space*

*$\mathcal{A}$  is the finite action space*

*$R(x, a) \in [R_{\min}, R_{\max}]$  is a bounded deterministic<sup>1</sup> reward generated by being in state  $x$  and selecting action  $a$*

*$P(\cdot|x, a)$  is the transition probability distribution*

*$\gamma \in [0, 1)$  is a discount factor*

We will denote  $x$  or  $x_t$  as the states visited in time  $t$  and  $x'$  or  $x_{t+1}$  states visited in time  $t + 1$ . **unclear: weird? unify?**

**Definition 2.** *A stationary (or Markovian) policy is a mapping from states to actions  $\pi : \mathcal{X} \rightarrow \mathcal{A}$ .*

**todo:** deterministic cost doesn't go together with distributional :(

### 2.2.2 Return

The ultimate goal of a reinforcement learning agent is to maximize some notion of reward, which is captured by the return. The two most commonly considered types of returns are the sum of rewards, or the mathematically convenient expected discounted reward. In this thesis we focus on the latter.

We define the return  $Z^\pi(x)$  as a random variable representing the discounted reward along a trajectory generated by the MDP by following policy  $\pi$ , starting at state  $x$ :

$$Z^\pi(x) = \sum_{t=0}^{\infty} \gamma^t R(x_t, a_t) \quad (2.1)$$

$$x_t \sim p(\cdot|x_{t-1}, a_{t-1}), a_t \sim \pi, x_0 = x$$

<sup>1</sup> All the presented results are extendable to the case with random reward and with reward being a function of the transition  $R(x, a, a')$ . We avoid these extension to keep the results readable.

As a useful notation, we denote  $Z^\pi(x, a)$  as the random variable representing the discounted reward along a trajectory generated by first selecting action  $a$  and then following policy  $\pi$ .

$$Z^\pi(x, a) = \sum_{t=0}^{\infty} \gamma^t R(x_t, a_t) \quad (2.2)$$

$$x_t \sim p(\cdot | x_{t-1}, a_{t-1}), a_t \sim \pi, x_0 = x, a_0 = a$$

We will sometimes omit the superscript  $\pi$  when the policy is clear from the context or is unimportant.

### 2.2.3 Bellman equations

The *value function*  $V^\pi : \mathcal{X} \rightarrow \mathbb{R}$  of policy  $\pi$  describes the expected return received from state  $x \in \mathcal{X}$  and acting according to  $\pi$ :

$$V^\pi(x) = \mathbb{E}[Z^\pi(x)] = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(x_t, a_t)\right] \quad (2.3)$$

The *action-value function*  $Q^\pi : \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R}$  of policy  $\pi$  describes the expected return from taking action  $a \in \mathcal{A}$  from state  $x \in \mathcal{X}$ , then acting according to  $\pi$ :

$$Q^\pi(x, a) = \mathbb{E}[Z^\pi(x, a)] = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(x_t, a_t)\right] \quad (2.4)$$

Fundamental to reinforcement learning is the use of Bellman's equation [8] to describe the value and action-value functions by a recursive relationship:

$$V^\pi(x) = \mathbb{E}[R(x, \pi(x))] + \gamma \mathbb{E}_{p, \pi}[V^\pi(x')] \quad (2.5)$$

$$Q^\pi(x, a) = \mathbb{E}[R(x, \pi(x))] + \gamma \mathbb{E}_{p, \pi}[V^\pi(x')] \quad (2.6)$$

As stated before, we are typically interested in maximizing the expected return. The most common approach for doing so involves the optimality equation

$$Q^*(x, a) = \mathbb{E}[R(x, a)] + \gamma \mathbb{E}_p \left[ \max_{a' \in \mathcal{A}} Q^*(x', a') \right] \quad (2.7)$$

This equation has a unique fixed point  $Q^*$  - the optimal value function, corresponding to the set of optimal policies  $\Pi^*$  ( $\pi^*$  is optimal if  $\mathbb{E}_{a \sim \pi^*} Q^*(x, a) = \max_a Q^*(x, a)$ ). We view value functions as vectors in  $\mathbb{R}^{\mathcal{X} \times \mathcal{A}}$ , and the expected reward function as one such vector. In this context, the *Bellman operator*  $\mathcal{T}^\pi$  and *optimality operator*  $\mathcal{T}$  are

$$\mathcal{T}^\pi Q(x, a) := \mathbb{E}[R(x, a)] + \gamma \mathbb{E}_{P, \pi}[Q(x', a')] \quad (2.8)$$

$$\mathcal{T} Q(x, a) := \mathbb{E}[R(x, a)] + \gamma \mathbb{E}_P \left[ \max_{a' \in \mathcal{A}} Q(x', a') \right] \quad (2.9)$$

These operators are useful as they describe the expected behaviour of popular learning algorithms such as SARSA and Q-Learning [39]. In particular they are both contraction mappings (see below), and their repeated application to some initial  $Q_0$  converges exponentially to  $Q^\pi$  or  $Q^*$ , respectively [9].

### 2.2.4 Contraction

**unclear:** use something else than  $X$  for sets? An important concept used in convergence analysis of reinforcement learning algorithms is that of a contraction.

**Definition 3.** A **fixed point** of a mapping  $T : X \rightarrow X$  of a set  $X$  into itself is  $x \in X$  which is mapped onto itself, that is  $Tx = x$ .

Let  $X = (X, d)$  be a metric space ( $d$  is a metric on  $X$ ). A mapping  $T$  is called a **contraction** on  $X$  if there exists a positive real number  $\gamma < 1$  such that for all  $x, y \in X$   $d(Tx, Ty) \leq \gamma d(x, y)$ .

**Theorem 1** (Banach Fixed Point Theorem, Contraction Theorem). Consider a metric space  $X = (X, d)$ , where  $X \neq \emptyset$ . Suppose that  $X$  is complete (every Cauchy sequence in  $X$  converges) and let  $T$  be a contraction on  $X$ . Then  $T$  has precisely one fixed point.

See e.g. Kreyszig [22] for a complete treatment of all mentioned concepts.

The takeaway for reinforcement learning is, that if  $T$  is a contraction, by recursively applying  $x_{k+1} = Tx_k$  we converge to the single fixed point of this operator. This is used in proofs for various RL operators, usually in combination with the infinity norm.

## 2.3 DISTRIBUTIONAL REINFORCEMENT LEARNING

In contrast to standard reinforcement learning, where we model the expected return, in distributional reinforcement learning [many] we aim to model the full distribution of the return. This is advantageous in cases where we want to e.g. model parametric uncertainty [...] or design risk-sensitive algorithms [30][29]. Bellemare, Dabney, and Munos [6] also argue, that the distributional approach is beneficial even in the case when we optimize the expected value, as the distribution gives us more information which helps the now commonly used approximate algorithms (such as DQN [28]).

At the core of the distributional approach lies the recursive equation of the return distribution:

$$\begin{aligned} Z(x, a) &\stackrel{D}{=} R(x, a) + \gamma Z(x', a') \\ x_t &\sim p(\cdot | x_{t-1}, a_{t-1}), a_t \sim \pi, x_0 = x, a_0 = a \end{aligned} \tag{2.10}$$

where  $\stackrel{D}{=}$  denotes that random variables on both sides of the equation share the same probability distribution.

