Finite-Difference Schemes (pt. 2)

- Physical Models for Sound Synthesis -

Silvin Willemsen - March 12, 2021

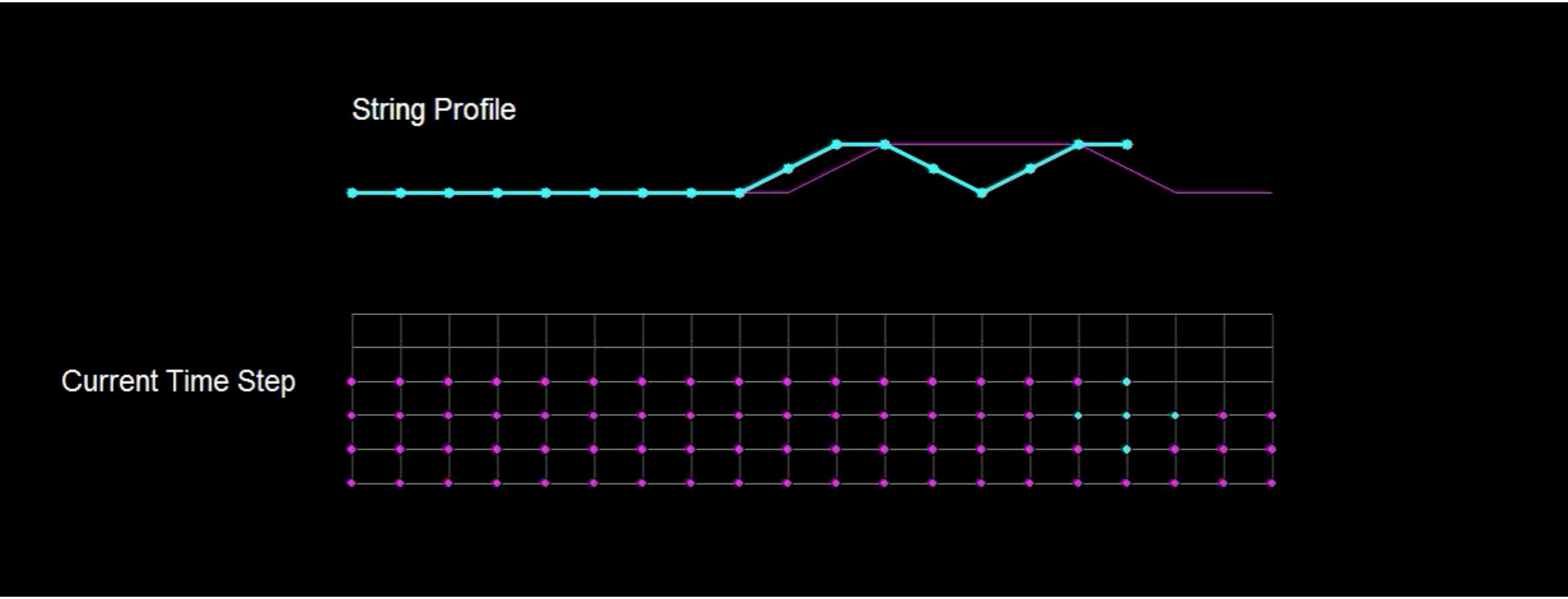
Agenda

- Recap last lecture
- Damped 1D wave
- Ideal Bar
- (Damped) Stiff String
- Towards complete instruments
  - ↳ Connecting models

Recap

Recap

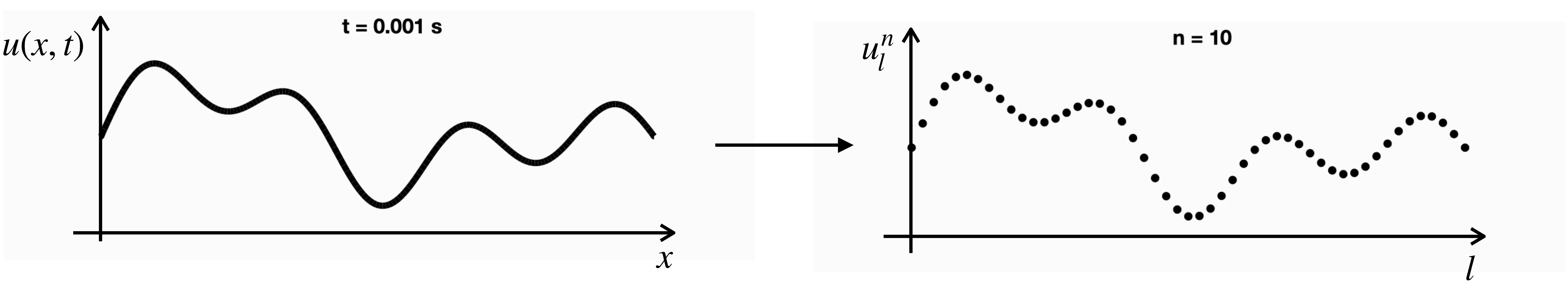
- FDSs subdivide continuous systems into points in space and time.



University of Edinburgh, NESS: Next Generation Sound Synthesis, 2016, url: http://www.ness.music.ed.ac.uk/project#3

Recap

- FDSs subdivide continuous systems into points in space and time.
- The state of a (1D) system is denoted by $u = u(x,t)$ which becomes grid function u_l^n when discretised.



- FDSs subdivide continuous systems into points in space and time.
- The state of a (1D) system is denoted by $u = u(x,t)$ which becomes grid function u_l^n when discretised.
- Steps to implementing a (continuous) PDE:

1. Discretise PDE to FDS. $u_{tt} = \ldots \rightarrow \delta_{tt}u_l^n = \ldots$
2. Expand operators. $\rightarrow \dfrac{1}{k^2}\left(u_l^{n+1} - 2u_l^n + u_l^{n-1}\right) = \ldots$
3. Solve for u_l^{n+1} $\rightarrow u_l^{n+1} = 2u_l^n - u_l^{n-1} + k^2(\ldots)$
4. Define domain and boundary conditions. $N = \ldots \qquad l = [0, \ldots, N]$ $u_0^n = u_N^n = 0$

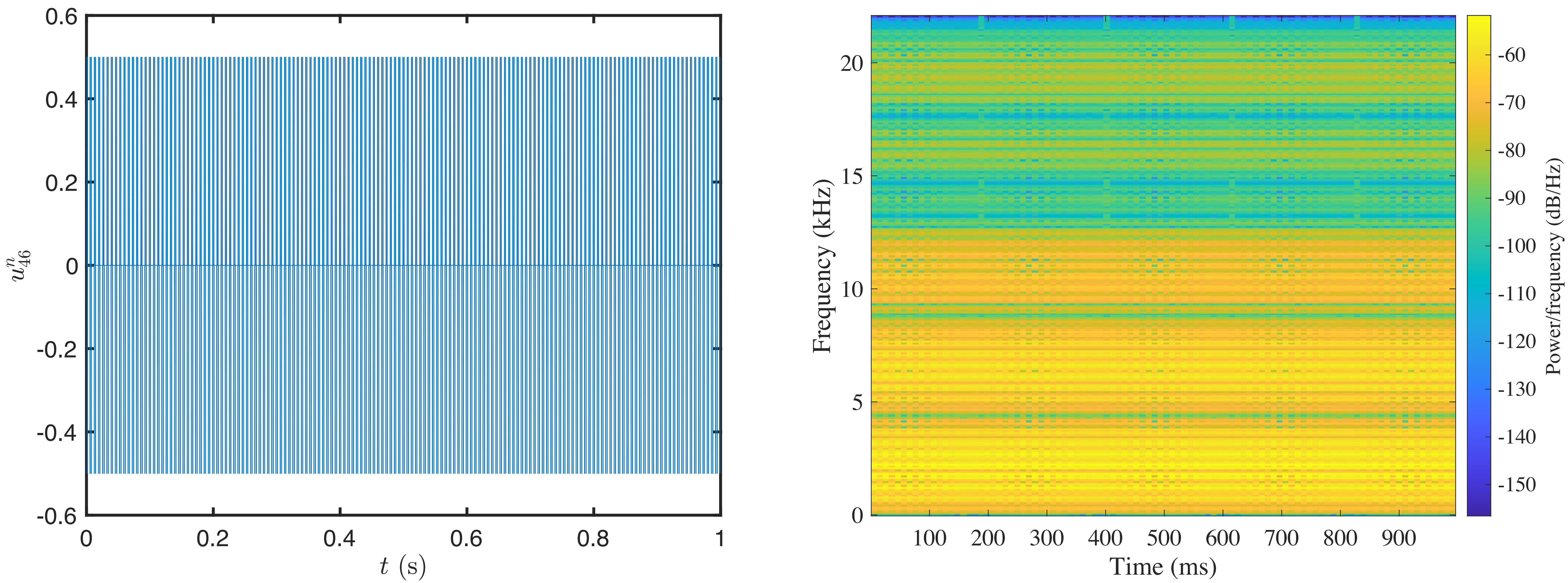
Damped 1D wave equation

Damped 1D wave equation

Who wants to show their implementation? :)

