Elasto-Plastic Calculations

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1 Introduction

Like in the dafx paper, you can make comments like this! Also you Stefania :)

2 Continuous-time

The PDE for the bowed stiff string looks like:

$$u_{tt} = c^2 u_{xx} - \kappa^2 u_{xxxx} - 2\sigma_0 u_t + 2\sigma_1 u_{txx} - J(x_\mathrm{B}) f(v,z)/\rho A \tag{1}$$

where

$$f(v,z) = s_0 z + s_1 \dot{z} + s_2 v + s_3 w \tag{2}$$

and

$$\dot{z}(v,z) = v\left[1 - \alpha(v,z)\frac{z}{z_\mathrm{ss}(v)}\right] \tag{3}$$

with adhesion map

$$\alpha(v,z)\begin{cases} \left.\begin{matrix} 0 & |z| \le z_\mathrm{ba} \\ \alpha_\mathrm{m}(v,z) & z_\mathrm{ba} < |z| < z_\mathrm{ss}(v) \\ 1 & |z| \ge z_\mathrm{ss}(v) \end{matrix}\right\} \text{if } \operatorname{sgn}(v) = \operatorname{sgn}(z) \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } \operatorname{sgn}(v) \neq \operatorname{sgn}(z), \end{cases} \tag{4}$$

the transition between the elastic and plastic behaviour

$$\alpha_\mathrm{m} = \frac{1}{2}\left[1 + \sin\big(\operatorname{sgn}(z)\Phi\big)\right] \quad \text{where} \quad \Phi = \pi\frac{z - \operatorname{sgn}(z)\frac{1}{2}(|z_\mathrm{ss}(v)| + z_\mathrm{ba})}{|z_\mathrm{ss}(v)| - z_\mathrm{ba}}, \tag{5}$$

and the steady-state function

$$z_\mathrm{ss} = \frac{\operatorname{sgn}(v)}{s_0}\left[f_\mathrm{C} + (f_\mathrm{S} - f_\mathrm{C})e^{-(v/v_\mathrm{S})^2}\right]. \tag{6}$$

3 Discrete-time

The FDS for the bowed string looks like:

$$\delta_{tt} u_l^n = c^2\delta_{xx}u_l^n - \kappa^2\delta_{xxxx}u_l^n - 2\sigma_0\delta_{t\cdot}u_l^n + 2\sigma_1\delta_{t-}\delta_{xx}u_l^n - J(x_\mathrm{B}^n)f(v^n,z^n)/\rho A, \tag{7}$$

where $x_\mathrm{B} \in [0,\ldots,L]$. If we use a truncating function rather than an interpolator and using the following notation

$$I_0(x_\mathrm{B}^n)u_l^n \Rightarrow u_{x_\mathrm{B}}^n \tag{8}$$

the following description for the relative velocity at the bowing point x_{B} can be defined

$$v^n = \delta_{t\cdot} u^n_{x_{\text{B}}} - v^n_{\text{B}} \quad \Rightarrow \quad \delta_{t\cdot} u^n_{x_{\text{B}}} = v^n + v^n_{\text{B}}. \tag{9}$$

Then, equation (7) can be rewritten at the bowing point $l = x_{\text{B}}$:

$$\delta_{tt} u^n_{x_{\text{B}}} = c^2 \delta_{xx} u^n_{x_{\text{B}}} - \kappa^2 \delta_{xxxx} u^n_{x_{\text{B}}} - 2\sigma_0 (v^n + v^n_{\text{B}}) + 2\sigma_1 \delta_{t-}\delta_{xx} u^n_{x_{\text{B}}} - f(v^n, z^n)/\rho A h, \tag{10}$$

where we need to iteratively solve for two unknown variables: the relative velocity between the bow and the string v and the mean bristle displacement z of the bow. The excitation (or bowing) function is defined as

$$f(v^n, z^n) = s_0 z^n + s_1 \dot{z}^n + s_2 v^n + s_3 w^n \ . \tag{11}$$

We can use $\alpha^n = \alpha(v^n, z^n)$ and $z^n_{\text{ss}} = z_{\text{ss}}(v^n)$ as the discrete counterparts of (4) and (6) respectively, to define the discrete counterpart of $\dot{z}$ in (3)

$$\dot{z}^n = \dot{z}(v^n, z^n) = v^n \left[1 - \alpha^n \frac{z^n}{z^n_{\text{ss}}} \right] . \tag{12}$$

