Brass

Silvin Willemsen

March 2020

1 Introduction

This document shows the work done and documentation on the brass part of the PhD.

Main references are [2] and [4].

2 Variable cross-section

2.1 Preamble

Schemes with variable cross-section behave best (have the greatest band-width) when stretched coordinates are used. Following Section 5.3 of [2], we can devine a new coordinate $\alpha = \alpha(x)$ where the map $x \rightarrow \alpha$ is "smooth and one-to-one" (i.e. if $x_1 \leq x_2$ then $\alpha_1 \leq \alpha_2$). Partial derivatives are as follows:

$$\partial_x \quad \longrightarrow \quad \alpha' \partial_\alpha \qquad \text{and} \qquad \partial_x^2 \quad \longrightarrow \quad \alpha' \partial_\alpha \left(\alpha' \partial_\alpha \right), \tag{1}$$

with $\alpha' = \partial \alpha / \partial x$ which describes the rate-of-change of grid spacing with respect to the original grid spacing. In discrete time, a centered approximation for the second spatial derivative is

$$\delta_{xx} \quad \longrightarrow \quad \alpha' \delta_{\alpha+}\big((\mu_{\alpha-}\alpha')\delta_{\alpha-}\big). \tag{2}$$

This operator applied to a grid function u_l^n and expanded yields

$$\frac{\alpha'_l}{2h^2}\Big((\alpha'_{l+1} + \alpha'_l)u_{l+1}^n + (\alpha'_l + \alpha'_{l-1})u_{l-1}^n - (\alpha'_{l+1} + 2\alpha'_l + \alpha'_{l-1})u_l^n\Big), \tag{3}$$

or

$$\frac{\alpha'_l}{h^2}\Big((\mu_{\alpha+}\alpha'_l)u_{l+1}^n + (\mu_{\alpha-}\alpha'_l)u_{l-1}^n - 2(\mu_{\alpha\alpha}\alpha'_l)u_l^n\Big) \tag{4}$$

2.1.1 String of varying cross-section

The PDE of an ideal string with a varying cross-section is

$$\epsilon^2 \partial_t^2 u = c_0^2 \partial_x^2 u, \tag{5}$$

where $c_0^2 = T_0/\rho A_0$ is the reference wave-speed (in m/s) and $\epsilon = \epsilon(x)$ is a spatially varying factor (in ?unit?). Using Eq. (2) where $\alpha' \rightarrow \epsilon$ we can rewrite Eq. (5) to

$$\epsilon^2 \partial_t^2 u = c_0^2 \epsilon \partial_\alpha (\epsilon \partial_\alpha u), \tag{6}$$

$$\epsilon \partial_t^2 u = c_0^2 \partial_\alpha (\epsilon \partial_\alpha u), \tag{7}$$

2.2 Webster's equation

The first main difference between the 1D brass PDE and the 1D wave equation is the possibility of having a variable cross-section. Following Section 19.3 from [4], the PDE for a 1D (axially symmetric) acoustic tube with variable cross-section is (also known as *Webster*'s equation)

$$S\partial_t^2\Psi = c^2\partial_x(S\partial_x\Psi), \tag{8}$$

with *acoustic potential* $\Psi = \Psi(x,t)$ (m^2/s), $S = S(x)$ is the cross sectional area (m^2) and wave speed c (m/s).

2.3 Discretisation

Introducing interleaved gridpoints at $n-1/2$ and $n+1/2$ for S, a we can discretise Eq. (8) (following [4]) to

$$\bar{S}_l\delta_{tt}\Psi_l^n = c^2\delta_{x+}(S_{l-1/2}(\delta_{x-}\Psi_l^n)), \tag{9}$$

where

$$\bar{S}_l = \mu_{t+}S_{l-1/2} = \frac{S_{l+1/2}+S_{l-1/2}}{2}. \tag{10}$$

The right side of the equation in (9) contains an operator applied to two grid functions (S and Ψ) multiplied onto each other. In order to expand this, we need to use the product rule (Eq. (2.23) in [2]) which is

$$\delta_{x+}(u_lw_l) = (\delta_{x+}u_l)(\mu_{x+}w_l) + (\mu_{x+}u_l)(\delta_{x+}w_l). \tag{11}$$

In the case of (9), $u_l \triangleq S_{l-1/2}$ and $w_l \triangleq \delta_{x-}\Psi_l^n$. Expanding (retaining the notation for $\bar{S}_l$) and solving for Ψ_l^{n+1} yields (Appendix A.1)

$$\Psi_l^{n+1} = 2(1-\lambda^2)\Psi_l^n - \Psi_l^{n-1} + \frac{\lambda^2 S_{l+1/2}}{\bar{S}_l}\Psi_{l+1}^n + \frac{\lambda^2 S_{l-1/2}}{\bar{S}_l}\Psi_{l-1}^n, \tag{12}$$

which is identical to Eq. (19.51) in [4].

2.4 Boundary Conditions

The choices for boundary conditions in an acoustic tube are open and closed, defined as [4]

$$\begin{aligned}\partial_t\Psi &= 0 \text{ (open, Dirichlet)}\\ \partial_x\Psi &= 0 \text{ (closed, Neumann)},\end{aligned} \tag{13}$$

at the ends of the tube. This might be slightly counter-intuitive as in the case of a string "closed" might imply the "clamped" or Dirichlet boundary condition. The opposite can be intuitively shown imagining a wave front with a positive acoustic potential moving through a tube and hitting a closed end. What comes back is also a wave front with a positive acoustic potential, i.e., the sign of the potential does not flip, which also happens using the free or Neumann condition for the string.

In this case we follow [2, Chapter 9] and use the following

$$\partial_x\Psi(0,t) = 0 \quad \text{and} \quad \partial_t\Psi(L,t) = 0 \tag{14}$$

i.e. closed at the left end and open at the right end. In discrete time we have two choices for the closed condition

$$\begin{aligned}\delta_{x\cdot}\Psi_0^n = 0 \;&\Rightarrow\; \Psi_{-1}^n = \Psi_1^n \quad \text{(centered)}\\ \delta_{x-}\Psi_0^n = 0 \;&\Rightarrow\; \Psi_{-1}^n = \Psi_0^n \quad \text{(non-centered)}\end{aligned} \tag{15}$$

At the left boundary we can now solve Eq. (12) for the centered case:

$$\begin{aligned}
\Psi_0^{n+1} &= 2(1-\lambda^2)\Psi_0^n - \Psi_0^{n-1} + \frac{\lambda^2 S_{1/2}}{\bar{S}_0}\Psi_1^n + \frac{\lambda^2 S_{-1/2}}{\bar{S}_0}\Psi_{-1}^n \\
\Psi_0^{n+1} &= 2(1-\lambda^2)\Psi_0^n - \Psi_0^{n-1} + \frac{\lambda^2 (S_{1/2}+S_{-1/2})}{\bar{S}_0}\Psi_1^n \\
\Psi_0^{n+1} &= 2(1-\lambda^2)\Psi_0^n - \Psi_0^{n-1} + 2\lambda^2\Psi_1^n,
\end{aligned}$$

and the non-centered case

$$\Psi_0^{n+1} = 2(1-\lambda^2)\Psi_0^n - \Psi_0^{n-1} + \frac{\lambda^2 S_{1/2}}{\bar{S}_0}\Psi_1^n + \frac{\lambda^2 S_{-1/2}}{\bar{S}_0}\Psi_0^n. \tag{16}$$

As can be seen from the equations above, we need undefined points $\bar{S}_0$ and $S_{-1/2}$. At the left boundary, we set $\bar{S}_0 = S_0$ from which, we can calculate $S_{-1/2}$:

$$S_0 = \frac{1}{2}(S_{1/2} + S_{-1/2}) \ \Rightarrow\ S_{-1/2} = 2S_0 - S_{1/2} \tag{17}$$

The same can be done for the right boundary ($\bar{S}_N = S_N$) if this is chosen to be anything else but open (e.g., closed or radiating – see Section 2.4.1):

$$S_N = \frac{1}{2}(S_{N+1/2} + S_{N-1/2}) \ \Rightarrow\ S_{N+1/2} = 2S_N - S_{N-1/2}. \tag{18}$$

For now though, we follow the conditions given in (14) and we can simply set the right boundary to its initial state

$$\Psi_N^n = \Psi_N^0 \tag{19}$$

which is normally 0. A more realistic open end is a radiating one, which can be found below.

2.4.1 Radiating end

We can change the condition presented in Eq. (14) to a radiating end,

$$\partial_x \Psi(L,t) = -a_1 \partial_t \Psi(L,t) - a_2 \Psi(L,t) \tag{20}$$

where [2]

$$a_1 = \frac{1}{2(0.8216)^2 c} \quad \text{and} \quad a_2 = \frac{L}{0.8216\sqrt{S_0 S(1)/\pi}}. \tag{21}$$

taken from [1] and are valid for a tube terminating on an infinite plane. The terms in Eq. (20) are a damping and an inertia term where a_1 is a loss coefficient and a_2 is the **inertia coefficient**. The centered and non-centered case are defined as

$$\begin{aligned}
\delta_{x\cdot}\Psi_N^n = 0 \ &\Rightarrow\ \Psi_{N+1}^n = \Psi_{N-1}^n \quad \text{(centered)} \\
\delta_{x+}\Psi_N^n = 0 \ &\Rightarrow\ \Psi_{N+1}^n = \Psi_N^n \qquad \text{(non-centered)}
\end{aligned} \tag{22}$$

First, we solve Eq. (20) for the centered (Eq. (9.16) in [2])

$$\delta_{x\cdot}\Psi_N^n = -a_1\delta_{t\cdot}\Psi_N^n - a_2\mu_{t\cdot}\Psi_N^n \tag{23}$$

which can be expanded and solved for Ψ_{N+1}^n according to

$$\begin{aligned}
\frac{1}{2h}(\Psi_{N+1}^n - \Psi_{N-1}^n) &= -\frac{a_1}{2k}(\Psi_N^{n+1} - \Psi_N^{n-1}) - \frac{a_2}{2}(\Psi_N^{n+1} + \Psi_N^{n-1}) \\
\Psi_{N+1}^n &= h\left(-\frac{a_1}{k}(\Psi_N^{n+1} - \Psi_N^{n-1}) - a_2(\Psi_N^{n+1} + \Psi_N^{n-1})\right) + \Psi_{N-1}^n,
\end{aligned} \tag{24}$$

which can be substituted into Eq. (12) (Appendix A.2)

$$\Psi_N^{n+1} = \frac{2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\left(\frac{a_1}{k} - a_2\right)\Psi_N^{n-1} + 2\lambda^2\Psi_{N-1}^n}{\left(1 + \left(\frac{a_1}{k} + a_2\right)\frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\right)}. \tag{25}$$