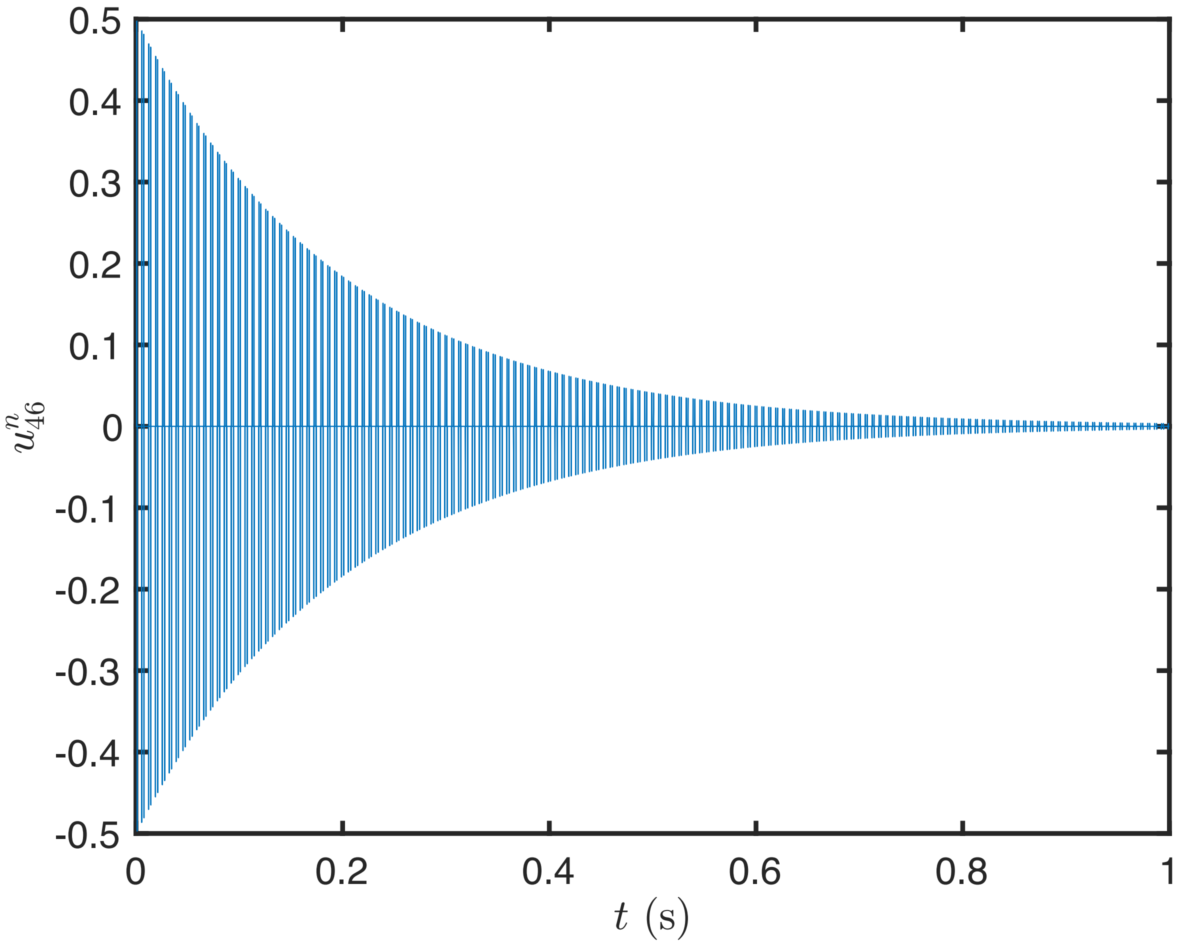
Sound examples

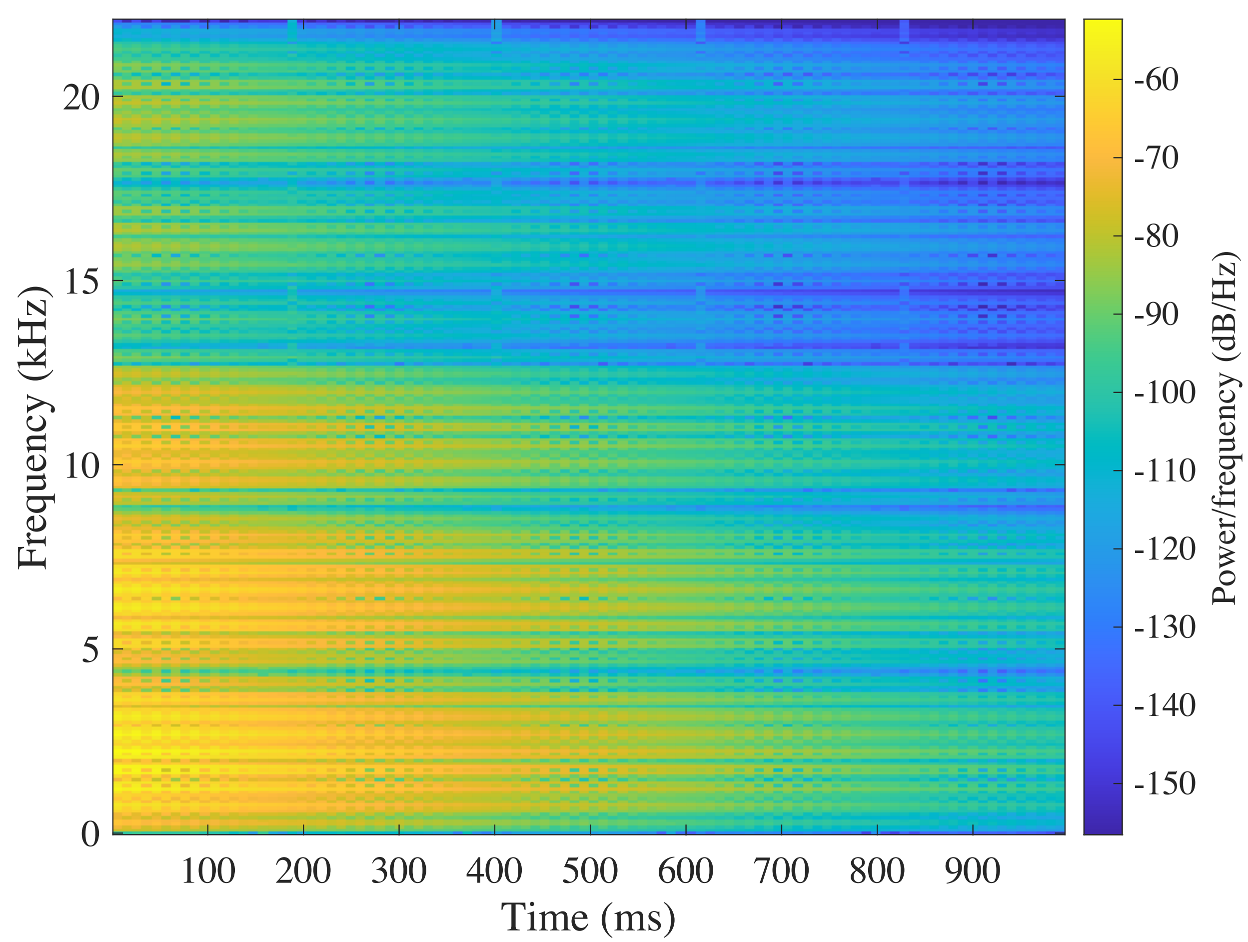
$$c = 300 \qquad L = 1 \qquad \sigma_0 = 0 \qquad \sigma_1 = 0$$



Sound examples

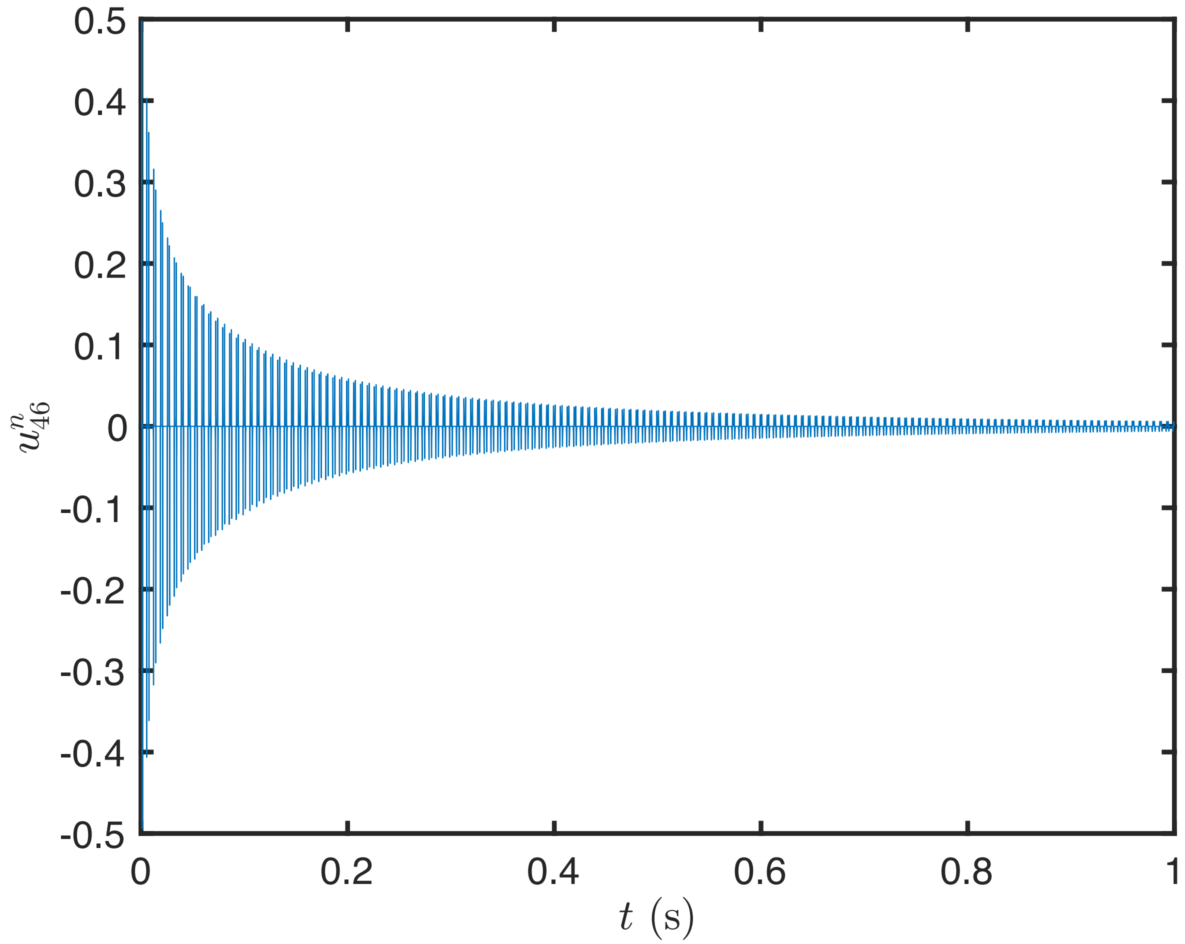
$$c = 300 \qquad L = 1 \qquad \sigma_0 = 5 \qquad \sigma_1 = 0$$

