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# The Emulated Ensemble

Real-Time Simulation of Musical Instruments using  
Finite-Difference Time-Domain Methods

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Ph.D. Dissertation  
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# Curriculum Vitae

Silvin Willemsen



Here is the CV text.

**Curriculum Vitae**

# Abstract

Digital musical instruments exist in large quantities and numerous strategies to virtualise traditional instruments have been developed. Although one could create digital musical instruments using pre-recorded samples of their real-life counterparts, the playability and interaction of the instruments will not be captured. Instead, a simulation of the underlying physics of the instrument could be created, and is much more flexible to player interaction. This *physical model* will allow a musician to be much more expressive when playing the digital instrument than if static samples were to be used. Using ad hoc hardware to control the simulation could potentially make the simulated instrument feel identical to the original.

Applications of physical modelling for musical instruments include simulating instruments that are unplayable as they are too rare or vulnerable. A model of the underlying physics of the instrument could potentially resurrect the instrument making it available to the public again. Furthermore, as a simulation is not restricted by the laws of physics, one could extend the possibilities of the original instrument. Properties such as the material or geometry of an instrument could be dynamically changed which broadens the range of expression of the musician. One could even imagine physically impossible musical instruments which still exhibit a natural sound due to the underlying models.

In this project, finite-difference time-domain FDTD methods have been chosen, as they have an advantage in terms of generality and flexibility regarding the systems they can model. A drawback of these methods is that they are quite computationally expensive, and although many highly accurate models based on these methods have existed for years, the computing power to run them in real time has only recently become available. The main challenge is thus to run the simulations in real time to allow for proper player interaction.

This work presents the development and real-time implementation of various physical models of traditional musical instruments based on FDTD methods. These instruments include the trombone, the violin and more obscure instruments such as the hurdy gurdy and the tromba marina. Furthermore, a novel method is presented that paves the way for dynamic parameters in

## Abstract

FDTD-based musical instrument simulations allowing for physically impossible instrument manipulations. Finally, this work doubles as an aid for beginners in the field of musical instrument simulations based on FDTD methods, and aims to provide a low-entry-level explanation of the literature and theory that the physical models are based on.

# Resumé

Der findes et enormt antalt digitale instrumenter og der findes adskillige strategier til virtualisering af traditionelle instrumenter. Selvom man kunne skabe digitale musikinstrumenter ved hjælp af lydoptagelser af deres virkelige modstykke, ville instrumenternes spilbarhed og interaktion ikke blive fanget i processen. I stedet kunne man implementere en simulering af instrumentets underliggende fysik, hvilken ville være mere fleksibel ift. spillerinteraktion. Denne fysiske model ville gøre det muligt for en musiker at være mere udtryksfuld når han eller hun spiller det digitale instrument, end hvis der bruges statiske lydoptagelser. Brug af ad hoc hardware til at styre simuleringen kunne antageligt få det simulerede instrument til at føles identisk med originalen. Anvendelser af fysisk modellering af musikinstrumenter inkluderer at simulere instrumenter der ikke kan spilles, da de er for sjeldne eller sårbare. En model af instrumentets underliggende fysik kunne potentielt genoplive instrumentet og gøre det tilgængeligt for offentligheden igen. Ydermere kunne man forbedre det originale instrument, eftersom en simulering ikke er begrænset af fysikkens love. Egenskaber som instruments materiale eller geometri kunne ændres dynamisk og udvide musikerens udtryksmuligheder. Man kunne endda forestille sig fysisk umulige musikinstrumenter, der stadig udviser en naturlig lyd på grund af de underliggende modeller. Der findes mange fysiske modelleringsteknikker, hvor FDTD (finite-difference time-domain) metoder har en fordel med hensyn til generalitet og fleksibilitet, samt til de systemer de kan modellere. En ulempe ved disse metoder er at de er beregningstunge, og selvom der har eksisteret nøjagtige modeller baseret på disse metoder i årevis, er regnekraften til at køre dem i realtid først blevet tilgængelig for nylig. Den største udfordring er således at køre simuleringerne i realtid, sådan at man kan opnå en livagtig interaktion for udøveren. Dette afhandling gennemgår udviklingen og realtidsimplementeringen af forskellige fysiske modeller af traditionelle musikinstrumenter baseret på FDTD metoder. Disse instrumenter inkluderer trombone, violin og mindre alment kendte instrumenter såsom drejelire og tromba marina. Desuden præsenteres en ny metode, der baner vejen for dynamiske parametre i FDTD-baserede musikinstrumentsimuleringer, der muliggør instrumentmanipulationer ellers umulige i den virkelige verden.

## Resumé

Endelig fungerer denne afhandling som et hjælpemiddel til begyndere inden for simuleringer af musikinstrumenter baseret på FDTD-metoder, og sigter mod at give en letfordøjelig forklaring af den litteratur og teori, som de fysiske modeller er baseret på.

# Preface

Starting this Ph.D. project, I did not have a background in mathematics, physics or computer science, which were three equally crucial components in creating the result of this project. After the initial steep learning curve of notation and terminology, I was surprised to find that the methods used for physical modelling are actually quite straightforward!

Of course it should take a bit of time to learn these things, but

Many concepts that seemed impossible at the beginning

I feel that the literature lacks a lot of the intuition needed for readers without a background in any of these topics. Rather, much of the literature I came across assumes that the reader has a degree in at least one of the aforementioned topics. Stefan Bilbao's seminal work *Numerical Sound Synthesis*, which is the most complete work to date describing how to physically model musical instruments using finite-difference time-domain methods says that "A strong background in digital signal processing, physics, and computer programming is essential." Even though some basic calculus knowledge is assumed to understand the concepts used in this work, a degree in any of the aforementioned topics is (hopefully) unnecessary. **Furthermore, some experience with MATLAB and C++ is assumed for the code examples**

Some working titles: *Physical Modelling for Dummies* *Physical Modelling for the faint-hearted*

Also, I came across a lot of "it can be shown that's without derivations. This is why I decided to write this work a bit more pedagogical, and perhaps more elaborate than what could be expected.

I believe that anyone with some basic skills in mathematics and programming is able to create a simulation based on physics within a short amount of time, given the right tools, which I hope that this dissertation could be.

The knowledge dissemination of this dissertation is thus not only limited to the research done and publications made over the course of the project, but also its pedagogical nature hopefully allowing future (or past) students to benefit from.

As with a musical instrument itself, a low entry level, a gentle learning curve along with a high virtuosity level is desired. Take a piano, for instance.

Most will be able to learn a simple melody — such as “Frère Jacques” — in minutes, but to become virtuous requires years of practice.

This is the way I wanted to write this dissertation: easy to understand the basic concepts, but many different aspects touched upon to allow for virtuosity. Hopefully by the end, the reader will at least grasp some highly complex concepts in the fields of mathematics, physics and computer science (which will hopefully take less time than it takes to become virtuous at playing the piano).

As Smith states in his work *Physical Audio Signal Processing* [1] “All we need is Newton”, and indeed, all Newton’s laws of motion will make their appearance in this document.

I wanted to show my learning process and (hopefully) explain topics such as *Energy Analysis*, *Stability Analysis*, etc. in a way that others lacking the same knowledge will be able to understand.

Make physical modelling more accessible to the non-physicist. Also supports reproducibility of science and lowers the entry level

*Interested in physically impossible manipulations of now-virtual instruments.*

**Could be fun to include this :)** : “I’m a musician. Will I be out of a job if you keep making physical models?” Physical modelling is not here to replace the original instruments and the musicians playing them. Instead, it can be used as a tool to understand the physics of existing instruments and possibly go beyond. Simulated instruments are not restricted by physics anymore and could provide new ways of expression for the musician.

## Acknowledgements

I would like to thank my mom..

# List of Publications

Listed below are the publications made during the PhD project, (co)authored by the PhD student. These are grouped by: the main publications, which are also included in Part V, papers with a supervisory role, and finally, other publications from various collaborative efforts.

## Main Publications

- [A] S. Willemesen, N. Andersson, S. Serafin, and S. Bilbao, “Real-time control of large-scale modular physical models using the sensel morph,” in *Proceedings of the 16th Sound and Music Computing (SMC) Conference*, 2019, pp. 275–280.
- [B] S. Willemesen, S. Bilbao, N. Andersson, and S. Serafin, “Physical models and real-time control with the sensel morph,” in *Proceedings of the 16th Sound and Music Computing (SMC) Conference*, 2019, pp. 95–96.
- [C] S. Willemesen, S. Bilbao, and S. Serafin, “Real-time implementation of an elasto-plastic friction model applied to stiff strings using finite difference schemes,” in *Proceedings of the 22nd International Conference on Digital Audio Effects (DAFx-19)*, 2019, pp. 40–46.
- [D] S. Willemesen, S. Serafin, S. Bilbao, and M. Duccheschi, “Real-time implementation of a physical model of the tromba marina,” in *Proceedings of the 17th Sound and Music Computing (SMC) Conference*, 2020, pp. 161–168.
- [E] S. Willemesen, R. Paisa, and S. Serafin, “Resurrecting the tromba marina: A bowed virtual reality instrument using haptic feedback and accurate physical modelling,” in *Proceedings of the 17th Sound and Music Computing (SMC) Conference*, 2020, pp. 300–307.
- [F] S. Willemesen, A.-S. Horvath, and M. Nascimben, “Digidrum: A haptic-based virtual reality musical instrument and a case study,” in *Proceedings of the 17th Sound and Music Computing (SMC) Conference*, 2020, pp. 292–299.

## Other Publications

- [G] S. Willemsen, S. Bilbao, M. Ducceschi, and S. Serafin, "Dynamic grids for finite-difference schemes in musical instrument simulations," in *Proceedings of the 23rd International Conference on Digital Audio Effects (DAFx2020in21)*, 2021.
- [H] S. Willemsen, S. Bilbao, M. Ducceschi, and S. Serafin, "A physical model of the trombone using dynamic grids for finite-difference schemes," in *Proceedings of the 23rd International Conference on Digital Audio Effects (DAFx2020in21)*, 2021.

## Publications with a Supervisory Role

- [S1] R. S. Alecu, S. Serafin, S. Willemsen, E. Parravicini, and S. Lucato, "Embouchure interaction model for brass instruments," in *Proceedings of the 17th Sound and Music Computing (SMC) Conference*, 2020, pp. 153–160.
- [S2] T. Lasickas, S. Willemsen, and S. Serafin, "Real-time implementation of the shamisen using finite difference schemes," in *Proceedings of the 18th Sound and Music Computing (SMC) Conference*, 2021, pp. 100–107.
- [S3] M. G. Onofrei, S. Willemsen, and S. Serafin, "Implementing physical models in real-time using partitioned convolution: An adjustable spring reverb," in *Proceedings of the 18th Sound and Music Computing (SMC) Conference*, 2021, pp. 108–114.
- [S4] M. G. Onofrei, S. Willemsen, and S. Serafin, "Real-time implementation of a friction drum inspired instrument using finite difference schemes," in *Proceedings of the 23rd International Conference on Digital Audio Effects (DAFx2020in21)*, 2021.

## Other Publications

- [O1] J. M. Hjerrild, S. Willemsen, and M. G. Christensen, "Physical models for fast estimation of guitar string, fret and plucking position," *IEEE Workshop on Applications of Signal Processing to Audio and Acoustics (WASPAA)*, pp. 155–159, 2019.
- [O2] K. Prawda, S. Willemsen, S. Serafin, and V. Välimäki, "Flexible real-time reverberation synthesis with accurate parameter control," in *Proceedings of the 23rd International Conference on Digital Audio Effects (DAFx-20)*, 2020.
- [O3] M. Ducceschi, S. Bilbao, S. Willemsen, and S. Serafin, "Linearly-implicit schemes for collisions in musical acoustics based on energy quadratisation," *Journal of the Acoustical Society of America (JASA)*, vol. 149, pp. 3502–3516, 2021.

# List of Symbols

The list of symbols found below contains often-used symbols in the thesis in the context that they are normally used. Depending on the context they might carry a different meaning ( $y$  being displacement of the lip-reed in Chapter ?? but the vertical spatial coordinate for 2D systems in fx. Chapter 6). Some might also be accompanied by a subscript in the main document

Symbol	Description	Unit
$\alpha$	Fractional part of	
$A$	Cross-sectional area of string	$\text{m}^2$
$c$	Wave speed	$\text{m}/\text{s}$
$\frac{d^n}{dt^n}$	$n^{\text{th}}$ order derivative with respect to $t$	-
$\partial_t^n$	$n^{\text{th}}$ order partial derivative with respect to $t$	-
$\delta_{t+}, \delta_{t-}, \delta_t.$	Forward, backward and centred difference in time operator	-
$\delta_{x+}, \delta_{x-}, \delta_x.$	Forward, backward and centred difference in space operator	-
$\delta_{tt}$	Second order difference in time operator	-
$\delta_{xx}$	Second order difference in space operator	-
$\delta_{xxxx}$	Fourth order difference in space operator	-
$\mu_{t+}, \mu_{t-}, \mu_t.$	Forward, backward and centred average in time operator	-
$\mu_{x+}, \mu_{x-}, \mu_x.$	Forward, backward and centred average in space operator	-
$\mu_{tt}$	Second order average in time operator	-
$\mu_{xx}$	Second order average in space operator	-
$E$	Young's Modulus	$\text{Pa} (\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-2})$
$f$	Force	N
$f$	Frequency	Hz

## Other Publications

Symbol	Description	Unit
$f_s$	Sample rate	Hz
$F$	Scaled force	depends on system
$h$	Grid spacing	m
$H$	Membrane / Plate thickness	m
$I$	Area moment of inertia	$m^4$
$l$	Spatial index to grid function	-
$L$	Length	m
$k$	Time step ( $= 1/f_s$ )	s
$K$	Spring coefficient	N/m
$\kappa$	Stiffness coefficient	$m^2/s$ (1D) $m^4 \cdot s^{-2}$ (2D)
$n$	Sample index to grid function	-
$N$	Number of points string	-
$\mathbb{N}^0$	Set of non-negative integers	-
$p$	Pressure	Pa
$r$	Radius	m
$S$	Cross-sectional area (brass)	$m^2$
$t$	Time	s
$T$	Tension	N (1D) N/m (2D)
$u$	State variable	m
$v$	Particle velocity	m/s
$x$	Spatial dimension (horizontal for 2D systems)	m
$y$	Vertical spatial dimension	m
$\gamma$	Scaled wave speed	$s^{-1}$
$\lambda$	Courant number for 1D wave eq. ( $= ck/h$ )	-
$\mu$	Stiffness free parameter	-
$\nu$	Poisson's ratio	-
$\eta$	Relative displacement spring	m
$\rho$	Material density	$kg \cdot m^{-3}$

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## Operations

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$\Im(\cdot)$	Imaginary part of
$\Re(\cdot)$	Real part of

## Other Publications

Symbol	Description	Unit
$\lfloor \cdot \rfloor$	Flooring operation	
$\lceil \cdot \rceil$	Ceiling operation	
<hr/>		
<b>Subscripts</b>		
c	Connection	
p	Plate	
s	String	

## Other Publications

# List of Abbreviations

Abbreviation	Definition
1D	one-dimensional or one dimension
2D	two-dimensional or two dimensions
3D	three-dimensional
DoF	Degrees of freedom
DSP	Digital signal processing
Eq.	Equation
Eqs.	Equations
FD	Finite-difference
FDTD	Finite-difference time-domain
LTI	Linear time-invariant
ODE	Ordinary differential equation
PDE	Partial differential equation
VR	Virtual reality

## Other Publications

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# **Part I**

# **Introduction**



# Chapter 1

## Physical Modelling of Musical Instruments

At the time of writing, an uncountable number of digital musical instruments exists. This range encompasses both digital keyboards that can create sounds from various real (and non-real) instruments, as well as digital instrument plug-ins used by music producers in Digital Audio Workstations (DAWs). Many of these digital instruments are based on samples, or recordings, of their real-life counterparts, while others use computationally cheap methods to generate sounds, some inspired by physical musical instruments. The earliest sound synthesis techniques date back to the late 1950s where Max Mathews proposed a technique called wavetable synthesis [2]. Not long after, in the 1960s, efficient sinusoidal-based and filter-based sound synthesis techniques such as additive synthesis, subtractive synthesis, and FM (frequency modulation) synthesis were invented [3, 4]. The latter became widely popular through the Yamaha DX7 synthesiser created in 1983 that synthesised sounds based solely on this technique [5]. Through a simple change of variables, FM synthesis can generate sounds ranging from brass instruments to drums.