In the *policy evaluation* setting [39] we are interested in the value function  $V^\pi$  associated with a given policy  $\pi$ . The analogue here is the value distribution  $Z^\pi$ . Note that  $Z^\pi$  describes the intrinsic randomness of the agent's interactions with its environment, rather than some measure of uncertainty about the environment itself.

We view the reward function as a random vector  $R \in \mathcal{Z}$ , and define the transition operator  $P^\pi : \mathcal{Z} \rightarrow \mathcal{Z}$

$$\begin{aligned} P^\pi Z(x, a) &\stackrel{D}{=} Z(x', a') \\ x' &\sim p(\cdot | x, a), a' \sim \pi \end{aligned} \tag{2.11}$$

We define the distributional Bellman operator  $\mathcal{T}^\pi : \mathcal{Z} \rightarrow \mathcal{Z}$  as

$$\mathcal{T}^\pi Z(x, a) \stackrel{D}{=} R(x, a) + \gamma P^\pi Z(x, a). \quad (2.12)$$

We emphasize that this is a distributional equation and the distributional bellman operator is therefore fundamentally different from the standard bellman operator.

Bellemare, Dabney, and Munos [6] have shown, that the distributional bellman operator  $\mathcal{T}^\pi$  is not a contraction in the commonly used KL divergence [24], but is a contraction in the infinity Wasserstein metric which we describe below, as it will become useful as a tool for evaluating algorithms in the rest of the thesis. Another important fact is, that the bellman optimality operator  $\mathcal{T} : \mathcal{Z} \rightarrow \mathcal{Z}$

$$\mathcal{T}Z = \mathcal{T}^\pi Z \quad \text{for some } \pi \text{ in the set of greedy policies} \quad (2.13)$$

is not a contraction in any metric. The distribution does not converge to a fixed point, but rather to a sequence of optimal (in terms of expected value) policies. We encourage the interested reader to explore these topics in the excellent papers by Bellemare, Dabney, and Munos [6] and Dabney et al. [17].

### 2.3.1 The Wasserstein Metric

One of the tools for analysis of distributional approaches to reinforcement learning is the Wasserstein metric  $d_p$  between cumulative distribution functions (see e.g. Bickel and Freedman [10]). The metric has recently gained in popularity and was used e.g. in unsupervised learning [2, 7]. Unlike the Kullback-Leibler divergence, which strictly measures change in probability, the Wasserstein metric reflects the underlying geometry between outcomes.

For  $F, G$  two c.d.fs over the reals, it is defined as

$$d_p(F, G) := \inf_{U, V} \|U - V\|_p,$$

where the infimum is taken over all pairs of random variables  $(U, V)$  with respective cumulative distributions  $F$  and  $G$ . The infimum is attained by the inverse c.d.f. transform of a random variable  $\mathcal{U}$  uniformly distributed on  $[0, 1]$ :

$$d_p(F, G) = \|F^{-1}(\mathcal{U}) - G^{-1}(\mathcal{U})\|_p.$$

For  $p < \infty$  this is more explicitly written as

$$d_p(F, G) = \left( \int_0^1 |F^{-1}(u) - G^{-1}(u)|^p du \right)^{1/p}. \quad (2.14)$$

meaning it is an integral over the difference in the quantile functions of the random variables. This will become important later, as the quantile function has a direct connection to the CVaR objective (??).

**unclear:** maybe visuals, if it becomes important

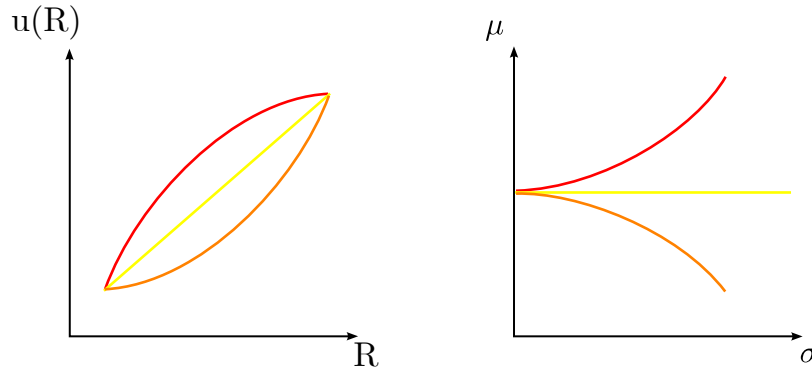


Figure 2.2: Standard in economic literature, these figures depict the differences between risk-averse (red), risk-neutral (yellow) and risk-seeking (orange) behaviors. Left: A subjective utility  $u(R)$ , based on the reward, is concave for risk-averse behaviors. Right: Risk-averse utility contour lines in standard deviation-expected value space are upward sloped.

## 2.4 RISK-SENSITIVITY

The standard reinforcement learning agent that maximizes the expected reward which we discussed in the previous chapter does not take risk into account. Indeed in a mathematical sense, the shape of the reward distribution is unimportant as wildly different distributions may have the same expectation. This unfortunately is not the case in the real world, where there exist catastrophic losses - an investment company may have a good strategy that yields profit in expectation, but if the strategy is too risky and the company's capital drops under zero this investment strategy is useless. This leads us to defining risk, which describes the potential of gaining or losing reward and is therefore more expressive than a simple expectation.

The finance literature differentiates between three risk-related types of behavior, namely *risk-neutral*, *risk-averse* and *risk-seeking*. We offer the following example to illustrate the differences between mentioned behaviors: Imagine you are facing two options, either (a) you get \$1 or (b) you get \$5 with 90% probability, but lose \$35 with 10% probability. A risk-neutral agent wouldn't differentiate between the two choices, as the expected value of reward is the same. A risk-averse agent would prefer option (a), as there is a risk of high losses in option (b). Risk-seeking agent would pick (b). The difference between the risk-sensitive behaviors can be visualized as in Figure ??.

The desired behavior for most critical applications is risk-averse and indeed it is the behavior of choice for financial institutions [46], citepbasel2013fundamental. It has also been suggested in the neuroscience literature, that humans use risk-averse behaviors when making decisions [36].

As we stated in Section 1.1 and will formally state in Section 2.5, we are interested in reinforcement learning that maximizes a certain risk-averse objective. Below we formally describe the metrics used to measure risk which we then use to formulate the exact problem tackled in this thesis.

### 2.4.1 Value-at-Risk

Value-at-risk (VaR, see e.g. Wipplinger [46]) is one of the most popular tools used to estimate exposure to risk used risk management and financial control.

Let  $Z$  be a random variable representing reward, with cumulative distribution function (c.d.f.)  $F(z) = \mathbb{P}(Z \leq z)$ . The Value-at-Risk at confidence level  $\alpha \in (0, 1)$  is the  $\alpha$ -quantile of  $Z$ , i.e.

$$\text{VaR}_\alpha(Z) = F^{-1}(\alpha) = \sup \{z | F(z) \leq \alpha\} \quad (2.15)$$

We will use the notation  $\text{VaR}_\alpha(Z)$ ,  $F^{-1}(\alpha)$  interchangeably, often explicitly denoting the random variable of inverse c.d.f. as  $F_Z^{-1}(\alpha)$ .