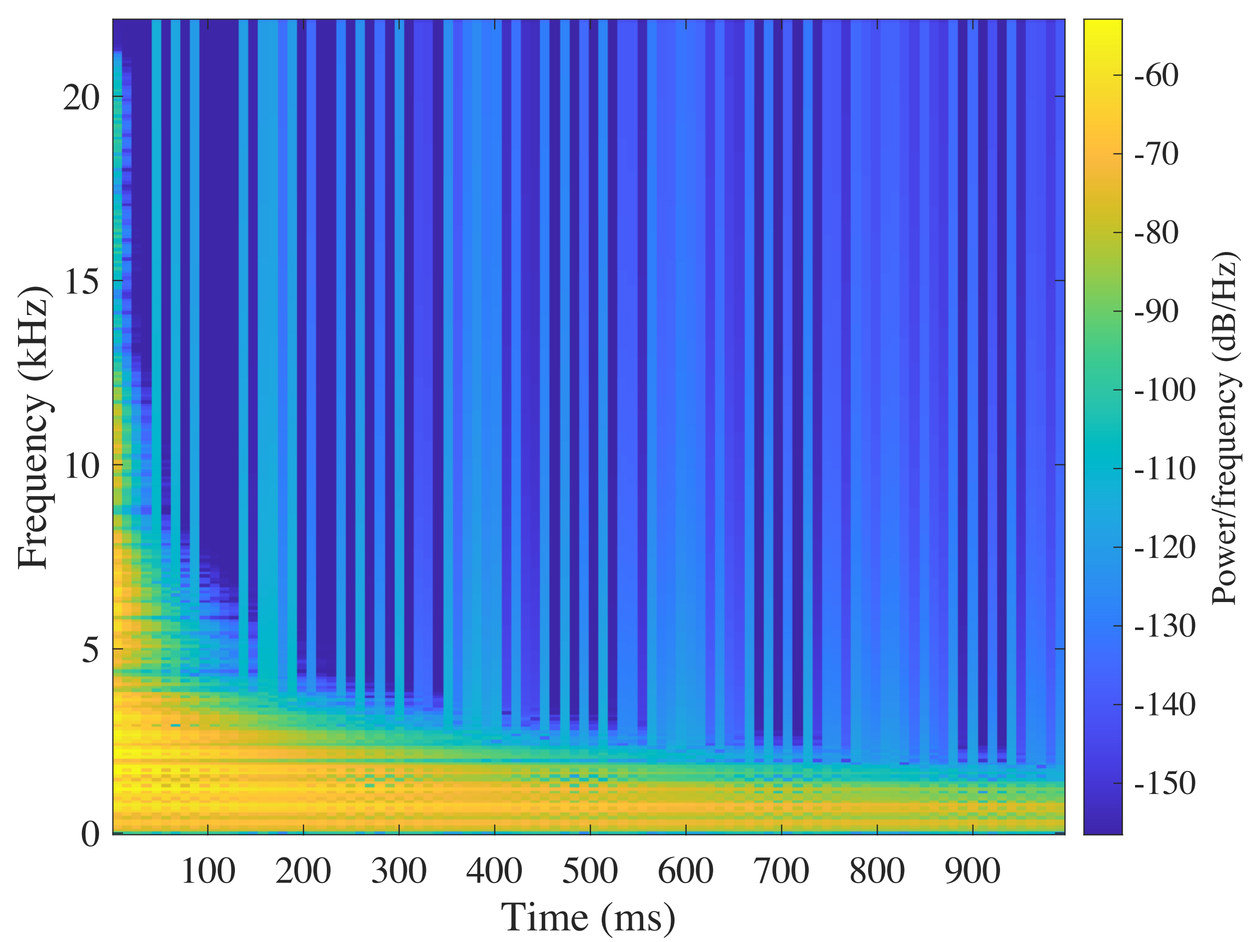




Sound examples

$$c = 300 \qquad L = 1 \qquad \sigma_0 = 1 \qquad \sigma_1 = 0.005$$





Damped 1D wave equation

What choices do we have for discretising

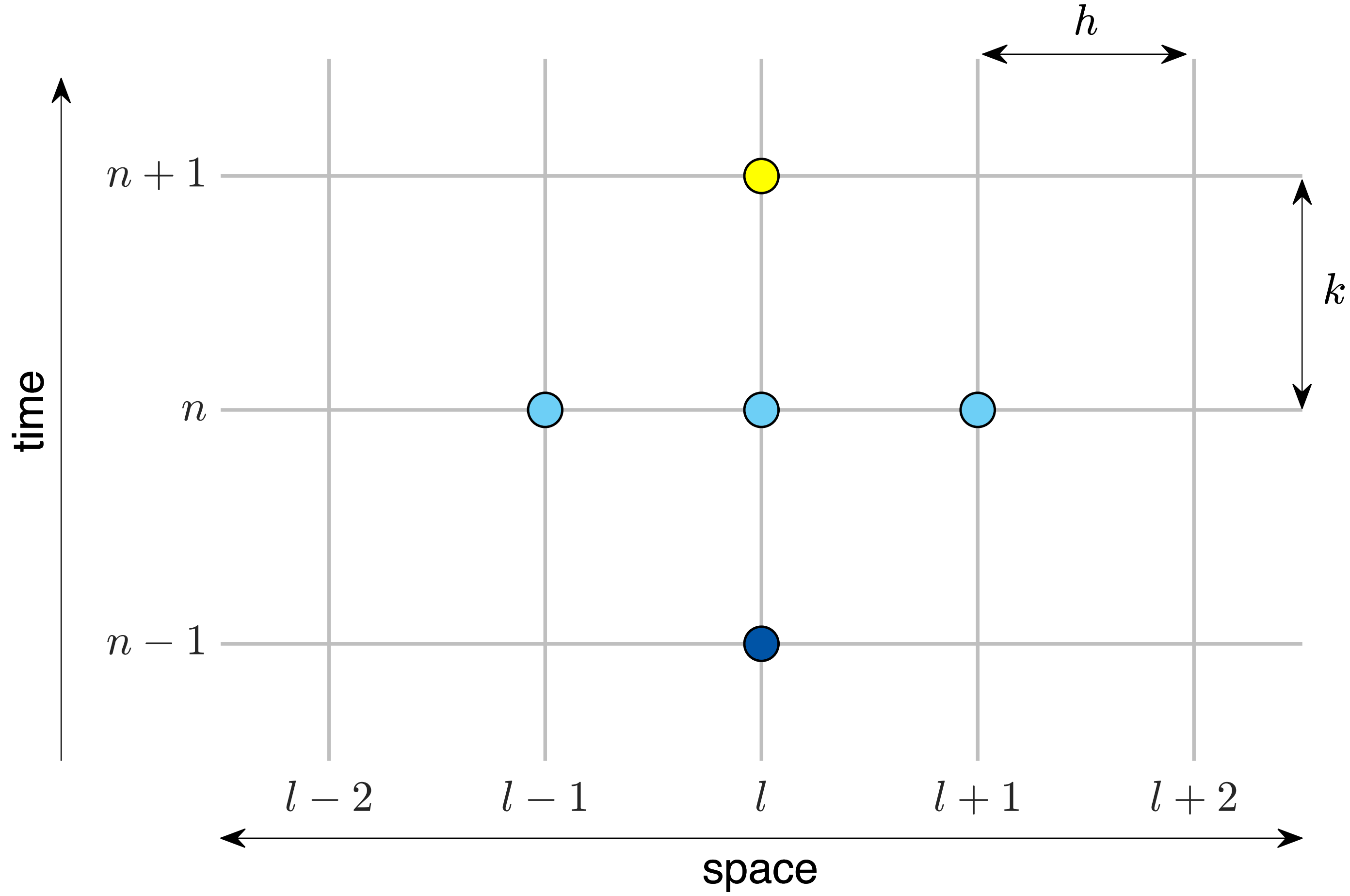
$$u_{tt} = c^2 u_{xx} - 2\sigma_0 u_t + 2\sigma_1 u_{txx}$$

and why?

Damped 1D wave equation

It can be useful to talk about a FDS in terms of a stencil, or the 'region of operation'.

The stencil of $\delta_{tt}u_l^n = c^2\delta_{xx}u_l^n$ can be visualised as

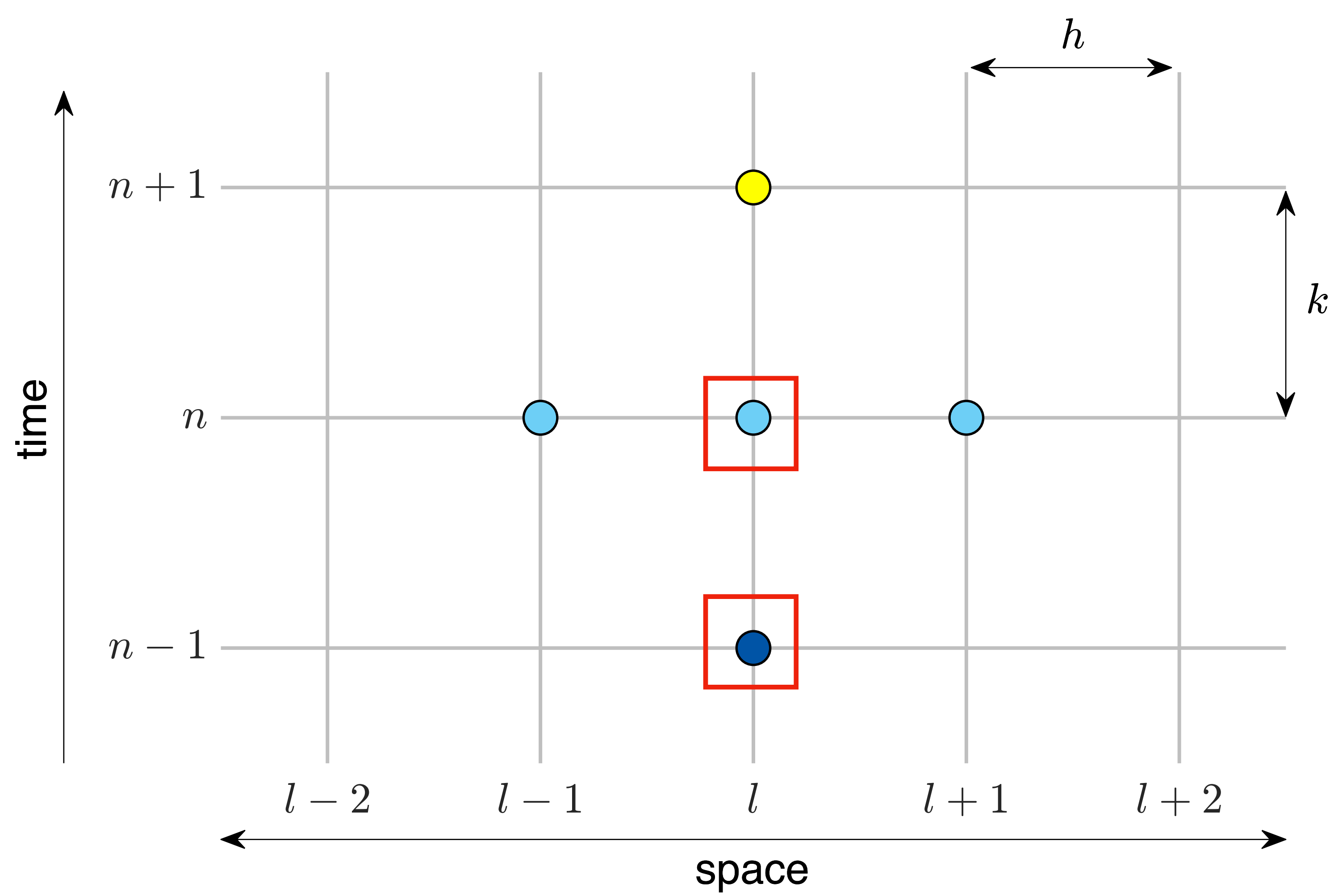


$$u_l^{n+1} = 2u_l^n - u_l^{n-1} + \lambda^2(u_{l+1}^n - 2u_l^n + u_{l-1}^n)$$

Damped 1D wave equation

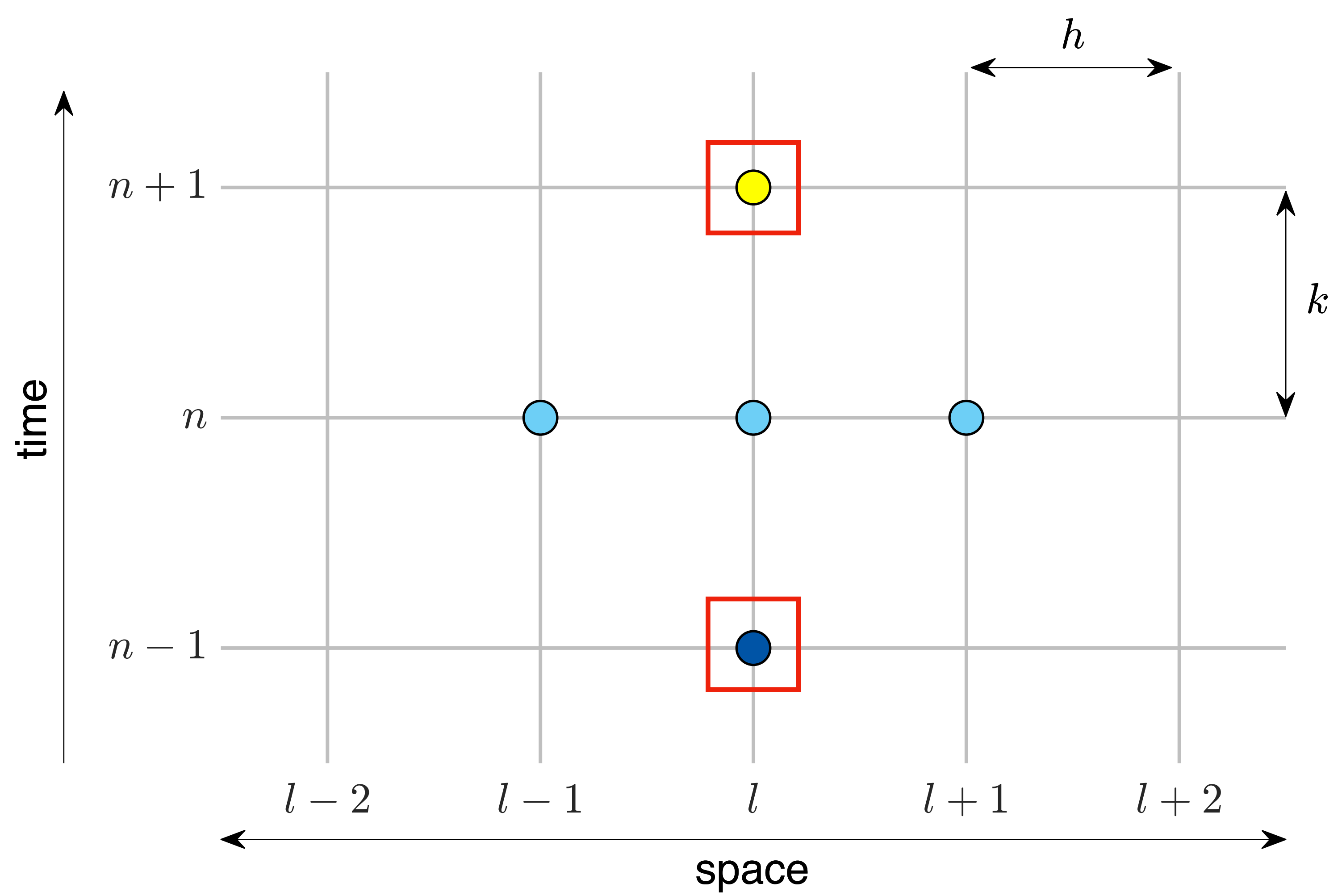
What about the the stencil of $\delta_{tt} u_l^n = c^2 \delta_{xx} u_l^n - 2\sigma_0 \delta_{t-} u_l^n$?