These techniques are referred to as *spectral modelling* methods, where the manipulation of sinusoids or filtering noise would produce (harmonic) sounds, that could be perceived by the listener as originating from a physical instrument. This top-down approach which starts at the perception of the listener, has advantages in terms of computational efficiency, but is quite limited by the systems it can model [6].

As computing power increased over the last few decades, using *physical models* rather than samples or spectral modelling methods gained an increased popularity. Physical modelling, in the context of sound and music, is a way to generate sound based on physical processes, including string vibrations in a guitar, air propagation in a trumpet, or even the reflections in a concert

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hall. When compared to spectral modelling, this is a bottom-up approach that attempts to model the sound from the source.

This work focuses on simulating<sup>1</sup> traditional musical instruments using physical modelling. The interest in physically modelling traditional musical instruments is twofold: 1) sound generation, and 2) understanding of the underlying physical processes. The main focus of this PhD project has been the former. One of the reasons why one would use physical models rather than samples of the real instrument, is that a model is much more flexible to player control. Consider the violin as an example, where the performer controls the bow force, velocity and position along the string, as well as the finger determining the pitch of the string. A physical model can generate the sound in real time based on these performance parameters. If samples were to be used, every single combination of these parameters would need to be recorded in order to capture the entire instrument.<sup>2</sup> A more in-depth reasoning behind using physical models for sound generation will be given in Section 1.4.

This chapter continues by giving a brief overview of the history of physical modelling for sound synthesis.

## 1.1 A brief history

Most likely the very first example of a physically modelled musical sound is the “Bicycle Built for Two” by Kelly, Lochbaum, and Matthews in 1961<sup>3</sup>. It uses what later got known as the Kelly-Lochbaum vocal-tract model to generate a voice and was published the year thereafter [9].

The very first musical instrument simulations were based on discretisation of differential equations using *finite-difference time-domain* (FDTD) methods. These were carried out around 1970 by Hiller and Ruiz [10, 11, 12] and applied to the wave equation to simulate string sounds. The sound generation, however, was far from real-time and it took several minutes to generate only 1 second of sound. In 1983, Cadoz et al. introduced CORDIS, a real-time sound generating system based on *mass-spring networks* [13]. The first physical model of the bowed string was due to McIntyre et al. in their 1983 publication [14]. In the same year Karplus and Strong devised an extremely efficient way to generate a string sound in [15] later known as the Karplus-Strong algorithm. Based on these ideas, Smith coined the term *digital waveguides* around the late 1980s and early 1990s in [16, 17] and continued to develop the method [1]. Around

---

<sup>1</sup>The term *emulated* is only used in the title of this work (because of the alliteration), but is synonymous to *simulated* in this context.

<sup>2</sup>This was actually done in 2019 for four century-old bowed-string instruments in the Italian city of Cremona. The recordings lasted five weeks, six days a week, eight hours a day, and required the city center to be as quiet as possible to record the instruments [7, 8].

<sup>3</sup><http://ccrma.stanford.edu/~jos/wav/daisy-klm.wav>

## 1.2. Exciter-resonator approach

the same time, Adrien in [18] and later Morrison and Adrien in [19] introduced *modal synthesis*, a way to synthesise an object's sound by decomposing it into its modes of vibration.

Although more techniques have been developed in the last 20-30 years, most of the developments in the field of physical modelling for musical instruments are based on those presented in this section. Before moving on to further details about these methods in Section 1.3, a modular approach to subdivide a musical instrument will be presented.

## 1.2 Exciter-resonator approach

Nearly any musical instrument can be subdivided into a resonator component and an exciter component, both of which can be simulated individually. This modular approach to musical instruments was first introduced by Borin, De Poli and Sarti in [20] and later developed by De Poli and Rocchesso in [21] and is used to structure this work. Examples of resonator-exciter combinations are the violin and the bow, or the trumpet and the lips of the player.

A resonator is a passive system, in this project mostly assumed to be linear, and does not emit sound unless triggered by an external source. Exciters can be seen as these external sources, and generally have a nonlinear element.<sup>4</sup> Exciters insert energy into a resonator and cause it to vibrate and emit sound, and the method of excitation greatly influences the sound of the resonator. In the real world, the interaction between the exciter and the resonator is bidirectional. In other words, the exciter not only affects the state of the resonator, but the resonator affects the exciter as well. For the most part, this is also what is attempted to model in this work.

The next section will talk about various techniques that can be used to implement the resonator. Details on excitation modelling are left for Part III.

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## 1.3 Physical modelling techniques

The time-evolution of dynamic systems, including that of musical instruments, can be well described by partial differential equations (PDEs) [22, 23]. Examples of dynamic systems are a guitar string, a drum-membrane, or air propagation in a concert hall; three very different concepts, but all based on the same

<sup>4</sup>The difference between linear and non-linear systems is in their response to input level or amplitude. The behaviour of linear systems does not change with the level of the input. Instead, it only scales (linearly) with the input level, i.e., an input to a linear system with twice the amplitude yields an output of twice the amplitude. The behaviour of nonlinear systems, however, does change depending on the level of the input. Although linear systems are rarely found in the real world, under low amplitude excitations most systems can still be considered linear and their nonlinear effects can be ignored.

types of equations of motion. Many of these equations and other knowledge currently available on the physics of musical instruments have been collected by Fletcher and Rossing in [22]. Though these equations are very powerful, only few have a closed-form solution, and in order for them to be implemented, they need to be approximated. In the past decades, much research has been done on implementing these PDEs to model and simulate different musical instruments. Great overviews of implementation techniques are given by, for example, Vesa Välimäki et al. in [24] and Julius O. Smith in [6, 1].

The most popular physical modelling techniques that are described in this literature can be found below:

*Modal Synthesis* decomposes a system into a series of uncoupled ‘modes of vibration’ and can be seen as a physically-based additive synthesis technique. First used in a musical context by Morrison and Adrien in [19], it is a technique that is still used today due to its computational efficiency, especially when simulating higher-dimensional systems such as (two-dimensional) plates or (three-dimensional) rooms. It is especially effective when used to describe a linear system with a small number of long-resonating modes [25, 6]. When used to describe nonlinear systems, however, the modes become ‘coupled’ and the system will quickly become more computationally expensive. Recent developments using the FAUST programming language allow a 3D-mesh model of any three-dimensional object to directly be decomposed into its modes of vibration and used as a sound-generating physical model [26].

*Finite-Difference Time Domain* methods (FDTD) aim to solve PDEs by approximating them with difference equations, discretising a continuous system into grid-points in space and time. In a musical context, this technique was first used for the case of string vibration by Hiller and Ruiz in [10, 11, 12] and later by Chaigne in [27, 28]. Bilbao extensively describes this method in [23, 25]. Although computationally expensive, especially when working with higher-dimensional systems, this technique could potentially accurately model any system, whether it is linear or nonlinear, time-invariant or time-variant.

*Digital Waveguide Modelling* (or Digital Waveguides (DWGs)) is a technique that discretises wave propagation and scattering. The technique was first presented by Smith in [16, 17], and is mostly used for one-dimensional systems, such as strings and acoustic tubes and decomposes their system into travelling wave components. This technique has also been used in higher-dimensional systems, but is superior in efficiency when used in the one-dimensional case [24]. Some authors have combined DWGs with FDTD methods (such as in [29, 30]) to accurately model nonlinear behaviour while maintaining high-speed implementation.

### 1.3. Physical modelling techniques

*Mass-spring networks* can be similar in nature to FDTD methods, but treat each grid point as an individual mass connected to other masses through springs in a network. Pioneered in a musical context by Cadoz in [31, 13, 32] it is currently being further developed by Leonard and Villeneuve in a real-time, interactive environment [33, 34].

## Discussion

This work focuses on physical modelling using FDTD methods. The main advantage of these methods is that they are extremely general and flexible in terms of the types and number of systems they can model. They allow any set of PDEs to be directly numerically simulated without making any assumptions regarding travelling wave solutions or modes. Moreover, FDTD methods allow for various PDEs, fx. a violin body and four strings, to be connected in a fairly straightforward manner. DWGs, for example, assume a travelling wave solution, which makes complex nonlinear effects extremely hard to model using this technique. To use modal synthesis for modelling a PDE, it requires the system to have closed-form or analytical solution. If this is not available, (finite-element) analysis of the system could be performed to obtain the modal shapes and frequencies of the system. This in itself is very computationally expensive and requires a lot of storage if the modal data needs to be saved.

Maybe add table of pros and cons OR when to use which

The main drawback of FDTD methods is the fact that they require great attention to numerical stability of the solution [23]. For a wrong choice of parameters, the implemented system could become unstable and “explode”<sup>5</sup>. Stability analysis as well as energy analysis techniques are invaluable in the process of ensuring a stable implementation and much attention to this will be given throughout this work.

check all of this

A final drawback of using FDTD methods is that – especially for higher-dimensional systems – they are much more computationally heavy than other methods, such as DWGs or modal synthesis techniques. The bright side – if one believes in Moore’s law [35] – is that it can be assumed that computing power will continue to increase and that within several years, running high-quality simulations of musical instruments based on FDTD methods in real time, will not be an issue.

---

<sup>5</sup>I learned the hard way that one should always implement a limiter when working with real-time physical models to avoid dangerously loud sounds.

## 1.4 Applications of physical modelling

It might not be clear why one would go through all the hassle of modelling musical instruments? Could we not use a recording of the original instrument and play that back at the right moment? Or taking another step back, why not buy a real instrument and learn to play that instead? This section aims to answer those questions, by providing some applications of physical modelling for musical instrument simulations. *Very informal section, but I kinda like it*

:)

Maybe move to  
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### 1.4.1 Samples vs. physical modelling

Despite the existence of many techniques to simulate musical instruments mentioned in the previous section, the bulk of the currently available digital musical instruments are still based on samples. This is mainly due to the computational power needed to generate sounds as opposed to simple playback of recordings. Furthermore, digital musical instruments based on samples, have an optimally realistic sound. As the output of the digitised instrument is exactly that of the original instrument, the digital version should thus sound indistinguishable from the original.

That said, it can be argued that these are the only advantages of using samples over physical models in this context. Samples are in essence static and unable to adapt to changes in performance; the recording is made by one player with one playing style or technique, using one specific microphone to record the sample, etc. Even if one accepts this, capturing the entire interaction space of an instrument is nearly impossible. Imagine recording a violin with every single combination of bowing force, velocity, position, duration and other aspects such as vibrato, pizzicato. Even if a complete sample library could be created, this would contain an immense amount of data and take an incredible amount of time to record.

Using physical models to simulate the musical instrument instead, allows the sound to be generated on the spot based on all the aforementioned interaction parameters. One is not stuck to a single recording of the instrument and, given the right tools or controller, one can alter the sound just as one can with its real-life counterpart.

A drawback of physical models is that, in order to generate a realistic sound, a highly accurate physical description of the original instrument is needed. Apart from (potentially) taking a lot of time to develop this model and tuning its parameters, the eventual implementation will be (much) more computationally expensive than if samples were to be used. Generally, the more accurate the model is, and thus the more true the sound is to the original, the higher the computational cost becomes.

## 1.5. Project objectives and main contributions

The main trade-off between samples and physical models is thus storage versus speed, or hard-disk versus CPU. Whether one method should be used over the other depends on the situation. If efficiency is required and the lack of flexibility in the sound is not an issue, samples might be the better choice. If, on the other hand, one wants to create a full digital version of a traditional instrument that responds to player-interaction in the same way as the original instrument would, a physical model should be chosen instead.

### 1.4.2 Resurrect old or rare instruments

Many instruments exist that are too old, too rare, or too valuable to be played. Some live behind museum glass only to be looked at by visitors, never to be played again. In these cases, it might even be hard to record samples of the musical instrument. If, however, the physics (geometry and material properties) of the instrument are available, a physical model of the instrument could be created, bringing its sound back to life.

However, applications of physical modelling are not limited to old or rare instruments. Popular musical instruments also require maintenance and might need to be replaced after years of usage. A simulation of these instruments will not age, unless that is of course desired and included in the model.

### 1.4.3 Go beyond what is physically possible

As a digital simulation is not restricted by the laws of physics of the real world, this opens up a substantial amount of possibilities. Musical instrument simulations make it possible for parameters like shape, size, material properties, etc. to be dynamically changed, which is physically impossible or very hard to do. A physical model of a violin could potentially change size and ‘morph’ into a cello while the simulation is running and a player is interacting with it. New ways of interaction and expression could be devised that control the physics of the instrument, expanding the range of possibilities for the musician.