The same can be done for the non-centered case (Eq. (9.15) in [2])

$$\delta_{x+}\Psi_N^n = -a_1\delta_{t\cdot}\Psi_N^n - a_2\mu_{t\cdot}\Psi_N^n \tag{26}$$

which when solved for Ψ_{N+1}^n yields

$$\begin{aligned}\frac{1}{h}(\Psi_{N+1}^n - \Psi_N^n) &= -\frac{a_1}{2k}(\Psi_N^{n+1} - \Psi_N^{n-1}) - \frac{a_2}{2}(\Psi_N^{n+1} + \Psi_N^{n-1})\\ \Psi_{N+1}^n &= h\left(-\frac{a_1}{2k}(\Psi_N^{n+1} - \Psi_N^{n-1}) - \frac{a_2}{2}(\Psi_N^{n+1} + \Psi_N^{n-1})\right) + \Psi_N^n.\end{aligned} \tag{27}$$

Substituted into Eq. (12) yields (Appendix A.3)

$$\Psi_N^{n+1} = \frac{2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\left(\frac{a_1}{2k} - \frac{a_2}{2}\right)\Psi_N^{n-1} + \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}\Psi_N^n + \frac{\lambda^2 S_{N-1/2}}{\bar{S}_N}\Psi_{N-1}^n}{\left(1 + \left(\frac{a_1}{2k} + \frac{a_2}{2}\right)\frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\right)}. \tag{28}$$

2.5 Energy

In continuous time, the rate of change of the energy of a PDE can be obtained by taking the inner product with the first-order time derivative of the state Say something here about obtaining units of power "in this case, this means to multiply with the first time-derivative of the state Ψ. Using state Ψ, this yields

$$\frac{d\mathfrak{H}}{dt} = \langle \partial_t\Psi, \text{PDE}\rangle \quad \text{where} \quad \mathfrak{H} = \mathfrak{T} + \mathfrak{V}, \tag{29}$$

with total energy (or Hamiltonian) $\mathfrak{H}$, kinetic energy $\mathfrak{T}$ and potential energy $\mathfrak{V}$.

In discrete time, the energy of a FDS can be obtained by taking the inner product the equation with $\delta_{t\cdot}\Psi_l^n$ according to

$$\delta_{t+}\mathfrak{h} = \langle \delta_{t\cdot}\Psi_l^n, \text{FDS}\rangle \quad \text{where} \quad \mathfrak{h} = \mathfrak{t} + \mathfrak{v}, \tag{30}$$

where $\mathfrak{h}$, $\mathfrak{t}$ and $\mathfrak{v}$ are the discrete counterparts of $\mathfrak{H}$, $\mathfrak{T}$ and $\mathfrak{V}$.

With a boundary term present, Eq. (30) becomes

$$\delta_{t+}\mathfrak{h} = \mathfrak{b}. \tag{31}$$

In the case of the radiating boundary – with an inertia and damping term – the energy equation is of the form

$$\delta_{t+}(\mathfrak{h} + \mathfrak{h}_\text{b}) = -\mathfrak{q}_\text{b}, \tag{32}$$

where $\mathfrak{h}_\text{b}$ is the energy stored by the boundary and $\mathfrak{q}_\text{b}$ is the loss energy.

2.5.1 Kinetic Energy

The (discrete) kinetic energy of the system $\mathfrak{t}$ of Webster's equation in Eq. (9) is found by taking the inner product of the left side of the equation with $\delta_{t\cdot}\Psi$ according to

$$\delta_{t+}\mathfrak{t} = \langle \delta_{t\cdot}\Psi_l^n, \bar{S}_l\delta_{tt}\Psi_l^n\rangle_{\mathcal{D}}, \tag{33}$$

where domain $\mathcal{D} \in [0, N]$. As $\bar{S}_l$ is merely a coefficient here, we can use identity (2.22a) from [2] and rewrite to

$$\delta_{t+}\left(\frac{1}{2}\sum_{\mathcal{D}} h\bar{S}_l(\delta_{t-}\Psi_l^n)^2\right) \quad \Rightarrow \quad \delta_{t+}\left(\frac{1}{2}\sum_{\mathcal{D}} h(\sqrt{\bar{S}_l}\delta_{t-}\Psi_l^n)^2\right). \tag{34}$$

We place $\bar{S}_l$ as a square-root in the equation, so that we can rewrite it as a norm over a domain as

$$\mathfrak{t} = \frac{1}{2}\|\sqrt{\bar{S}_l}\delta_{t-}\Psi_l^n\|_{\mathcal{D}}^2 \tag{35}$$

just like Eq. (9.14) in [2]. When the boundaries of the scheme are centered we need to use a primed inner product (Eq. (5.23) in [2]) as Bilbao explains right below Eq. (5.28) in [2]:

$$\langle f, g\rangle'_{\mathcal{D}} = \sum_{l=1}^{N-1} hf_l g_l + \frac{h}{2}f_0 g_0 + \frac{h}{2}f_N g_N, \tag{36}$$

However, in the spatially varying case (which we are dealing with here), a more general weighted inner product needs to be used, which for a domain $\mathcal{D} = [0, \ldots, N]$ is defined as (as given in Eq. (5.38) in [2])

$$\langle f, g\rangle_{\mathcal{D}}^{\epsilon_\text{l},\epsilon_\text{r}} = \sum_{l=1}^{N-1} hf_l g_l + \frac{\epsilon_\text{l}}{2}hf_0 g_0 + \frac{\epsilon_\text{r}}{2}hf_N g_N, \tag{37}$$

where free parameters $\epsilon_\text{l}, \epsilon_\text{r} > 0$ 'tune' the weighting of the boundaries. If $\epsilon_\text{l} = \epsilon_\text{r} = 1$, Eq. (37) reduces to the primed inner product from Eq. (36). For the kinetic energy, we now have

$$\mathfrak{t} = \frac{1}{2}\left(\|\sqrt{\bar{S}_l}\delta_{t-}\Psi_l^n\|_{\mathcal{D}}^{\epsilon_\text{l},\epsilon_\text{r}}\right)^2, \tag{38}$$

which, when expanded, yields

$$\begin{aligned}\mathfrak{t} &= \frac{1}{2}\left(\sqrt{\sum_{l=1}^{N-1} h\left(\sqrt{\bar{S}_l}\delta_{t-}\Psi_l^n\right)^2 + \frac{\epsilon_\text{l}}{2}h\left(\sqrt{\bar{S}_0}\delta_{t-}\Psi_0^n\right)^2 + \frac{\epsilon_\text{r}}{2}h\left(\sqrt{\bar{S}_N}\delta_{t-}\Psi_N^n\right)^2}\right)^2\\ &= \frac{1}{2}\left(\sum_{l=1}^{N-1} h\bar{S}_l(\delta_{t-}\Psi_l^n)^2 + \frac{\epsilon_\text{l}}{2}h\bar{S}_0(\delta_{t-}\Psi_0^n)^2 + \frac{\epsilon_\text{r}}{2}h\bar{S}_N(\delta_{t-}\Psi_N^n)^2\right).\end{aligned} \tag{39}$$

The values of ϵ_l and ϵ_r can be calculated by performing the same energy analysis techniques on the right side of the equation to obtain the potential energy and the boundary terms, and will be given below.

2.5.2 Potential Energy

It can be shown that (9) is equal to

$$\bar{S}_l\delta_{tt}\Psi_l^n = c^2\delta_{x-}\big(S_{l+1/2}(\delta_{x+}\Psi_l^n)\big), \tag{40}$$

i.e. changing the signs of the operators at the left side of the equation. Using ($\mu_{x+}S = S_{l+1/2}$), we define for the non-centered case

$$c^2\langle\delta_{t\cdot}\Psi_l^n, \delta_{x-}((\mu_{x+}S)(\delta_{x+}\Psi_l^n))\rangle_{\mathcal{D}}, \tag{41}$$

which, using summation by parts (see Appendix C) can be written as (see Appendix D.1)

$$\underbrace{-c^2\langle(\mu_{x+}S)\delta_{t\cdot}\delta_{x+}\Psi_l^n, \delta_{x+}\Psi_l^n\rangle_{\underline{\mathcal{D}}}}_{\delta_{t+}\mathfrak{v}} + \underbrace{c^2\Big((\delta_{t\cdot}\Psi_N^n)(\mu_{x+}S_N)(\delta_{x+}\Psi_N^n) - (\delta_{t\cdot}\Psi_0^n)(\mu_{x-}S_0)(\delta_{x-}\Psi_0^n)\Big)}_{\mathfrak{b}}, \tag{42}$$

with boundary energy term $\mathfrak{b}$ as seen in Eq. (31) (for more detail, see Section 2.5.3). Notice that $\mu_{x+}S$ has been moved to the other side of the inner product according to match Eq. (9.14) in [2]. We can also take $\mu_{x+}S$ out of the inner product (as it is a coefficient here)

$$-c^2(\mu_{x+}S)\langle\delta_{t\cdot}\delta_{x+}\Psi_l^n, \delta_{x+}\Psi_l^n\rangle_{\underline{\mathcal{D}}}, \tag{43}$$

to use Eq. (2.22b) to end up with

$$\delta_{t+}\mathfrak{v} = \delta_{t+}\left(-\frac{c^2}{2}(\mu_{x+}S)\langle\delta_{x+}\Psi_l^n, e_{t-}\delta_{x+}\Psi_l^n\rangle_{\underline{\mathcal{D}}}\right), \tag{44}$$

where (again, as S acts as a coefficient) $\mu_{x+}S$ can be inserted back into the inner product

$$\mathfrak{v} = -\frac{c^2}{2}\langle(\mu_{x+}S)\delta_{x+}\Psi_l^n, e_{t-}\delta_{x+}\Psi_l^n\rangle_{\underline{\mathcal{D}}}. \tag{45}$$

See Appendix B for a proof of this.