$$\ldots - 2\sigma_0/k\left(u_l^n - u_l^{n-1}\right)$$



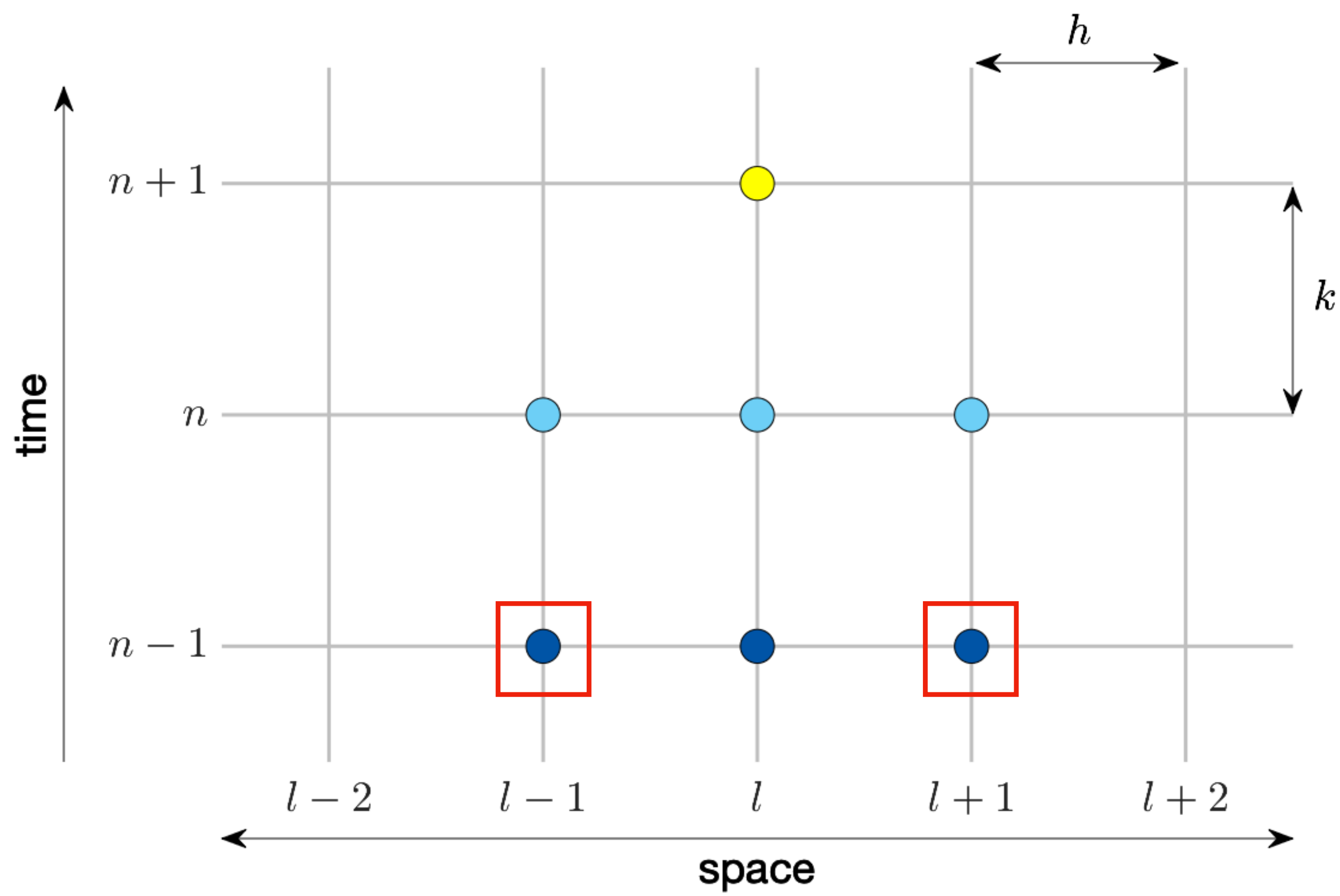
Does anything change for $\delta_{tt}u_l^n = c^2\delta_{xx}u_l^n - 2\sigma_0\delta_{t\cdot}u_l^n$?

$$\ldots - \sigma_0/k\left(u_l^{n+1} - u_l^{n-1}\right)$$

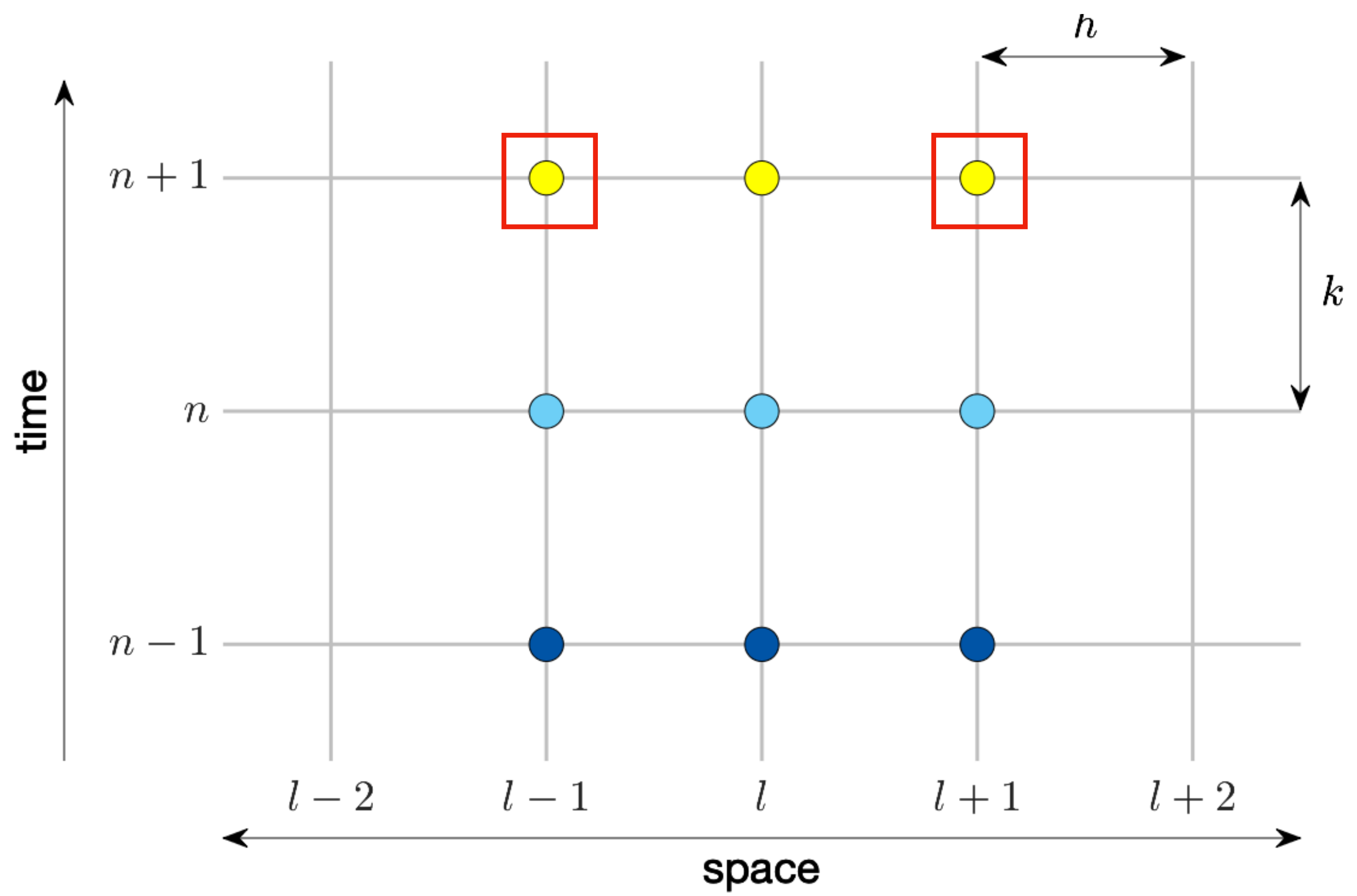


What about $\delta_{tt}u_l^n = c^2\delta_{xx}u_l^n - 2\sigma_0\delta_{t\cdot}u_l^n + 2\sigma_1\delta_{t-}\delta_{xx}u_l^n$?

$$\ldots + 2\sigma_1/kh^2\left(u_{l+1}^n - 2u_l^n + u_{l-1}^n - \boxed{u_{l+1}^{n-1}} + 2u_l^{n-1} - \boxed{u_{l-1}^{n-1}}\right)$$



And finally $\delta_{tt}u_l^n = c^2\delta_{xx}u_l^n - 2\sigma_0\delta_{t\cdot}u_l^n + 2\sigma_1\delta_{t\cdot}\delta_{xx}u_l^n$



implicit!

Ideal Bar

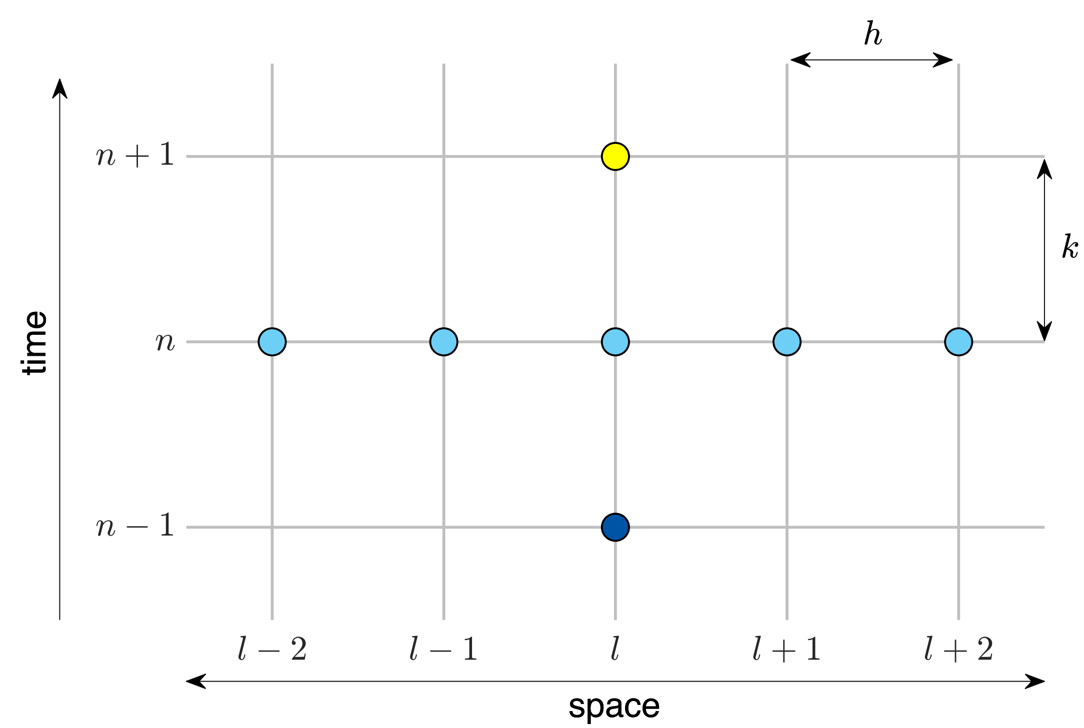
The PDE of the ideal bar is defined as follows:

$$\boxed{u_{tt} = -\kappa^2 u_{xxxx}}$$

with stiffness coefficient κ and 4th-order spatial derivative:

$$u_{xxxx} \approx \delta_{xxxx} u_l^n = \delta_{xx}\delta_{xx} u_l^n = \frac{1}{h^4}\left(u_{l+2}^n - 4u_{l+1}^n + 6u_l^n - 4u_{l-1}^n + u_{l-2}^n\right)$$