Furthermore, different instrument components can be combined to create hybrid instruments. For example, one could bow the air in a trumpet, or lip-excite a string (similar to what Smith states in [6]). This could potentially result in unique sounds that can only be created using physical models.

## 1.5 Project objectives and main contributions

This section presents several research questions and provides the main objectives and contributions of the project.

### **How can computationally expensive physical models be made playable in real-time?**

Even though physical modelling has been a popular research field in the past few decades, relatively little research has been done on making the models work in real-time, i.e., ‘playable’ [36]. Several virtual string instruments and different electric pianos have been made real-time by Pfeifle and Bader in [37, 38, 39]. The authors used field programmable gate arrays (FPGAs) for implementing models based on FDTD methods. Furthermore, Roland’s V-series use COSM (Composite Object Sound Modelling) technology [40] that implement real-time physical models in hardware instruments. In the NESS project, Stefan Bilbao and his team focused on implementing systems using FDTD methods in real-time using parallelisation techniques and the GPU [41, 42].

The biggest challenge in real-time audio applications, as opposed to those only involving graphics for example, is that the sample rate required is extremely high. As Nyquist’s sampling theory states, a sampling rate of at least 40 kHz is necessary to produce frequencies up to the human hearing limit of 20 kHz [43]. Most graphics applications are made with a temporal sample rate (mostly referred to as frames per second (FPS)) of around 60 Hz [44], which is orders of magnitude smaller than the auditory sample rate. For comparison, for a commonly used auditory sample rate of 44100 Hz, running a simulation for audio requires 735x as many iterations as if this simulation was done for graphics only.

The main objective of this work is to implement physical models using FDTD methods in real time without the need of special hardware, i.e., on a regular personal computer or laptop. The objective is not to renew the underlying models themselves, but novel combinations of existing were made to simulate relatively unknown instruments as test cases for this objective. The instruments modelled over the course of this project are the esraj (Bowed Sitar), hammered dulcimer and hurdy gurdy presented in paper [A], the tromba marina presented in paper [D] and the trombone presented in paper [H], all implemented in real time using FDTD methods. An extended summary of these papers can be found in Part ??.

### **How can (the sound of) traditional instruments be extended upon?**

As mentioned in Section 1.4.3, using physical modelling to simulate real-life instruments relieves the physical limitations that the real world imposes on them. As FDTD methods are quite rigid, dynamically changing parameters while the instrument simulation is running, is a challenge. Other techniques, such as modal synthesis, are much more suitable for this, but come with the drawbacks mentioned in Section 1.3. Therefore, one of the main objectives of

## 1.6. Thesis outline

this project was to devise a method to allow parameters in musical instrument simulations based on FDTD methods to be dynamically varied.

Indeed, during this PhD project, a novel method was devised to smoothly change parameters over time, introducing this to FDTD methods. This method was published in [G] and will be elaborated on in Chapter ??.

### How can the now-virtual instruments be controlled in an expressive way?

A great challenge in musical instrument simulations is their control. In many physical instruments, one interacts immediately with the sound-creating object, such as a string on a guitar or a membrane on a drum. This allows the musician to be much more expressive than if they only used the keyboard and mouse. Expressivity, however, is not the only thing that makes an instrument interesting and enjoyable to play. The interaction with a musical instrument simulations could feel very 'dry' or unnatural as there is no haptic feedback; something present in (nearly) all physical musical instruments.

The last objective of this PhD project is thus to find ways to control the instrument simulations in an expressive way. Over the course of this PhD project, the Sensel Morph, or Sensel for short, has been used extensively [45]. The Sensel is a controller containing ca. 20,000 pressure sensors in a small area that allow for highly expressive control of the instruments. This controller has been used in papers [A], [B], [C] and [D].

Although the Sensel allows for more expressive control than a keyboard and mouse, it does not resemble any of the interaction paradigms of the original instruments. It was thus attempted to include a controller that would be more suited for controlling the musical instrument simulation and allow for a more intuitive control. For one of the projects, a virtual-reality implementation of the tromba marina is controlled by the PHANTOM Omni [46] which is a six-degrees-of-freedom haptic device. The controller contains a hand-held pen-like object that is attached to a robotic arm and can be linked to a virtual environment to provide force and vibrotactile feedback through the arm based on this environment. Paper [E] presents this project containing an evaluation and be touched upon in Chapter ??.

## 1.6 Thesis outline

**not done:** Although a collection of papers format has been used for the structure of this thesis, the introductory part to the papers has been written in a monographic style. This hybrid format

maybe chapters in parentheses such that the story is more fluent

This thesis is divided into several parts which in their turn are divided in chapters. See Figure 1.1 for a visual overview of the thesis structure.

## Part I: Introduction

This part introduces the field of physical modelling for musical instruments in Chapter 1, by giving a brief history of the field and providing background for the project. Furthermore, the project objectives and contributions to the field are detailed. Chapter 2 provides a thorough introduction to finite-difference time-domain methods using simple sound-generating systems as examples, after which Chapter 3 introduces several analysis techniques in a tutorial-like fashion.

## Part II: Resonators

The resonator component of a musical instrument, as introduced in Section 1.2, can – for most instruments – be further decomposed into more basic resonators. In order to model the violin, for example, one can decompose the entire resonator into four strings and its body. Chapter 4 presents a model for the stiff string, Chapter 5 introduces acoustic tubes that can be used to model brass instruments and Chapter 6 introduces two-dimensional systems such as membranes and plates which can be used to simulate simplified instrument bodies.

## Part III: Exciters

As stated in Section 1.2, the excitation greatly determines the behaviour of the resonator. This part presents various ways in which the resonators introduced in Part II can be excited. Chapter 7 introduces ‘physically inspired’ excitations, Chapter 8 introduces the bow and presents the contribution made in paper [C], and finally, Chapter 9 presents the lip reed used to excite brass instruments.

## Part IV: Interactions

As mentioned before, most musical instruments consist of many individual resonators, and to properly model these, their interaction must be taken into account. This part describes two different ways that the resonators can interact with each other; connections between various resonators are presented in Chapter 10 and collisions between them in Chapter 11.

The above parts are used as an introduction for the main contributions of the PhD project and – with the exception of Chapter 8 – do not contain any novelty. Much effort has been put in explaining the existing methods and models from the literature used in this project in a way that is slightly more in-depth and pedagogical than might be common for a PhD thesis (specifically Part I). It is the hope of the PhD student that going this extra mile could make this

## 1.6. Thesis outline

work (and the above parts in particular) be a contribution in itself: to put this research field into reach for beginners in the field of physical modelling for sound synthesis using FDTD methods without the need of much experience in the fields of physics, mathematics or computer science.

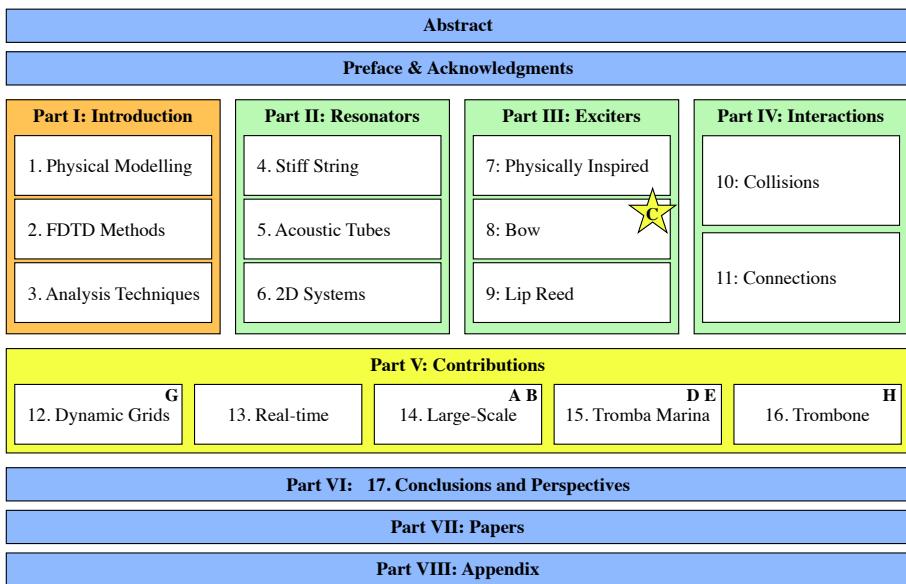
### **Part ??: Contributions**

This part contains extended summaries of the main contributions of the PhD project. Chapter ?? summarises paper [G] and extends it by providing some design considerations. Chapter ?? explains the considerations necessary for real-time implementation of physical models. Chapters ??, ?? and ??, summarise papers [A], [D] and [H] respectively and extend the papers mainly with more implementation details.

### **Part ??: Conclusions and Perspectives**

This part concludes the thesis and puts the contributions into context of the physical modelling field. Future perspectives and possible continuations of this work are given as well.

Finally, **Part V: Papers** contains the main publications made over the course of this PhD project and an appendix appears in **Part VI: Appendix**.



**Fig. 1.1:** The outline of this thesis. The contributions made throughout the PhD project are marked in yellow. Most are collected in Part ??, though the novel work done on the bow will already appear in Chapter 8. Marked in green are the parts that describe the physical models which the contributions are based on. The basics of the methods used for these models are introduced in Part I marked in orange. The chapters that can be seen as an extended summary of the papers in Part V are indicated by the letter of the respective paper.

## Chapter 2

# Introduction to Finite-Difference Time-Domain Methods

*"Since Newton, mankind has come to realize that the laws of physics are always expressed in the language of differential equations."*  
- Steven Strogatz

This chapter introduces some important concepts needed to understand finite-difference time-domain (FDTD) methods. These techniques are what the implementation of the physical models presented later on in this document are based on. By means of a simple mass-spring system and the 1D wave equation, the notation and terminology used throughout this document will be explained. Unless denoted otherwise, the theory presented in this chapter and the notation have been taken from [23].

### 2.1 Differential equations

Differential equations are used to describe the motion of dynamic systems, including vibrations in musical instruments. In this work, these equations are used to describe, among others, the movement of a string, an instrument body and the air pressure in an acoustic tube.

A characteristic feature of these equations is that, rather than an absolute value or *state* of a system, such as displacement from the equilibrium of a string, or the pressure in a tube, the time derivative of its state – its velocity – or the second time derivative – its acceleration – is described. From this,

the absolute state of the system can then be computed. This state is usually described by the variable  $u$  which is always a function of time, i.e.,  $u = u(t)$ . If the system is distributed in space,  $u$  also becomes a function of space, i.e.,  $u = u(x, t)$ , or with two spatial dimensions,  $u = u(x, y, t)$ , etc. Though this work only describes systems of up to two spatial dimensions, one can easily extend to three dimensions [47] and potentially higher-dimensional systems. See Section 2.1.1 for more information on dimensions.

If  $u$  is univariate, and only a function of time, the differential equation that describes the motion of the system is called an *ordinary differential equation* (ODE). Various ways to describe the second derivative in time of  $u$ , or the acceleration of  $u$  are

$$\begin{aligned} \frac{d^2u}{dt^2} & \quad (\text{Leibniz's notation}), \\ \ddot{u} & \quad (\text{Newton's notation}), \\ D_t^2 u & \quad (\text{Euler's notation}). \end{aligned}$$

Leibniz's notation could be considered the most standard notation but is not necessarily compact. Newton's notation on the other hand allows for an ultra compact notation using dots above the function to denote time-derivatives. For this reason, Newton's notation will be used for ODEs in isolation. The drawback of this notation is that it can only be used for univariate functions. Finally, Euler's notation indicates a derivative using an operator which can be applied to a function.

If  $u$  is also a function of at least one spatial dimension, the equation of motion is called a *partial differential equation* (PDE). The literature uses different types of notation for taking (continuous-time) partial derivatives. Applied to a state variable  $u$  these can look like

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} & \quad (\text{Leibniz's notation}), \\ u_{tt} & \quad (\text{subscript notation}), \\ \partial_t^2 u & \quad (\text{Euler's notation}), \end{aligned}$$

where the subscript notation could be seen as the partial derivative counterpart to Newton's notation due to its compactness. In the remainder of this document, Euler's notation will be used for PDEs, due to their similarity to operators in discrete time (introduced in Section 2.2.2). Also, it allows for more compactness when creating bigger operators with multiple (connected) systems (see e.g. Chapter ??). Moreover, state-of-the-art literature in the field of FDTD methods for sound synthesis use this notation as well [25].

### 2.1.1 Dimensions and degrees of freedom

All objects in the physical world are three-dimensional (3D) as they have a non-zero width, length and depth. Moreover, these objects can move in these three dimensions and thus have three translational *degrees of freedom (DoF)* (the three rotational DoF are ignored here). To reduce the complexity of the models describing physical systems as well as computational complexity (computational cost), simplifications can be made to reduce both the dimensionality of the spatial distribution of a physical object as well as that of the translational DoF.

Generally, the spatial distribution of an object can be simplified if one (or more) of the dimensions are small, relative to the wavelengths of interest. A guitar string, for instance, has much greater length than its width or depth and can therefore be reduced to a one-dimensional (1D) system. If a 3D description were to be kept, the relative displacement between two locations on one cross-section along the length of the string would be taken into account. One could imagine that this displacement will always be orders of magnitude smaller than the relative displacement of two points along the string length and is thus negligible. Similarly, the thickness of a drum membrane is much smaller than its length and width and can therefore be simplified to a two-dimensional (2D) system.<sup>1</sup>

The translational DoF, on the other hand, describe now many “coordinates” a state variable includes. In much of the literature on FDTD methods in the field of musical acoustics, the state variable only has one coordinate. In most string models, for example, only the transverse displacement in one polarisation is considered (see Chapter 4) and the other polarisation as well as the longitudinal motion of the string (motion along the string length) are ignored. In other words, every point along the string can only move up and down, not side-to-side and not forward and back. Although this greatly simplifies the system at hand and reduces computational complexity, this is not what happens in reality. Nonlinear effects such as pitch glides due to tension modulation caused by high-amplitude string vibration are not present in the simplified model and have not been included in this project.

Work has been done on strings with dual (transverse) polarisation by Desvages [48] and Desvages and Bilbao [49] using FDTD methods. Models including longitudinal string vibration, where the longitudinal and transversal displacements are coupled can be found in [23, 50]. In [33], Villeneuve and Leonard present a mass-spring network where the state of every individual mass has three translational DoF. Due to these additional DoF, these networks do capture the aforementioned effects, but greatly increase the computational complexity of the models.