For the centered case, we need to use the weighted inner product as defined in (37):

$$c^2\langle\delta_{t\cdot}\Psi_l^n, \delta_{x-}((\mu_{x+}S)(\delta_{x+}\Psi_l^n))\rangle_{\mathcal{D}}^{\epsilon_{\mathrm{l}},\epsilon_{\mathrm{r}}}, \tag{46}$$

which, after summation by parts becomes (see Appendix D.3) (*Note that* $\mathfrak{b}_l$ *is subtracted*)

$$\underbrace{-c^2\langle(\mu_{x+}S)\delta_{t\cdot}\delta_{x+}\Psi_l^n, \delta_{x+}\Psi_l^n\rangle_{\underline{\mathcal{D}}}}_{\delta_{t+}\mathfrak{v}} + \mathfrak{b}_{\mathrm{r}} - \mathfrak{b}_{\mathrm{l}}, \tag{47}$$

where

$$\mathfrak{b}_{\mathrm{r}} = c^2(\delta_{t\cdot}\Psi_N^n)\left(\frac{\epsilon_{\mathrm{r}}}{2}S_{N+1/2}(\delta_{x+}\Psi_N^n) + \left(1-\frac{\epsilon_{\mathrm{r}}}{2}\right)S_{N-1/2}(\delta_{x-}\Psi_N^n)\right), \tag{48a}$$

$$\mathfrak{b}_{\mathrm{l}} = c^2(\delta_{t\cdot}\Psi_0^n)\left(\frac{\epsilon_{\mathrm{l}}}{2}S_{-1/2}(\delta_{x-}\Psi_0^n) + \left(1-\frac{\epsilon_{\mathrm{l}}}{2}\right)S_{1/2}(\delta_{x+}\Psi_0^n)\right). \tag{48b}$$

For the special cases of $\epsilon_{\mathrm{r}} = S_{N-1/2}/\mu_{xx}S_N$ and $\epsilon_{\mathrm{l}} = S_{1/2}/\mu_{xx}S_0$ the boundary terms become strictly dissipative (see Appendix D.3 for the (painstaking) derivation of this)

$$\mathfrak{b}_{\mathrm{r}} = c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}(2-\epsilon_{\mathrm{r}})(\delta_{x\cdot}\Psi_N^n) \tag{49a}$$

$$\mathfrak{b}_{\mathrm{l}} = c^2(\delta_{t\cdot}\Psi_0^n)S_{1/2}(2-\epsilon_{\mathrm{l}})(\delta_{x\cdot}\Psi_0^n) \tag{49b}$$

2.5.3 Boundaries

Recalling the boundary term from Eq. (42)

$$\mathfrak{b} = \underbrace{c^2(\delta_{t\cdot}\Psi_N^n)(\mu_{x+}S_N)(\delta_{x+}\Psi_N^n)}_{\mathfrak{b}_{\mathrm{r}}\text{ (right boundary)}} - \underbrace{c^2(\delta_{t\cdot}\Psi_0^n)(\mu_{x-}S_0)(\delta_{x-}\Psi_0^n)}_{\mathfrak{b}_{\mathrm{l}}\text{ (left boundary)}} \tag{50}$$

We first consider the left boundary. Recalling Eq. (15), we see that for the non-centered case, $\Psi_{-1}^n = \Psi_0^n$. This means that $(\delta_{x-}\Psi_0^n) = 0$ yielding

$$\mathfrak{b}_{\mathrm{l}} = c^2(\delta_{t\cdot}\Psi_0^n)(S_{-1/2})(0) = 0. \tag{51}$$

For the centered case, $\Psi_{-1}^n = \Psi_1^n$. Substituting this into the equation for the left boundary yields

$$\mathfrak{b}_{\mathrm{l}} = \frac{c^2}{2kh}(\Psi_0^{n+1} - \Psi_0^{n-1})(S_{-1/2})(\Psi_0^n - \Psi_1^n) \tag{52}$$

Continuing with the right boundary it can be seen that the lossless case, where $(\delta_{t\cdot}\Psi_N^n) = 0$ results in

$$\mathfrak{b}_{\mathrm{r}} = c^2(0)(S_{N+1/2})(\delta_{x+}\Psi_N^n) = 0. \tag{53}$$

The damped condition, however, is much more interesting. In the non-centered case, we can substitute this into the definition found in Eq. (26) to get

$$\mathfrak{b}_{\mathrm{r}} = c^2(\delta_{t\cdot}\Psi_N^n)(S_{N+1/2})(-a_1\delta_{t\cdot}\Psi_N^n - a_2\mu_{t\cdot}\Psi_N^n), \tag{54}$$

and using the identity

$$(\delta_{t\cdot}u)(\mu_{t\cdot}u) = \frac{1}{2}\delta_{t\cdot}(u)^2, \tag{55}$$

we end up with

$$\mathfrak{b}_\mathrm{r} = c^2 S_{N+1/2}\left(-a_1(\delta_{t\cdot}\Psi_N^n)^2 - \frac{a_2}{2}\delta_{t\cdot}(\Psi_N^n)^2\right). \tag{56}$$

As this boundary both has damping with energy term $\mathfrak{q}_\mathrm{b}$ and inertia with stored energy $\mathfrak{h}_\mathrm{b}$ it can be written in the form found in Eq. (32). First, substituting $\mathfrak{b}_\mathrm{r}$ into Eq. (31) yields

$$\delta_{t+}\mathfrak{h} = c^2 S_{N+1/2}\left(-a_1(\delta_{t\cdot}\Psi_N^n)^2 - \frac{a_2}{2}\delta_{t\cdot}(\Psi_N^n)^2\right) \tag{57}$$

$$\delta_{t+}\mathfrak{h} + \frac{c^2 S_{N+1/2}a_2}{2}\delta_{t+}\mu_{t-}(\Psi_N^n)^2 = -c^2 S_{N+1/2}a_1(\delta_{t\cdot}\Psi_N^n)^2 \tag{58}$$

$$\begin{gathered}\delta_{t+}\left(\mathfrak{h} + \mathfrak{h}_\mathrm{b}\right) = -\mathfrak{q}_\mathrm{b} \quad \text{where} \\ \mathfrak{h}_\mathrm{b} = \frac{c^2 S_{N+1/2}a_2}{2}\mu_{t-}(\Psi_N^n)^2 \quad \text{and} \quad \mathfrak{q}_\mathrm{b} = c^2 S_{N+1/2}a_1(\delta_{t\cdot}\Psi_N^n)^2,\end{gathered} \tag{59}$$

which is exactly what we find in [2] underneath Eq. (9.15).

For the centered case, recalling Eqs. (23) and (49a) we can follow similar steps to end up at

$$\mathfrak{b}_\mathrm{r} = c^2 S_{N-1/2}(2-\epsilon_\mathrm{r})(-a_1(\delta_{t\cdot}\Psi_N^n)^2 - \frac{a_2}{2}\delta_{t\cdot}(\Psi_N^n)^2). \tag{60}$$

When written in the form found in Eq. (32) we obtain

$$\begin{gathered}\delta_{t+}\left(\mathfrak{h} + \mathfrak{h}_\mathrm{b}\right) = -\mathfrak{q}_\mathrm{b} \quad \text{where} \\ \mathfrak{h}_\mathrm{b} = \frac{c^2 S_{N-1/2}(2-\epsilon_\mathrm{r})a_2}{2}\mu_{t-}(\Psi_N^n)^2 \quad \text{and} \quad \mathfrak{q}_\mathrm{b} = c^2 S_{N-1/2}(2-\epsilon_\mathrm{r})a_1(\delta_{t\cdot}\Psi_N^n)^2,\end{gathered} \tag{61}$$

3 First-order systems

Until now we have looked at the second-order PDE shown in Eq. (8). However, according to [7], (apart from being useful for a "simpler discussion" of boundary conditions) a system of first-order PDEs equivalent to Webster's equation was shown to be more accurate [8]. Furthermore, the state of the art [7, 5] works with these schemes and allow to more easily add viscothermal losses.

The system is defined as follows:

$$\frac{S}{\rho_0 c^2}\partial_t p = -\partial_x(Sv) \tag{62a}$$

$$\rho_0\partial_t v = -\partial_x p \tag{62b}$$

where pressure $p = p(x,t) = \rho_0\partial_t\Psi$ (N/m^2) and particle velocity $v = v(x,t) = -\partial_x\Psi$ (m/s) with air density ρ_0. Indeed it can be shown by substituting these definitions into Eq. (62a), Webster's equation is obtained again:

$$\frac{S}{\rho_0 c^2}\partial_t(\rho_0\partial_t\psi) = -\partial_x(S(-\partial_x\Psi)) \quad \Longrightarrow \quad S\partial_t^2\psi = c^2\partial_x(S\partial_x\Psi).$$

3.1 Discretisation

As done in [7], it is useful to place either p or v on an interleaved grid; both in space and time. Following Harrison, we place v on this interleaved grid. Accordingly, system (62) is discretised as

$$\frac{\bar{S}_l}{\rho_0 c^2}\delta_{t+}p_l^n = -\delta_{x-}(S_{l+1/2}v_{l+1/2}^{n+1/2}), \tag{63a}$$

$$\rho_0\delta_{t-}v_{l+1/2}^{n+1/2} = -\delta_{x+}p_l^n, \tag{63b}$$

after which the update schemes become

$$p_l^{n+1} = p_l^n - \frac{\rho\lambda}{\bar{S}_l}(S_{l+1/2}v_{l+1/2}^{n+1/2} - S_{l-1/2}v_{l-1/2}^{n-1/2}), \tag{64a}$$

$$v_{l+1/2}^{n+1/2} = v_{l+1/2}^{n-1/2} - \frac{\lambda}{\rho_0 c}(p_{l+1}^n - p_l^n), \tag{64b}$$

where (again) $\lambda = ck/h$.

3.2 Boundary Conditions

In the first-order PDE case, the boundary conditions are defined as follows

$$\begin{aligned} p_0^n &= 0, & p_N^n &= 0, & &\text{(Dirichlet)},\\ \mu_{x-}(S_{1/2}v_{1/2}^n) &= 0, & \mu_{x+}(S_{N-1/2}v_{N-1/2}^n) &= 0, & &\text{(Neumann)}, \end{aligned}$$

for all n.

