Hess-Smith Panel method

Alessandro Chiarini, Andrea Rausa

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Politecnico di Milano

Overview

- Arguments
  - Hess-Smith Panel method (2D airfoil)
  - Weissinger (3D wing)
  - Multi-element airfoils and ground effect
- Methodology
  - Theory
  - Code developing + doubts clarification

Hess-Smith Panel method

Why?

The thin airfoil theory provides a very valuable insight into the

- generation of lift
- Kutta condition
- effect of the camber distribution on C_p, C_ℓ and C_m

but..

- it ignores the effects of the thickness distribution on C_ℓ and C_m
- Airfoils with high camber or large thickness violate the assumptions of the thin airfoil theory: the prediction accuracy quickly degrades

Therefore, in some cases we need something more! $\rightarrow$ PANEL METHODS

Hypothesis

We consider:

- Incompressible flow
- Inviscid flow (viscosity is confined to small regions, i.e. in a thin boundary layer and/or in the wake region)
- Irrotational flow

Therefore:

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi$$

and the continuity equation $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ leads to the Laplace equation

$$\boldsymbol{\nabla}^2\phi = 0.$$

Hypothesis

Therefore we are considering the following problem:

$$\begin{cases} \boldsymbol{\nabla}^2 \phi = 0 \\ \frac{\partial \phi}{\partial \boldsymbol{n}} = \boldsymbol{b}_n \quad \text{on the body.} \\ \boldsymbol{\nabla} \phi \to U_\infty \quad \text{at } \boldsymbol{r} \to \infty \end{cases}$$

Solution strategy

The Laplace equation is linear. Therefore, solution can be superposed:

$$\phi = \sum_{k=1}^{N} c_k \phi_k.$$

Complex flows can be obtained as superimposition of elementary solutions opportunely located:

- Uniform flow

$$\phi_\infty = Ux + Vy$$

- Source/Sink

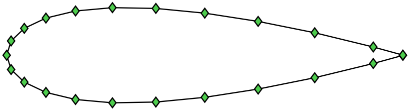
$$\phi_s = \frac{\sigma}{2\pi} \log\left(\sqrt{x^2 + y^2}\right)$$

- Vortex

$$\phi_v = \frac{\Gamma}{2\pi} \tan^{-1}\left(\frac{y}{x}\right)$$

Solution strategy

In panel methods singularities are opportunely located on panels (segments in the $2D$ case) that approximate the solid boundary shape.



And their strength is obtained by imposing:

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{b}_n \\ \text{Kutta Condition} \end{cases}$$

Panel methods

Advantages:

- Easy to implement
- Low computational cost
- Flexible and adaptable for complex geometries
- It does not need to create mesh for the flow field

Disadvantages:

- It does not work for separated flows or whenever viscous flows are important

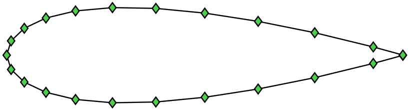
The Hess-Smith panel method

It is based on a distribution of sources and vortices on the surface body:

$$\phi = \phi_\infty + \phi_s + \phi_v$$

where:

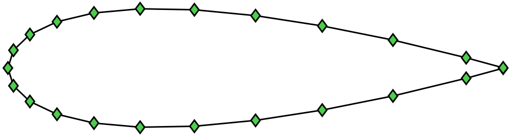
- The vortex strength is constant over the whole body
- The source strength varies from panel to panel

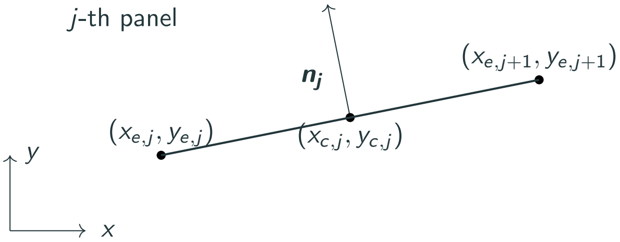


Hess-Smith overview

- Solid surface discretisation in N panels
- Uniform planar source and vortex distribution on each panel
- Sources strength varies for different panels, vortex intensity is the same
- $N+1$ unknowns: N source strengths q_j and 1 vortex intensity γ
- It sets $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{b}_n$ at the centre of every panel, i.e. at the control points: N equations
- It imposes the Kutta condition at the trailing edge: 1 equation
- It solves a linear system to obtain the unknowns q_j and γ
- It computes the pressure distribution and the aerodynamic loads