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<sup>1</sup>In this work, ‘1D’ and ‘2D’ will also be used to abbreviate ‘one dimension’ and ‘two dimensions’.

Although the dimensionality reduction ignores some of the physical processes, surprisingly realistic sounding models can be made despite these simplifications. Due to computational considerations, all models used in this work thus only have 1 translational DoF.

### Notation

When describing the state of a system, the spatial dimensions it is distributed over appear in the argument of the state variable. For example, the state of a 2D system with 1 translational DoF, is written as  $u(x, y, t)$ .

The translational DoF, on the other hand, determines the number of coordinates that the state variable describes. A 1D system with 3 translational DoF can thus be written as  $\mathbf{u}(x, t)$  where  $\mathbf{u}$  is a vector containing the coordinates for all three translational DoF.

#### 2.1.2 Ranges of definition and domains

When modelling physical systems, one needs to provide a *range of definition* over which they are defined. For a 1D system  $u = u(x, t)$ , ranges of definition must be given for  $x$  and  $t$ . Usually, the temporal range  $t \geq 0$ , meaning that the system is defined for non-negative time.

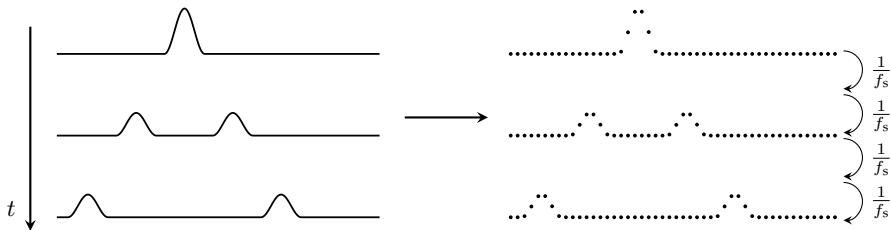
In space, the range of definition is usually referred to as a (spatial) *domain*, denoted by the symbol  $\mathcal{D}$ . Using the example above,  $x$  may be defined over  $\mathcal{D}$ , which is written as  $x \in \mathcal{D}$ . For analysis purposes, infinite domains ( $\mathcal{D} = \mathbb{R} = (-\infty, \infty)$ ) or semi-infinite domains ( $\mathcal{D} = \mathbb{R}^+ = [0, \infty)$ ) may be used, but for implementation purposes, a finite domain needs to be established. For higher dimensional systems, one needs to define higher dimensional domains. A 2D system  $u = u(x, y, t)$ , for simplicity assumed to be rectangular, may be defined over ‘horizontal domain’  $\mathcal{D}_x$  and ‘vertical domain’  $\mathcal{D}_y$ , which are both 1D domains. The system is then defined for  $(x, y) \in \mathcal{D}$  where  $\mathcal{D} = \mathcal{D}_x \times \mathcal{D}_y$ .

## 2.2 Discretisation using FDTD methods

Differential equations are powerful tools to describe the motion of physical systems. Despite this, only few of these have a closed-form, or analytical, solution. More complex systems require methods that do not perfectly solve, but rather *approximate* the solutions to these equations. FDTD methods are the most straightforward approach to numerically approximate differential equations. These methods are considered to be among the most general and flexible techniques in terms of the systems they can model, and frankly, relatively simple to understand once some familiarity with them is obtained. The main concern with these methods is the numerical stability of the eventual

approximation. Conditions for stability can be mathematically derived and will be introduced in Section 3.3.

FDTD methods essentially subdivide a continuous differential equation into discrete points in time and space, a process called *discretisation*. Once an ODE or PDE is discretised using these methods it is now called a *finite-difference (FD) scheme* which approximates the original differential equation. See Figure 2.1. In the following, for generality and ease of explanation, a 1D system will be used, and (again) the theory and notation follows [23].



**Fig. 2.1:** A continuous PDE is discretised to a FD scheme. In a PDE, time passes continuously, whereas for a FD scheme, time passes in finite increments with a duration of the reciprocal of the sample rate  $f_s$ .

### 2.2.1 Grid functions

The first step to approximate continuous PDEs, is to define a discrete *grid* over time and space. See Figure 2.2. A system described by state  $u = u(x, t)$  defined over time  $t$  and one spatial dimension  $x$ , can be discretised to a *grid function*  $u_l^n$ . Here, integers  $l$  and  $n$  describe the spatial and temporal indices respectively, and arise from the discretisation of the continuous variables  $x$  and  $t$ , according to  $x = lh$  and  $t = nk$ . The spatial step  $h$ , also called the *grid spacing* describes the distance (in m) between two neighbouring *grid points*, and is closely related to the stability of the FD scheme. The temporal step  $k$ , or *time step*, is the time (in s) between two consecutive temporal indices and can be calculated  $k = 1/f_s$  for a sample rate  $f_s$  (in Hz). In many audio applications  $f_s = 44100$  Hz, which is the sample rate that will be used in this work (unless denoted otherwise).

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As mentioned in Section 2.1.2, a 1D system needs to be defined over a temporal range of definition and one spatial domain. In discrete time,  $t \geq 0$  is discretised to  $n \in \mathbb{N}^0$ .<sup>2</sup> The spatial domain  $\mathcal{D}$  can be subdivided into  $N$  equal sections, or intervals, of length  $h$  (see Figure 2.2). The grid points describing the state of the system are placed at the edge of each interval, including the end points. The spatial range of interest then becomes  $l \in \{0, \dots, N\}$  and the

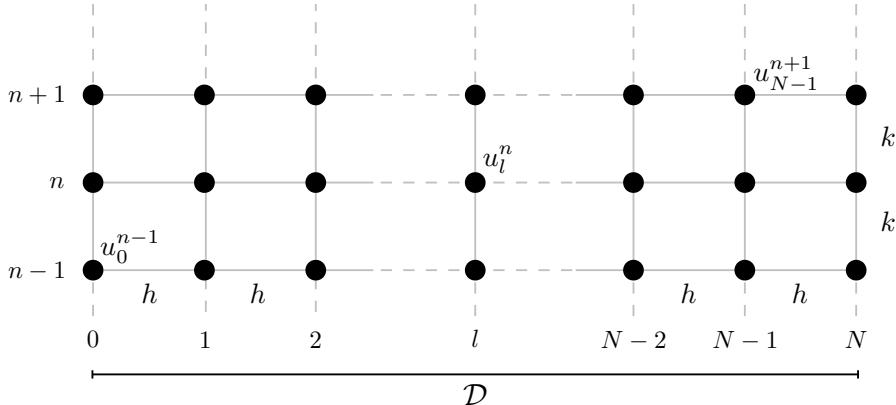
<sup>2</sup>In this work,  $\mathbb{N}^0$  is used to denote the set of non-negative integers ( $\mathbb{N}^0 = 0, 1, 2, \dots$ ).

total number of grid points is  $N + 1$ , which is one more than the number of intervals.

To summarise, for a 1D system

$$u(x, t) \cong u_l^n \quad \text{with} \quad x = lh \quad \text{and} \quad t = nk,$$

$$l \in \{0, \dots, N\} \quad \text{and} \quad n \in \mathbb{N}^0.$$



**Fig. 2.2:** The spatio-temporal grid that appears when a 1D system  $u(x, t)$  with  $x \in \mathcal{D}$  is discretised to a grid function  $u_l^n$ . The spatial domain  $\mathcal{D}$  is divided into  $N$  intervals of length  $h$  and spatial range of interest  $l = \{0, \dots, N\}$ . Time is subdivided into time steps of duration  $k$  and together with the discretised domain, forms a grid over space and time. Some grid points are labelled with the appropriate grid function.

## 2.2.2 Finite-difference operators

Now that the state variable has a discrete counterpart, this leaves the derivatives to be discretised, or approximated. We start by introducing *shift operators* that can be applied to a grid function and ‘shift’ its indexing, either temporally or spatially. Forward and backward shifts in time, together with the identity operation are

$$e_{t+}u_l^n = u_l^{n+1}, \quad e_{t-}u_l^n = u_l^{n-1}, \quad \text{and} \quad 1u_l^n = u_l^n. \quad (2.1)$$

Similarly, forward and backward shifts in space are

$$e_{x+}u_l^n = u_{l+1}^n, \quad \text{and} \quad e_{x-}u_l^n = u_{l-1}^n. \quad (2.2)$$

These shift operators are rarely used in isolation, though they do appear in energy analysis techniques detailed in Section 3.4. The operators do, however, form the basis of commonly used *finite-difference (FD) operators*. The first-order

many figures  
for shift and  
FD operators

## 2.2. Discretisation using FDTD methods

derivative in time can be discretised in three different ways. The forward, backward and centred difference operators are

$$\partial_t \approx \begin{cases} \delta_{t+} \triangleq \frac{1}{k} (e_{t+} - 1), \\ \delta_{t-} \triangleq \frac{1}{k} (1 - e_{t-}), \\ \delta_t \triangleq \frac{1}{2k} (e_{t+} - e_{t-}), \end{cases} \quad (2.3a)$$

$$(2.3b)$$

$$(2.3c)$$

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where “ $\triangleq$ ” means “equal to by definition”. These operators can then be applied to grid function  $u_l^n$  to get

$$\partial_t u \approx \begin{cases} \delta_{t+} u_l^n \frac{1}{k} (u_l^{n+1} - u_l^n), \\ \delta_{t-} u_l^n \frac{1}{k} (u_l^n - u_l^{n-1}), \\ \delta_t u_l^n = \frac{1}{2k} (u_l^{n+1} - u_l^{n-1}), \end{cases} \quad (2.4a)$$

$$(2.4b)$$

$$(2.4c)$$

and all approximate the first-order time derivative of  $u$ . Note that the centred difference has a division by  $2k$  as the time difference between  $n + 1$  and  $n - 1$  is, indeed, twice the time step.

Similar operators exist for a first-order derivative in space, where the forward, backward and centred difference are

$$\partial_x \approx \begin{cases} \delta_{x+} \triangleq \frac{1}{h} (e_{x+} - 1), \\ \delta_{x-} \triangleq \frac{1}{h} (1 - e_{x-}), \\ \delta_x \triangleq \frac{1}{2h} (e_{x+} - e_{x-}), \end{cases} \quad (2.5a)$$

$$(2.5b)$$

$$(2.5c)$$

figure here vi-  
sualising op-  
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and when applied to  $u_l^n$  are

$$\partial_x u \approx \begin{cases} \delta_{x+} u_l^n \frac{1}{h} (u_{l+1}^n - u_l^n), \\ \delta_{x-} u_l^n \frac{1}{h} (u_l^n - u_{l-1}^n), \\ \delta_x u_l^n = \frac{1}{2h} (u_{l+1}^n - u_{l-1}^n). \end{cases} \quad (2.6a)$$

$$(2.6b)$$

$$(2.6c)$$

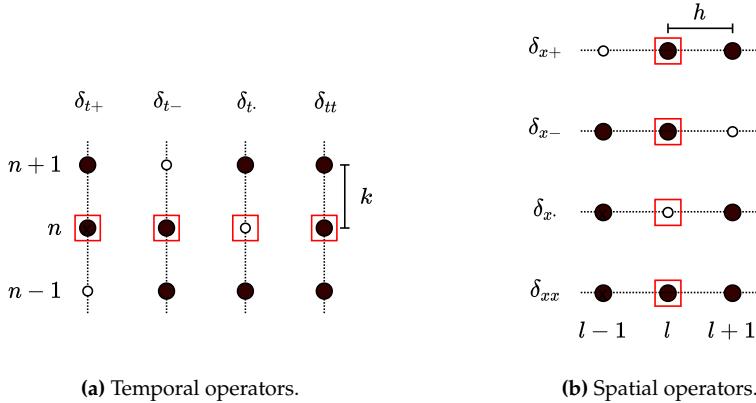
Higher-order differences can be approximated through a composition of first-order difference operators where their definitions are multiplied.<sup>3</sup> The second-order difference in time may be approximated using

$$\partial_t^2 \approx \delta_{t+} \delta_{t-} = \delta_{tt} \triangleq \frac{1}{k^2} (e_{t+} - 2 + e_{t-}), \quad (2.7)$$

where “2” is the identity operator applied twice. This can be done similarly for the second-order difference in space

$$\partial_x^2 \approx \delta_{x+} \delta_{x-} = \delta_{xx} \triangleq \frac{1}{h^2} (e_{x+} - 2 + e_{x-}), \quad (2.8)$$

both of which can be applied to a grid function  $u_l^n$  in a similar fashion. Figure 2.3 shows the *stencils* of the operators introduced above. A stencil shows the grid points needed to perform the operation of a FD operator.



**Fig. 2.3:** The stencils of various FD operators applied to the grid point highlighted with a red square. Black grid points are used in the calculation, and white grid points are not. The averaging operators follow the same pattern.

in energy analysis, inter-leaved grids, etc.

Also useful are averaging operators, all of which approximate the identity operation. The temporal forward, backward and centred averaging operators are

$$1 \approx \begin{cases} \mu_t \triangleq \frac{1}{2} (e_{t+} + 1), \\ \mu_t \triangleq \frac{1}{2} (1 + e_{t-}), \\ \mu_t \triangleq \frac{1}{2} (e_{t+} + e_{t-}). \end{cases} \quad (2.9a)$$

$$(2.9b)$$

$$(2.9c)$$

Notice how these definitions are different than the difference operators in (2.3): the terms in the parentheses are added rather than subtracted, and rather than a division by the time step  $k$  there is a division by 2. Finally, the centred averaging operator does not have an extra division by 2 as in (2.3c). Applied to  $u_l^n$ , Eqs. (2.9) become

$$u_l^n \approx \begin{cases} \mu_{t+} u_l^n \triangleq \frac{1}{2} (u_l^{n+1} + u_l^n), \\ \mu_{t-} u_l^n \triangleq \frac{1}{2} (u_l^n + u_l^{n-1}), \\ \mu_t u_l^n \triangleq \frac{1}{2} (u_l^{n+1} + u_l^{n-1}). \end{cases} \quad (2.10a)$$

$$(2.10b)$$

$$(2.10c)$$

Similarly, spatial averaging operators are

$$1 \approx \begin{cases} \mu_{x+} \triangleq \frac{1}{2} (e_{x+} + 1), \\ \mu_{x-} \triangleq \frac{1}{2} (1 + e_{x-}), \\ \mu_x \triangleq \frac{1}{2} (e_{x+} + e_{x-}), \end{cases} \quad (2.11a)$$

$$(2.11b)$$

$$(2.11c)$$

---

<sup>3</sup>Alternatively, one could first apply one operator to a grid function, expand it, and apply the other operator to all individual grid functions in the result of the first expansion thereafter.