Note on notation: In the risk-related literature, it is common to work with losses instead of rewards. The Value-at-Risk is then defined as the  $1 - \alpha$  quantile. The notation we use reflects the use of reward in reinforcement learning rather than losses and this sometimes leads to the need of reformulating some definitions or theorems. While these reformulations may differ in notation, they characterize the same underlying ideas.

### 2.4.2 Conditional Value-at-Risk

Conditional Value-at-Risk (CVaR, see Rockafellar and Uryasev [33, 34]), sometimes called Expected Shortfall (ES), Average Value-at-Risk (AVaR) or Tail Value-at-Risk (TVaR), is a risk measure that aims to fix inadequacies of measuring risk introduced by Value-at-Risk. Firstly, it has the desirable mathematical properties of monotonicity, translation invariance, positive homogeneity and subadditivity (see Artzner et al. [3]), which makes CVaR computation much easier compared to VaR. It's properties were also recently identified as suitable for measuring risk in robotics [26]. Another strong point of CVaR is that unlike VaR, it is able to distinguish between large and catastrophic losses. For these reasons, CVaR is starting to replace VaR as a standard measure for risk in financial applications [16] and beyond [something].

The Conditional Value-at-Risk (CVaR) at confidence level  $\alpha \in (0, 1)$  is defined as the expected reward of outcomes worse than the  $\alpha$ -quantile ( $\text{VaR}_\alpha$ ):

$$\text{CVaR}_\alpha(Z) = \frac{1}{\alpha} \int_0^\alpha F_Z^{-1}(\beta) d\beta = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(Z) d\beta \quad (2.16)$$

Rockafellar and Uryasev [33] also showed that CVaR is equivalent to the solution of

$$\text{CVaR}_\alpha(Z) = \max_s \left\{ \frac{1}{\alpha} \mathbb{E} [(Z - s)^-] + s \right\} \quad (2.17)$$

where  $(x)^- = \min(x, 0)$  represents the negative part of  $x$  and in the optimal point  $s^* = \text{VaR}_\alpha(Z)$

$$\text{CVaR}_\alpha(Z) = \frac{1}{\alpha} \mathbb{E} [(Z - \text{VaR}_\alpha(Z))^-] + \text{VaR}_\alpha(Z) \quad (2.18)$$

The last definition we will need is the dual formulation of (2.17)

$$\text{CVaR}_\alpha(Z) = \min_{\xi \in \mathcal{U}_{\text{CVaR}}(\alpha, P(\cdot | x, a))} \mathbb{E}_\xi[Z] \quad (2.19)$$

$$\mathcal{U}_{\text{CVaR}}(\alpha, P(\cdot | x, a)) = \left\{ \xi : \xi(\omega) \in \left[0, \frac{1}{\alpha}\right], \int_\omega \xi(\omega) \mathbb{P}(\omega) d\omega = 1 \right\} \quad (2.20)$$



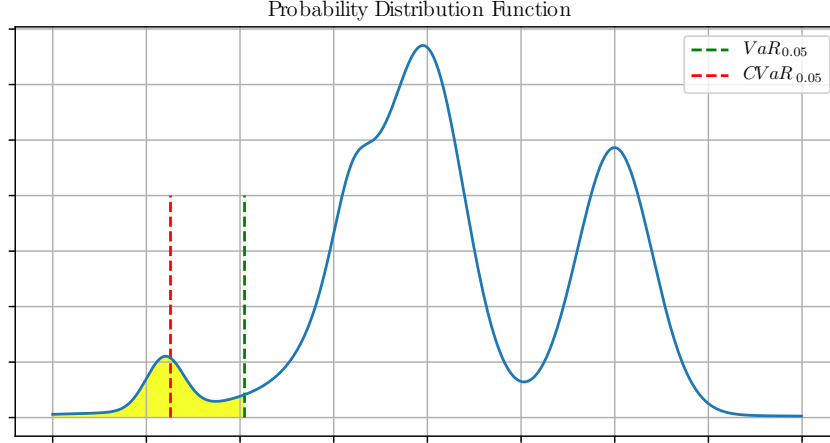


Figure 2.3: Value-at-Risk and Conditional Value-at-Risk of a general probability distribution with the integral  $\alpha = 0.05$  marked in yellow. The main flaw of the VaR metric is clearly visible here, as we could shift the leftmost 'mode' of the distribution into minus infinity and the VaR would remain unchanged, while CVaR would change with the shift.

For a treatment of duality, see e.g. Boyd and Vandenberghe [13].

## 2.5 PROBLEM FORMULATION

The problem tackled in this thesis considers reinforcement learning with optimization of the CVaR objective. Unlike the expected value criterion, it is insufficient to consider only stationary policies, and we must work with general history-dependent policies. We define them formally below.

**Definition 4** (History-Dependent Policies). *Let the space of admissible histories up to time  $t$  be  $H_t = H_{t-1} \times \mathcal{A} \times \mathcal{X}$  for  $t \geq 1$ , and  $H_0 = \mathcal{X}$ . A generic element  $h_t \in H_t$  is of the form  $h_t = (x_0, a_0, \dots, x_{t-1}, a_{t-1})$ . Let  $\Pi_{H,t}$  be the set of all history-dependent policies with the property that at each time  $t$  the randomized control action is a function of  $h_t$ . In other words,  $\Pi_{H,t} = \{\pi_0 : H_0 \rightarrow \mathbb{P}(\mathcal{A}), \dots, \pi_t : H_t \rightarrow \mathbb{P}(\mathcal{A})\}$ . We also let  $\Pi_H = \lim_{t \rightarrow \infty} \Pi_{H,t}$  be the set of all history-dependent policies.*

The risk-averse objective we wish to address for a given confidence level  $\alpha$  is the following

$$\max_{\pi \in \Pi_H} \text{CVaR}_\alpha(Z^\pi(x_0)) \quad (2.21)$$

where  $Z^\pi(x_0)$  coincides with definition (??).

In words, our goal is to find a general policy  $\pi^* \in \Pi_H$ , that maximizes conditional value-at-risk of the return, starting in state  $x_0$ . We emphasize the importance of the starting state since, unlike the expected value, the CVaR objective is not time-consistent.

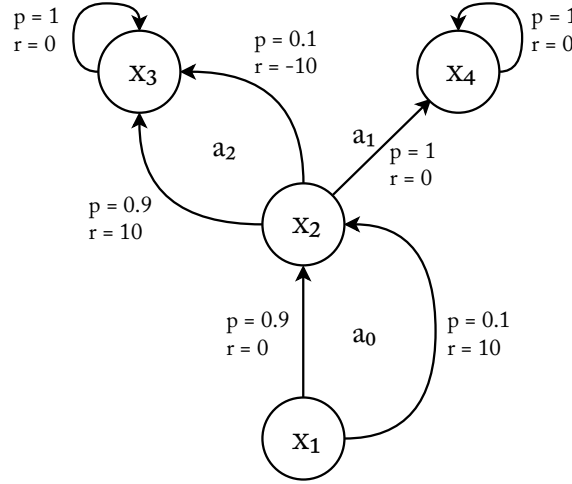


Figure 2.4: MDP showing time-inconsistency of the CVaR objective.