Geometrical discretisation





- $(x_{e,j}, y_{e,j})$ and $(x_{e,j+1}, y_{e,j+1})$ denote the extrema of the $j-$th panel
- $(x_{c,j}, y_{c,j})$ indicates the centre of the $j-$th panel, i.e. the control point

Potential velocity

According to this method the velocity of the flow field is expressed as:

$$\boldsymbol{u}(x,y) = \underbrace{\boldsymbol{U}_\infty}_{\text{Uniform flow}} + \underbrace{\sum_{j=1}^{N} \boldsymbol{u}_j^s(x,y) q_j}_{\text{Source induced velocity}} + \underbrace{\gamma \sum_{j=1}^{N} \boldsymbol{u}_j^v(x,y)}_{\text{Vortex induced velocity}}$$

and to evaluate the $N+1$ unknowns, i.e. the N source strengths q_j and the 1 vortex strength γ, we impose

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{b}_n \quad \text{at the control point of every panel} \\ \text{Kutta condition} \end{cases}$$

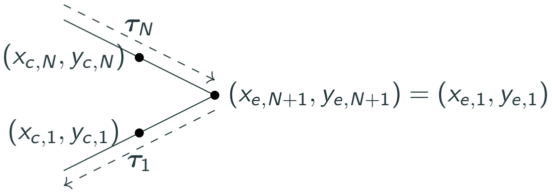
Computation of q_j and γ

We set the normal boundary condition at each panel at its control point

$$\boldsymbol{u}(x_{c,i}, y_{c,i}) \cdot \boldsymbol{n_i} = 0 \quad \text{i=1,..N}$$

and the Kutta condition which requires the flow to leave smoothly the trailing edge. It corresponds to require that the tangential components of the velocity on the two panels adjacent to the trailing edge to be equal:

$$\boldsymbol{u}(x_{c,1}, y_{c,1}) \cdot \boldsymbol{\tau}_1 + \boldsymbol{u}(x_{c,N}, y_{c,N}) \cdot \boldsymbol{\tau}_N = 0$$



The linear system

It leads to the following linear system:

$$\begin{bmatrix} \boldsymbol{A}^s & \boldsymbol{a}^v \\ (\boldsymbol{c}^s)^T & c^v \end{bmatrix} \begin{bmatrix} \boldsymbol{q} \\ \gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}^s \\ b^v \end{bmatrix}$$

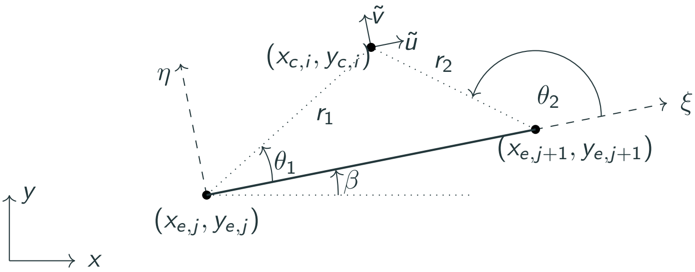
$$A^s_{ij} = \boldsymbol{u}^s_j(x_{c,i}, y_{c,i}) \cdot \boldsymbol{n}_i \qquad a^v_i = \sum_{j=1}^{N} \boldsymbol{u}^v_j(x_{c,i}, y_{c,i}) \cdot \boldsymbol{n}_i$$

$$c^s_j = \boldsymbol{u}^s_j(x_{c,1}, y_{c,1}) \cdot \boldsymbol{\tau}_1 + \boldsymbol{u}^s_j(x_{c,N}, y_{c,N}) \cdot \boldsymbol{\tau}_N \qquad c^v = \sum_{j=1}^{N} \boldsymbol{u}^v_j(x_{c,1}, y_{c,1}) \cdot \boldsymbol{\tau_1} + \boldsymbol{u}^v_j(x_{c,N}, y_{c,N}) \cdot \boldsymbol{\tau_N}$$

$$b^s_i = -\boldsymbol{U}_\infty \cdot \boldsymbol{n}_i \qquad b^v = -\boldsymbol{U}_\infty(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_N)$$

where $\boldsymbol{u}^s_j(x_{c,i}, y_{c,i})$ and $\boldsymbol{u}^v_j(x_{c,i}, y_{c,i})$ are the velocity a unitary constant source and vortex distribution on the $j-$th panel induced on the control point of the $i-$th panel:

Computation of $u_j^s(x_{c,i}, y_{c,i})$ and $u_j^v(x_{c,i}, y_{c,i})$

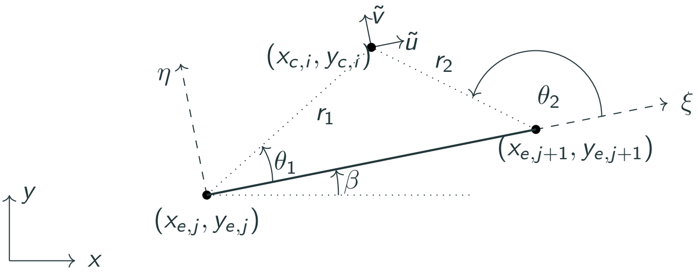


In the local reference frame, the unit-strength source distribution on the $j-$th panel induce on the control point of the $i-$th panel:

$$\tilde{u}_j^s(\xi_{c,i}, \eta_{c,i}) = \frac{1}{2\pi} \int_0^{l_j} \frac{\xi_{c,i} - t}{(\xi_{c,i} - t)^2 + \eta_{c,i}^2} dt = -\frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right)$$

$$\tilde{v}_j^s(\xi_{c,i}, \eta_{c,i}) = \frac{1}{2\pi} \int_0^{l_j} \frac{\eta_{c,i}}{(\xi_{c,i} - t)^2 + \eta_{c,i}^2} dt = \frac{\theta_2 - \theta_1}{2\pi}$$

Computation of $u_j^s(x_{c,i}, y_{c,i})$ and $u_j^v(x_{c,i}, y_{c,i})$

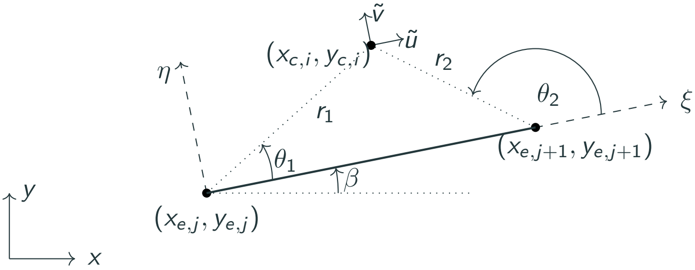


In the local reference frame, the unit-strength vortex distribution on the $j-$th panel induce on the control point of the $i-$th panel:

$$\tilde{u}_j^v(\xi_{c,i}, \eta_{c,i}) = \frac{1}{2\pi} \int_0^{l_j} \frac{\eta_{c,i}}{(\xi_{c,i} - t)^2 + \eta_{c,i}^2} dt = \frac{\theta_2 - \theta_1}{2\pi}$$

$$\tilde{v}_j^v(\xi_{c,i}, \eta_{c,i}) = -\frac{1}{2\pi} \int_0^{l_j} \frac{\xi_{c,i} - t}{(\xi_{c,i} - t)^2 + \eta_{c,i}^2} dt = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right)$$

Computation of $u_j^s(x_{c,i}, y_{c,i})$ and $u_j^v(x_{c,i}, y_{c,i})$



You have to bring the induced velocity back to the (x, y) global reference frame:

$$\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\boldsymbol{u}} = \underbrace{\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}}_{\boldsymbol{Q}^T} \underbrace{\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}}_{\tilde{\boldsymbol{u}}}$$

Computation of $\boldsymbol{u}_j^s(x_{c,i}, y_{c,i})$ and $\boldsymbol{u}_j^v(x_{c,i}, y_{c,i})$