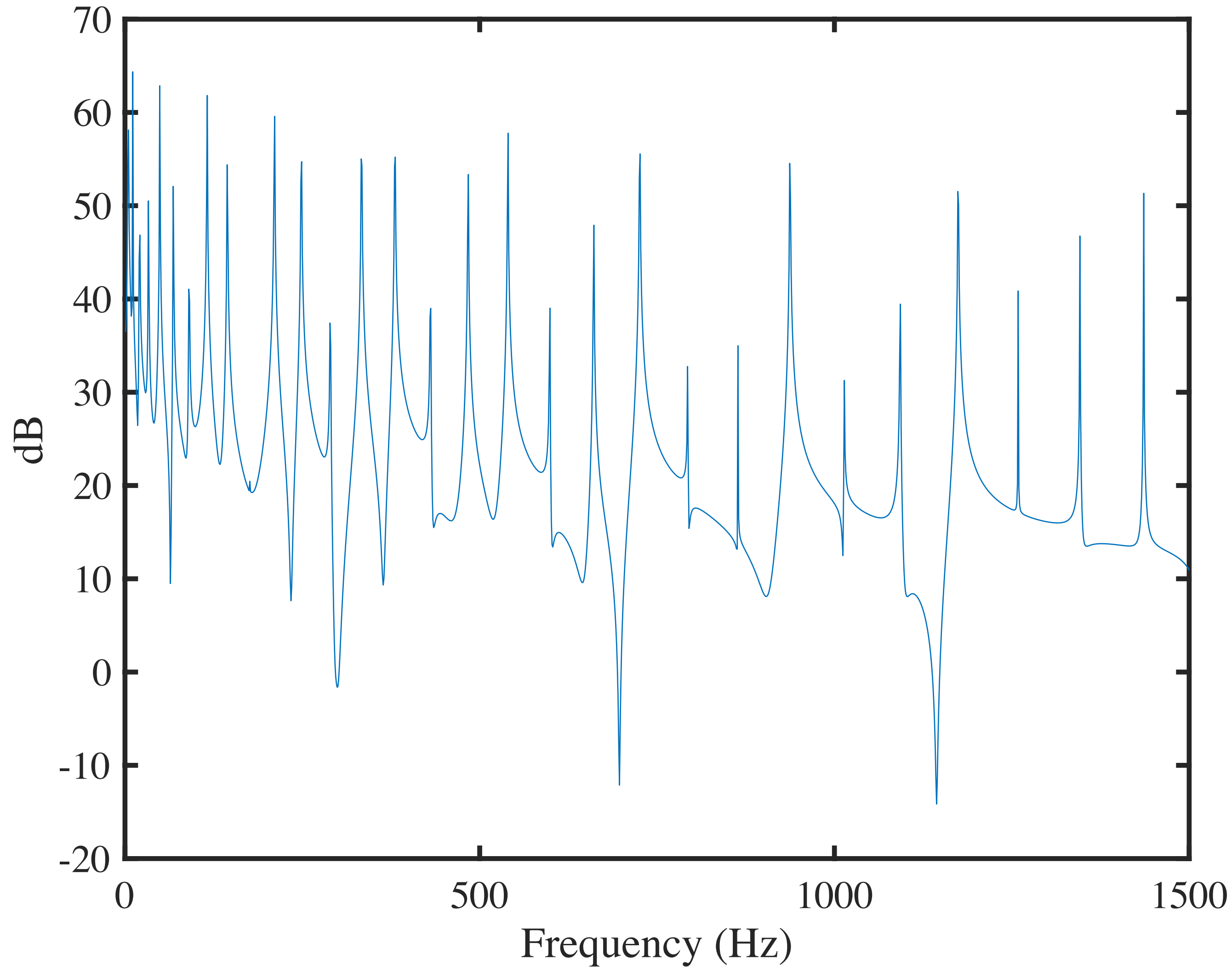
Stencil?



Ideal Bar

The 4th-order derivative causes dispersion:

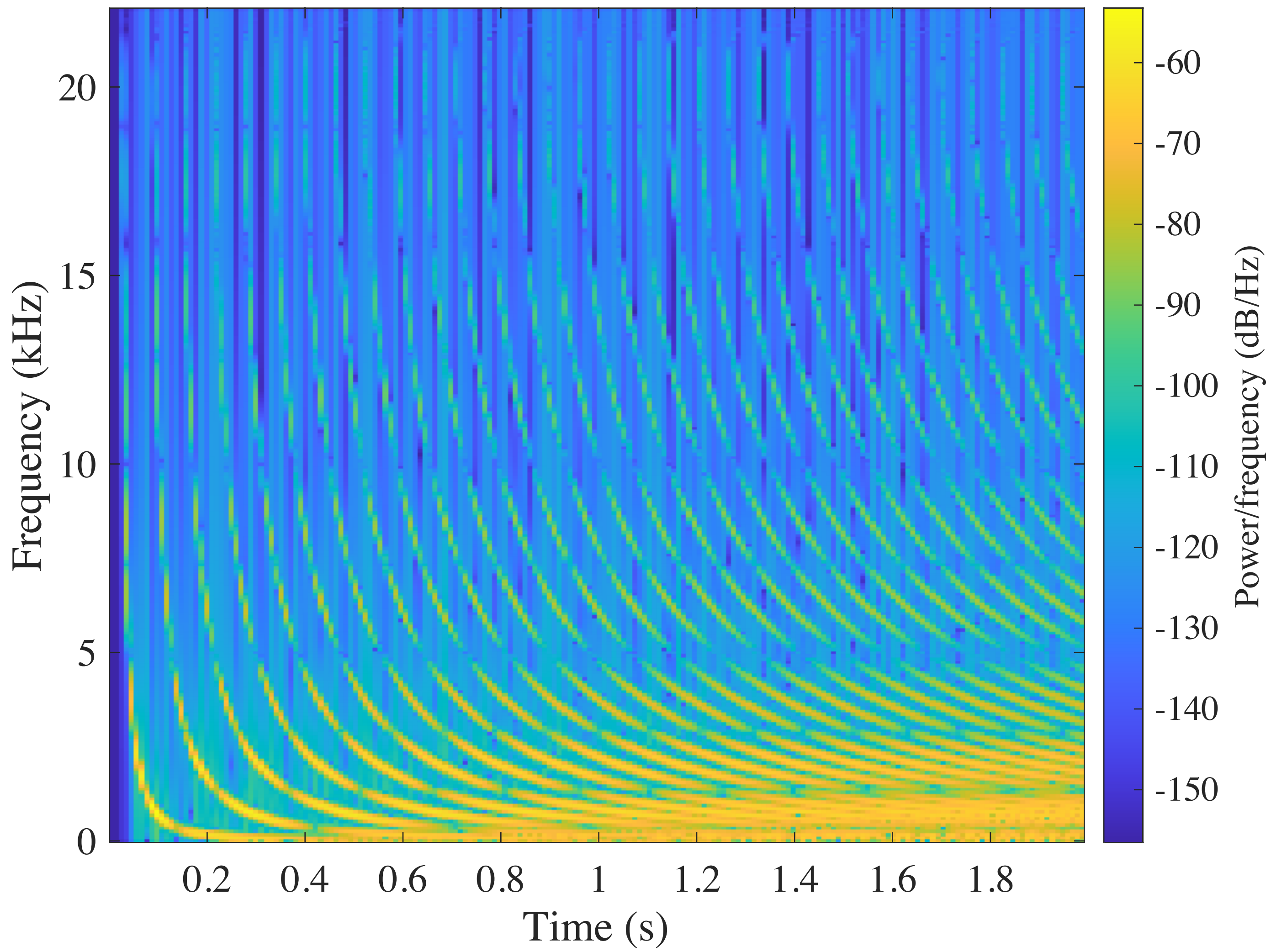
- higher frequencies travel faster than lower ones



Ideal Bar

Very low values of κ can be used to model a spring reverb

- $\kappa = 0.006, \;\; \sigma_0 = 0.1, \;\; \sigma_1 = 0.000001$



or star wars
sound effects

Ideal Bar

For the 1D wave equation, our range of calculation was $l = [1, \ldots, N-1]$.

In the case of the bar, we seem to need points outside our defined region! These are referred to as virtual grid points.

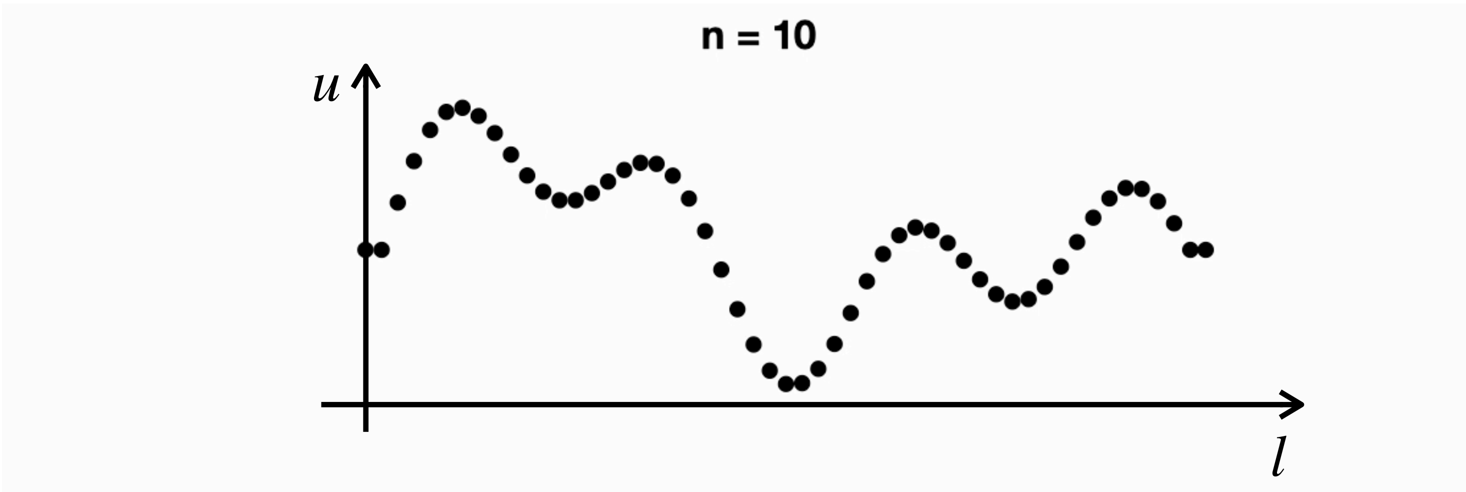
At the boundaries ($l = 0, N$), the following boundary conditions can be defined:

- Clamped $\quad u = u_x = 0$
- Simply supported $\quad u = u_{xx} = 0$
- Free $\quad u_{xx} = u_{xxx} = 0$

From these, we can obtain definitions for the undefined points.

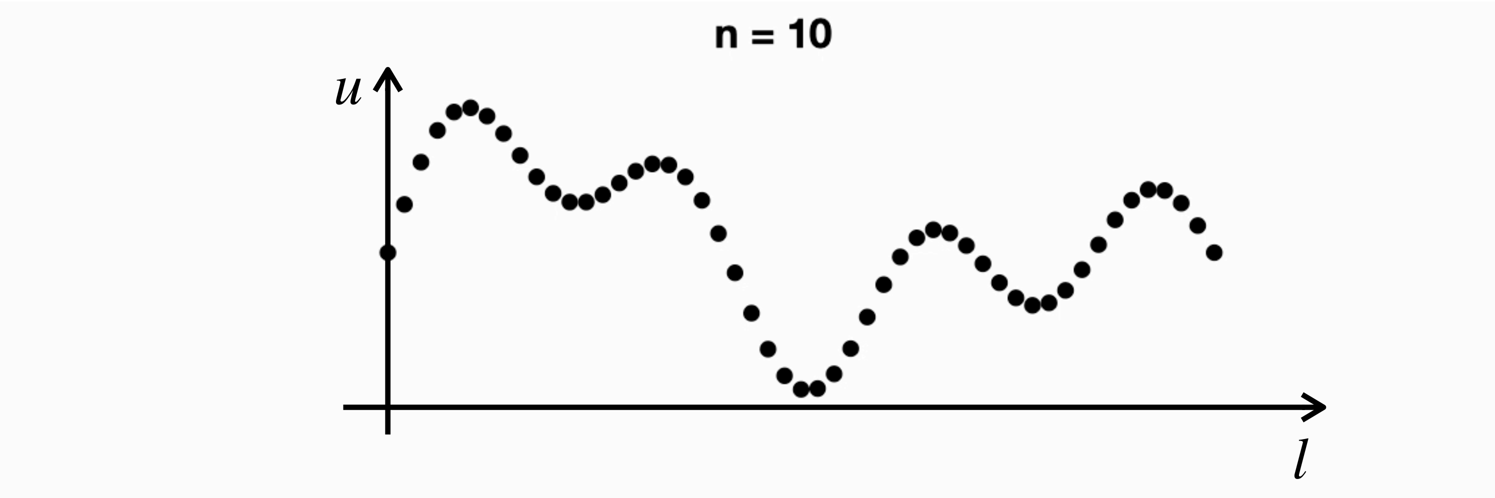
Ideal Bar

The clamped boundary condition looks like



where we only need to calculate the points $l = [2, \ldots, N-2]$.