3.3 Energy

For the energy analysis of the first-order system presented above, [7] is followed. Instead of taking an inner product with $\delta_{t\cdot}\Psi$ we take an inner product with $\mu_{t+}p$ instead. Using the primed inner product from Eq. (36)[1] for Eq. (62a) and after taking all terms to the left-hand side, this yields

$$\delta_{t+}\mathfrak{h} = \frac{1}{\rho_0 c^2}\langle \mu_{t+}p_l^n, \bar{S}_l\delta_{t+}p_l^n\rangle'_{\mathcal{D}} + \langle \mu_{t+}p_l^n, \delta_{x-}(S_{l+1/2}v_{l+1/2}^{n+1/2})\rangle'_{\mathcal{D}}. \tag{65}$$

From now on, the following superscripts and subscripts will be assumed unless denoted otherwise: n and l for p, l for $\bar{S}$, $l+1/2$ for S and $l+1/2$ and $n+1/2$ for v. After integration by parts of the last term, Eq. (65) becomes (also see App. D.2)

$$\begin{aligned} \delta_{t+}\mathfrak{h} &= \frac{1}{\rho_0 c^2}\langle \mu_{t+}p, \bar{S}\delta_{t+}p\rangle'_{\mathcal{D}} - \langle \mu_{t+}\delta_{x+}p, Sv\rangle_{\underline{\mathcal{D}}} + \mathfrak{h}_\text{b} \quad \text{where} \\ \mathfrak{h}_\text{b} &= (\mu_{t+}p_N^n)\mu_{x+}(S_{N-1/2}v_{N-1/2}^n) - (\mu_{t+}p_0^n)\mu_{x-}(S_{1/2}v_{1/2}^n). \end{aligned} \tag{66}$$

Then, Eq. (62b) can be substituted into (66) to get

$$\delta_{t+}\mathfrak{h} = \frac{1}{\rho_0 c^2}\langle \mu_{t+}p, \bar{S}\delta_{t+}p\rangle'_{\mathcal{D}} - \langle \mu_{t+}(-\rho_0\delta_{t-}v), Sv\rangle_{\underline{\mathcal{D}}} + \mathfrak{h}_\text{b} \tag{67}$$

$$= \frac{1}{\rho_0 c^2}\langle \mu_{t+}p, \bar{S}\delta_{t+}p\rangle'_{\mathcal{D}} + \rho_0\langle \delta_{t\cdot}v, Sv\rangle_{\underline{\mathcal{D}}} + \mathfrak{h}_\text{b}. \tag{68}$$

Then we can use the following identity for the first term

$$\langle \delta_{t+}u, \mu_{t+}u\rangle_{\mathcal{D}} = \delta_{t+}\left(\frac{1}{2}\|u\|_{\mathcal{D}}^2\right) \tag{69}$$

and the following identity for the second term

$$\langle \delta_{t\cdot}u, u\rangle_{\mathcal{D}} = \delta_{t+}\left(\frac{1}{2}\langle u, e_{t-}u\rangle_{\mathcal{D}}\right) \tag{70}$$

to get

$$\begin{aligned} &\mathfrak{h} = \mathfrak{t} + \mathfrak{v} \quad \text{where} \\ &\mathfrak{t} = \frac{1}{2\rho_0 c^2}\left(\|\sqrt{\bar{S}}p\|'_{\mathcal{D}}\right)^2 \quad \text{and} \quad \mathfrak{v} = \frac{\rho_0}{2}\langle Sv, e_{t-}v\rangle_{\underline{\mathcal{D}}}. \end{aligned} \tag{71}$$

[1]We can use the primed rather than the weighted inner product here as the boundary conditions are defined as found in Section 3.2.

3.4 Viscothermal Losses

Until now, only lossless tubes have been considered, but in reality, losses due to viscothermal effects occur. In frequency domain, the PDEs in (62) can be written as

$$SY\hat{p} = -\partial_x(S\hat{v}) \tag{72a}$$

$$Z\hat{v} = -\partial_x\hat{p} \tag{72b}$$

where the shunt admittance Y and series impedance Z, collectively called *immitances*, are defined as

$$Y = \frac{j\omega}{\rho_0 c^2}(1 + (\gamma - 1)F_\mathrm{t}), \quad \text{and} \quad Z = \frac{j\omega\rho_0}{1 - F_\mathrm{v}}, \tag{73}$$

with ratio of specific heats γ and

$$F_\mathrm{t} = \phi\left(\sqrt{-j}r_\mathrm{t}\right), \quad F_\mathrm{v} = \phi\left(\sqrt{-j}r_\mathrm{v}\right), \quad \phi(\xi) = \frac{2}{\xi}\frac{J_1(\xi)}{J_0(\xi)}\ , \tag{74}$$

where J_0 and J_1 are Bessel functions of the zeroth and first order respectively. Furthermore, thermal and viscous boundary layer thicknesses are defined as

$$r_\mathrm{t} = \nu r_\mathrm{v}, \quad r_\mathrm{v} = r\sqrt{\frac{\rho_0\omega}{\eta}} \tag{75}$$

with root of Prandtl number ν radius r and shear viscosity η.

It is useful to divide these into a lossless and a lossy part. We can define[2]

$$Y = Y_0 + Y_\mathrm{t}, \quad Z = Z_0 + Z_\mathrm{t}, \tag{76}$$

with lossless immitances

$$Y_0 = \frac{j\omega}{\rho_0 c^2}, \quad \text{and} \quad Z_0 = j\omega\rho_0, \tag{77}$$

and lossy immitances

$$Y_\mathrm{t} = \frac{j\omega}{\rho_0 c^2}(\gamma - 1)F_\mathrm{t}, \quad Z_\mathrm{t} = j\omega\rho_0\frac{F_\mathrm{v}}{1 - F_\mathrm{v}}. \tag{78}$$

where Y_t is due to thermal effects and Z_v is due to viscous effects.

Only considering the lossless case for now ($Y_\mathrm{t} = Z_\mathrm{t} = 0$), we can see that when using Eq. (76) in Eqs. (72) the latter reduces to the PDEs in (62) (knowing that a multiplication with $j\omega$ in the frequency domain is a first-order derivative (see Appendix **??**)).

4 Lip excitation

To excite the system we can use a pulse train. A more physical approach, that is bidirectional, is to model a lip as a mass-spring system that interacts with the left boundary of the tube (see Figure **??**). Following [7] we get

$$M\frac{d^2y}{dt^2} = -M\omega_0^2 y - M\sigma\frac{dy}{dt} + S_\mathrm{r}\Delta p, \tag{79}$$

with displacement of the lip reed from equilibrium $y = y(t)$, lip-mass M (kg) natural angular frequency of the lip reed $\omega_0 = \sqrt{K/M}$ (rad/s), spring stiffness of the lip K (N/m), loss parameter σ (s^{-1}), effective surface area of the lip S_r (m^2) and

$$\Delta p = P_\mathrm{m} - p(0, t) \tag{80}$$

is the difference between the pressure in the mouth P_m (kPa) and the pressure in the mouth piece $p(t, 0)$ (kPa). This pressure difference causes a volume flow velocity following the Bernoulli equation

[2]...by rewriting the impedance from Eq. (73) as $Z = j\omega\rho_0\left(\frac{1-F_\mathrm{v}+F_\mathrm{v}}{1-F_\mathrm{v}}\right) = j\omega\rho_0\left(1 + \frac{F_\mathrm{v}}{1-F_\mathrm{v}}\right)$.

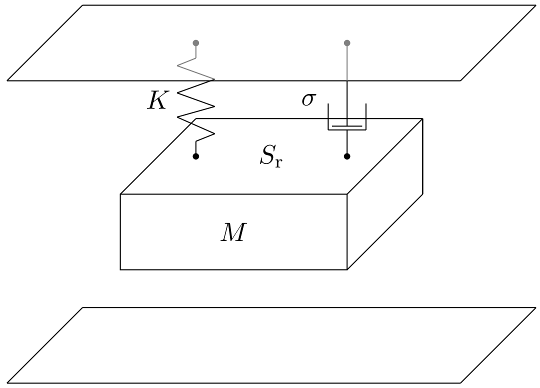

Figure 1: Lipsystem

$$U_{\mathrm{B}} = w[y + H_0]_+ \mathrm{sgn}(\Delta p)\sqrt{\frac{2|\Delta p|}{\rho_0}}, \tag{81}$$

with effective lip-reed width w (m), static equilibrium separation H_0 (m) and $[x]_+ = 0.5(x + |x|)$ describes the “positive part of“”. Notice that when $y + H_0 \leq 0$, the lips are closed and the volume velocity U_{B} is 0. Another volume flow is generated by the lip reed itself according to

$$U_{\mathrm{r}} = S_{\mathrm{r}}\frac{dy}{dt}. \tag{82}$$

Assuming that the volume flow velocity is conserved we define the total air volume entering the system is defined as

$$S(0)v(0,t) = U_{\mathrm{B}}(t) + U_{\mathrm{r}}(t). \tag{83}$$

4.1 Discrete Time

Placing y, Δp, and thereby U_{B} and U_{r} on the interleaved temporal grid (but on the non-interleaved spatial grid), we discretise the equations above to get the following system

$$\begin{cases} M\delta_{tt}y^{n+1/2} & = -M\omega_0^2\mu_{t\cdot}y^{n+1/2} - M\sigma\delta_{t\cdot}y^{n+1/2} + S_{\mathrm{r}}\Delta p^{n+1/2} & \text{(84a)} \\ \Delta p^{n+1/2} & = P_{\mathrm{m}} - \mu_{t+}p_0^n & \text{(84b)} \\ U_{\mathrm{B}}^{n+1/2} & = w[y^{n+1/2} + H_0]_+\mathrm{sgn}(\Delta p^{n+1/2})\sqrt{\frac{2|\Delta p^{n+1/2}|}{\rho_0}} & \text{(84c)} \\ U_{\mathrm{r}}^{n+1/2} & = S_{\mathrm{r}}\delta_{t\cdot}y^{n+1/2} & \text{(84d)} \\ \mu_{x-}(S_{1/2}^{n+1/2}v_{1/2}^{n+1/2}) & = U_{\mathrm{B}}^{n+1/2} + U_r^{n+1/2} & \text{(84e)} \end{cases}$$