- Step 1: determine the local coordinate of the control point of the $i-$th panel $(\xi_{c,i}, \eta_{c,i})$

$$\underbrace{\begin{bmatrix} \xi_{c,i} \\ \eta_{c,i} \end{bmatrix}}_{\boldsymbol{\xi}_{c,i}} = \underbrace{\begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}}_{\boldsymbol{Q}} \underbrace{\begin{bmatrix} x_{c,i} \\ y_{c,i} \end{bmatrix}}_{\boldsymbol{x}_{c,i}}$$

- Step 2: evaluate $\tilde{\boldsymbol{u}}^s$ and $\tilde{\boldsymbol{u}}^v$
- Step 3: Go back to the global reference frame

$$\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\boldsymbol{u}} = \underbrace{\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}}_{\boldsymbol{Q}^T} \underbrace{\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}}_{\tilde{\boldsymbol{u}}}$$

Aerodynamic loads

- Circulation

$$\Gamma = \sum_{i=1}^{N} l_i \gamma$$

- Pressure coefficient (Bernoulli equation)

$$C_{p,i} = 1 - \frac{(\boldsymbol{u}_i \cdot \boldsymbol{\tau}_i)^2}{|\boldsymbol{U}_\infty|^2}$$

where

$$\boldsymbol{u}_i \cdot \boldsymbol{\tau}_i = \left(\boldsymbol{U}_\infty + \sum_{j=1}^{N} \boldsymbol{u}_j^s(x_{c,i}, y_{c,i}) q_j + \gamma \sum_{j=1}^{N} \boldsymbol{u}_j^v(x_{c,i}, y_{c,i}) \right) \cdot \boldsymbol{\tau}_i$$

Aerodynamic loads

- Lift coefficient (pressure integration)

$$C_\ell = \sum_{i=1}^{N} C_{p,i} \frac{l_i}{c} \boldsymbol{n_i} \cdot \boldsymbol{n}_{U_\infty}$$

- Lift coefficient (Kutta-Joukowsky theorem)

$$L = \rho \boldsymbol{U}_\infty \times \boldsymbol{\Gamma}, \quad C_\ell = -2 \frac{\Gamma}{U_\infty}$$

- Leading edge moment coefficient

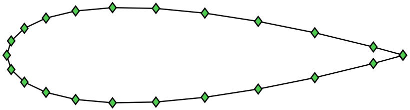
$$C_{M,LE} = -\sum_{i=1}^{N} C_{p,i} \frac{l_i}{c^2} \left(\boldsymbol{r}_{c,i} \times \boldsymbol{n}_{c,i} \right) \cdot \hat{\boldsymbol{z}}$$

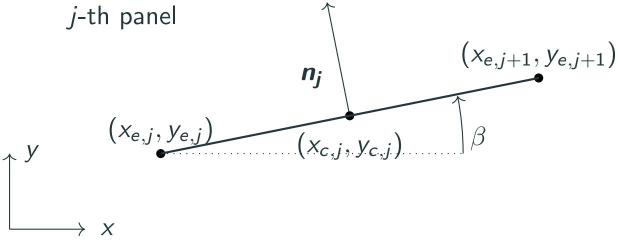
Implementation

- Step 1: Geometry panelisation
- Step 2: Build functions that provide the velocity that the unity-strength source and vortex distributions of the $j-$th panel induce on the control point of the $i-$th panel
- Step 3: Build the linear system and solve for $\boldsymbol{q}$ and γ
- Step 4: Compute the flow field, the pressure distribution and the aerodynamic loads

Implementation

- Step 1: Geometry panelisation





- Subdivide the geometry in N panels (uniform distribution, half-cosine distribution etc ..)
- Determine the extrema of the panels $(x_{e,i}, y_{e,i})$ with $i = 1..N+1$ and the control points $(x_{c,i}, y_{c,i})$
- Determine the normal and tangent vectors $\boldsymbol{n} = (-\sin(\beta), \cos(\beta))$, $\boldsymbol{\tau} = (\cos(\beta), \sin(\beta))$.