At the bowing position we can rewrite (10) using the following identities

$$\delta_{tt} u^n_l = \frac{2}{k}\left(\delta_{t\cdot} u^n_l - \delta_{t-} u^n_l\right) \quad \text{and} \quad \delta_{t\cdot} u^n_l = v^n + v^n_{\text{B}}, \tag{13}$$

resulting in

$$\frac{2}{k} v^n + \frac{2}{k} v^n_{\text{B}} - \frac{2}{k}\delta_{t-} u^n_l = c^2 \delta_{xx} u^n_l - \kappa^2 \delta_{xxxx} u^n_l - 2\sigma_0 v^n - 2\sigma_0 v^n_{\text{B}} + 2\sigma_1 \delta_{t-}\delta_{xx} u^n_l - f(v^n, z^n)/\rho A h. \tag{14}$$

This can be rewritten to

$$g_1(v^n, z^n) = \left(\frac{2}{k} + 2\sigma_0\right) v^n + \frac{s_0 z^n + s_1 \dot{z}^n + s_2 v^n + s_3 w^n}{\rho A h} + b^n = 0 \tag{15}$$

where

$$b^n = \frac{2}{k} v^n_{\text{B}} - \frac{2}{k}\delta_{t-} u^n_l - c^2 \delta_{xx} u^n_l + \kappa^2 \delta_{xxxx} u^n_l + 2\sigma_0 v^n_{\text{B}} - 2\sigma_1 \delta_{t-}\delta_{xx} u^n_l \tag{16}$$

can be pre-computed. Using a^n in (18), a second function can be defined

$$g_2(v^n, z^n) = \dot{z}^n - a^n = 0, \tag{17}$$

where

$$a^n = (\mu_{t-})^{-1} \delta_{t-} z^n = \frac{2}{k}(z^n - z^{n-1}) - a^{n-1} \tag{18}$$

Changed the forward to backward operators. Still a bilinear transform right? describes the bilinear transform of z. This function, together with (15) can be used for multivariate newton raphson:

$$\begin{bmatrix} v^n_{(i+1)} \\ z^n_{(i+1)} \end{bmatrix} = \begin{bmatrix} v^n_{(i)} \\ z^n_{(i)} \end{bmatrix} - \begin{bmatrix} \frac{\partial g_1}{\partial v} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial v} & \frac{\partial g_2}{\partial z} \end{bmatrix}^{-1} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{19}$$

where the derivatives are defined as follows (see Section 3.1 for derivations of some of these):

$$\frac{\partial g_1}{\partial v} = \frac{2}{k} + 2\sigma_0 + \frac{s_2}{\rho A h} \tag{20}$$

$$\frac{\partial g_1}{\partial z} = \frac{s_0}{\rho A h} + \frac{2 s_1}{k \rho A h} \tag{21}$$

$$\frac{\partial g_2}{\partial v} = 1 - z^n \left(\frac{(\alpha^n + \frac{\partial \alpha^n}{\partial v} v^n) z^n_{\text{ss}} - \frac{\partial z^n_{\text{ss}}}{\partial v} \alpha^n v^n}{(z^n_{\text{ss}})^2} \right) \tag{22}$$

$$\frac{\partial g_2}{\partial z} = -\frac{v^n}{z^n_{\text{ss}}} \left(\frac{\partial \alpha^n}{\partial z} z^n + \alpha^n \right) - \frac{2}{k} \tag{23}$$

with

$$\frac{\partial \alpha^n}{\partial v} = \frac{\partial |z_{\mathrm{ss}}^n|}{\partial v} \frac{z_{\mathrm{ba}} - |z^n|}{(|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}})^2} \frac{\pi}{2} \cos\big(\operatorname{sgn}(z^n)\Phi\big) \tag{24}$$

$$\frac{\partial \alpha^n}{\partial z} = \frac{\operatorname{sgn}(z^n)\pi \cos\big(\operatorname{sgn}(z^n)\Phi\big)}{2(|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}})} \tag{25}$$

$$\frac{\partial z_{\mathrm{ss}}^n}{\partial v} = -\frac{2v^n \operatorname{sgn}(v^n)}{v_{\mathrm{S}}^2 s_0}(f_{\mathrm{S}} - f_{\mathrm{C}})e^{-(v^n/v_{\mathrm{S}})^2} \tag{26}$$

3.1 Derivations

Derivative of α with respect to v: $\frac{\partial \alpha^n}{\partial v}$

$$\begin{aligned}
\frac{\partial \alpha^n}{\partial v} &= \frac{\partial}{\partial v}\left(\frac{1}{2}\big[1 + \sin(\operatorname{sgn}(z^n)\Phi)\big]\right) \\
&= \frac{\partial}{\partial v}\Big(\operatorname{sgn}(z^n)\Phi\Big)\frac{1}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big) \\
&= \frac{\partial}{\partial v}\left(\frac{\operatorname{sgn}(z^n)\pi\big(z^n - \operatorname{sgn}(z^n)\frac{1}{2}(|z_{\mathrm{ss}}^n| + z_{\mathrm{ba}})\big)}{|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}}}\right)\frac{1}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big)
\end{aligned}$$