5 Time-varying System

The PDE for the time-varying version of Webster’s equation is (Eq. (9.23) in [2])

$$\partial_t(S\partial_t\Psi) = c^2\partial_x(S\partial_x\Psi). \tag{85}$$

which can be discretised as (right below Eq. (9.23) in [2])

$$\delta_{t+}\big((\mu_{t-}\bar{S}_l^n)(\delta_{t-}\Psi_l^n)\big) = c^2\delta_{x+}\big((\mu_{tt}S_{l-1/2}^n)(\delta_{x-}\Psi_l^n)\big) \tag{86}$$

The update scheme for this time varying system can be written as

$$\begin{aligned} (\mu_{t+}\bar{S}_l^n)\Psi_l^{n+1} = &\Big(2\mu_{tt}\bar{S}_l^n - \lambda^2(\mu_{tt}S_{l+1/2}^n + \mu_{tt}S_{l-1/2}^n)\Big)\Psi_l^n \\ &+ \lambda^2\Big((\mu_{tt}S_{l+1/2}^n)\Psi_{l+1}^n + (\mu_{tt}S_{l-1/2})\Psi_{l-1}^n\Big) - (\mu_{t-}\bar{S}_l^n)\Psi_l^{n-1} \end{aligned} \tag{87}$$

6 Excitation

To excite the system, an input impedance can be defined as follows

$$\delta_{x\cdot}\Psi_0^n = -v_{\text{in}}^n. \tag{88}$$

Rewriting to

$$\frac{1}{2h}(\Psi_1^n - \Psi_{-1}^n) = -v_{\text{in}}^n \quad \Rightarrow \quad \Psi_{-1}^n = \Psi_1^n + 2hv_{\text{in}}^n \tag{89}$$

and substituting this in Eq. (12) at $l = 0$ yields

$$\begin{aligned}
\Psi_0^{n+1} &= 2(1-\lambda^2)\Psi_0^n - \Psi_0^{n-1} + \frac{\lambda^2 S_{1/2}}{\bar{S}_0}\Psi_1^n + \frac{\lambda^2 S_{-1/2}}{\bar{S}_0}\Psi_1 + \frac{2h\lambda^2 S_{-1/2}}{\bar{S}_0}v_{\text{in}}^n,\\
\Psi_0^{n+1} &= 2(1-\lambda^2)\Psi_0^n - \Psi_0^{n-1} + \frac{2\lambda^2(S_{1/2}+S_{-1/2})}{S_{1/2}+S_{-1/2}}\Psi_1^n + \frac{2h\lambda^2 S_{-1/2}}{\bar{S}_0}v_{\text{in}}^n,\\
\Psi_0^{n+1} &= 2(1-\lambda^2)\Psi_0^n - \Psi_0^{n-1} + 2\lambda^2\Psi_1^n + \frac{2h\lambda^2 S_{-1/2}}{\bar{S}_0}v_{\text{in}}^n,
\end{aligned} \tag{90}$$

which is identical to Eq. (19.58) in [4].

7 Damping

As said in [3], a fractional derivative can be used for viscothermal damping. This, in discrete time can be approximated using the bilinear transform (or Tustin's transformation).

Following [6]:

$$(\omega(z^{-1}))^r = \left(\frac{2}{k}\right)^r \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^r. \tag{91}$$

Then, if we consider $r = 3$:

$$\begin{aligned}
(\omega(z^{-1}))^3 &= \left(\frac{2}{k}\right)^3 \left(\frac{(1-z^{-1})^3}{(1+z^{-1})^3}\right)\\
&= \left(\frac{2}{k}\right)^3 \left(\frac{1-z^{-1}-2z^{-1}+2z^{-2}+z^{-2}-z^{-3}}{1+z^{-1}+2z^{-1}+2z^{-2}+z^{-2}+z^{-3}}\right)\\
&= \left(\frac{2}{k}\right)^3 \left(\frac{1-3z^{-1}+3z^{-2}-z^{-3}}{1+3z^{-1}+3z^{-2}+z^{-3}}\right)
\end{aligned} \tag{92}$$

We can do the same if we follow Eq. (2) from [6] with $n = 3$:

$$\begin{aligned}
(\omega(z^{-1}))^3 &= \left(\frac{2}{k}\right)^3 \frac{A_3(z^{-1},3)}{A_3(z^{-1},-3)}\\
&= \left(\frac{2}{k}\right)^3 \frac{-\frac{1}{3}3z^{-3}+\frac{1}{3}3^2z^{-2}-3z^{-1}+1}{-\frac{1}{3}(-3)z^{-3}+\frac{1}{3}(-3)^2z^{-2}-(-3)z^{-1}+1}\\
&= \left(\frac{2}{k}\right)^3 \frac{-z^{-3}+3z^{-2}-3z^{-1}+1}{z^{-3}+3z^{-2}+3z^{-1}+1},
\end{aligned} \tag{93}$$

which is equivalent to (19).

7.1 Muir-recursion

The Muir-recursion is described as

$$A_n(z^{-1},r) = A_{n-1}(z^{-1},r) - c_n z^n A_{n-1}(z,r) \quad \text{with} \quad A_0(z^{-1},r) = 1 \tag{94}$$

and

$$c_n = \begin{cases} r/n & n \text{ is odd,} \\ 0 & n \text{ is even.} \end{cases} \tag{95}$$

If we let $n = 3$ we get

$$\begin{aligned} A_3(z^{-1}, r) &= A_2(\underbrace{(z^{-1})^{-1}}_{z}, r) - \frac{r}{3}(z^{-1})^3 A_2(z^{-1}, r) \\ &= A_1((z)^{-1}, r) - \frac{r}{3} z^{-3} A_1((z^{-1})^{-1}, r) \\ &= A_0((z^{-1})^{-1}, r) - r(z^{-1})^1 A_0(z^{-1}, r) - \frac{r}{3} z^{-3} \left(A_0((z)^{-1}, r) - rz^1 A_0(z, r)\right) \\ &(if\ A_0(z, r) = 1) \\ &= 1 - rz^{-1} - \frac{r}{3} z^{-3} (1 - rz) \\ &= 1 - rz^{-1} + \frac{r^2}{3} z^{-2} - \frac{r}{3} z^{-3} \end{aligned} \tag{96}$$

7.2 Implementation of Muir-recursion

Classes and member variables

- `Z`

  `double coeff`

  `int power`

- `Equation<M>`

  `std::vector<M> zs` (of length M+1)

  `int zSign` (that only takes the values -1 and 1).

There are a few observations that I made that helped the implementation

-

7.2.1 The `swapCoeffs (int amount)` function

This function is called when a `Z` is multiplied onto an `Equation`, i.e. the last term of Eq. (94) ($z^n A_{n-1}(z, r)$). I observed that in this term of the recursion the following relationship is true:

$$\text{For any } z^p \cdot A(z, r), \quad p = p_{A+} + 2 \cdot \text{sgn}(p_{A+}), \tag{97}$$

where p_{A+} is the largest power whether positive or negative present in A (i.e. furthest away from 0). For example, the following could occur in the recursion:

$$z^3(A_1(z^{-1}, r)) = z^3(1 - rz^{-1}) = z^3 - rz^2. \tag{98}$$

In this case $p_{A+} = -1$, and $p = 3$ (as can also be seen from the relationship in (97)).

Now, the implementation of this allows us to swap coefficients of the `Z`'s in the z-vector around

$$\text{swap-around index} = |p|/2, \tag{99}$$

where p is – again – the power of the z multiplied onto the equation. As this power is always odd, the swap-around index is an integer-and-a-half. For the above example, the `Z` is z^3 and the `Equation` is $A_1(z^{-1}, r) = 1 - rz^{-1}$. The coefficients-vector of the `Equation`, `{1, -r, 0, 0}` (with negative powers) can now be flipped around index abs(3) / 2 = 1.5, so between index 1 and 2. If this is done and the signs of the powers of the `Z`'s in the vector are flipped, the coefficient-vector of the solution looks like `{0, 0, -r, 1}` (with positive powers), which is shown in the Eq. (98), i.e. $0z^0 + 0z^1 - rz^2 + 1z^3$.

References

[1] M. Atig, J.-P. Dalmont, and J. Gilbert. Termination impedance of open-ended cylindrical tubes at high sound pressure level. *Comptes Rendus Mécanique*, 332:299–304, 2004.

[2] Stefan Bilbao. *Numerical Sound Synthesis*. John Wiley & Sons, 2009.

[3] Stefan Bilbao and John Chick. Finite difference time domain simulation for the brass instrument bore. *The Journal of the Acoustical Society of America*, 2013.

[4] Stefan Bilbao, Brian Hamilton, R. Harrison, and A Torin. Finite-difference schemes in musical acoustics: A tutorial. *Springer handbook of systematic musicology*, 2018.

[5] Stefan Bilbao and Reginald Harrison. Passive time-domain numerical models of viscothermal wave propagation in acoustic tubes of variable cross section. *Journal of the Acoustical Society of America*, 2016.

[6] Yang Quan Chen and Kevin L. Moore. Discretization schemes for fractional-order differentiators and integrators. *IEEE Transactions on Circuits and Systems*, 2002.

[7] Reginald Langford Harrison-Harsley. *Physical Modelling of Brass Instruments using Finite-Difference Time-Domain Methods*. PhD thesis, University of Edinburgh, 2018.

[8] Alberto Torin. *Percussion Instrument Modelling in 3D: Sound Synthesis Through Time Domain Numerical Simulation*. PhD thesis, University of Edinburgh, 2015.