Implementation

- Step 2: build functions that provide the velocity that the unity-strength source and vortex distributions of the $j-$th panel induce on the control point of the $i-$th panel

Input	$\longrightarrow$	Output
$(x_{c,i}, y_{c,i})$		$\boldsymbol{u}_j^s(x_{c,i}, y_{c,i})$
$(x_{e,j}, y_{e,j})$		$\boldsymbol{u}_j^v(x_{c,i}, y_{c,i})$
$(x_{e,j+1}, y_{e,j+1})$		

Implementation

- A: determine the local coordinate of the control point of the $i-$th panel $(\xi_{c,i}, \eta_{c,i})$

$$\underbrace{\begin{bmatrix} \xi_{c,i} \\ \eta_{c,i} \end{bmatrix}}_{\boldsymbol{\xi}_{c,i}} = \underbrace{\begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}}_{\boldsymbol{Q}} \underbrace{\begin{bmatrix} x_{c,i} \\ y_{c,i} \end{bmatrix}}_{\boldsymbol{x}_{c,i}}$$

- B: evaluate $\tilde{\boldsymbol{u}}^s$ and $\tilde{\boldsymbol{u}}^v$
- C: Go back to the global reference frame

$$\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\boldsymbol{u}} = \underbrace{\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}}_{\boldsymbol{Q}^T} \underbrace{\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}}_{\tilde{\boldsymbol{u}}}$$

Implementation

- Step 3: Build the linear system and solve for $\boldsymbol{q}$ and γ.

$$\begin{bmatrix} \boldsymbol{A}^s & \boldsymbol{a}^v \\ (\boldsymbol{c}^s)^T & c^v \end{bmatrix} \begin{bmatrix} \boldsymbol{q} \\ \gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}^s \\ b^v \end{bmatrix}$$

$$A^s_{ij} = \boldsymbol{u}^s_j(x_{c,i}, y_{c,i}) \cdot \boldsymbol{n}_i \qquad a^v_i = \sum_{j=1}^{N} \boldsymbol{u}^v_j(x_{c,i}, y_{c,i}) \cdot \boldsymbol{n}_i$$

$$c^s_j = \boldsymbol{u}^s_j(x_{c,1}, y_{c,1}) \cdot \boldsymbol{\tau}_1 + \boldsymbol{u}^s_j(x_{c,N}, y_{c,N}) \cdot \boldsymbol{\tau}_N \qquad c^v = \sum_{j=1}^{N} \boldsymbol{u}^v_j(x_{c,1}, y_{c,1}) \cdot \boldsymbol{\tau}_1 + \boldsymbol{u}^v_j(x_{c,N}, y_{c,N}) \cdot \boldsymbol{\tau}_N$$

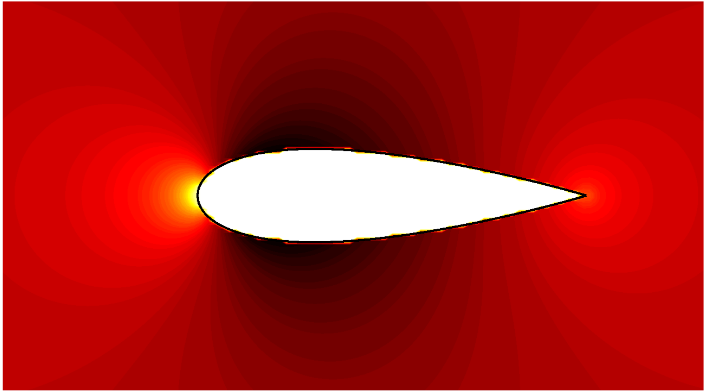
$$b^s_i = -\boldsymbol{U}_\infty \cdot \boldsymbol{n}_i \qquad b^v = -\boldsymbol{U}_\infty(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_N)$$

Implementation

- Step 4: Compute the flow field, the pressure distribution and the aerodynamic loads

Input	$\longrightarrow$	Output
$\boldsymbol{q}$		$\boldsymbol{u}$
γ		$C_{p,i}$
Geometry		C_ℓ, $C_{M,LE}$

Example: Naca 0024



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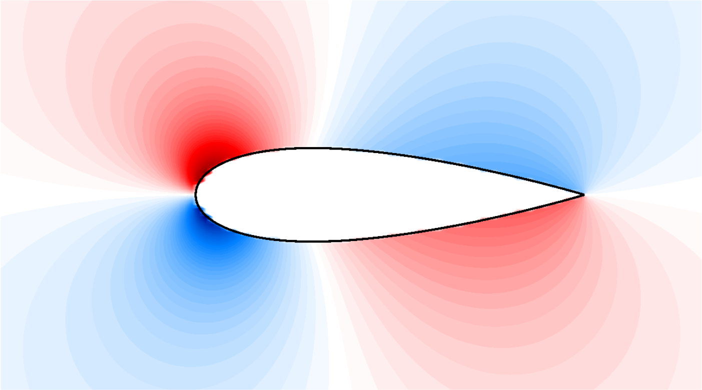