### 2.5.1 Time-consistency

There exist several definitions of time-consistency [11, 31]. Informally, if the criterion is time-consistent, we can limit ourselves to the space of stationary policies, as the optimal policy is part of this space. On the other hand, non-stationary policies may be required to solve a time-inconsistent problem.

We provide the following example to show that the CVaR criterion is indeed time-inconsistent. On Figure 2.4 we can see an MDP with starting state  $x_1$  with a single action  $a_0$ , followed by state  $x_2$  where the agent can choose between actions  $a_1, a_2$ ; states  $x_3, x_4$  are terminal. We now compare three policies  $\pi_1(x_1) = a_1, \pi_2(x_1) = a_2$  and a non-stationary policy  $\pi_3$  that chooses  $a_2$ , unless the agent was lucky and received the reward 10 when transitioning from state  $x_1$ . Let us examine the CVaR objective with  $\alpha = 0.19$ :

$$\begin{aligned} \text{CVaR}_{0.19}(Z^{\pi_1}(x_1)) &= \frac{0.19 \cdot 0}{0.19} = 0 \\ \text{CVaR}_{0.19}(Z^{\pi_2}(x_1)) &= \frac{0.09 \cdot (-10) + 0.01 \cdot 0 + 0.09 \cdot 10}{0.19} = 0 \\ \text{CVaR}_{0.19}(Z^{\pi_3}(x_1)) &= \frac{0.09 \cdot (-10) + 0.1 \cdot 10}{0.19} \doteq 0.526 \end{aligned}$$

By examining the results, we can see that the non-stationary policy  $\pi_3$  is better than any stationary one, confirming CVaR as a time-inconsistent objective, explaining the need for a history-dependent policy in our problem definition 2.21.

**unclear:** this is for  $\gamma = 1$ , different gamma would introduce ugly numbers **todo:** mention the  $\alpha$  inconsistency part

### 2.5.2 Robustness

An important motivational point for the CVaR objective (2.21) is its relationship with robustness. Chow et al. [15] have shown that optimizing the objective is equivalent to being robust to model perturbations. Thus, by minimizing CVaR, the decision

maker also guarantees robustness to modeling errors. For completeness, we repeat the formulation of the equivalence relation below.

Let  $(x_0, a_0, \dots, x_T)$  be a trajectory in a finite-horizon MDP problem. The total probability of the trajectory is  $P(x_0, a_0, \dots, x_T) = P(x_0)P(x_1|x_0, a_0) \cdots P(x_T|x_{T-1}, a_{T-1})$ . For each step  $1 \leq t \leq T$  consider a perturbed transition matrix  $\hat{P} = P \circ \delta_t$  where  $\delta_t \in \mathbb{R}^{\mathcal{X} \times \mathcal{A} \times \mathcal{X}}$  and  $\circ$  is the element-wise product under the condition that  $\hat{P}$  is a stochastic matrix. Let  $\Delta_t$  be the set of perturbation matrices that satisfy this condition and  $\Delta = \Delta_1 \times \cdots \times \Delta_T$  be the set of all possible perturbations to the trajectory distribution.

We now impose a budget constraint on the perturbations as follows. For some budget  $\eta \geq 1$ , we consider the constraint

$$\delta_1(x_1|x_0)\delta_2(x_2|x_1) \cdots \delta_T(x_T|x_{T-1}) \leq \eta \quad (2.22)$$

Essentially, the product in (2.22) states that *the worst cannot happen at each time*. Instead, the perturbation budget has to be split (multiplicatively) along the trajectory. We note that (2.22) is in fact a constraint on the perturbation matrices, and we denote by  $\Delta_\eta \subset \Delta$  the set of perturbations that satisfy this constraint with budget  $\eta$ . Then the following holds (Proposition 1 of [15])

$$\text{CVaR}_{\frac{1}{\eta}}(R_{0,T}(x_1, \dots, x_T)) = \inf_{(\delta_1, \dots, \delta_T) \in \Delta_\eta} \mathbb{E}_{\hat{P}}[R_{0,T}(x_1, \dots, x_T)], \quad (2.23)$$

where  $R_{0,T}(x_1, \dots, x_T)$  denotes the random variable representing the reward along the particular trajectory and  $\mathbb{E}_{\hat{P}}[\cdot]$  denotes expectation with respect to a Markov chain with transitions  $\hat{P}_t$ .

## 2.6 LITERATURE SURVEY

Risk-sensitive MDPs have been studied thoroughly in the past, with different risk-related objectives. Due to its good computational properties, earlier efforts focused on exponential utility [19], the max-min criterion [**minmax**] or e.g. maximizing the mean with constrained variance [38]. A comprehensive overview of the different objectives can be found in Garcia and Fernández [18], together with a unified look on the different methods used in safe reinforcement learning. Among CVaR-related objectives, some publications focus on optimizing the expected value with a CVaR constraint [12, 32].

Recently, for the reasons explained above, several authors have investigated the exact objective (2.21). A considerable effort has gone towards policy-gradient [40] and Actor-Critic [21] algorithms with the CVaR objective. [14, 41] present useful ways of computing the CVaR gradients with parametric models and have shown the practicality and scalability of these approaches on interesting domains such as the well-known game of Tetris. An important setback of these methods is their limitation of the hypothesis space to the class of stationary policies, meaning they can only reach a *local* minimum of our objective. Similar policy gradient methods have also been investigated in the context of general coherent measures, a class of risk measures encapsulating many used measures including CVaR. Tamar et al. present a policy gradient algorithm [43] and a gradient-based Actor-Critic algorithm [43]. Again, these algorithms only converge in local extremes.

Some authors have also tried to sidestep the time-consistency issue of CVaR by either focusing on a time-consistent subclass of coherent measures [???], limiting the hypothesis space to time-consistent policies or reformulating the CVaR objective in a time-consistent way [27].

Bäuerle and Ott [5] use a state space extensions and show that this new extended state space contains globally optimal policies. Unfortunately, the fact that the state-space is continuous, the design of a solution algorithm is challenging.

The approach of Chow et al. [15] also uses a continuous augmented state-space but unlike Bäuerle and Ott [5], this continuous state is shown to have bounded error when a particular linear discretization is used. The only flaw of this approach is the requirement of a linear program computation, which we address in the next chapter.

**todo:** morimura

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## VALUE ITERATION WITH CVAR

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Value iteration is a standard algorithm for maximizing expected discounted reward used in reinforcement learning. In this chapter we extend the results of Chow et al. [15], who have recently proposed an approximate value iteration algorithm for CVaR MDPs.