A Derivations

A.1 Webster's Update Equation

$$
\begin{aligned}
\frac{\bar{S}_l}{k^2}(\Psi_l^{n+1} - 2\Psi_l^n + \Psi_l^{n-1}) &= c^2\left((\delta_{x+}S_{l-1/2})(\mu_{x+}\delta_{x-}\Psi_l^n) + (\mu_{x+}S_{l-1/2})(\delta_{x+}\delta_{x-}\Psi_l^n)\right)\\
\Psi_l^{n+1} - 2\Psi_l^n + \Psi_l^{n-1} &= \frac{c^2k^2}{\bar{S}_l}\Bigg(\frac{1}{h}(S_{l+1/2} - S_{l-1/2})\frac{1}{2h}\overbrace{(\Psi_{l+1}^n - \Psi_{l-1}^n)}^{\mu_{x+}\delta_{x-}\Psi_l^n = \delta_{x\cdot}\Psi_l^n}\\
&\qquad + \frac{1}{2}(S_{l+1/2} + S_{l-1/2})\frac{1}{h^2}(\Psi_{l+1}^n - 2\Psi_l^n + \Psi_{l-1}^n)\Bigg)\\
\Psi_l^{n+1} &= 2\Psi_l^n - \Psi_l^{n-1} + \frac{\overbrace{\lambda^2}^{\lambda = \frac{ck}{h}}}{2\bar{S}_l}\Big(S_{l+1/2}\Psi_{l+1}^n - S_{l+1/2}\Psi_{l-1}^n - S_{l-1/2}\Psi_{l+1}^n + S_{l-1/2}\Psi_{l-1}^n\\
&\quad + S_{l+1/2}\Psi_{l+1}^n + S_{l+1/2}\Psi_{l-1}^n + S_{l-1/2}\Psi_{l+1}^n + S_{l-1/2}\Psi_{l-1}^n - 2(S_{l+1/2} + S_{l-1/2})\Psi_l^n\Big)\\
\Psi_l^{n+1} &= 2\Psi_l^n - \Psi_l^{n-1} + \frac{\lambda^2}{2\bar{S}_l}\left(2S_{l+1/2}\Psi_{l+1}^n + 2S_{l-1/2}\Psi_{l-1}^n - 4\bar{S}_l\Psi_l^n\right)\\
\Psi_l^{n+1} &= 2\Psi_l^n - \Psi_l^{n-1} + \frac{\lambda^2 S_{l+1/2}}{\bar{S}_l}\Psi_{l+1}^n + \frac{\lambda^2 S_{l-1/2}}{\bar{S}_l}\Psi_{l-1}^n - 2\lambda^2\Psi_l^n\\
\Psi_l^{n+1} &= 2(1-\lambda^2)\Psi_l^n - \Psi_l^{n-1} + \frac{\lambda^2 S_{l+1/2}}{\bar{S}_l}\Psi_{l+1}^n + \frac{\lambda^2 S_{l-1/2}}{\bar{S}_l}\Psi_{l-1}^n, \qquad (100)
\end{aligned}
$$

A.2 Centered Radiation

$$
\begin{aligned}
\Psi_N^{n+1} &= 2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}\Psi_{N+1}^n + \frac{\lambda^2 S_{N-1/2}}{\bar{S}_N}\Psi_{N-1}^n \\
\Psi_N^{n+1} &= 2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} \\
&\quad + \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}\left[h\left(-\frac{a_1}{k}(\Psi_N^{n+1} - \Psi_N^{n-1}) - a_2(\Psi_N^{n+1} + \Psi_N^{n-1})\right) + \Psi_{N-1}^n\right] \\
&\quad + \frac{\lambda^2 S_{N-1/2}}{\bar{S}_N}\Psi_{N-1}^n \\
\Psi_N^{n+1} &= 2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\left[\left(-\frac{a_1}{k} - a_2\right)\Psi_N^{n+1} + \left(\frac{a_1}{k} - a_2\right)\Psi_N^{n-1}\right] \\
&\quad + \frac{\lambda^2(S_{N+1/2} + S_{N-1/2})}{\bar{S}_N}\Psi_{N-1}^n \\
\left(1 + \left(\frac{a_1}{k} + a_2\right)\frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\right)\Psi_N^{n+1} &= 2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\left(\frac{a_1}{k} - a_2\right)\Psi_N^{n-1} + 2\lambda^2\Psi_{N-1}^n \\
\Psi_N^{n+1} &= \frac{2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\left(\frac{a_1}{k} - a_2\right)\Psi_N^{n-1} + 2\lambda^2\Psi_{N-1}^n}{\left(1 + \left(\frac{a_1}{k} + a_2\right)\frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\right)} \qquad (101)
\end{aligned}
$$

A.3 Non-centered Radiation

$$
\begin{aligned}
\Psi_N^{n+1} &= 2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}\Psi_{N+1}^n + \frac{\lambda^2 S_{N-1/2}}{\bar{S}_N}\Psi_{N-1}^n \\
\Psi_N^{n+1} &= 2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} \\
&\quad + \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}\left[h\left(-\frac{a_1}{2k}(\Psi_N^{n+1} - \Psi_N^{n-1}) - \frac{a_2}{2}(\Psi_N^{n+1} + \Psi_N^{n-1})\right) + \Psi_N^n\right] \\
&\quad + \frac{\lambda^2 S_{N-1/2}}{\bar{S}_N}\Psi_{N-1}^n \\
\Psi_N^{n+1} &= 2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\left[\left(-\frac{a_1}{2k} - \frac{a_2}{2}\right)\Psi_N^{n+1} + \left(\frac{a_1}{k} - \frac{a_2}{2}\right)\Psi_N^{n-1}\right] \\
&\quad + \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}\Psi_N^n + \frac{\lambda^2 S_{N-1/2}}{\bar{S}_N}\Psi_{N-1}^n \\
\left(1 + \left(\frac{a_1}{2k} + \frac{a_2}{2}\right)\frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\right)\Psi_N^{n+1} &= 2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\left(\frac{a_1}{2k} - \frac{a_2}{2}\right)\Psi_N^{n-1} \\
&\quad + \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}\Psi_N^n + \frac{\lambda^2 S_{N-1/2}}{\bar{S}_N}\Psi_{N-1}^n \\
\Psi_N^{n+1} &= \frac{2(1-\lambda^2)\Psi_N^n - \Psi_N^{n-1} + \frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\left(\frac{a_1}{2k} - \frac{a_2}{2}\right)\Psi_N^{n-1} + \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}\Psi_N^n + \frac{\lambda^2 S_{N-1/2}}{\bar{S}_N}\Psi_{N-1}^n}{\left(1 + \left(\frac{a_1}{2k} + \frac{a_2}{2}\right)\frac{h\lambda^2 S_{N+1/2}}{\bar{S}_N}\right)} \\
&\qquad (102)
\end{aligned}
$$

B Inner Product with S

$$
\begin{aligned}
& -c^2\langle \delta_{t\cdot}\delta_{x+}\Psi_l^n, \mu_{x+}S\delta_{x+}\Psi_l^n\rangle_{\mathcal{D}}, \\
\iff\quad & -c^2\sum_{\mathcal{D}} h(\delta_{t\cdot}\delta_{x+}\Psi_l^n)(\mu_{x+}S\delta_{x+}\Psi_l^n) \\
\iff\quad & -c^2\sum_{\mathcal{D}} \frac{h}{4kh^2}\left(\Psi_{l+1}^{n+1}-\Psi_{l+1}^{n-1}-\Psi_l^{n+1}+\Psi_l^{n-1}\right)\left(S_{l+1}+S_l\right)\left(\Psi_{l+1}^n-\Psi_l^n\right) \\
\iff\quad & -c^2\sum_{\mathcal{D}} \frac{h}{4kh^2}\big(\Psi_{l+1}^{n+1}\Psi_{l+1}^n-\Psi_{l+1}^{n+1}\Psi_l^n-\Psi_l^{n+1}\Psi_{l+1}^n+\Psi_l^{n+1}\Psi_l^n \\
& \qquad\qquad -\Psi_{l+1}^n\Psi_{l+1}^{n-1}+\Psi_{l+1}^n\Psi_l^{n-1}+\Psi_l^n\Psi_{l+1}^{n-1}-\Psi_l^n\Psi_l^{n-1}\big)\left(S_{l+1}+S_l\right) \\
\iff\quad & -c^2\sum_{\mathcal{D}} \frac{h}{4h^2}\delta_{t+}\left(\Psi_{l+1}^n\Psi_{l+1}^{n-1}-\Psi_{l+1}^n\Psi_l^{n-1}-\Psi_l^n\Psi_{l+1}^{n-1}+\Psi_l^n\Psi_l^{n-1}\right)\left(S_{l+1}+S_l\right) \\
\iff\quad & \delta_{t+}\left(-c^2\sum_{\mathcal{D}}\frac{h}{2}(\delta_{x+}\Psi_l^n)(\delta_{x+}\Psi_l^{n-1})(\mu_{x+}S)\right) \\
\iff\quad & \delta_{t+}\left(-\frac{c^2}{2}\langle(\mu_{x+}S)\delta_{x+}\Psi_l^n, e_{t-}\Psi_l^n\rangle_{\mathcal{D}}\right),
\end{aligned}
\tag{103}
$$

which (except for the use of c instead of γ) is equivalent to Eq. (9.14).

C Summation by parts (proving equations (5.25) and (5.26))

The general form of summation by parts of two functions f and g, the latter of which has a single spatial derivative is as follows (Eq. 5.25) in [2]):

$$
\langle f, \delta_{x+}g\rangle_{\mathcal{D}} = \sum_{l=d_-}^{d_+} hf_l\frac{1}{h}(g_{l+1}-g_l) = -\langle\delta_{x-}f, g\rangle_{\mathcal{D}} + f_{d_+}g_{d_++1} - f_{d_--1}g_{d_-} \tag{104}
$$

Proving this, we can use an example domain of $\mathcal{D}\in[0,...,3]$:

$$
\begin{aligned}
\sum_{l=0}^{3} hf_l\frac{1}{h}(g_{l+1}-g_l) &= f_0g_1-f_0g_0+f_1g_2-f_1g_1+f_2g_3-f_2g_2+f_3g_4-f_3g_3 \\
&= g_0(f_{-1}-f_0)+g_1(f_0-f_1)+g_2(f_1-f_2)+g_3(f_2-f_3)+f_3g_4-f_{-1}g_0 \\
&= -g_0(f_0-f_{-1})-g_1(f_1-f_0)-g_2(f_2-f_1)-g_3(f_3-f_2)+f_3g_4-f_{-1}g_0 \\
&= -\sum_{l=0}^{3} g_l(f_l-f_{l-1})+f_3g_4-f_{-1}g_0 \\
&= -\sum_{l=0}^{3} h\delta_{x-}f_lg_l+f_3g_4-f_{-1}g_0
\end{aligned}
\tag{105}
$$

Then replacing the applied domain with the general form yields

$$
-\sum_{l=d_-}^{d_+} h\delta_{x-}f_lg_l + f_{d_+}g_{d_++1} - f_{d_--1}g_{d_-} \quad\Rightarrow\quad -\langle\delta_{x-}f, g\rangle_{\mathcal{D}} + f_{d_+}g_{d_++1} - f_{d_--1}g_{d_-} \tag{106}
$$

which is the general form in (104).