The simply supported boundary condition looks like



where we need to calculate the points $l = [1, \ldots, N-1]$.

For the simply supported boundary condition, we need to find definitions for virtual grid points when calculating the scheme at $l = 1$ and $l = N-1$:

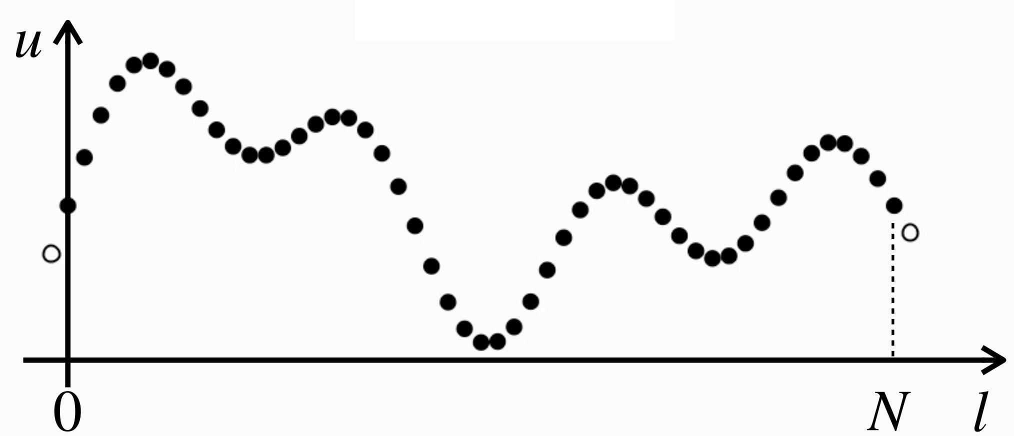
$$\ldots - \frac{\kappa^2}{h^4}\left(u_3^n - 4u_2^n + 6u_1^n - 4u_0 + \boxed{u_{-1}^n}\right) \qquad \ldots - \frac{\kappa^2}{h^4}\left(\boxed{u_{N+1}^n} - 4u_N^n + 6u_{N-1}^n - 4u_{N-2} + u_{N-3}^n\right)$$

We can discretise $u = u_{xx} = 0$ and solve for these points at $l = 0, N$:

$$u_l^n = \boxed{\delta_{xx}u_l^n = 0} \xrightarrow{\text{expand at } l=0} \frac{1}{h^2}(u_1^n - 2u_0^n + u_{-1}^n) = 0 \xrightarrow{u_0^n = 0} u_{-1}^n = -u_1^n$$

$$\ldots \; u_{N+1}^n = -u_{N-1}^n$$

Visualised, this looks something like:



Ideal Bar

So the expansion of $\delta_{xxxx}u_l^n$ at $l = 1$

$$\ldots - \frac{\kappa^2}{h^4}\left(u_3^n - 4u_2^n + 6u_1^n - 4u_0 + u_{-1}^n\right)$$

$$u_{-1}^n = -u_1^n$$

becomes

$$\ldots - \frac{\kappa^2}{h^4}\left(u_3^n - 4u_2^n + 5u_1^n - 4u_0\right)$$

The same goes for $l = N - 1$.

Let's implement the ideal bar using the simply supported boundary condition.

Advice: start with clamped ($l = [2, \ldots, N-2]$)

The (unexpanded) FDS is:

$$\delta_{tt} u_l^n = -\kappa^2 \delta_{xxxx} u_l^n$$

with

$$\delta_{xxxx} u_l^n = \frac{1}{h^4} \left(u_{l+2}^n - 4u_{l+1}^n + 6u_l^n - 4u_{l-1}^n + u_{l-2}^n \right)$$

Boundary condition: $u_l^n = \delta_{xx} u_l^n = 0$ at $l = 0, N$

$\implies u_{-1}^n = -u_1^n$ and $u_{N+1}^n = -u_{N-1}^n$

For stability, use:

$$h = \sqrt{2\kappa k}$$

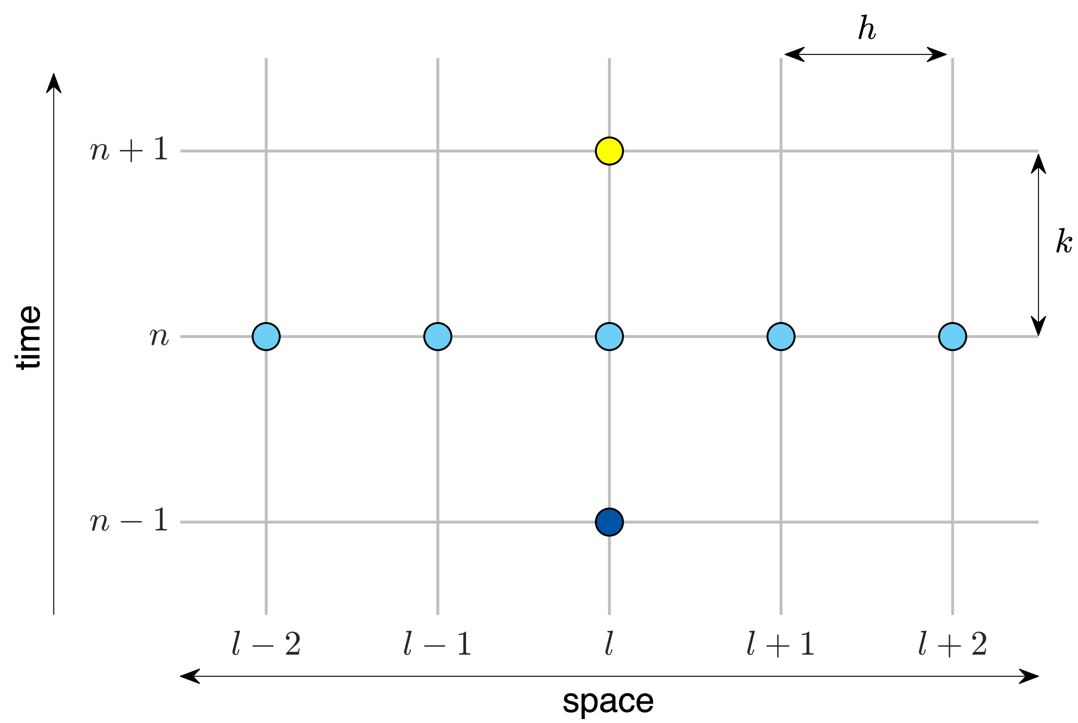
Stiff String

Stiff String

Basically a combination of the 1D wave equation and the ideal bar.

$$\boxed{u_{tt} = c^2 u_{xx} - \kappa^2 u_{xxxx}}$$

Stencil:



For stability, use:

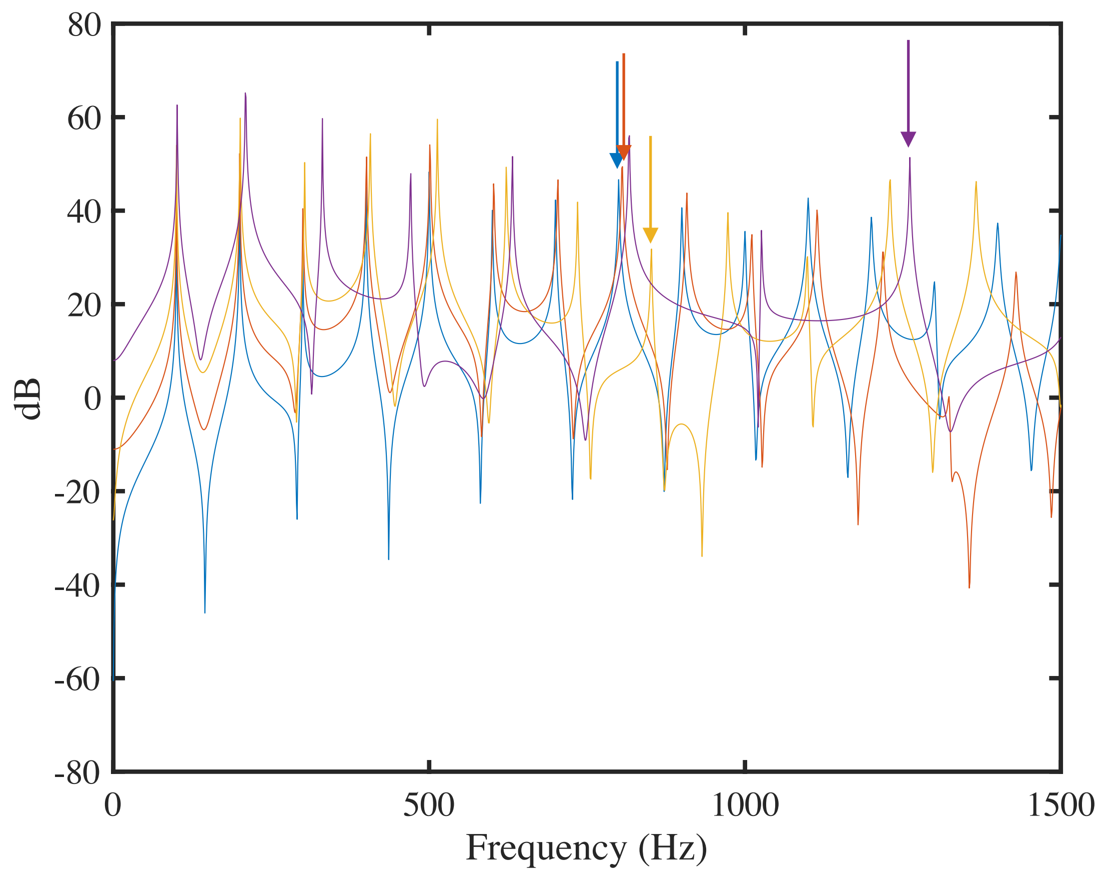
$$h = \sqrt{\frac{c^2k^2 + \sqrt{c^4k^4 + 16\kappa^2k^2}}{2}}$$

Damped Stiff String

We can (again) simply add terms to the PDE:

$$\boxed{u_{tt} = c^2 u_{xx} - \kappa^2 u_{xxxx} - 2\sigma_0 u_t + 2\sigma_1 u_{txx}}$$

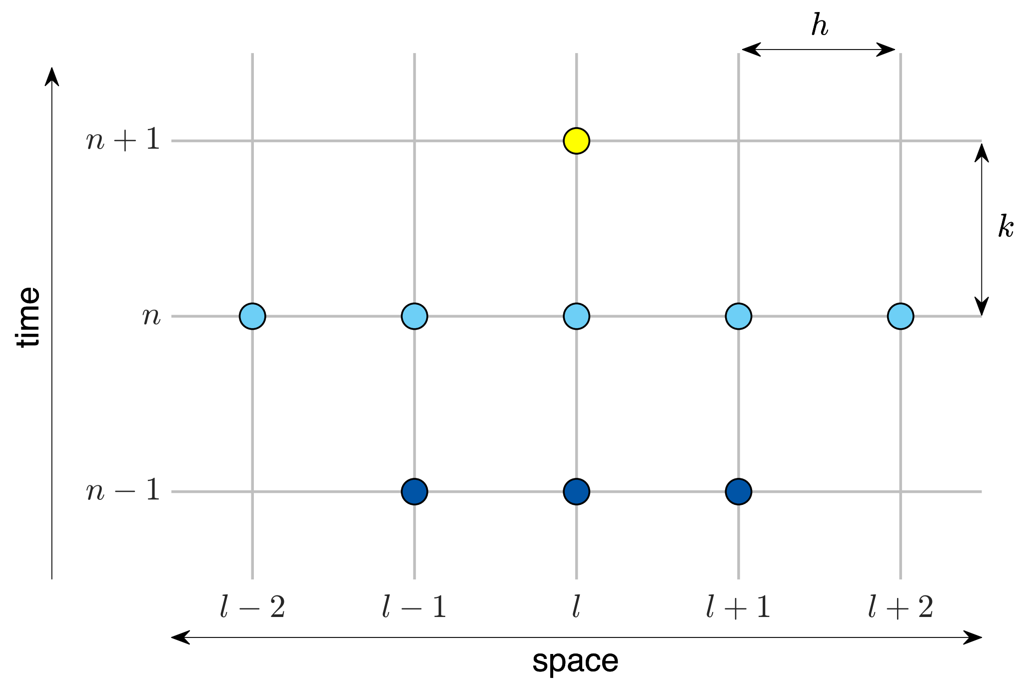
Sound examples: $c = 200,\ L = 1,\ \sigma_0 = 1,\ \sigma_1 = 0.005,\ \kappa = 10$