The original algorithm requires the computation of a linear program in each step of the value iteration procedure. Utilizing a connection between the used  $\alpha\text{CVaR}_\alpha$  function and the quantile function, we sidestep the need for this computation and propose a linear-time version of the algorithm, making CVaR value iteration feasible for much larger MDPs.

After reminding the reader of the standard value iteration algorithm, we present the original algorithm in Section 3.2. The improved algorithm is presented in Section 3.3, followed by section Section 3.4, where we test the algorithm on selected environments.

### 3.1 VALUE ITERATION

Value iteration [39] is a well-known algorithm for computing the optimal (action-)value function and hereby finding the optimal policy. Let us remind ourselves of the Bellman optimality operator  $\mathcal{T}$  (2.9):

$$\mathcal{T}Q(x, a) := \mathbb{E}[R(x, a)] + \gamma \mathbb{E}_P \left[ \max_{a' \in \mathcal{A}} Q(x', a') \right]$$

or rewritten for the value function  $V$

$$\mathcal{T}V(x) := \max_a \{ \mathbb{E}[R(x, a)] + \gamma \mathbb{E}_P[V(x')] \} \quad (3.1)$$

As stated before,  $\mathcal{T}$  is a contraction (Section ??). This means that by repeatedly applying the operator we eventually converge to the optimal point, since we converge and the definition holds in this point. This leads to the formulation of the *Value Iteration* algorithm. The only difference between theory and practice is the introduction of a small parameter  $\epsilon$  that allows us to check the converge and end the algorithm when we reach a certain precision, as the contraction converges only in the limit.

See Algorithm 1.

### 3.2 CVAR VALUE ITERATION

Chow et al. [15] present a dynamic programming formulation for the CVaR MDP problem (2.21). As CVaR is a time-inconsistent measure, their method requires an