In the same way, we can prove Eq. (5.26):

$$\begin{aligned}\langle f,\delta_{x+}g\rangle_{\underline{\mathcal{D}}} = \sum_{l=0}^{2} hf_l\delta_{x+}g_l &= f_0g_1 - f_0g_0 + f_1g_2 - f_1g_1 + f_2g_3 - f_2g_2\\ &= -g_1(f_1-f_0) - g_2(f_2-f_1) - g_3(f_3-f_2) + f_3g_3 - f_0g_0\\ &= -\sum_{l=1}^{3} h\delta_{x-}f_lg_l + f_3g_3 - f_0g_0\end{aligned} \tag{107}$$

Again, going general yields

$$-\sum_{l=d_-+1}^{d_+} h\delta_{x-}f_lg_l + f_{d_+}g_{d_+} - f_{d_-}g_{d_-} \quad\Rightarrow\quad -\langle\delta_{x-}f,g\rangle_{\overline{\mathcal{D}}} + f_{d_+}g_{d_+} - f_{d_-}g_{d_-} \tag{108}$$

C.1 Proving (5.27)

Using again an example domain of $\mathcal{D}\in[0,\dots,3]$

$$\begin{aligned}\langle f,\delta_{xx}g\rangle_{\mathcal{D}} &= \sum_{l=0}^{3} hf_l\frac{1}{h^2}(g_{l+1}-2g_l+g_{l-1})\\ &= \frac{1}{h}\Big(f_0g_{-1} - 2f_0g_0 + f_0g_1 + f_1g_0 - 2f_1g_1 + f_1g_2 + f_2g_1 - 2f_2g_2 + f_2g_3 + f_3g_2 - 2f_3g_3 + f_3g_4\Big)\\ &= \frac{1}{h}\Big(f_0g_{-1} + g_0(f_{-1}-2f_0+f_1) - g_0f_{-1} + g_1(f_0-2f_1+f_2)+\\ &\qquad g_2(f_1-2f_2+f_3) + g_3(f_2-2f_3+f_4) - g_3f_4 + f_3g_4\Big)\\ &= \frac{1}{h}\Big(\sum_{l=0}^{3} h^2g_l\delta_{xx}f_l + f_0g_{-1} - g_0f_{-1} - g_3f_4 + f_3g_4\Big)\\ &= \sum_{l=0}^{3} hg_l\delta_{xx}f_l + \frac{1}{h}\Big(-f_0g_0 + f_0g_{-1} + f_0g_0 - f_{-1}g_0 + f_3g_4 - f_3g_3 - f_4g_3 + f_3g_3\Big)\\ &= \langle\delta_{xx}f,g\rangle_{\mathcal{D}} - f_0\delta_{x-}g_0 + g_0\delta_{x-}f_0 + f_3\delta_{x+}g_3 - g_3\delta_{x+}f_3\end{aligned} \tag{109}$$

C.2 The same with a primed inner product

The primed inner product from [2] is defined as

$$\langle f^n,g^n\rangle'_{\mathcal{D}} = \sum_{d_-+1}^{d_+-1} hf_l^ng_l^n + \frac{h}{2}f_{d-}^ng_{d-}^n + \frac{h}{2}f_{d+}^ng_{d+}^n, \tag{110}$$

so essentially a regular inner product with the outer terms scaled by 1/2.

Using the same case as above we get

$$
\begin{aligned}
\langle f, \delta_{xx} g\rangle'_{\mathcal{D}} &= \sum_{l=1}^{2} h f_l \frac{1}{h^2}(g_{l+1} - 2g_l + g_{l-1}) + \frac{h}{2} f_0 \frac{1}{h^2}(g_1 - 2g_0 + g_{-1}) + f_3 \frac{1}{h^2}(g_4 - 2g_3 + g_2) \\
&= \frac{1}{h}\Big(\frac{1}{2}(f_0 g_1 - 2f_0 g_0 + f_0 g_{-1}) + f_1 g_2 - 2f_1 g_1 + f_1 g_0 \\
&\qquad + f_2 g_3 - 2f_2 g_2 + f_2 g_1 + \frac{1}{2}(f_3 g_4 - 2f_3 g_3 + f_3 g_2)\Big) \\
&= \frac{1}{h}\Big(g_1(f_2 - 2f_1 + f_0) - \frac{1}{2} f_0 g_1 + g_2(f_3 - 2f_2 + f_1) - \frac{1}{2} f_3 g_2 \\
&\qquad - f_0 g_0 + \frac{1}{2} f_0 g_{-1} - f_3 g_3 + \frac{1}{2} f_3 g_4 + f_1 g_0 + f_2 g_3)\Big) \\
&= \frac{1}{h}\left(\sum_{l=1}^{2} h^2 g_l \delta_{xx} f_l + \frac{1}{2} f_0 g_{-1} - f_0 g_0 - \frac{1}{2} f_0 g_1 + f_1 g_0 + f_2 g_3 - \frac{1}{2} f_3 g_2 - f_3 g_3 + \frac{1}{2} f_3 g_4\right) \\
&= \langle \delta_{xx} f, g\rangle'_{\mathcal{D}} - \frac{h}{2} g_0 \delta_{xx} f_0 - \frac{h}{2} g_3 \delta_{xx} f_3 \\
&\qquad + \frac{1}{h}\left(\frac{1}{2} f_0 g_{-1} - f_0 g_0 - \frac{1}{2} f_0 g_1 + f_1 g_0 + f_2 g_3 - \frac{1}{2} f_3 g_2 - f_3 g_3 + \frac{1}{2} f_3 g_4\right) \\
&= \langle \delta_{xx} f, g\rangle'_{\mathcal{D}} - \frac{h}{2} g_0 \frac{1}{h^2}(f_1 - 2f_0 + f_{-1}) - \frac{h}{2} g_3 \frac{1}{h^2}(f_4 - 2f_3 + f_2) \\
&\qquad + \frac{1}{h}\left(\frac{1}{2} f_0 g_{-1} - f_0 g_0 - \frac{1}{2} f_0 g_1 + f_1 g_0 + f_2 g_3 - \frac{1}{2} f_3 g_2 - f_3 g_3 + \frac{1}{2} f_3 g_4\right) \\
&= \langle \delta_{xx} f, g\rangle'_{\mathcal{D}} + \frac{1}{h}\Big(-\frac{1}{2} f_1 g_0 + f_0 g_0 - \frac{1}{2} f_{-1} g_0 - \frac{1}{2} f_4 g_3 + f_3 g_3 - \frac{1}{2} f_2 g_3 \\
&\qquad + \frac{1}{2} f_0 g_{-1} - f_0 g_0 - \frac{1}{2} f_0 g_1 + f_1 g_0 + f_2 g_3 - \frac{1}{2} f_3 g_2 - f_3 g_3 + \frac{1}{2} f_3 g_4\Big) \\
&= \langle \delta_{xx} f, g\rangle'_{\mathcal{D}} + \frac{1}{2h}\Big(f_1 g_0 - f_{-1} g_0 - f_4 g_3 + f_2 g_3 - f_0 g_1 + f_0 g_{-1} - f_3 g_2 + f_3 g_4\Big) \\
&= \langle \delta_{xx} f, g\rangle'_{\mathcal{D}} + \frac{1}{2h}(f_1 - f_{-1}) g_0 - \frac{1}{2h}(g_1 - g_{-1}) f_0 + \frac{1}{2h}(g_4 - g_2) f_3 - \frac{1}{2h}(f_4 - f_2) g_3 \\
&= \langle \delta_{xx} f, g\rangle'_{\mathcal{D}} + g_0 \delta_{x\cdot} f_0 - f_0 \delta_{x\cdot} g_0 - g_3 \delta_{x\cdot} f_3 + f_3 \delta_{x\cdot} g_3
\end{aligned} \tag{111}
$$

which is identical to the identity presented in Problem 5.8 in [2]. Note that the difference with the regular inner product is that the boundary terms contain a centered derivative rather than a non-centered one.

D Potential energy derivation

D.1 ...for the non-centered case

Recalling Eq. (41) and disregarding the multiplication with c^2 for now, we set $f = \delta_{t\cdot}\Psi$ and $g = (\mu_{x+}S)(\delta_{x+}\Psi)$ and domain $\mathcal{D} \in [0, N]$ which yields

$$\begin{aligned}
\langle f, \delta_{x-}g\rangle_{\mathcal{D}} = \sum_{l=0}^{N} hf_l\delta_{x-}g_l &= f_0(g_0 - g_{-1}) + f_1(g_1 - g_0) + \dots f_N(g_N - g_{N-1}) \\
&= -g_0(f_1 - f_0) - g_1(f_2 - f_1) - \dots g_{N-1}(f_N - f_{N-1}) + f_N g_N - f_0 g_{-1} \\
&= -\sum_{l=0}^{N-1} h\delta_{x+}f_l g_l + f_N g_N - f_0 g_{-1} \\
&= -\langle \delta_{x+}f, g\rangle_{\underline{\mathcal{D}}} + f_N g_N - f_0 g_{-1} \\
\\
&= -\langle \delta_{x+}\delta_{t\cdot}\Psi, (\mu_{x+}S)\delta_{x+}\Psi\rangle_{\underline{\mathcal{D}}} + (\delta_{t\cdot}\Psi_N)(\mu_{x+}S_N)(\delta_{x+}\Psi_N) \\
&\qquad - (\delta_{t\cdot}\Psi_0)\underbrace{(\mu_{x+}S_{-1})}_{(\mu_{x-}S_0)}\underbrace{(\delta_{x+}\Psi_{-1})}_{(\delta_{x-}\Psi_0)}
\end{aligned} \tag{112}$$