(Explicit) FDS is:

$$\delta_{tt}u_l^n = c^2\delta_{xx}u_l^n - \kappa^2\delta_{xxxx}u_l^n - 2\sigma_0\delta_{t\cdot}u_l^n + 2\sigma_1\delta_{t-}\delta_{xx}u_l^n$$

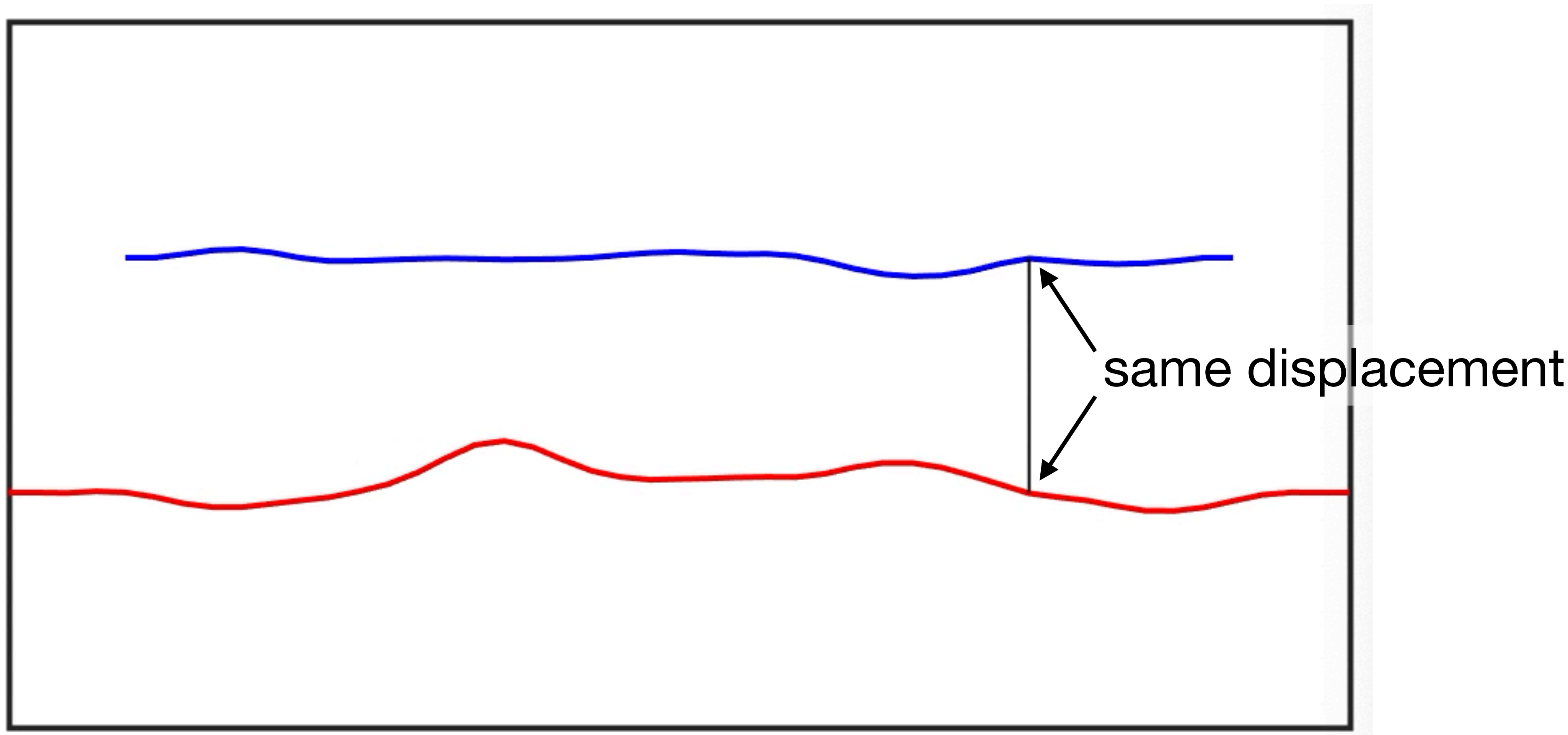
Stencil?



Stability: $$h = \sqrt{\frac{c^2k^2 + 4\sigma_1 k + \sqrt{(c^2k^2 + 4\sigma_1 k)^2 + 16\kappa^2k^2}}{2}}$$

Towards complete instruments

We can connect different models



Towards complete instruments

When connecting different models, it is important to write out all physical parameters.

- Undamped ideal bar:

$$\rho A u_{tt} = -EIu_{xxxx}$$

- Stiffness parameter $\kappa = \sqrt{EI/\rho A}$

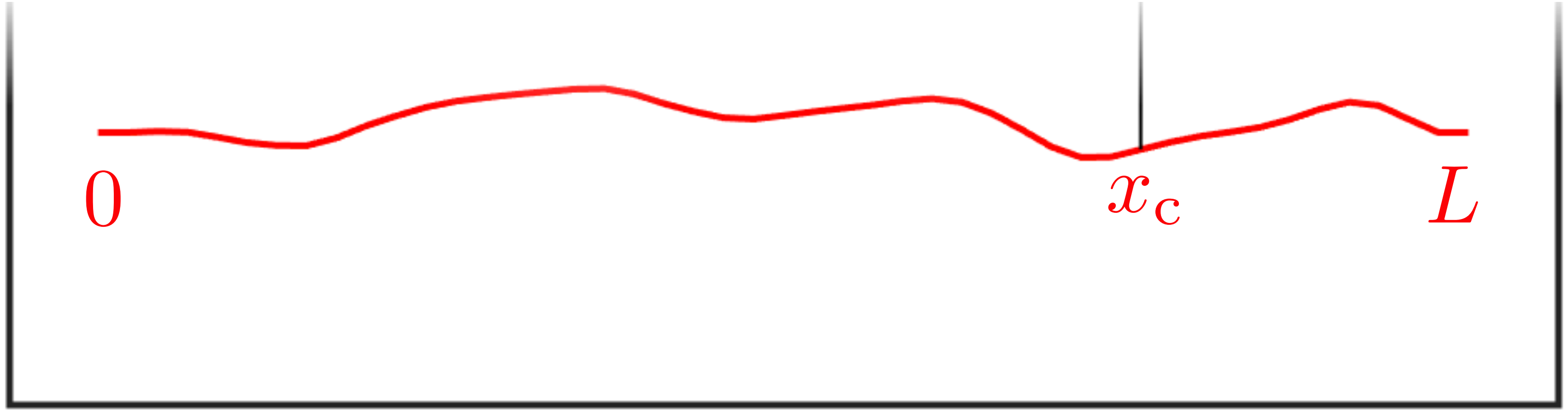
A and I can be calculated from radius r (for bar with circular cross-section): $A = \pi r^2$ and $I = \pi r^4/4$.

Symbol	**Description**	**Unit**
ρ	material density	$\mathrm{kg/m^3}$
A	cross-sectional area	$\mathrm{m^2}$
E	Young's modulus	Pa
I	moment of inertia	$\mathrm{m^4}$

We can then add some connection force f:

$$\rho A u_{tt} = -EIu_{xxxx} + \delta(x - x_{\mathrm{c}})f$$

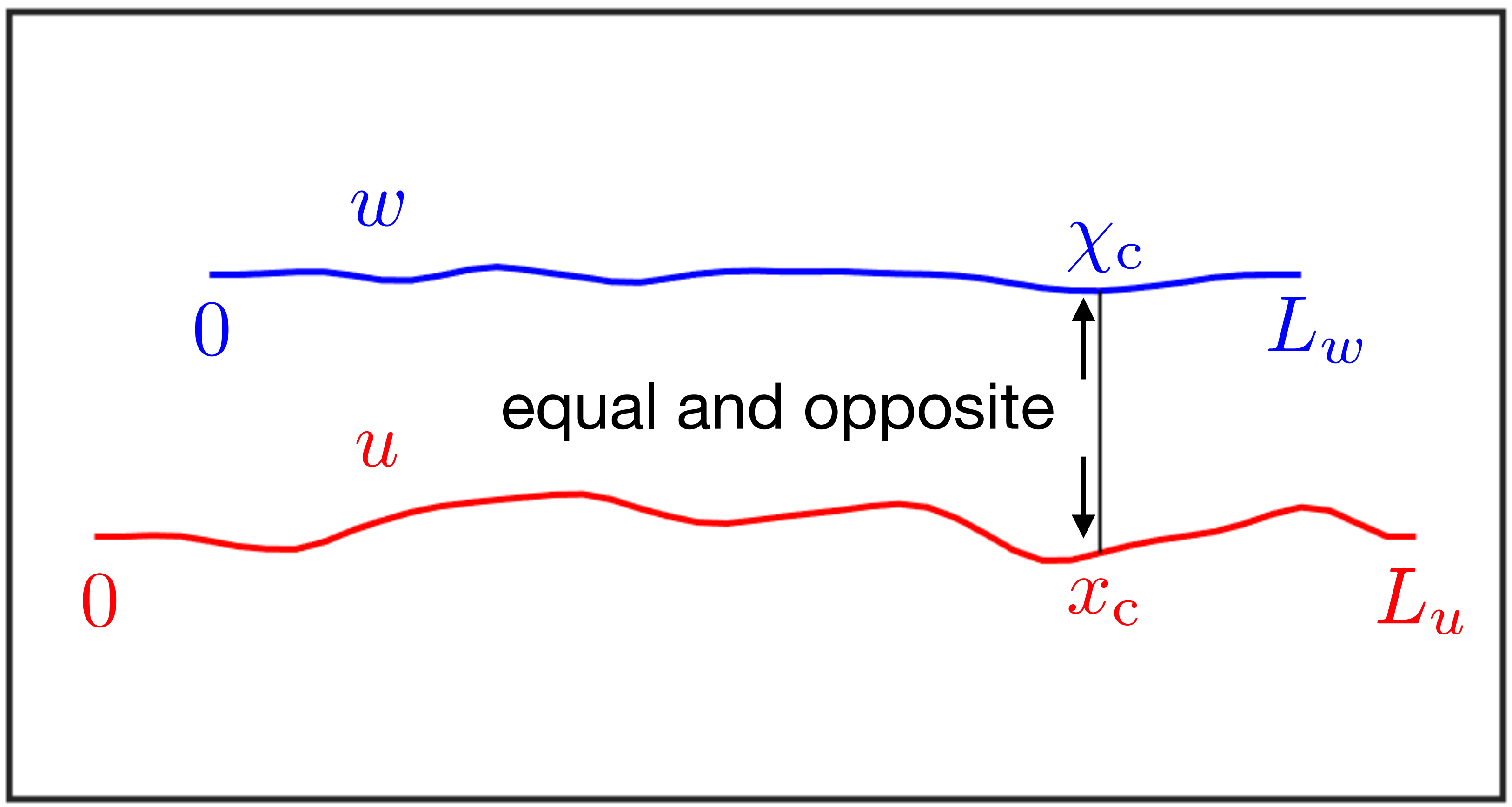
where $\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$ applies force f to location $x_{\mathrm{c}} \in \mathcal{D}$.