**Algorithm 1** Value Iteration

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```

Initialize  $V$  arbitrarily (e.g.  $V(x) = 0$  for all  $x \in \mathcal{X}$ )
repeat
   $v = V(x)$ 
   $\Delta = 0$ 
  for each  $x \in \mathcal{X}$  do
     $V(x) = \max_a \{r(x, a) + \gamma \sum_{s'} p(s'|x, a) V(x')\}$ 
     $\Delta = \max \{\Delta, |v - V(x)|\}$ 
  end for
until  $\Delta < \epsilon$ 
Output a deterministic policy  $\pi \approx \pi^*$ :
   $\pi(x) = \arg \max_a \{r(x, a) + \gamma \sum_{s'} p(s'|x, a) V(x')\}$ 

```

---

extension of the state space. A Value Iteration type algorithm is then applied on this extended space and Chow et al. [15] proved it's convergence.

We repeat their key ideas and results bellow, as they form a basis for our contributions presented in later sections. The results are presented with our notation introduced in Chapter 2, which differs slightly from the paper, but the core ideas remain the same.

### 3.2.1 Bellman Equation for CVaR

The results of Chow et al. [15] heavily rely on the CVaR decomposition theorem (Lemma 22, [31]):

**unclear:** repeat the original theorems in full?

$$CVaR_\alpha(Z^\pi(x, a)) = \min_{\xi \in \mathcal{U}_{CVaR}(\alpha, P(\cdot|x, a))} \sum_{x'} p(x'|x, a) \xi(x') CVaR_{\xi(x')\alpha}(Z^\pi(x')) \quad (3.2)$$

where the risk envelope  $\mathcal{U}_{CVaR}(\alpha, P(\cdot|x, a))$  coincides with the dual definition of CVaR (2.20). The theorem states that we can compute the  $CVaR_\alpha(Z^\pi(x, a))$  as the minimal (or worst-case) weighted combination of  $CVaR_\alpha(Z^\pi(x'))$  under a probability distribution perturbed by  $\xi(x')$ .

Note that the decomposition requires only the representation of CVaR at different confidence levels and not the whole distribution at each level, which we might be tempted to think because of the time-inconsistency issue.

Chow et al. [15] extend the decomposition theorem by defining the *CVaR value function*  $V(x, y)$  with an augmented state-space  $\mathcal{X} \times \mathcal{Y}$  where  $\mathcal{Y} = (0, 1]$  is an additional continuous state that represents the different confidence levels.

$$V(x, y) = \max_{\pi \in \Pi_H} CVaR_y(Z^\pi(x)) \quad (3.3)$$

Similar to standard dynamic programming, it is convenient to work with operators defined on the space of value functions. This leads to the following definition of the CVaR Bellman operator  $\mathbf{T} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Y}$ :

$$\mathbf{TV}(x, y) = \max_a \left[ R(x, a) + \gamma \min_{\xi \in \mathcal{U}_{CVaR}(\alpha, P(\cdot|x, a))} \sum_{x'} p(x'|x, a) \xi(x') V(x', y\xi(x')) \right]$$

$$(3.4)$$

or in our simplified notation:

$$\mathbf{T}CVaR_y(Z(x)) = \max_a [R(x, a) + \gamma CVaR_y(Z(x, a))] \quad (3.5)$$

Chow et al. [15] further showed (Lemma 3) that the operator  $\mathbf{T}$  is a contraction and also preserves the convexity of  $yCVaR_t$ . The optimization problem (3.2) is a convex one and therefore has a unique solution. Additionally, the fixed point of this contraction is the optimal  $V^*(x, y) = \max_{\pi \in \Pi} CVaR_y(Z^\pi(x, y))$  ([15], Theorem 4).

The value-function  $V^*$  can then be used to extract the optimal policy  $\pi^*$  of the original problem (2.21), using the following theorem.

**Theorem 2** (Optimal Policies, Theorem 5 in [15]). *Let  $\pi_H^* = \{\mu_0, \mu_1, \dots\} \in \Pi_H$  be a history-dependent policy recursively defined as:*

$$\mu_k(h_k) = u^*(x_k, y_k), \quad \forall k \geq 0, \quad (3.6)$$

with initial conditions  $x_0$  and  $y_0 = \alpha$ , and state transitions

$$x_k \sim P(\cdot | x_{k-1}, u^*(x_{k-1}, y_{k-1})), \quad y_k = y_{k-1} \xi_{x_{k-1}, y_{k-1}, u^*}^*(x_k), \quad \forall k \geq 1, \quad (3.7)$$

where the stationary Markovian policy  $u^*(x, y)$  and risk factor  $\xi_{x, y, u^*}^*(\cdot)$  are solution to the max-min optimization problem in the CVaR Bellman operator  $\mathbf{T}[V^*](x, y)$ . Then,  $\pi_H^*$  is an optimal policy for problem (2.21) with initial state  $x_0$  and CVaR confidence level  $\alpha$ .

Naive value iteration with operator  $\mathbf{T}$  is unfortunately unusable in practice, as the state space is continuous in  $y$ . The solution proposed in [15] is then to represent the convex  $yCVaR_y$  as a piecewise linear function.

### 3.2.2 Value Iteration with Linear Interpolation

Given a set of  $N(x)$  interpolation points  $\mathbf{Y}(x) = \{y_1, \dots, y_{N(x)}\}$ , we can interpolate the  $yV(x, y)$  function on these points, i.e.

$$\mathcal{I}_x[V](y) = y_i V(x, y_i) + \frac{y_{i+1} V(x, y_{i+1}) - y_i V(x, y_i)}{y_{i+1} - y_i} (y - y_i),$$

where  $y_i = \max \{y' \in \mathbf{Y}(x) : y' \leq y\}$ . The interpolated Bellman operator  $\mathbf{T}_{\mathcal{I}}$  is then also a contraction and has a bounded error ([15], Theorem 7). **todo: bounded -> linear in  $\theta$**

$$\mathbf{T}_{\mathcal{I}}V(x, y) = \max_a \left[ R(x, a) + \gamma \min_{\xi \in \mathcal{U}_{CVaR}(\alpha, P(\cdot | x, a))} \sum_{x'} p(x' | x, a) \frac{\mathcal{I}_{x'}[V](y \xi(x'))}{y} \right] \quad (3.8)$$

The full value iteration procedure is presented in Algorithm 2.

This algorithm can be used to find an approximate global optimum in any MDP. There is however the issue of computational complexity. As the algorithm stands, the straightforward approach is to solve each iteration of (3.8) as a linear program, since the problem is convex and piecewise linear, but this is not practical, as the LP computation can be demanding and is therefore not suitable for large state-spaces.

**unclear: maybe formulate the LP exactly?**

In the next section we aim to find a different way of computing the optimization problem.

**Algorithm 2** CVaR Value Iteration with Linear Interpolation**1: Given:**

- $N(x)$  interpolation points  $\mathbf{Y}(x) = \{y_1, \dots, y_{N(x)}\} \in [0, 1]^{N(x)}$  for every  $x \in \mathcal{X}$  with  $y_i < y_{i+1}$ ,  $y_1 = 0$  and  $y_{N(x)} = 1$ .
- Initial value function  $V_0(x, y)$  that satisfies:
  1.  $yV_0(x, y)$  is convex in  $y$  for all  $x$
  2.  $yV_0(x, y)$  is continuous in  $y$  for all  $x$

**2: For**  $t = 1, 2, \dots$ 

- For each  $x \in \mathcal{X}$  and each  $y_i \in \mathbf{Y}(x)$ , update the value function estimate as follows:

$$V_t(x, y_i) = \mathbf{T}_{\mathcal{X}}[V_{t-1}](x, y_i),$$

**3: Set** the converged value iteration estimate as  $\hat{V}^*(x, y_i)$ , for any  $x \in \mathcal{X}$ , and  $y_i \in \mathbf{Y}(x)$ .

**3.3 EFFICIENT COMPUTATION USING QUANTILE REPRESENTATION**

We present our original contributions in this section. We first hint at a connection between the  $y\text{CVaR}_y$  function and the quantile function of the underlying distribution. We then use this connection to formulate a faster computation of the value iteration step, resulting in the first linear-time algorithm for solving CVaR MDPs with bounded error.

Consider the following facts regarding the CVaR computation. Firstly, it is unimportant *how* we compute  $\text{CVaR}_y(Z(x, a))$  present in the CVaR Bellman operator (3.5). Secondly, note that any discrete distribution has a piecewise linear  $y\text{CVaR}_y$  function [34], and this statement is two-sided: any a piecewise linear  $y\text{CVaR}_y$  function can be seen as representing a certain discrete distribution.

The relation between  $y\text{CVaR}_y$  and the underlying distribution is best described in the following equation:

$$\frac{\partial}{\partial y} y\text{CVaR}_y(Z) = \frac{\partial}{\partial y} \int_0^y \text{VaR}_\beta(Z) d\beta = \text{VaR}_y(Z) \quad (3.9)$$

We see that the quantile function of  $Z$  is extractable from the  $y\text{CVaR}_y$  function by differentiating over  $y$ . The opposite direction also holds:

$$y\text{CVaR}_y(Z) = \int_0^y \text{VaR}_\beta(Z) d\beta + c \quad (3.10)$$

where we have  $c = 0$  because of a starting condition  $y\text{CVaR}_y \Big|_{y=0} = 0$ .



### 3.3.1 CVaR Computation via Quantile Representation

We propose the following procedure: instead of using linear programming for the CVaR computation, we use the underlying distributions represented by the  $\alpha\text{CVaR}_\alpha$  function to compute CVaR.

The computation of CVaR of a discrete probability mixture is a linear-time process as we show bellow. The general steps of the computation are as follows

1. transform  $y\text{CVaR}_y$  of each possible state transition to a discrete probability distribution function using (3.9)
2. combine these to to a distribution representing the full state-action distribution
3. compute  $y\text{CVaR}_y$  for all atoms using (3.10)

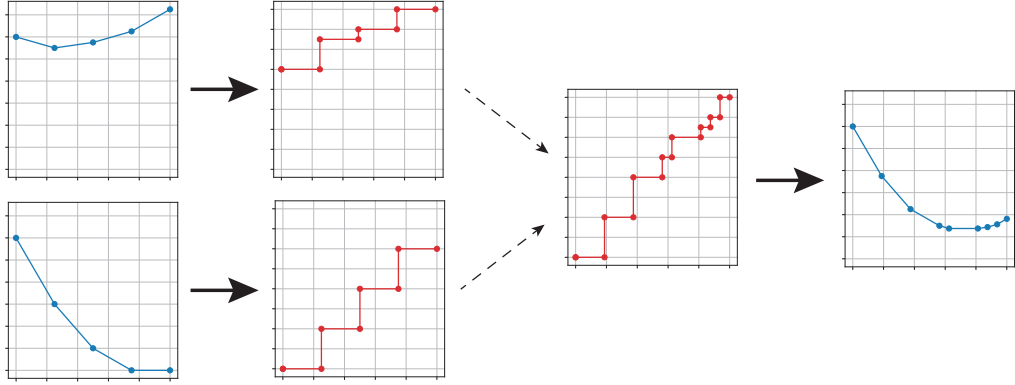


Figure 3.1: Visualization of the CVaR Value Iteration for a single state and action with two transition states. Thick arrows represent the conversion between  $\alpha\text{CVaR}_\alpha$  and the quantile function.