D.2 ..using the primed inner product

Again, using domain $\mathcal{D} \in [0, \dots, N]$ and $N = 3$, we get

$$\begin{aligned}
\langle f, \delta_{x-}g\rangle'_{\mathcal{D}} &= \sum_{l=1}^{N-1} hf_l\delta_{x-}g_l + \frac{h}{2}f_0\delta_{x-}g_0 + \frac{h}{2}f_3\delta_{x-}g_3 \\
&= \frac{h}{2}f_0\frac{1}{h}(g_0 - g_{-1}) + hf_1\frac{1}{h}(g_1 - g_0) + hf_2\frac{1}{h}(g_2 - g_1) + \frac{h}{2}f_3\frac{1}{h}(g_3 - g_2) \\
&= \frac{1}{2}f_0g_0 - \frac{1}{2}f_0g_{-1} + f_1g_1 - f_1g_0 + f_2g_2 - f_2g_1 + \frac{1}{2}f_3g_3 - \frac{1}{2}f_3g_2 \\
&= -g_0(f_1 - f_0) - \frac{1}{2}f_0g_0 - g_1(f_2 - f_1) - g_2(f_3 - f_2) + \frac{1}{2}f_3g_2 - \frac{1}{2}f_0g_{-1} + \frac{1}{2}f_3g_3 \\
&= -\sum_{l=0}^{N-1} hg_l\delta_{x+}f_l - \frac{1}{2}f_0g_0 - \frac{1}{2}f_0g_{-1} + \frac{1}{2}f_3g_3 + \frac{1}{2}f_3g_2 \\
&= -\langle \delta_{x+}f, g\rangle_{\underline{\mathcal{D}}} - f_0(\mu_{x-}g_0) + f_3(\mu_{x-}g_3)
\end{aligned} \tag{113}$$

Then, filling in the $f = \delta_{t\cdot}\Psi$ and $g = (\mu_{x+}S)(\delta_{x+}\Psi)$ yields

$$-\langle \delta_{x+}\delta_{t\cdot}\Psi, (\mu_{x+}S)(\delta_{x+}\Psi)\rangle_{\underline{\mathcal{D}}} - (\delta_{t\cdot}\Psi_0^n)\mu_{x-}\big((\mu_{x+}S_0)(\delta_{x+}\Psi_0^n)\big) + (\delta_{t\cdot}\Psi_N^n)\mu_{x-}\big((\mu_{x+}S_N)(\delta_{x+}\Psi_N^n)\big) \tag{114}$$

which is close (!) but unfortunately doesn't give us the right answer.

D.3 (scratch that) ...using the more general weighed inner product! AKA solving Problem 9.5 AKA ... for the centered case

Recalling Eq. (41), but now applying the weighted inner product found in Eq. (37) (with $f = \delta_{t\cdot}\Psi$ and $g = S_{l+1/2}(\delta_{x+}\Psi)$) yields

$$\begin{aligned}
\langle f, \delta_{x-} g\rangle_{\mathcal{D}}^{\epsilon_\mathrm{l},\epsilon_\mathrm{r}} &= \sum_{l=1}^{N-1} h f_l \frac{1}{h}(g_l - g_{l-1}) + \frac{\epsilon_\mathrm{l}}{2} h f_0 \frac{1}{h}(g_0 - g_{-1}) + \frac{\epsilon_\mathrm{r}}{2} h f_N \frac{1}{h}(g_N - g_{N-1}) \\
&= f_1 g_1 - f_1 g_0 + f_2 g_2 - f_2 g_1 + \ldots + f_{N-1} g_{N-1} - f_{N-1} g_{N-2} \\
&\quad + \frac{\epsilon_\mathrm{l}}{2} f_0 (g_0 - g_{-1}) + \frac{\epsilon_\mathrm{r}}{2} f_N (g_N - g_{N-1}) \\
&= -g_0(f_1 - f_0) - f_0 g_0 - g_1(f_2 - f_1) - \ldots - g_{N-1}(f_N - f_{N-1}) + f_N g_{N-1} \\
&\quad + \frac{\epsilon_\mathrm{l}}{2} f_0 (g_0 - g_{-1}) + \frac{\epsilon_\mathrm{r}}{2} f_N (g_N - g_{N-1}) \\
&= -\langle \delta_{x+} f, g\rangle_{\underline{\mathcal{D}}} - f_0 g_0 + \frac{\epsilon_\mathrm{l}}{2} f_0 (g_0 - g_{-1}) + f_N g_{N-1} + \frac{\epsilon_\mathrm{r}}{2} f_N (g_N - g_{N-1}).
\end{aligned} \tag{115}$$

Then, only looking at the boundaries, filling in the definitions for f and g and recalling the multiplication with c^2 yields

$$\begin{aligned}
\mathfrak{b} &= -c^2 (\delta_{t\cdot}\Psi_0^n)\Big(S_{1/2}(\delta_{x+}\Psi_0^n)\Big) + \frac{\epsilon_\mathrm{l}}{2}(\delta_{t\cdot}\Psi_0^n)\Big(S_{1/2}(\delta_{x+}\Psi_0^n) - S_{-1/2}\overbrace{(\delta_{x+}\Psi_{-1}^n)}^{(\delta_{x-}\Psi_0^n)}\Big) \\
&\quad + c^2(\delta_{t\cdot}\Psi_N^n)\Big(S_{N-1/2}\underbrace{(\delta_{x+}\Psi_{N-1}^n)}_{(\delta_{x-}\Psi_N^n)}\Big) + \frac{\epsilon_\mathrm{r}}{2}(\delta_{t\cdot}\Psi_N^n)\Big(S_{N+1/2}(\delta_{x+}\Psi_N^n) - S_{N-1/2}\underbrace{(\delta_{x+}\Psi_{N-1}^n)}_{(\delta_{x-}\Psi_N^n)}\Big) \\
&= c^2(\delta_{t\cdot}\Psi_N^n)\left(\frac{\epsilon_\mathrm{r}}{2} S_{N+1/2}(\delta_{x+}\Psi_N^n) + \left(1 - \frac{\epsilon_\mathrm{r}}{2}\right) S_{N-1/2}(\delta_{x-}\Psi_N^n)\right) \\
&\quad - c^2(\delta_{t\cdot}\Psi_0^n)\left(\frac{\epsilon_\mathrm{l}}{2} S_{-1/2}(\delta_{x-}\Psi_0^n) + \left(1 - \frac{\epsilon_\mathrm{l}}{2}\right) S_{1/2}(\delta_{x+}\Psi_0^n))\right)
\end{aligned} \tag{116}$$

which is what is shown in Problem 9.5.

Then, for the the centered radiating boundary condition shown in Eq. (9.16) in [2] to be strictly dissipative we need to make the special choice for $\epsilon_\mathrm{r} = S_{N-1/2}/\mu_{xx} S_N$. Only considering the right boundary and continuing with this choice of ϵ_{r} we get

$$\begin{aligned}
\mathfrak{b}_{\mathrm{r}} &= c^2(\delta_{t\cdot}\Psi_N^n)\left(\frac{S_{N-1/2}}{2\mu_{xx}S_N}S_{N+1/2}(\delta_{x+}\Psi_N^n)+\left(1-\frac{S_{N-1/2}}{2\mu_{xx}S_N}\right)S_{N-1/2}(\delta_{x-}\Psi_N^n)\right)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(\frac{S_{N+1/2}}{2\mu_{xx}S_N}(\delta_{x+}\Psi_N^n)+\left(1-\frac{S_{N-1/2}}{2\mu_{xx}S_N}\right)(\delta_{x-}\Psi_N^n)\right)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(1-\frac{S_{N-1/2}}{2\mu_{xx}S_N}\right)\left(\frac{\frac{S_{N+1/2}(\delta_{x+}\Psi_N^n)}{2\mu_{xx}S_N}}{\left(1-\frac{S_{N-1/2}}{2\mu_{xx}S_N}\right)}+\delta_{x-}\Psi_N^n\right)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(1-\frac{\epsilon_{\mathrm{r}}}{2}\right)\left(\frac{\frac{S_{N+1/2}(\delta_{x+}\Psi_N^n)}{2\mu_{xx}S_N}}{\left(\frac{2\mu_{xx}-S_{N-1/2}}{2\mu_{xx}S_N}\right)}+\delta_{x-}\Psi_N^n\right)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(1-\frac{\epsilon_{\mathrm{r}}}{2}\right)\left(\frac{S_{N+1/2}(\delta_{x+}\Psi_N^n)}{2\mu_{xx}S_N-S_{N-1/2}}+\delta_{x-}\Psi_N^n\right)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(1-\frac{\epsilon_{\mathrm{r}}}{2}\right)\left(\frac{S_{N+1/2}(\delta_{x+}\Psi_N^n)}{S_{N+1/2}+S_{N-1/2}-S_{N-1/2}}+\delta_{x-}\Psi_N^n\right)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(1-\frac{\epsilon_{\mathrm{r}}}{2}\right)(\delta_{x+}\Psi_N^n+\delta_{x-}\Psi_N^n)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(1-\frac{\epsilon_{\mathrm{r}}}{2}\right)\left(\frac{1}{h}\left(\Psi_{N+1}^n-\Psi_N^n+\Psi_N^n-\Psi_{N-1}^n\right)\right)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(1-\frac{\epsilon_{\mathrm{r}}}{2}\right)2(\delta_{x\cdot}\Psi_N^n)\\
&= c^2(\delta_{t\cdot}\Psi_N^n)S_{N-1/2}\left(2-\epsilon_{\mathrm{r}}\right)(\delta_{x\cdot}\Psi_N^n)
\end{aligned}\tag{117}$$

The same can be done for $\mathfrak{b}_{\mathrm{l}}$ with $\epsilon_{\mathrm{l}} = S_{1/2}/\mu_{xx}S_0$ to get

$$\mathfrak{b}_{\mathrm{l}} = c^2(\delta_{t\cdot}\Psi_0^n)S_{1/2}\left(2-\epsilon_{\mathrm{l}}\right)(\delta_{x\cdot}\Psi_0^n)\tag{118}$$

E Derivatives in the frequency domain