Towards complete instruments

Introducing a second bar $w = w(\chi, t)$:

$$\begin{cases} \rho_1 A_1 u_{tt} = -E_1 I_1 u_{xxxx} + \delta(x - x_\mathrm{c}) f \\ \rho_2 A_2 w_{tt} = -E_2 I_2 w_{\chi\chi\chi\chi} \boxed{-} \delta(\chi - \chi_\mathrm{c}) f \end{cases}$$



Towards complete instruments

Introducing a second bar $w = w(\chi, t)$:

$$\begin{cases} \rho_1 A_1 u_{tt} = -E_1 I_1 u_{xxxx} + \delta(x - x_\mathrm{c}) f \\ \rho_2 A_2 w_{tt} = -E_2 I_2 w_{\chi\chi\chi\chi} - \delta(\chi - \chi_\mathrm{c}) f \end{cases}$$

We can discretise the system as follows:

$$\begin{cases} \rho_1 A_1 \delta_{tt} u_l^n = -E_1 I_1 \delta_{xxxx} u_l^n + J(x_\mathrm{c}) f \\ \rho_2 A_2 \delta_{tt} w_l^n = -E_2 I_2 \delta_{\chi\chi\chi\chi} w_l^n - J(\chi_\mathrm{c}) f \end{cases}$$

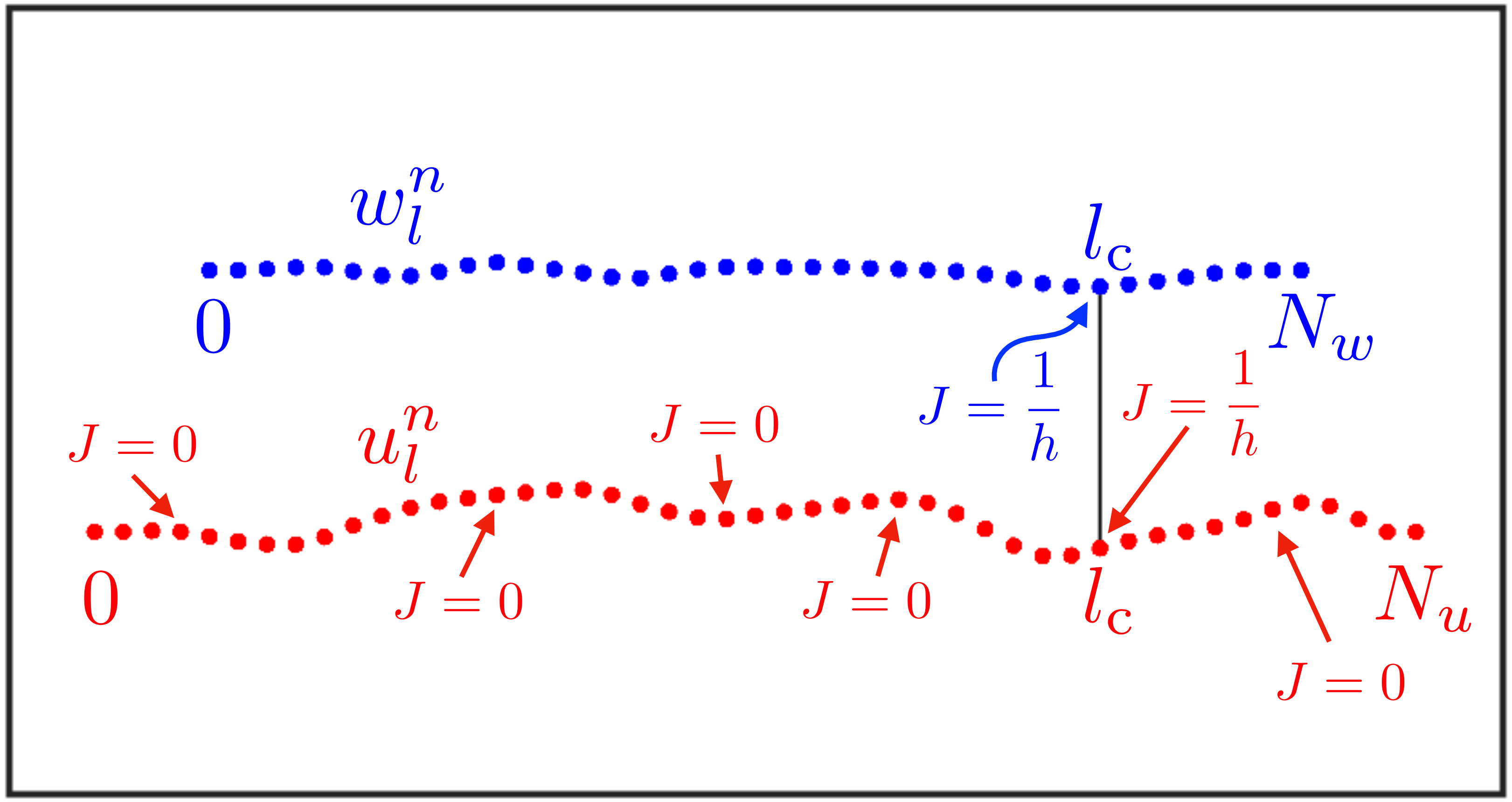
where spreading operator

$$J(x_i) = \begin{cases} \frac{1}{h} & l = l_i = \mathrm{round}(x_i/h) \\ 0 & \text{otherwise} \end{cases}$$

is the discrete version of $\delta(x_i)$.

We can discretise the system as follows:

$$\begin{cases} \rho_1 A_1 \delta_{tt} u_l^n = -E_1 I_1 \delta_{xxxx} u_l^n + J(x_c) f \\ \rho_2 A_2 \delta_{tt} w_l^n = -E_2 I_2 \delta_{\chi\chi\chi\chi} w_l^n - J(\chi_c) f \end{cases}$$



We can discretise the system as follows:

$$\begin{cases} \rho_1 A_1 \delta_{tt} u_l^n = -E_1 I_1 \delta_{xxxx} u_l^n + J(x_\mathrm{c}) f \\ \rho_2 A_2 \delta_{tt} w_l^n = -E_2 I_2 \delta_{\chi\chi\chi\chi} w_l^n - J(\chi_\mathrm{c}) f \end{cases}$$

Now we need to solve for f.

The first step is to divide by ρA

$$\begin{cases} \delta_{tt} u_l^n = -\kappa_1^2 \delta_{xxxx} u_l^n + J(x_\mathrm{c}) \dfrac{f}{\rho_1 A_1} \\ \delta_{tt} w_l^n = -\kappa_2^2 \delta_{\chi\chi\chi\chi} w_l^n - J(\chi_\mathrm{c}) \dfrac{f}{\rho_2 A_2} \end{cases}$$

importance of using physical parameters

Then, we look at the schemes at their connection locations

$$\begin{cases} \delta_{tt} u_{l_c}^n = -\kappa_1^2 \delta_{xxxx} u_{l_c}^n + \dfrac{f}{\rho_1 A_1 h_1} \\ \delta_{tt} w_{l_c}^n = -\kappa_2^2 \delta_{\chi\chi\chi\chi} w_{l_c}^n - \dfrac{f}{\rho_2 A_2 h_2} \end{cases}$$

weight of one grid point

Towards complete instruments

Then, we look at the schemes at their connection locations

$$\begin{cases} \delta_{tt}u_{l_c}^n = -\kappa_1^2\delta_{xxxx}u_{l_c}^n + \frac{f}{\rho_1 A_1 h_1} \\ \delta_{tt}w_{l_c}^n = -\kappa_2^2\delta_{\chi\chi\chi\chi}w_{l_c}^n - \frac{f}{\rho_2 A_2 h_2} \end{cases}$$

expand, and solve for $u_{l_c}^{n+1}$ and $w_{l_c}^{n+1}$

$$\begin{cases} u_{l_c}^{n+1} = 2u_{l_c}^n - u_{l_c}^{n-1} - \kappa_1^2 k^2 \delta_{xxxx}u_{l_c}^n + \frac{fk^2}{\rho_1 A_1 h_1} \\ w_{l_c}^{n+1} = 2w_{l_c}^n - w_{l_c}^{n-1} - \kappa_2^2 k^2 \delta_{\chi\chi\chi\chi}w_{l_c}^n - \frac{fk^2}{\rho_2 A_2 h_2} \end{cases}$$

left unexpanded for brevity

How we proceed next depends on the connection type

- Rigid: $u_{l_c}^n = w_{l_c}^n$ for all n
- Spring-like: $f = -K\mu_{t\cdot}\eta^n$ with $\eta^n = u_{l_c}^n - w_{l_c}^n$

Towards complete instruments

$$\begin{cases} u_{l_c}^{n+1} = 2u_{l_c}^n - u_{l_c}^{n-1} - \kappa_1^2 k^2 \delta_{xxxx} u_{l_c}^n + \dfrac{fk^2}{\rho_1 A_1 h_1} \\ w_{l_c}^{n+1} = 2w_{l_c}^n - w_{l_c}^{n-1} - \kappa_2^2 k^2 \delta_{\chi\chi\chi\chi} w_{l_c}^n - \dfrac{fk^2}{\rho_2 A_2 h_2} \end{cases}$$

Using a rigid connection, $u_{l_c}^n = w_{l_c}^n$ for all n, we also know that $u_{l_c}^{n+1} = w_{l_c}^{n+1}$

$$\cancel{2u_{l_c}^n} - \cancel{u_{l_c}^{n-1}} - \kappa^2 k^2 \delta_{xxxx} u_{l_c}^n + \frac{fk^2}{\rho_1 A_1 h_1} = \cancel{2w_{l_c}^n} - \cancel{w_{l_c}^{n-1}} - \kappa^2 k^2 \delta_{\chi\chi\chi\chi} w_{l_c}^n - \frac{fk^2}{\rho_2 A_2 h_2}$$

$$\left(\frac{k^2}{\rho_1 A_1 h_1} + \frac{k^2}{\rho_2 A_2 h_2}\right) f = -\kappa^2 k^2 \delta_{\chi\chi\chi\chi} w_{l_c}^n + \kappa^2 k^2 \delta_{xxxx} u_{l_c}^n$$

$$f = \frac{-\kappa_2^2 k^2 \delta_{\chi\chi\chi\chi} w_{l_c}^n + \kappa_1^2 k^2 \delta_{xxxx} u_{l_c}^n}{\frac{k^2}{\rho_1 A_1 h_1} + \frac{k^2}{\rho_2 A_2 h_2}}$$

Implementation

1. Calculate schemes without connection forces (the usual)

intermediate

$$\begin{cases} u_l^{\mathrm{I}} = 2u_l^n - u_l^{n-1} - \kappa_1^2 k^2 \delta_{xxxx} u_l^n \\ w_l^{\mathrm{I}} = 2w_l^n - w_l^{n-1} - \kappa_2^2 k^2 \delta_{\chi\chi\chi\chi} w_l^n \end{cases}$$

```
uNext(range) = ...
wNext(range) = ...
```

where $u_{l_c}^{\mathrm{I}} + \dfrac{fk^2}{\rho_1 A_1 h_1} = u_{l_c}^{n+1}$ and $w_{l_c}^{\mathrm{I}} - \dfrac{fk^2}{\rho_2 A_2 h_2} = w_{l_c}^{n+1}$.

2. Calculate force using intermediate states

$$u_{l_c}^{\mathrm{I}} + \frac{fk^2}{\rho_1 A_1 h_1} = w_{l_c}^{\mathrm{I}} - \frac{fk^2}{\rho_2 A_2 h_2} \implies f = \frac{w_{l_c}^{\mathrm{I}} - u_{l_c}^{\mathrm{I}}}{\left(\frac{k^2}{\rho_1 A_1 h_1} + \frac{k^2}{\rho_2 A_2 h_2}\right)}$$

```
force = ...
```

3. Add force to intermediate states to get states at $n+1$

$$u_{l_c}^{n+1} = u_{l_c}^{\mathrm{I}} + \frac{fk^2}{\rho_1 A_1 h_1} \qquad w_{l_c}^{n+1} = w_{l_c}^{\mathrm{I}} - \frac{fk^2}{\rho_2 A_2 h_2}$$

```
uNext(lcu) =
uNext(lcu) + ...
wNext(lcw) =
wNext(lcw) + ...
```