## 2.2. Discretisation using FDTD methods

and when applied to  $u_l^n$

$$u_l^n \approx \begin{cases} \mu_{x+} u_l^n - \frac{1}{2} (u_{l+1}^n + u_l^n), \\ \mu_{x-} u_l^n - \frac{1}{2} (u_l^n + u_{l-1}^n), \end{cases} \quad (2.12a)$$

$$u_l^n \approx \begin{cases} \mu_x u_l^n - \frac{1}{2} (u_{l+1}^n + u_{l-1}^n). \end{cases} \quad (2.12b)$$

$$u_l^n \approx \begin{cases} \mu_x u_l^n - \frac{1}{2} (u_{l+1}^n + u_{l-1}^n). \end{cases} \quad (2.12c)$$

Finally, using forward and backward averaging operators, second-order temporal and spatial averaging operators can be created according to

$$1 \approx \mu_{tt} = \mu_{t+} + \mu_{t-} \triangleq \frac{1}{4} (e_{t+} + 2 + e_{t-}), \quad (2.13)$$

and

$$1 \approx \mu_{xx} = \mu_{x+} + \mu_{x-} \triangleq \frac{1}{4} (e_{x+} + 2 + e_{x-}). \quad (2.14)$$

Operators and derivatives in 2D will be discussed in Chapter 6.

### Accuracy

As FDTD methods approximate continuous systems, the resulting solution is rarely 100% accurate. To determine the accuracy of the FD operators above, one can perform a *Taylor series analysis*. The Taylor series is an infinite sum and its expansion of a function  $f$  about a point  $a$  is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a) \quad (2.15)$$

where superscript  $(n)$  denotes the  $n^{\text{th}}$  derivative of  $f$  with respect to  $x$ . The analysis will be performed on the temporal operators in this section, but also applies to the spatial operators presented.

Using continuous function  $u = u(t)$  and following Bilbao's "slight abuse of notation" in [23], one may apply FD operators to continuous functions according to

$$\delta_{t+} u(t) = \frac{u(t+k) - u(t)}{k}. \quad (2.16)$$

Assuming that  $u$  is infinitely differentiable,  $u(t+k)$ , i.e.,  $u$  at the next time step (in continuous time), can be approximated using a Taylor series expansion of  $u$  about  $t$  according to

$$u(t+k) = u(t) + k\dot{u} + \frac{k^2}{2}\ddot{u} + \frac{k^3}{6}\dddot{u} + \mathcal{O}(k^4). \quad (2.17)$$

Here, (following Newton's notation introduced in Section 2.1) the dot describes a single temporal derivative and  $\mathcal{O}$  includes additional terms in the expansion. The power of  $k$  in the argument of  $\mathcal{O}$  describes the order of accuracy, the higher

the power of  $k$  the more accurate the approximation. Equation (2.17) can be rewritten to

$$\frac{u(t+k) - u(t)}{k} = \dot{u} + \frac{k}{2}\ddot{u} + \frac{k^2}{6}\dot{\ddot{u}} + \mathcal{O}(k^3),$$

and using Eq. (2.16) can be written to

$$\delta_{t+}u(t) = \dot{u} + \mathcal{O}(k). \quad (2.18)$$

This says that the forward difference operator approximates the continuous first order derivative with an additional error term that depends on  $k$ . As the power of  $k$  in  $\mathcal{O}$ 's argument is 1, the forward operator is first-order accurate. One can also observe that, as expected, the error gets smaller as the time step  $k$  gets smaller and indicates that higher sample rates result in more accurate simulations (through  $k = 1/f_s$ ).

One can arrive at a similar result for the backward operator. Applying Eq. (2.3b) to  $u(t)$  yields

$$\delta_{t-}u(t) = \frac{u(t) - u(t-k)}{k}. \quad (2.19)$$

One can then approximate  $u(t-k)$  by performing a Taylor series expansion of  $u$  about  $t$  according to

$$u(t-k) = u(t) + (-k)\dot{u} + \frac{(-k)^2}{2}\ddot{u} + \frac{(-k)^3}{6}\dot{\ddot{u}} + \mathcal{O}(k^4), \quad (2.20)$$

$$\begin{aligned} \frac{u(t-k) - u(t)}{k} &= -\dot{u} + \frac{k}{2}\ddot{u} - \frac{k^2}{6}\dot{\ddot{u}} + \mathcal{O}(k^3), \\ \delta_{t-}u(t) &= \dot{u} + \mathcal{O}(k). \end{aligned} \quad (2.21)$$

Notice that the sign of  $\mathcal{O}$  does not matter.

Applying the centred operator in Eq. (2.3c) to  $u(t)$  yields

$$\delta_{t.}u(t) = \frac{u(t+k) - u(t-k)}{2k}, \quad (2.22)$$

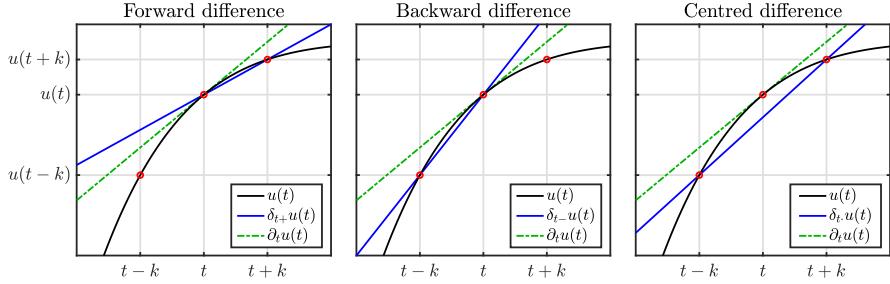
indicating that to find the order of accuracy for this operator, both Eqs. (2.17) and (2.20) are needed. Subtracting these and substituting their definitions yields

$$\begin{aligned} u(t+k) - u(t-k) &= 2k\dot{u} - \frac{2k^3}{6}\dot{\ddot{u}} + 2\mathcal{O}(k^5), \\ \frac{u(t+k) - u(t-k)}{2k} &= \dot{u} + \mathcal{O}(k^2), \\ \delta_{t.}u(t) &= \dot{u} + \mathcal{O}(k^2), \end{aligned} \quad (2.23)$$

and shows that the centred difference operator is second-order accurate.

As a first-order derivative indicates the *slope* of a function, the differences in accuracy between the above operators can be visualised as in Figure 2.4. It

## 2.2. Discretisation using FDTD methods



**Fig. 2.4:** The accuracy of the forward, backward and centred difference operators in (2.3) visualised. The centred difference operator much more closely approximates the derivative, or the slope, of  $u$  at  $t$  when compared to the forward and backward difference operators.

can be observed that the derivative approximation – the slope – of the centred operator matches much more closely the true derivative of  $u$  at  $t$ .

Higher-order differences, such as the second-order difference in time operator in Eq. (2.7) can also be applied to  $u(t)$  to get

$$\delta_{tt}u(t) = \frac{u(t+k) - 2u(t) + u(t-k)}{k^2}, \quad (2.24)$$

and can be proven to be second-order accurate by adding Eqs. (2.17) and (2.20):

$$\begin{aligned} u(t+k) + u(t-k) &= 2u(t) + k^2\ddot{u} + \mathcal{O}(k^4), \\ \frac{u(t+k) - 2u(t) + u(t-k)}{k^2} &= \ddot{u} + \mathcal{O}(k^2), \\ \delta_{tt}u(t) &= \ddot{u} + \mathcal{O}(k^2). \end{aligned} \quad (2.25)$$

The accuracy of averaging operators can be found in the same way and follow a similar pattern.

$$\begin{aligned} \mu_{t+}u(t) &= u(t) + \mathcal{O}(k), & \mu_{t-}u(t) &= u(t) + \mathcal{O}(k), \\ \mu_{t\cdot}u(t) &= u(t) + \mathcal{O}(k), & \mu_{tt}u(t) &= u(t) + \mathcal{O}(k^2). \end{aligned} \quad (2.26)$$

### 2.2.3 Identities

For working with FD schemes, either for implementation or analysis, it can be extremely useful to rewrite the operators presented above to equivalent versions of themselves. These are called *identities* and for future reference, some useful ones are listed below:

$$\delta_{tt} = \frac{2}{k} (\delta_{t\cdot} - \delta_{t-}), \quad (2.27a)$$

$$\delta_{t\cdot} = \delta_{t+}\mu_{t-} = \delta_{t-}\mu_{t+}, \quad (2.27b)$$

$$\mu_{t+} = \frac{k}{2}\delta_{t+} + 1. \quad (2.27c)$$

see whether  
the negative  
version of  
identity (2.27c)  
is also used  
later on

That these equalities hold, can easily be proven by expanding the operators defined in Section 2.2.2. Naturally, these identities also hold for spatial operators by simply substituting the ‘ $t$ ’ subscripts for ‘ $x$ ’.

## 2.3 The mass-spring system

Though a complete physical modelling field on their own (see Chapter 1), mass-spring systems are also sound-generating systems and lend themselves well to illustrating and explaining FDTD methods in practice. Starting with the continuous-time ODE, the current section follows the discretisation process to a FD scheme using the operators described in Section 2.2.2. Finally, the scheme is rewritten to an update equation that can be implemented and the output of the system is shown.

### 2.3.1 Continuous time

Using dots to indicate a temporal derivative, the ODE of a simple mass-spring system is defined as

$$M\ddot{u} = -Ku, \quad (2.28)$$

where  $u = u(t)$  is the distance from the equilibrium position (in m),  $M > 0$  is the mass of the mass (in kg) and  $K \geq 0$  is the spring constant (in N/m). Equation (2.28) can be written as

$$\ddot{u} = -\omega_0^2 u, \quad (2.29)$$

with angular frequency (in rad/s)

$$\omega_0 = \sqrt{K/M}. \quad (2.30)$$

This way of writing the mass-spring ODE is more compact and can more directly be related to the fundamental frequency  $f_0 = \omega_0/2\pi$  (in Hz) of the system.

Apart from the choices of  $K$  and  $M$ , the behaviour of the mass-spring system is determined by its *initial conditions*, being  $u(0)$  and  $\partial_t u(0)$ , i.e., the displacement and velocity of the mass at  $t = 0$ . If the initial conditions are non-zero, the path that the displacement of the mass follows over time is sinusoidal (see Figure 2.5), which is also why the mass-spring system is often referred to as the *simple harmonic oscillator*. The amplitude of the sinusoid is determined by the initial conditions, whereas the frequency is determined by  $M$  and  $K$ .

### 2.3. The mass-spring system

#### Intuition

The behaviour of the mass-spring system in Eq. (2.28) arises from two basic laws of physics: *Newton's second law* and *Hooke's law*.

move section up (stefan's comment)

Starting with Newton's second law – *force equals mass times acceleration* – and relating this to the variables used in Eq. (2.28) yields an expression for force

$$F = M\ddot{u}. \quad (2.31)$$

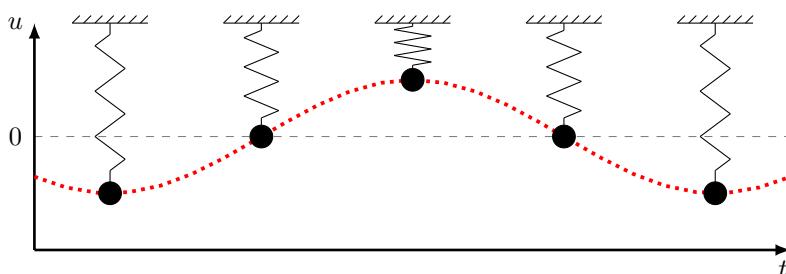
This equation in isolation can be used to, for example, calculate the force necessary to accelerate a mass of  $M$  kg to  $\ddot{u}$  m/s<sup>2</sup>. Next, the force generated by the spring follows Hooke's law:

$$F = -Ku, \quad (2.32)$$

which simply states that the force generated by a spring with stiffness  $K$  is negatively proportional to the value of  $u$ . In other words, the further the spring is extended (from the equilibrium  $u = 0$ ), the more force will be generated in the opposite direction. Finally, as the sole force acting on the mass is the one generated by the spring, the two expressions for the force  $F$  can be set equal to each other and yield the equation for the mass-spring system in (2.28).

The sinusoidal behaviour of the mass-spring system, or at least the fact that the mass "gets pulled back" to the equilibrium, is apparent from the minus-sign in Eq. (2.32). The frequency of the sinusoid, depends on the value of  $K$  as the "pull" happens to a higher degree for a higher spring stiffness. That the frequency of the system is also dependent on the mass  $M$  can be explained by the fact that a lighter object is more easily moved and vice versa, which is apparent from Eq. (2.31). In other words, the pull of the spring has a greater effect on the acceleration of a lighter object than a heavier one.

Finally, if  $u = 0$  there is no spring force present and the acceleration remains unchanged. This is exactly what Newton's first law states: if the net force acting on an object is zero, its velocity will be constant. If the mass is not in motion,



**Fig. 2.5:** Mass-spring system over time. The system follows a harmonic (sinusoidal) motion.

this means that it remains stationary. If it is, at the exact moment that  $u = 0$ , the velocity is unchanged.

### 2.3.2 Discrete time

Following the discretisation process introduced in Section 2.2, one can approximate the PDE in Eq. (2.28). The displacement of the mass is approximated using

$$u(t) \approx u^n, \quad (2.33)$$

with time  $t = nk$ , time step  $k = 1/f_s$ , sample rate  $f_s$  and temporal index and  $n \in \mathbb{N}^0$ . Note that the “grid function” does not have a subscript  $l$  as  $u$  is not distributed in space and is now simply called a *time series*.

Using the operators found in Section 2.2.2, Eq. (2.28) can be discretised as follows:

$$M\delta_{tt}u^n = -Ku^n, \quad (2.34)$$

which is the first appearance of a FD scheme in this work. Expanding the  $\delta_{tt}$  operator yields

$$\frac{M}{k^2} (u^{n+1} - 2u^n + u^{n-1}) = -Ku^n,$$

and solving for  $u^{n+1}$  results in the following recursion or *update equation*:

$$u^{n+1} = \left( 2 - \frac{Kk^2}{M} \right) u^n - u^{n-1}, \quad (2.35)$$

which can be implemented in a programming language such as MATLAB.

### 2.3.3 Implementation and output

A simple MATLAB script implementing the mass-spring system described in this section is shown in Appendix C.1. The most important part of the algorithm happens in a for-loop recursion, where update equation (2.35) is implemented. At the end of each loop, the system states are updated and prepared for the next iteration.