Quotient rule: $\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$

$$\begin{aligned}
&= \frac{-\frac{\pi}{2}\frac{\partial |z_{\mathrm{ss}}^n|}{\partial v}(|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}}) - \frac{\partial |z_{\mathrm{ss}}^n|}{\partial v}\Big(\operatorname{sgn}(z^n)\pi\big[z^n - \operatorname{sgn}(z^n)\frac{1}{2}(|z_{\mathrm{ss}}^n| + z_{\mathrm{ba}})\big]\Big)}{(|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}})^2}\frac{1}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big) \\
&= \frac{\partial |z_{\mathrm{ss}}^n|}{\partial v}\frac{-\frac{\pi}{2}(|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}}) + \frac{\pi}{2}(|z_{\mathrm{ss}}^n| + z_{\mathrm{ba}}) - \operatorname{sgn}(z^n)\pi z^n}{(|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}})^2}\frac{1}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big) \\
&= \frac{\partial |z_{\mathrm{ss}}^n|}{\partial v}\frac{z_{\mathrm{ba}} - |z^n|}{(|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}})^2}\frac{\pi}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big)
\end{aligned}$$

Derivative of absolute value: $\boxed{|f|' = \frac{f}{|f|}f' = \operatorname{sgn}(f)f'}$

$$= \operatorname{sgn}(z_{\mathrm{ss}})\frac{\partial z_{\mathrm{ss}}^n}{\partial v}\frac{z_{\mathrm{ba}} - |z^n|}{(|z_{\mathrm{ss}}^n| - z_{\mathrm{ba}})^2}\frac{\pi}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big) \tag{27}$$

Derivative of α with respect to z: $\frac{\partial \alpha^n}{\partial z}$

$$
\begin{aligned}
\frac{\partial \alpha^n}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{1}{2}\big[1+\sin(\operatorname{sgn}(z^n)\Phi)\big]\right) \\
&= \frac{\partial}{\partial z}\Big(\operatorname{sgn}(z^n)\Phi\Big)\frac{1}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big) \\
&= \frac{\partial}{\partial z}\left(\frac{\operatorname{sgn}(z^n)\pi\big(z^n-\operatorname{sgn}(z^n)\frac{1}{2}(|z^n_{\mathrm{ss}}|+z_{\mathrm{ba}})\big)}{|z^n_{\mathrm{ss}}|-z_{\mathrm{ba}}}\right)\frac{1}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big) \\
&= \frac{\operatorname{sgn}(z^n)\pi}{|z^n_{\mathrm{ss}}|-z_{\mathrm{ba}}}\frac{1}{2}\cos\big(\operatorname{sgn}(z^n)\Phi\big) \\
&= \frac{\operatorname{sgn}(z^n)\pi\cos\big(\operatorname{sgn}(z^n)\Phi\big)}{2(|z^n_{\mathrm{ss}}|-z_{\mathrm{ba}})}
\end{aligned} \tag{28}
$$

Derivative of z_{ss} with respect to v

$$
\begin{aligned}
\frac{\partial z^n_{\mathrm{ss}}}{\partial v} &= \frac{\partial}{\partial v}\left(\frac{\operatorname{sgn}(z^n)}{s_0}\Big(f_{\mathrm{C}}+(f_{\mathrm{S}}-f_{\mathrm{C}})e^{-(v^n/v_{\mathrm{S}})^2}\Big)\right) \\
&= \frac{\partial}{\partial v}\big(-(v^n/v_{\mathrm{S}})^2\big)\frac{\operatorname{sgn}(z^n)}{s_0}(f_{\mathrm{S}}-f_{\mathrm{C}})e^{-(v^n/v_{\mathrm{S}})^2} \\
&= \frac{-2v^n\operatorname{sgn}(z^n)}{v_{\mathrm{S}}^2 s_0}(f_{\mathrm{S}}-f_{\mathrm{C}})e^{-(v/v_{\mathrm{S}})^2}
\end{aligned} \tag{29}
$$