**unclear:** proof necessary? also, maybe it is already in the  $\xi$  proof

### 3.3.2 $\xi$ -computation

Similarly to Theorem 2, we need a way to compute the  $y_{k+1} = y_k \xi^*(x_k)$  to extract the optimal policy. Again, we can skip the LP computation by using the following intuition:  $y_{k+1}$  is the portion of  $Z(x_{k+1})$  that is present in  $\text{CVaR}_{y_k}(Z(x_k))$ . In the continuous case, it is the probability in  $Z(x_{k+1})$  before the  $\text{VaR}_{y_k}(Z(x_k))$  as we show bellow.

**todo:** proof for discrete distributions

**Theorem 3.** *Solution to minimization problem (3.2) can be computed without optimization by setting*

$$\xi(x') = \frac{F_{x'}(F_x^{-1}(\alpha))}{\alpha} \quad (3.11)$$

*Proof.* For simplification, we work only with two states:  $x'$  the actual sampled state and  $\bar{x}'$  representing the other states. The equation then simplifies to

$$\begin{aligned} CVaR_\alpha(x, a) &= \min_{\xi} p\xi CVaR_{\xi\alpha}(x') + (1-p)\frac{1-p\xi}{1-p} CVaR_{\frac{1-p\xi}{1-p}\alpha}(\bar{x}') \\ &= \min_{\xi} p\xi CVaR_{\xi\alpha}(x') + (1-p\xi) CVaR_{\frac{1-p\xi}{1-p}\alpha}(\bar{x}') \end{aligned} \quad (3.12)$$

To find the min we first find the first derivative<sup>1</sup> w.r.t.  $\xi$

$$\begin{aligned} \frac{\partial CVaR_\alpha}{\partial \xi} &= pCVaR_{\xi\alpha} + p\xi \frac{\partial CVaR_{\alpha\xi}}{\partial \xi} - pCVaR_{\frac{1-p\xi}{1-p}\alpha} + (1-p\xi) \frac{\partial CVaR_{\frac{1-p\xi}{1-p}\alpha}}{\partial \xi} \\ &= pCVaR_{\xi\alpha} + p\xi \left[ \frac{1}{\xi} VaR_{\xi\alpha} - \frac{1}{\xi} CVaR_{\xi\alpha} \right] - pCVaR_{\frac{1-p\xi}{1-p}\alpha} \\ &\quad + (1-p\xi) \left[ \frac{p}{1-p\xi} CVaR_{\frac{1-p\xi}{1-p}\alpha} - \frac{p}{1-p\xi} VaR_{\frac{1-p\xi}{1-p}\alpha} \right] \\ &= pCVaR_{\xi\alpha} + pVaR_{\xi\alpha} - pCVaR_{\xi\alpha} - pCVaR_{\xi\alpha} - pCVaR_{\frac{1-p\xi}{1-p}\alpha} \\ &\quad + CVaR_{\frac{1-p\xi}{1-p}\alpha} - pVaR_{\frac{1-p\xi}{1-p}\alpha} \\ &= pVaR_{\xi\alpha} - pVaR_{\frac{1-p\xi}{1-p}\alpha} \end{aligned} \quad (3.13)$$

By setting the derivative to 0 (to find the min), we get

$$VaR_{\xi\alpha}(x') = VaR_{\frac{1-p\xi}{1-p}\alpha}(\bar{x}') \quad (3.14)$$

By inserting claim (3.11) into (3.14) we get the symmetrical claim

$$\frac{1-p\xi}{1-p} = \xi(\bar{x}') = \frac{F_{\bar{x}'}(F_x^{-1}(\alpha))}{\alpha} \quad (3.15)$$

We rewrite (3.12) as (assuming  $\xi$  is the minimum point)

$$\begin{aligned} \frac{1}{\alpha} \int_0^\alpha F_x^{-1}(t) dt &= p\xi \frac{1}{\xi\alpha} \int_0^{\xi\alpha} F_{x'}^{-1}(t) dt + (1-p\xi) \frac{1-p}{(1-p\xi)\alpha} \int_0^{\frac{1-p\xi}{1-p}\alpha} F_{\bar{x}'}^{-1}(t) dt \\ &= p \frac{1}{\alpha} \int_0^{\xi\alpha} F_{x'}^{-1}(t) dt + (1-p) \frac{1}{\alpha} \int_0^{\frac{1-p\xi}{1-p}\alpha} F_{\bar{x}'}^{-1}(t) dt \end{aligned} \quad (3.16)$$

This must also hold if we multiply both sides by  $\alpha$

$$\int_0^\alpha F_x^{-1}(t) dt = p \int_0^{\xi\alpha} F_{x'}^{-1}(t) dt + (1-p) \int_0^{\frac{1-p\xi}{1-p}\alpha} F_{\bar{x}'}^{-1}(t) dt \quad (3.17)$$

<sup>1</sup> We used the following identities:

$$\frac{\partial CVaR_{\alpha\xi}}{\partial \xi} = \frac{1}{\xi} VaR_{\xi\alpha} - \frac{1}{\xi} CVaR_{\xi\alpha} \quad \frac{\partial CVaR_{\frac{1-p\xi}{1-p}\alpha}}{\partial \xi} = \frac{p}{1-p\xi} CVaR_{\frac{1-p\xi}{1-p}\alpha} - \frac{p}{1-p\xi} VaR_{\frac{1-p\xi}{1-p}\alpha}$$

And we take derivations w.r.t.  $\alpha$  of both sides

$$F_x^{-1}(\alpha) = p\xi F_{x'}^{-1}(\xi\alpha) + (1 - p\xi)F_{\bar{x}'}^{-1}\left(\frac{1 - p\xi}{1 - p}\alpha\right) \quad (3.18)$$

By inserting (3.11) we get

$$\begin{aligned} p\xi F_{x'}^{-1}(\xi\alpha) + (1 - p)\xi_2 F_{\bar{x}'}^{-1}(\xi_2\alpha) &= p\xi F_{x'}^{-1}(F_{x'}(F_x^{-1}(\alpha))) + (1 - p\xi)F_{\bar{x}'}^{-1}\left(F_{\bar{x}'}(F_x^{-1}(\alpha))\right) \\ &= p\xi F_x^{-1}(\alpha) + (1 - p\xi)F_x^{-1}(\alpha) = F_x^{-1}(\alpha) \end{aligned} \quad (3.19)$$

We've shown that the proposed solution (3.11) satisfies the minimization constraint (3.14) (= is a minimal point) and satisfies the dual decomposition (3.2). (This has been shown only in the differentiated form )

□

### 3.4 EXPERIMENTS

**todo:**  $|\mathcal{X}| \sim 1M$  tabular environment

**todo:** get matlab code from tamar

#### 3.4.1 *Cliffworld*

### 3.5 SUMMARY

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## Q-LEARNING WITH CVAR

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### 4.1 VAR-BASED POLICY IMPROVEMENT

#### 4.1.1 Policy Improvement

Recall the primal definition of CVaR (\*\*). Our goal can then be rewritten as

$$\max_{\pi} CVaR_{\alpha}^{\pi}(Z) = \max_{\pi} \max_s \frac{1}{\alpha} \mathbb{E} [(Z^{\pi} - s)^{-}] + s \quad (4.1)$$

It also holds [XXX] that for the maximum, it holds  $s^* = VaR_{\alpha}$

$$CVaR_{\alpha}(Z) = \max_s \left\{ \frac{1}{\alpha} \mathbb{E} [(Z - s)^{-}] + s \right\} = \frac{1}{\alpha} \mathbb{E} [(Z - VaR_{\alpha}(Z))^{-}] + VaR_{\alpha}(Z) \quad (4.2)$$

The main idea of the algorithm 3, partially explored in [5], is as follows: If we knew the value  $s^*$  in the solution to equation (\*\*), we could simplify the problem to maximize only

$$\max_{\pi} CVaR_{\alpha}(Z) = \max_{\pi} \frac{1}{\alpha} \mathbb{E} [(Z^{\pi} - s^*)^{-}] + s^* \quad (4.3)$$

Given that we have access to the return distributions, we can improve the policy by simply choosing an action that maximizes CVaR in the first state  $a_0 = \arg \max_{\pi} CVaR_{\alpha}(Z^{\pi}(x_0))$ . We can then, as an approximation, set  $s = VaR_{\alpha}(Z(x_0))$  and then only maximize the simpler criterion

$$\max_{\pi} CVaR_{\alpha}(Z) = \max_{\pi} \frac{1}{\alpha} \mathbb{E} [(Z^{\pi} - s^*)^{-}] + s^* \quad (4.4)$$

The algorithm can be seen as coordinate ascent; in the first phase (when we compute CVaR) we maximize w.r.t.  $s$  while keeping  $\pi$  fixed; in the second phase we fix  $s$  and maximize w.r.t.  $\pi$ .

In the following theorem, we show that this indeed leads to a monotonic improvement over the previous policy.

**Theorem 4.** *Let  $\pi$  be a fixed policy,  $\alpha \in (0, 1]$ . By following policy  $\pi'$  from algorithm 3, we will improve  $CVaR_{\alpha}(Z)$  in expectation:*

$$CVaR_{\alpha}(Z^{\pi}) \leq CVaR_{\alpha}(Z^{\pi'})$$

*Proof.* Let  $s^*$  be a solution to eq ?? . Then by optimizing  $\max_{\pi} \frac{1}{\alpha} \mathbb{E} [(Z - s^*)^-]$ , we will monotonely improve the optimization criterion 4.3.

$$CVaR_{\alpha}(Z^{\pi}) = \frac{1}{\alpha} \mathbb{E} [(Z^{\pi} - s^*)^-] + s^* \leq \max_{\pi'} \frac{1}{\alpha} \mathbb{E} [(Z^{\pi'} - s^*)^-] + s^*$$

Note the following facts:

$$Z_t = R_t + \gamma Z_{t+1} \quad (4.5)$$

$$\mathbb{E} [(Z_t - s)^-] = \mathbb{E} [(Z_t - s) \mathbf{1}(Z_t \leq s)] \quad (4.6)$$

$$\mathbb{E}[H(Z)] = \sum_i p_i \mathbb{E}[H(Z_i)] \quad (4.7)$$

The last equation holds if  $Z \sim p_i$  is a probability mixture for any function  $H$ . We can rewrite the criterion as

$$\begin{aligned} \mathbb{E} [(Z_t - s)^-] &= \mathbb{E} [(Z_t - s) \mathbf{1}(Z_t \leq s)] = \mathbb{E} \left[ (R_t + \gamma Z_{t+1} - s) \mathbf{1}(Z_{t+1} \leq \frac{s - R_t}{\gamma}) \right] \\ &= \sum_{x_{t+1}, r_t} P(x_{t+1}, r_t | x_t, a) \mathbb{E} \left[ (r_t + \gamma Z(x_{t+1}) - s) \mathbf{1}(Z(x_{t+1}) \leq \frac{s - r_t}{\gamma}) \right] \\ &= \sum_{x_{t+1}, r_t} P(x_{t+1}, r_t | x_t, a) \mathbb{E} \left[ \gamma \left( Z(x_{t+1}) - \frac{s - r_t}{\gamma} \right) \mathbf{1}(Z(x_{t+1}) \leq \frac{s - r_t}{\gamma}) \right] \\ &= \gamma \sum_{x_{t+1}, r_t} P(x_{t+1}, r_t | x_t, a) \mathbb{E} \left[ \left( Z(x_{t+1}) - \frac{s - r_t}{\gamma} \right) \mathbf{1}(Z(x_{t+1}) \leq \frac{s - r_t}{\gamma}) \right] \\ &= \gamma \sum_{x_{t+1}, r_t} P(x_{t+1}, r_t | x_t, a) \mathbb{E} \left[ \left( Z(x_{t+1}) - \frac{s - r_t}{\gamma} \right)^- \right] \end{aligned} \quad (4.8)$$

Now let's say we sampled reward  $\hat{r}_t$  and state  $\hat{x}_{t+1}$ , we are still trying to find a policy  $\pi^*$  that maximizes

$$\begin{aligned} \pi^* &= \arg \max_{\pi} \mathbb{E} [(Z_t - s)^- | \hat{x}_{t+1}, \hat{r}] \\ &= \arg \max_{\pi} \mathbb{E} \left[ \left( Z(\hat{x}_{t+1}) - \frac{s - \hat{r}_t}{\gamma} \right)^- \right] \end{aligned} \quad (4.9)$$

Where we ignored the unsampled states (since these are not a function of  $\hat{x}_{t+1}$ ) and the multiplicative constant  $\gamma$  that will not affect the maximum argument.

At the starting state, we set  $s = s^*$ . At each following state we select an action according to equation 4.9. By induction we maximize the criterion ?? in each step.  $\square$

Note that the resulting policy is nonstationary, however we do not need an extended state-space to follow this policy, it is only necessary to remember our previous  $s$ .

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**Algorithm 3** VaR-based policy improvement

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```

input  $\alpha, x_0, \gamma$ 
 $a = \arg \max_a CVaR_\alpha(Z(x_0, a))$ 
 $s = VaR_\alpha(Z(x_0, a))$ 
 $x_t, r_t = \text{envTransition}(x_0, a)$ 
while  $x_t$  is not terminal do
   $s = \frac{s - r_t}{\gamma}$ 
   $a = \arg \max_a \mathbb{E} [(Z(x_t, a) - s)^-]$ 
   $x_t, r_t = \text{envTransition}(x_t, a)$ 
end while