To be able to start the simulation of the scheme, the initial conditions given in Section 2.3.1 must be discretised at  $n = 0$ . As  $n$  is only defined for values greater than zero, the forward difference operator is used. A simple way to obtain a sinusoidal motion with an amplitude of 1, is to set the initial conditions as follows:

$$u^0 = 1 \quad \text{and} \quad \delta_{t+}u^0 = 0. \quad (2.36)$$

## 2.4. The 1D wave equation

The latter equality can be expanded and solved for  $u^1$  to obtain its definition:

$$\begin{aligned} \frac{1}{k} (u^1 - u^0) &= 0, \\ \xleftarrow{u^0=1} u^1 - 1 &= 0, \\ u^1 &= 1. \end{aligned}$$

In short, setting  $u^0 = u^1 \neq 0$  yields an oscillatory behaviour with an amplitude of 1. Note that any other non-zero initial condition will also yield oscillatory behaviour, but likely with a different amplitude.

The values for  $K$  and  $M$  are restricted by a stability condition

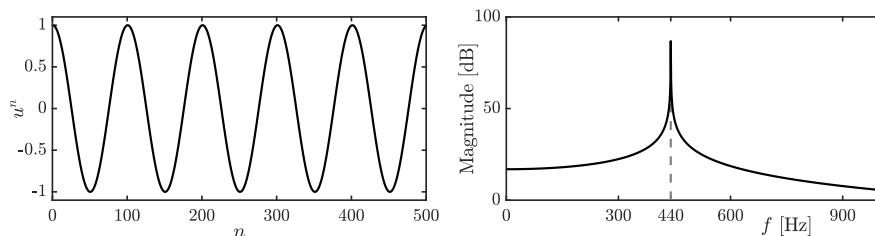
$$k < 2\sqrt{\frac{M}{K}}, \quad (2.37)$$

which will be elaborated on in Section 3.3. If this condition is not satisfied, the system will exhibit (exponential) growth and is *unstable*.

The output of the system can be obtained by ‘recording’ the displacement of the mass and listening to this at the given sample rate  $f_s$ . An example of this can be found in Figure 2.6 where the frequency of oscillation  $f_0 = 440$  Hz.

## 2.4 The 1D wave equation

Arguably the most important PDE in the field of physical modelling for sound synthesis is the 1D wave equation. It can be used to describe transverse vibration in an ideal string, longitudinal vibration in an ideal bar or the pressure in an acoustic tube (see Chapter 5). Although the behaviour of this equation alone does not appear in the real world as such – as no physical system is ideal – it is extremely useful as a test case and a basis for more complicated models.



**Fig. 2.6:** The time domain and frequency domain output of a mass-spring system with  $f_0 = 440$  Hz.

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### 2.4.1 Continuous time

The 1D wave equation is a PDE that describes the motion of a system distributed in one dimension of space. Consider the state of a 1D system  $u = u(x, t)$  of length  $L$  (in m) defined for time  $t \geq 0$  and  $x \in \mathcal{D}$  with  $\mathcal{D} = [0, L]$ . The PDE describing its motion is

$$\partial_t^2 u = c^2 \partial_x^2 u, \quad (2.38)$$

where  $c$  is the wave speed of the system (in m/s). If the PDE is used to model an ideal string, the wave speed can be defined as  $c = \sqrt{T/\rho A}$ , with tension  $T$  (in N), material density  $\rho$  (in kg/m<sup>3</sup>) and cross-sectional area  $A$ . If instead, it is used to model pressure in an acoustic tube  $c$  is the speed of sound in air. Figure 2.7 shows the wave propagation of the 1D wave equation excited using a raised cosine.

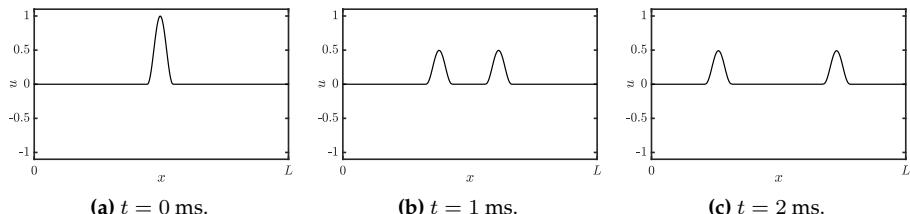


Fig. 2.7: Wave propagation in the 1D wave equation in Eq. (2.38) with  $c \approx 127$  m/s.

### Intuition

As with the mass-spring system in Section 2.3 the working of the PDE in (2.38) arises from Newton's second law, even though this connection might be less apparent.

The 1D wave equation in (2.38) states that the acceleration of  $u(x, t)$  at location  $x$  is determined by the second-order spatial derivative of  $u$  at that same location (scaled by a constant  $c^2$ ). In the case that  $u$  describes the transverse displacement of an ideal string, this second-order derivative denotes the *curvature* of this string. As  $c^2$  is always positive, the sign (or direction) of the acceleration is fully determined by the sign of the curvature. In other words, a ‘positive’ curvature at location  $x$  along the ideal string yields a ‘positive’ or upwards acceleration at that same location.

What a ‘positive’ or ‘negative’ curvature implies is more easily seen when we take a simple function describing a parabola,  $y(x) = x^2$ , and take its second derivative to get  $y''(x) = 2$ . The answer is a positive number which means that  $y$  has a positive curvature.

So, what does this mean for the 1D wave equation? As a positive curvature implies a positive or upwards acceleration as per Eq. (2.38),  $u$  with a positive curvature at a location  $x$  will start to move upwards and vice versa. Of course, the state of a physical system such as  $u$  will rarely have a perfect parabolic shape, but the argument still applies. See Figure 2.8 for a visualisation of the forces acting on  $u$  due to curvature.

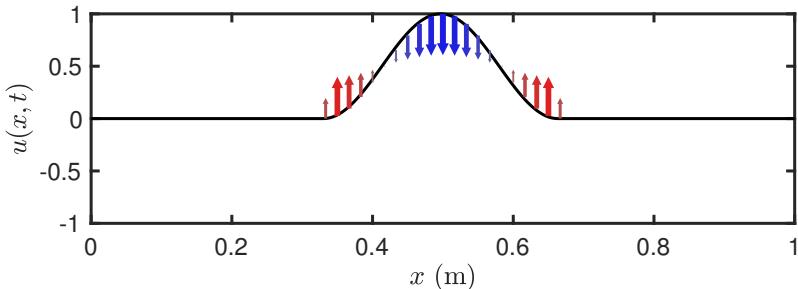
How the 1D wave equation relates to Newton's second law, becomes apparent by slightly rewriting Eq. (2.38). Recalling the definition of  $c$  for an ideal string, one can rewrite the 1D wave equation to

$$\rho A \partial_t^2 u = T \partial_x^2 u,$$

where  $\rho A$  describes the *mass per unit length* of the string. As the forces present in the system act on infinitesimally small portions of the string Newton's second law appears by a multiplication of  $dx$

$$\underbrace{\rho A \partial_t^2 u dx}_{ma} = \underbrace{T \partial_x^2 u dx}_F,$$

where  $\rho A dx$  is the mass of a (tiny) portion of the string of length  $dx$  (in m),  $\partial_t^2 u$  is the acceleration of that portion and  $T \partial_x^2 u dx$  describes the force acting on that portion, yielding Newton's second law.



**Fig. 2.8:** The forces acting on the 1D wave equation described by  $u(x, t)$  due to curvature. The arrows indicate the direction and magnitude of the force, and simultaneously the acceleration as these are connected through Eq. (2.38).

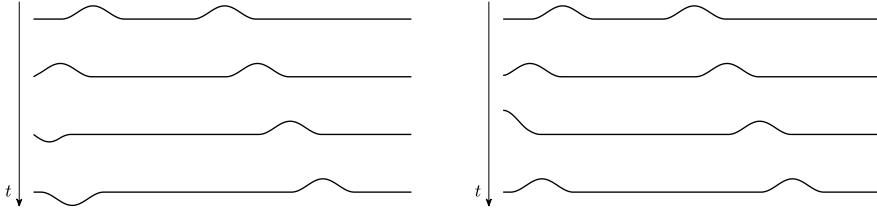
different wording in caption

### Boundary conditions

When a system is distributed in space, *boundary conditions* must be determined. Recalling that  $x$  is defined over domain  $\mathcal{D} = [0, L]$ , the boundaries, or end points of the system are located at  $x = 0$  and  $x = L$ . Two often-used alternatives for the boundary conditions are

$$u(0, t) = u(L, t) = 0 \quad (\text{Dirichlet, fixed}), \quad (2.39a)$$

$$\partial_x u(0, t) = \partial_x u(L, t) = 0 \quad (\text{Neumann, free}). \quad (2.39b)$$



(a) The Dirichlet boundary condition in Eq. (2.39a) fixes the boundary, which causes the incoming waves to invert.

(b) The Neumann or free boundary condition in Eq. (2.39b) fixes the slope at the boundary, causing the incoming waves to not invert.

**Fig. 2.9:** The behaviour of the 1D wave equation with (a) Dirichlet or (b) Neumann boundary conditions.

The Dirichlet boundary condition says that at the end points of the system, the state is 0 at all times. The Neumann condition on the other hand, says that rather the slope of these points needs to be 0, but that the end points are free to move transversely. In the former case, incoming waves invert after reaching the boundary whereas in the latter incoming waves are reflected un-inverted. See Figure 2.9.

If both boundaries of the 1D wave equation share the same condition, the fundamental frequency of the simulation can be calculated using

$$f_0 = \frac{c}{2L} . \quad (2.40)$$

## Scaling

As this work follows much of Bilbao's *Numerical Sound Synthesis* [23], it might be good to talk about a major discrepancy between the PDEs and FD schemes that appear there and those used here. Non-dimensionalisation, or *scaling*, is extensively used in [23] and much of the literature published around that time (fx. [51, 50]) and can be useful to reduce the number of parameters used to describe a system.

Scaling techniques normalise the domain  $x \in [0, L]$  to  $x' \in [0, 1]$  with  $x' = x/L$ . The 1D wave equation in (2.38) can then be rewritten to

$$\partial_t^2 u = \gamma^2 \partial_{x'x'} u, \quad (2.41)$$

where scaled wave speed  $\gamma = c/L$  has units of frequency. The scaling has removed the necessity for both  $c$  and  $L$  and simply specifying the scaled wave speed  $\gamma$  is enough to parametrise the behaviour of the system. The parameter

reduction gets more apparent for more complex systems and could greatly simplify the models used, at least in notation and parameter control.

Although this parameter reduction might be useful for resonators in isolation, when multiple resonators interact with each other (see Part IV), it is better to keep the systems dimensional. As a big part of this work includes interaction between multiple resonators, only dimensional systems will appear here.

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## 2.4.2 Discrete time

Coming back to the PDE presented in Eq. (2.38), we continue by finding a discrete-time approximation for it. As explained in Section 2.2.1, a continuous state variable  $u = u(x, t)$  can be discretised using  $x = lh$  with grid spacing  $h$  (in m) and  $t = nk$  with time step  $k$  (in s). The grid function  $u_l^n$  approximating  $u$  can then be indexed by spatial index  $l \in \{0, \dots, N\}$  with number of intervals between the grid points  $N$  and temporal index  $n \in \mathbb{N}^0$ . Continuing with the approximations of the derivatives in the 1D wave equation, the most straightforward discretisation of Eq. (2.38) is the following FD scheme

$$\delta_{tt} u_l^n = c^2 \delta_{xx} u_l^n. \quad (2.42)$$

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Other schemes exist (see e.g. [23]), but are excluded as they have not been used in this work. Expanding the operators using the definitions given in Section 2.2.2 yields

$$\frac{1}{k^2} (u_l^{n+1} - 2u_l^n + u_l^{n-1}) = \frac{c^2}{h^2} (u_{l+1}^n - 2u_l^n + u_{l-1}^n). \quad (2.43)$$

and solving for  $u_l^{n+1}$  yields

$$u_l^{n+1} = (2 - 2\lambda^2) u_l^n + \lambda^2 (u_{l+1}^n + u_{l-1}^n) - u_l^{n-1}. \quad (2.44)$$

Here,

$$\lambda = \frac{ck}{h} \quad (2.45)$$

is called the *Courant number* and plays a big role in stability and quality of the FD scheme. More specifically,  $\lambda$  needs to abide the (famous) Courant-Friedrichs-Lowy or *CFL condition* for short [52]

$$\lambda \leq 1, \quad (2.46)$$

which acts as a stability condition for scheme (2.42). More details on this are given in Section 2.4.4.

As  $c$ ,  $k$  and  $h$  are interdependent due to the CFL condition, it is useful to rewrite Eq. (2.46) in terms of known variables. As the time step  $k$  is based on

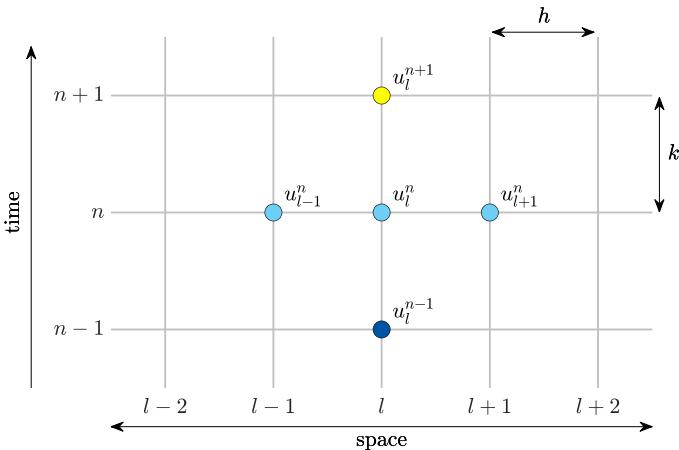
the sample rate and thus (usually) fixed, and  $c$  is a user-defined wave speed, the CFL condition can be rewritten in terms of the grid spacing  $h$ :

$$h \geq ck, \quad (2.47)$$

which, in implementation, is used as a stability condition for the scheme. See Section 3.3 for more information on how to derive a stability condition from a FD scheme.

### Stencil

As was done for several FD operators in Figure 2.3, it can be useful to visualise the *stencil*, or region of operation, of a FD scheme. A stencil of a scheme visualises what grid values are necessary to calculate the state at the next time step  $u_l^{n+1}$ . Figure 2.10 shows the stencil for scheme (2.42) and – in essence – visualises the various shifts of the grid function in Eq. (2.44). One could visualise this stencil to be placed on the left-most points of the grid shown in Figure 2.2. The update equation then iterates this stencil over the entire domain and calculates all values of  $u_l^{n+1}$  based on known values of  $u_l^n$  and  $u_l^{n-1}$ .