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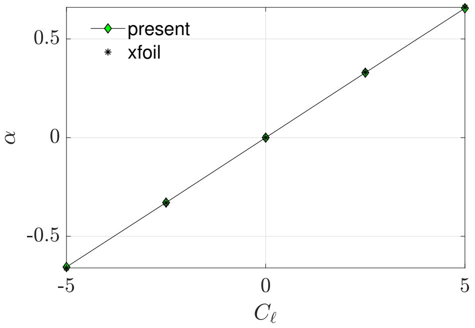
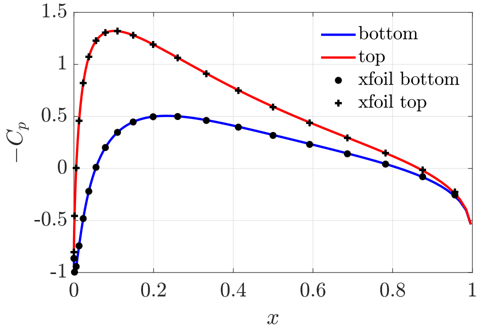
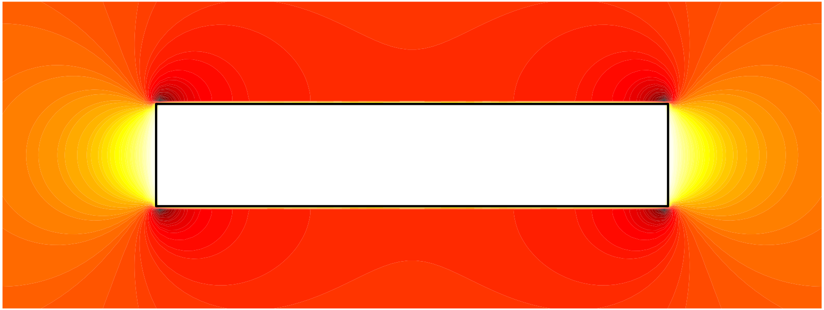




Example

It may be used also to create initial flow field for DNS simulations.



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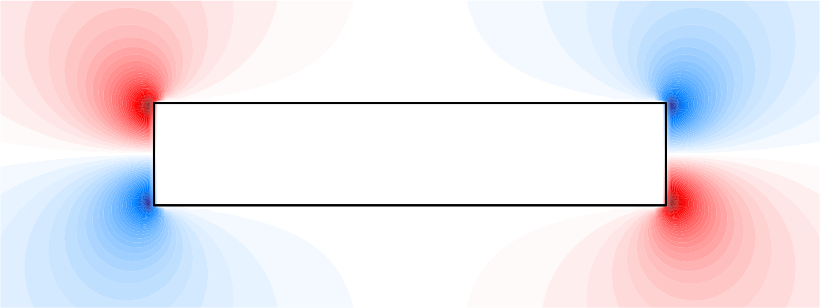

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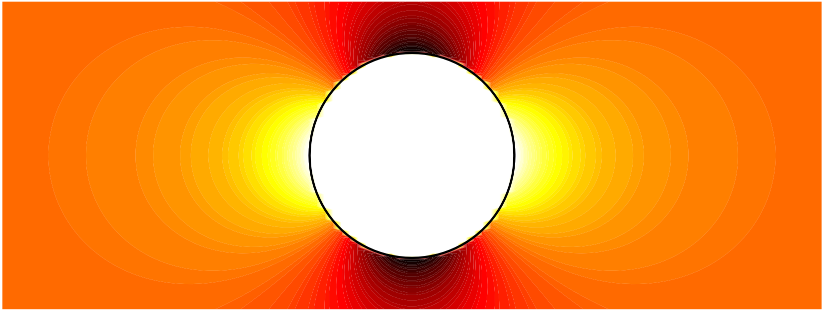

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