```

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4.1.2 *Repeated policy improvement*

This policy then could be evaluated again by the distributional Q-learning procedures, however we

4.2 TODO

4.3 TODO

4.4 TODO

4.5 EXPERIMENTS

**todo:** ai safety gridworld

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## APPROXIMATE Q-LEARNING WITH CVAR

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5.1 TODO

5.2 TODO

5.3 TODO

5.4 EXPERIMENTS

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## CONCLUSION

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Bäuerle and Ott [5] Bellemare, Dabney, and Munos [6] Chow et al. [15] Dabney et al. [17] Garcia and Fernández [18] Majumdar and Pavone [26] Morimura et al. [29] Morimura et al. [30] Pflug and Pichler [31] Rockafellar and Uryasev [33] Rockafellar and Uryasev [34] Majumdar and Pavone [26] Leike et al. [25] Amodei et al. [1] Shapiro [35] Artzner et al. [3] Tamar et al. [42] Sutton and Barto [39] Watkins and Dayan [45] Bellman [8] Tsitsiklis [44] Boyd and Vandenberghe [13] Kreyszig [22] Koenker and Hallock [20] Committee [16] Mnih et al. [28] Silver et al. [37] Bahdanau, Cho, and Bengio [4] Krizhevsky, Sutskever, and Hinton [23]



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## DECLARATION

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Put your declaration here.

*Prague, February, 2018*

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Silvestr Stanko

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