**Fig. 2.10:** The stencil, or region of operation, for the FD scheme in (2.42). The time steps of the various grid points are colour-coded by yellow ( $n + 1$ ), light blue ( $n$ ) and dark blue ( $n - 1$ ).

### Boundary conditions and virtual grid points

The end points of the discrete domain are located at  $l = 0$  and  $l = N$ . Substituting these locations into Eq. (2.44) shows that grid points outside of the

## 2.4. The 1D wave equation

defined domain are needed, namely  $u_{-1}^n$  and  $u_{N+1}^n$ . These can be referred to as *virtual grid points* and can be accounted for by discretising the boundary conditions in Eq. (2.39). Discretising these (using the most accurate centred spatial difference operator for the Neumann condition) yields

$$u_0^n = u_N^n = 0 \quad (\text{Dirichlet, fixed}), \quad (2.48a)$$

$$\delta_x u_0^n = \delta_x u_N^n = 0 \quad (\text{Neumann, free}). \quad (2.48b)$$

If Dirichlet boundary conditions are used, the states of the boundary points will always be zero and can therefore be excluded from the calculations. The range of calculation then simply becomes  $l \in \{1, \dots, N-1\}$  and no virtual grid points are needed when performing the update.

If, on the other hand, Neumann conditions are used, the range of calculation remains  $l \in \{0, \dots, N\}$  and definitions for the virtual grid points need to be found. Expanding the operators in Eq. (2.48b) and solving for  $u_{-1}^n$  and  $u_{N+1}^n$  provides the definitions for these virtual grid points based on values inside the defined domain:

$$\begin{aligned} \frac{1}{2h} (u_1^n - u_{-1}^n) &= 0, & \frac{1}{2h} (u_{N+1}^n - u_{N-1}^n) &= 0, \\ u_1^n - u_{-1}^n &= 0, & u_{N+1}^n - u_{N-1}^n &= 0, \\ u_{-1}^n &= u_1^n. & u_{N+1}^n &= u_{N-1}^n. \end{aligned}$$

At the boundaries, the update equation in Eq. (2.44) will then have the above definitions for the virtual grid points substituted and will become

$$u_0^{n+1} = (2 - 2\lambda^2) u_0^n + 2\lambda^2 u_1^n - u_0^{n-1}, \quad (2.49)$$

and

$$u_N^{n+1} = (2 - 2\lambda^2) u_N^n + 2\lambda^2 u_{N-1}^n - u_N^{n-1}, \quad (2.50)$$

at the left and right boundary respectively.

### 2.4.3 Implementation

This section provides information on the excitation and output of the system. See Appendix C.2 for a MATLAB implementation of the 1D wave equation.

#### Excitation

A simple way to excite the system is to initialise the state using a raised cosine, or Hann window. More information on excitations will be given in Part III, but for completeness, the formula for a discrete raised cosine will be given here.

The discrete raised cosine can be parametrised by its center location  $l_c$  and width  $w$  from which the start index  $l_s$  and end index  $l_e$  can be calculated, according to

$$l_s = l_c - \lfloor w/2 \rfloor \quad \text{and} \quad l_e = l_c + \lfloor w/2 \rfloor, \quad (2.51)$$

where  $\lfloor \cdot \rfloor$  denotes the flooring operation and needs to be used as all the above variables are integers. Furthermore, both  $l_s$  and  $l_e$  must fall into the defined spatial range of calculation. Then, a raised cosine with an amplitude of 1 can be calculated and used as an initial condition for the system according to

$$u_l^1 = u_l^0 = \begin{cases} 0.5 - 0.5 \cos\left(\frac{2\pi(l-l_s)}{w-1}\right), & l_s \leq l < l_e, \\ 0, & \text{otherwise.} \end{cases} \quad (2.52)$$

As done for the implementation of the mass-spring system in Section 2.3.3, both  $u_l^0$  and  $u_l^1$  are initialised with the same state, as to only have an initial displacement, and not an initial velocity.

In MATLAB, an easier way to obtain a raised cosine is to use the `hann(w)` function which returns a raised cosine (or Hann window) of width  $w$ .

### Output and modes

After the system is excited, one can retrieve the output of the system by selecting a grid point  $l_{\text{out}}$  and listening to that at the given sample rate  $f_s$ . An example using the parameters in Table 2.1 and Dirichlet boundary conditions is shown in Figure 2.11.

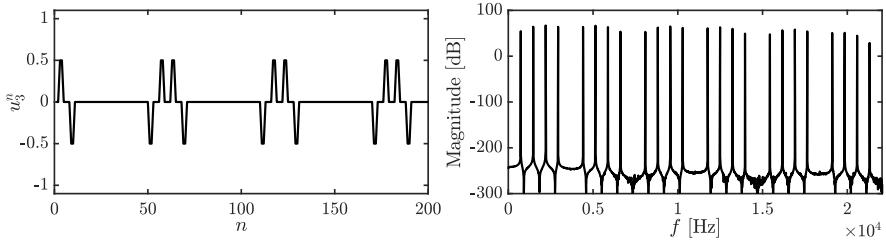
Name	Symbol (unit)	Value
<b>User-defined parameters</b>		
Length	$L$ (m)	1
Wave speed	$c$ (m/s)	1470
Sample rate	$f_s$ (Hz)	44100
<b>Derived parameters</b>		
Fundamental frequency	$f_0$ (Hz)	735
No. of intervals	$N$ (-)	30
Time step	$k$ (s)	$\approx 2.27 \cdot 10^{-5}$
Grid spacing	$h$ (m)	$\approx 0.033$
Courant number	$\lambda$ (-)	1
<b>Excitation and output</b>		
Center location	$l_c$ (-)	$0.2N$
Width	$w$ (-)	4
Output location	$l_{\text{out}}$	3

**Table 2.1:** Parameters used for 1D wave equation example used in this section. The user-defined parameters have been chosen such that  $\lambda = 1$ .

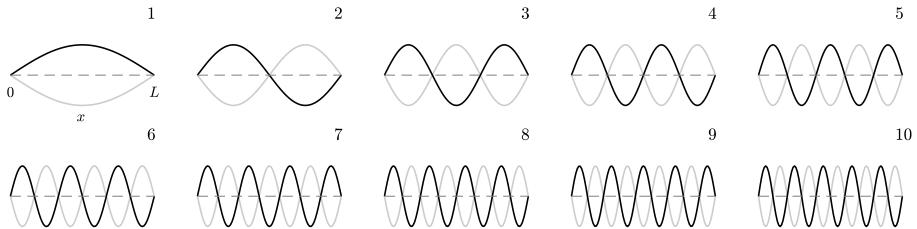
As can be seen from Figure 2.11, the output of the 1D wave equation contains many peaks in the frequency spectrum on top of the fundamental frequency. These are called *harmonic partials* or *harmonics* for short and arise

## 2.4. The 1D wave equation

from the various modes of vibration present in the system (see Figure 2.12). Although the PDE has not been implemented using modal synthesis (discussed in Chapter 1), the system can still be decomposed into different modes of vibration, each corresponding to a harmonic frequency. These modes are assumed to vibrate independently, and their weighted sum yields the eventual behaviour of the system.<sup>4</sup>



**Fig. 2.11:** The time-domain and frequency domain output of the 1D wave equation with  $f_0 = 735$  Hz and  $f_s = 44100$  Hz ( $N = 30$  and  $\lambda = 1$ ) and Dirichlet boundary conditions. The system is initialised with a raised cosine described in Eq. (2.52) with  $l_c = 0.2N$ ) and  $w = 4$  and the output is retrieved at  $l_{\text{out}} = 3$ .



**Fig. 2.12:** The first 10 modal shapes of the 1D wave equation with Dirichlet boundary conditions defined for  $x \in [0, L]$  (only shown for mode 1). The modes are normalised to have the same amplitude and vibrate at their respective modal frequencies with the extremes indicated by the black and the grey plot. The number of the shape can be determined by the number of antinodes present in the shape.

The number of modes present in the continuous PDE of the 1D wave equation is theoretically infinite. The number present in the discrete FD scheme, however, is determined by the number of moving points in the system. If Dirichlet boundary conditions are used, this means that there are  $N - 1$  modes, and  $N + 1$  modes for Neumann boundary conditions. If the CFL condition is satisfied with equality, the frequencies of these modes are integer multiples of

---

<sup>4</sup>Modes of the vibrating string were first discovered by Sauveur in 1701 who said that “especially at night” he observed “other small sounds” on top of the fundamental frequency and coined the terms ‘node’ and ‘harmonic’ [53].

the fundamental:  $f_m = m f_0$  for mode number  $m \in \{1, \dots, N-1\}$  for Dirichlet and  $m \in \{0, \dots, N\}$  when using Neumann boundary conditions. The frequency of the harmonics – and even the modal shapes – can be analytically derived using modal analysis as will be explained in Section 3.5.

The amplitude of the different modes depends on the excitation location (and type) and the output location. Figure 2.11, for example, seemingly shows that the system only exhibits 24 modes, rather than the 29 ( $N-1$ ) predicted. As the system is excited at  $0.2N$ , or in other words,  $1/5^{\text{th}}$  of the length of the system, this means that every  $5^{\text{th}}$  mode will be attenuated. To understand how and/or why this happens, one can refer to Figure 2.12 and see that every  $5^{\text{th}}$  modal shape has a node at  $1/5^{\text{th}}$  of its length. If the system is excited exactly there, this modal shape will not obtain any energy and will thus not resonate. Similarly, if the system is excited exactly in the middle, every  $2^{\text{nd}}$  modal frequency will be attenuated as there is a node present in the corresponding modal shape. The output would then only contain odd-numbered modes.

#### 2.4.4 Stability and simulation quality

As shown in Eq. (2.46), the Courant number needs to abide the CFL condition in order for the scheme to be stable. A system is regarded *unstable* if it exhibits (exponential) unbounded growth. If Neumann boundary conditions (free) are used, it is possible that the system drifts off over time. This does not mean that the system is unstable, it is actually entirely physically possible!<sup>5</sup>

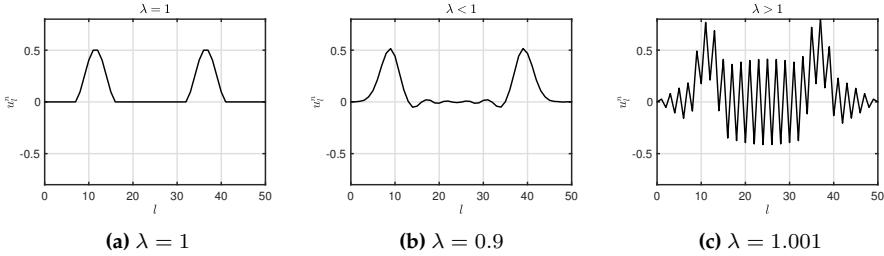
Besides stability, the value of  $\lambda$  is closely related to the quality of the simulation. If  $\lambda = 1$ , Eq. (2.42) is actually an exact solution to Eq. (2.38), which is quite uncommon in the realm of differential equations! See Figure 2.13a. Identically, if Eq. (2.47) is satisfied with equality, the FD scheme is an exact solution to the PDE, and if  $h$  deviates from this condition, the quality of the simulation decreases.

If  $\lambda < 1$ , the quality of the simulation decreases in an effect called *numerical dispersion*. Dispersion is a phenomenon where some frequencies travel faster through a medium than others, which is desired in some models (see fx. Chapter 4). Numerical dispersion, however, which is due to numerical inaccuracy, never is! Figure 2.13b shows an example when  $\lambda = 0.9$ , and one can observe that the wave propagation does not match the ideal case as Figure 2.13a shows. Moreover, bandlimiting effects occur, meaning that the highest frequency that the system can generate decreases. See Figure 2.14. Higher modes get ‘squished’ together and are not exact multiples of the fundamental anymore. Section 3.5 elaborates on how to calculate the exact modal frequencies of a FD implementation of the 1D wave equation.

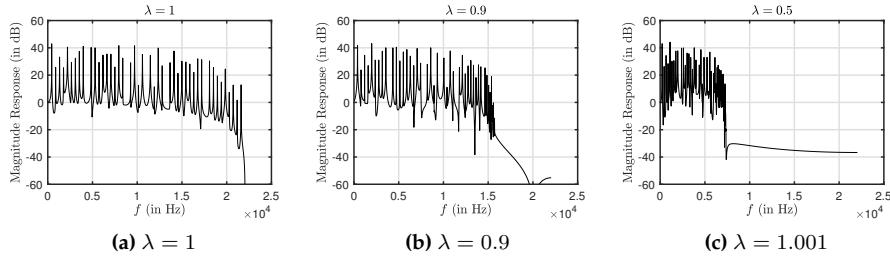
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<sup>5</sup>Imagine a ‘free’ guitar string where the ends are not connected to the nut and bridge of a guitar. The string can be taken far away from the guitar without it breaking or exploding.

## 2.4. The 1D wave equation



**Fig. 2.13:** Grid function  $u_i^n$  visualised  $\sim 100$  samples after excitation. (a) If  $\lambda = 1$ , the solution is exact. (b) If  $\lambda < 1$  dispersive behaviour shows. (c) If  $\lambda > 1$  the CFL condition in Eq. (2.46) is not satisfied and the system is unstable.



**Fig. 2.14:** Frequency spectra of the simulation output. The Courant number is set to (a)  $\lambda = 1$ , (b)  $\lambda = 0.9$  and (c)  $\lambda = 0.5$ . One can observe that for lower values of  $\lambda$  the bandwidth of the output decreases drastically.

Finally, if  $\lambda > 1$  the system becomes unstable. An example is shown in Figure 2.13c. Unstable behaviour usually comes in the form of high frequencies (around the Nyquist frequency of  $f_s/2$ ) growing without bounds.

In what situation would the stability condition then not be satisfied with equality? As mentioned in Section 2.2.1, a continuous domain  $\mathcal{D} = [0, L]$  for a system of length  $L$  needs to be divided into  $N$  equal sections of length  $h$  in the discretisation process. A logical step to calculate  $N$  would be to divide  $L$  by  $h$  calculated using Eq. (2.47) satisfied with equality to get the highest possible simulation quality. However, this calculation might not result in an integer value, which  $N$  should be! To stay as close to the stability condition as possible, the following calculations are performed in order:

$$h := ck, \quad N := \left\lceil \frac{L}{h} \right\rceil, \quad h := \frac{L}{N}, \quad \lambda := \frac{ck}{h}. \quad (2.53)$$

In other words, Eq. (2.47) is satisfied with equality and used to calculate integer  $N$ . After this,  $h$  is recalculated based on  $N$  and used to calculate the Courant number using Eq. (2.45). This process assures that  $N$  is an integer and that

the CFL condition is satisfied, though not necessarily with equality.