Derivative of g_2 with respect to v

$$
\begin{aligned}
\frac{\partial g_2}{\partial v} &= \frac{\partial}{\partial v}\left(v^n\left[1-\alpha^n\frac{z^n}{z^n_{\mathrm{ss}}}\right]-a^n\right) \\
&= 1-\frac{\partial}{\partial v}\left(\frac{\alpha^n v^n z^n}{z^n_{\mathrm{ss}}}\right)
\end{aligned}
$$

$$
\text{Quotient rule: } \boxed{\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}}
$$

$$
= 1-\frac{\frac{\partial}{\partial v}(\alpha^n v^n z^n)z^n_{\mathrm{ss}}-\frac{\partial z^n_{\mathrm{ss}}}{\partial v}\alpha^n v^n z^n}{(z^n_{\mathrm{ss}})^2} \tag{30}
$$

$$
\text{Product rule: } \boxed{\big(f\cdot g\big)' = f'g+fg'}
$$

$$
\begin{aligned}
&= 1-\left(\frac{(\alpha^n z^n+\frac{\partial \alpha^n}{\partial v}v^n z^n)z^n_{\mathrm{ss}}-\frac{\partial z^n_{\mathrm{ss}}}{\partial v}\alpha^n v^n z^n}{(z^n_{\mathrm{ss}})^2}\right) \\
&= 1-z^n\left(\frac{(\alpha^n+\frac{\partial \alpha^n}{\partial v}v^n)z^n_{\mathrm{ss}}-\frac{\partial z^n_{\mathrm{ss}}}{\partial v}\alpha^n v^n}{(z^n_{\mathrm{ss}})^2}\right)
\end{aligned}
$$

3.1.1 Derivative of g_2 with respect to z

$$
\begin{aligned}
\frac{\partial g_2}{\partial z} &= \frac{\partial}{\partial z}\left(v^n\left[1-\alpha^n\frac{z^n}{z_{\text{ss}}^n}\right]-a^n\right)\\
&= \frac{\partial}{\partial z}\left(-\frac{\alpha^n v^n z^n}{z_{\text{ss}}^n}\right)-\frac{2}{k}
\end{aligned}
\tag{31}
$$

$$
\text{Product rule: } \boxed{\left(f\cdot g\right)' = f'g + fg'}
$$

$$
= -\frac{v^n}{z_{\text{ss}}^n}\left(\frac{\partial \alpha^n}{\partial z}z^n+\alpha^n\right)-\frac{2}{k}
$$

4 Order of calculation

Essentially, the order of calculation happens from top to bottom in section 2

for t = *1:lengthSound* **do**
- calculate computable part b^n
- $i = 0$
- **while** $\epsilon <$ *tol* **do**
  - calculate..
  - 1. $z_{\text{ss}}(v^n_{(i)})$
  - 2. $\alpha(v^n_{(i)}, z^n_{(i)})$
  - 3. a^n
  - 4. $\dot{z}(v^n_{(i)}, z^n_{(i)})$
  - 5. g_1, g_2
  - 6. Calculate ϵ: $\epsilon = \left|\dot{z}(v^n_{(i)}, z^n_{(i)}) - \dot{z}(v^n_{(i-1)}, z^n_{(i-1)})\right|$
  - 7.–11. Compute derivatives of 1.–5. in the same order.
  - 12. Perform vector NR to obtain $v_{(i+1)}$ and $z_{(i+1)}$
  - 13. Increment i: $i = i + 1$
- **end**
- Calculate $\mathbf{u}^{n+1}$ using the values for v^n and z^n from the NR.

end

Algorithm 1: Pseudocode showing the current order of calculation.

5 Notes from before

Newton's method (or Newton-Raphson) is defined as

$$
x^{i+1} = x^i - \frac{g(x^i)}{g'(x^i)} \tag{32}
$$

where $g(x)$ is an arbitrary function dependent on to-be-calculated variable x. As we need to find the roots of (3), g with $x = \dot{z}$ is defined as

$$
g(\dot{z}) = v\left[1-\alpha(v,z)\frac{z}{z_{\text{ss}}(v)}\right]-\dot{z}(v,z) = 0, \tag{33}
$$

and

$$\dot{z}^{i+1} = \dot{z}^i - \frac{g(\dot{z}^i)}{g'(\dot{z}^i)} \tag{34}$$

where

$$g'(\dot{z}^i) = -0.2295\frac{dg(\dot{z}^i)}{dv} + \frac{k}{2}\frac{dg(\dot{z}^i)}{dz} - 1. \tag{35}$$

In the iteration, we use the newly calculated value for $\dot{z}$ and the value of z at the previous time step to calculate an estimate of z using the trapezoidal rule

$$z^n = z^{n-1} + \frac{k}{2}(\dot{z}^{i+1} + \dot{z}^{n-1}). \tag{36}$$

Inserting this into (15) we can caluclate v using

$$v = \frac{-s_0 z - s_1 \dot{z} - b}{s_2 + \frac{2}{k} + 2\sigma_0}. \tag{37}$$