To understand why  $h$  needs to be recalculated, consider the following example. Consider the 1D wave equation defined over domain  $\mathcal{D} = [0, L]$  where  $L = 1$ . Furthermore, we say that the system should produce a fundamental frequency of  $f_0 = 750$  Hz which requires a wave speed of  $c = 1500$  m/s according to Eq. (2.40). If we use the commonly-used sample rate of  $f_s = 44100$  Hz, and recalling that  $k = 1/f_s$ , these values can be filled into (2.47) satisfied with equality and yields  $h \approx 0.034$ . If we divide the length by the grid spacing, we get  $L/h = 29.4$ , meaning that exactly 29.4 intervals of size  $h$  fit in the domain  $\mathcal{D}$ . However, the number of intervals needs to be an integer and – using Eq. (2.53) – we get  $N = 29$ . If  $h$  is not recalculated according to (2.53), the total length will be 29 times the grid spacing  $h$ . This results in  $L \approx 0.986$  and is slightly less than the original length of 1. Although the CFL condition will be satisfied with equality, the fundamental frequency will be slightly higher than desired:  $f_0 \approx 760.34$  Hz. If  $h$  is recalculated based on  $N$ , then  $L$  and  $f_0$  will be unchanged, and the system will have the correct fundamental frequency. The Courant number  $\lambda \approx 0.986$  is still very close to satisfying the condition in Eq. (2.46), and the decrease in quality will be perceptually irrelevant – or at the very least, less perceptually relevant than the change in  $f_0$  if  $h$  is not recalculated.

## Intuition

It might not be immediately clear why a too low value for  $h$  might cause instability. Some intuition is provided in [23, Fig. 6.9], but here I would like to provide an alternative, hopefully more tangible way to see this.

In a FD implementation of the 1D wave equation, grid points can only affect their neighbours as seen in update equation (2.44). Using the values in Table 2.1 as an example,  $N = 30$  and if  $\lambda = 1$ , it takes exactly 30 samples, or iterations of Eq. (2.44), for a wave to travel from one boundary to the other.

If  $h$  were to be chosen to be twice as big so that there are only half as many intervals between the grid points as per Eq. (2.53) ( $N = 15$ ), the grid points could be set to ‘affect’ their neighbours to a lesser degree. This way, the wave still takes the same amount of time to travel between the boundaries and the fundamental frequency stays approximately the same. This is essentially what happens when  $\lambda < 1$  (in this case  $\lambda = 0.5$ ) and can be observed from the update equation in Eq. (2.44); the effect that the neighbouring grid points have on each other will indeed be less. The output of the system will have approximately the same fundamental frequency as if  $\lambda = 1$ , but its partials will be detuned due to numerical dispersion as explained in this section.

If, on the other hand,  $h$  were to be chosen to be twice as small so that there are twice as many intervals between grid points ( $N = 60$ ), it is impossible for the waves to travel from one boundary to the other in 30 samples. If they

could interact with their second neighbour, this would be possible, but the FD scheme in (2.42) does not allow for this. Indeed, as  $\lambda = 2$  in this case, the effect that the grid points have on each other will be disproportionate. In a way, grid points have too much energy that they can not lose to their neighbours, because their effect should have reached their second neighbour over the course of one sample. The way to solve this would be to halve the time step  $k$  (or double the sample rate  $f_s$ ), which would allow grid points to interact with their second neighbours over the course of once the old time step (as this is now divided into two time steps). This also shows in the fact that  $\lambda = 1$  again (as halving  $k$  cancels out halving  $h$ ) and grid points transfer their energy to their neighbours proportionately again.

figure?

### Possible solution

One of the main contributions of the PhD project is published in paper [G] and summarised in Chapter ??, where a ‘fractional’ number of intervals is introduced. This removes the necessity of the flooring operation in Eq. (2.53) and circumvents the recalculation of  $h$  to always satisfy the stability condition with equality while retaining the correct fundamental frequency.



# Chapter 3

# Analysis Techniques

This chapter provides some useful techniques to analyse FD schemes. Techniques to analyse PDEs also exist, but the focus here is of a practical nature and will especially revolve around the discrete schemes. This chapter can be seen as a ‘tutorial’ on how to use these techniques. Starting off with some necessary theory on matrices in a FDTD context and other mathematical tools, this chapter continues to introduce

- *Frequency domain analysis*, which can be used to determine stability conditions of (linear and time-invariant) FD schemes,
- *Energy analysis*, which can both be used to debug implementations of FD schemes, as well as determine stability conditions in a more general fashion, and
- *Modal analysis* which can be used to determine the modal frequencies (and damping per mode) that a FD scheme exhibits.

## 3.1 Matrices in a FDTD context

For several purposes, such as implementation in MATLAB and several analysis techniques described shortly, it is useful to write a FD scheme in *matrix form*.<sup>1</sup> Matrix multiplication when working with FDTD methods usually involves multiplying a square matrix (with equal rows and columns) onto a column vector. Consider a  $(N + 1) \times (N + 1)$  square matrix  $\mathbf{A}$  and a  $(N + 1) \times 1$  column vector  $\mathbf{u}$ . Multiplying these results in a  $(N + 1) \times 1$  column vector  $\mathbf{w}$ :

$$\mathbf{A}\mathbf{u} = \mathbf{w}. \quad (3.1)$$

---

<sup>1</sup>Appendix B provides some basic knowledge on matrices and linear algebra for those unfamiliar with this.

Expanding this operation results in

$$\underbrace{\begin{bmatrix} a_{00} & a_{01} & \dots & a_{0N} \\ a_{10} & a_{11} & \dots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N0} & a_{N1} & \dots & a_{NN} \end{bmatrix}}_A \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix}}_u = \underbrace{\begin{bmatrix} a_{00}u_0 + a_{01}u_1 + \dots + a_{0N}u_N \\ a_{10}u_0 + a_{11}u_1 + \dots + a_{1N}u_N \\ \vdots \\ a_{N0}u_0 + a_{N1}u_1 + \dots + a_{NN}u_N \end{bmatrix}}_w \quad (3.2)$$

where the indexing of the matrix elements starts at 0 rather than 1 here, as it relates better to operations used in a FDTD context.

### 3.1.1 FD operators in matrix form

FD operators approximating spatial derivatives and averages introduced in Section 2.2.2 can be written in matrix form and applied to a column vector  $\mathbf{u}^n$  containing the state of the system at time index  $n$ . These matrices are square and their sizes depend on the number of grid points the system is described for and on the boundary conditions. Not assuming a specific size for now, the FD operators in (2.6) can be written in matrix form according to

$$\mathbf{D}_{x+} = \frac{1}{h} \begin{bmatrix} \ddots & \ddots & & 0 \\ & -1 & 1 & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & & \\ & & & & -1 & \ddots & & \\ 0 & & & & & & \ddots & \ddots \end{bmatrix} \quad \mathbf{D}_{x-} = \frac{1}{h} \begin{bmatrix} \ddots & & & 0 \\ & \ddots & & 1 \\ & & -1 & 1 \\ & & & -1 & 1 \\ & & & & -1 & 1 \\ 0 & & & & & & \ddots & \ddots \end{bmatrix}$$

$$\mathbf{D}_{x\cdot} = \frac{1}{2h} \begin{bmatrix} \ddots & \ddots & & 0 \\ & \ddots & 0 & 1 \\ & & -1 & 0 & 1 \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 & \ddots \\ 0 & & & & & & \ddots & \ddots \end{bmatrix}$$

where the diagonal dots denote that the values on the respective diagonals continue until the top-left and bottom-right corners of the matrix. The 0s indicate that the rest of the values in the matrix are zeros.

Averaging operators  $\mu_{x+}$ ,  $\mu_{x-}$  and  $\mu_{x\cdot}$  are defined in a similar way:

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explain it?

### 3.1. Matrices in a FDTD context

$$\mathbf{M}_{x+} = \frac{1}{2} \begin{bmatrix} \ddots & \ddots & & \mathbf{0} \\ & 1 & 1 & & \\ & & 1 & 1 & & \\ & & & 1 & 1 & & \\ & & & & 1 & \ddots & \\ \mathbf{0} & & & & & & \ddots \end{bmatrix} \quad \mathbf{M}_{x-} = \frac{1}{2} \begin{bmatrix} \ddots & & & \mathbf{0} \\ & \ddots & 1 & & \\ & & 1 & 1 & & \\ & & & 1 & 1 & & \\ & & & & 1 & 1 & & \\ \mathbf{0} & & & & & & \ddots & \ddots \end{bmatrix}$$

$$\mathbf{M}_{x\cdot} = \frac{1}{2} \begin{bmatrix} \ddots & \ddots & & \mathbf{0} \\ & \ddots & 0 & 1 & & \\ & & 1 & 0 & 1 & & \\ & & & 1 & 0 & 1 & & \\ & & & & 1 & 0 & \ddots & \\ \mathbf{0} & & & & & & \ddots & \ddots \end{bmatrix}$$

It is important to notice that only spatial operators are written in this matrix form and then applied to state vectors at different time steps ( $\mathbf{u}^{n+1}$ ,  $\mathbf{u}^n$  and  $\mathbf{u}^{n-1}$ ).

Finally, the identity matrix is a matrix with only 1s on the diagonal and 0s elsewhere:

$$\mathbf{I} = \begin{bmatrix} \ddots & & & \mathbf{0} \\ & 1 & & & \\ & & 1 & & & \\ & & & 1 & & & \\ \mathbf{0} & & & & & & \ddots \end{bmatrix},$$

and has the following special property

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}.$$

#### 3.1.2 Schemes and update equations in matrix form

With the spatial operators in matrix form presented above, the FD scheme of the 1D wave equation in Eq. (2.42) can be written in matrix form.

If the Dirichlet boundary conditions in (2.48a) are used, the end points of the system do not have to be included in the calculation. The values of the grid function  $u_l^n$  for  $l \in \{1, \dots, N-1\}$  can then be stored in a column vector according to  $\mathbf{u}^n = [u_1^n, \dots, u_{N-1}^n]^T$ . Furthermore,  $(N-1) \times (N-1)$  matrix

$\mathbf{D}_{xx}$  is defined as

$$\mathbf{D}_{xx} = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & \mathbf{0} \\ 1 & -2 & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \\ & 1 & -2 & 1 & \\ \mathbf{0} & & 1 & -2 & \end{bmatrix}. \quad (3.3)$$

If instead, Neumann boundary conditions in Eq. (2.48a) are used, the values of  $u_l^n$  for the full range  $l \in \{0, \dots, N\}$  need to be stored as  $\mathbf{u}^n = [u_0^n, \dots, u_N^n]^T$  and the  $(N+1) \times (N+1)$  matrix  $\mathbf{D}_{xx}$  will be

$$\mathbf{D}_{xx} = \frac{1}{h^2} \begin{bmatrix} -2 & 2 & & & \mathbf{0} \\ 1 & -2 & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \\ & 1 & -2 & 1 & \\ \mathbf{0} & & 2 & -2 & \end{bmatrix}, \quad (3.4)$$

where the 2s in the top and bottom row correspond to the multiplication by 2 with  $u_1^n$  and  $u_{N-1}^n$  in update equations (2.49) and (2.50) respectively.

Regardless of the boundary conditions, the FD scheme in (2.42) can be written in matrix form as

$$\frac{1}{k^2} (\mathbf{u}^{n+1} - 2\mathbf{u} + \mathbf{u}^{n-1}) = c^2 \mathbf{D}_{xx} \mathbf{u}^n, \quad (3.5)$$

and rewritten to a matrix form of the update equation analogous to Eq. (2.44)

$$\mathbf{u}^{n+1} = (2\mathbf{I} + c^2 k^2 \mathbf{D}_{xx}) \mathbf{u}^n - \mathbf{u}^{n-1}. \quad (3.6)$$

The identity matrix is necessary here for correct matrix addition.

## 3.2 Mathematical tools and product identities

Some useful mathematical tools used for the energy analysis techniques presented in Section 3.4 will be shown here. The tools shown here can be applied to 1D systems. These will be extended to 2D systems in Chapter 6. Unless denoted otherwise, the notation and theory will follow [23].

### 3.2.1 Inner product

For two functions  $f = f(x, t)$  and  $g = g(x, t)$  defined for  $x \in \mathcal{D}$  where  $\mathcal{D} = [0, L]$ , their  $l_2$  inner product and  $l_2$  norm are defined as

$$\langle f, g \rangle_{\mathcal{D}} = \int_{\mathcal{D}} f g dx \quad \text{and} \quad \|f\|_{\mathcal{D}} = \sqrt{\langle f, f \rangle_{\mathcal{D}}}. \quad (3.7)$$

### 3.2. Mathematical tools and product identities

These functions do not have to be time-dependent (i.e., they can also simply be  $f(x)$  and  $g(x)$ ), but as all functions used in this work are in fact time-dependent, this is left for coherence. It is also important to note that these functions do not have to be ‘isolated’ state variables per se (such as  $u(x, t)$  used in the previous chapter), but could also be state variables with a derivative applied to it (such as  $\partial_t u(x, t)$ ).

The discrete inner product of any two (1D) functions  $f_l^n$  and  $g_l^n$  defined for  $l \in d$ , with discrete domain  $d = \{0, \dots, N\}$ , is

$$\langle f_l^n, g_l^n \rangle_d = \sum_{l=0}^N h f_l^n g_l^n, \quad (3.8)$$

where the multiplication by  $h$  is the discrete counterpart of  $dx$  in the continuous definition in (3.7). Also useful are the primed inner product

$$\langle f_l^n, g_l^n \rangle'_d = \sum_{l=1}^{N-1} h f_l^n g_l^n + \frac{h}{2} f_0^n g_0^n + \frac{h}{2} f_N^n g_N^n, \quad (3.9)$$

and the more general weighted inner product

$$\langle f_l^n, g_l^n \rangle_d^{\epsilon_l, \epsilon_r} = \sum_{l=1}^{N-1} h f_l^n g_l^n + \frac{\epsilon_l}{2} h f_0^n g_0^n + \frac{\epsilon_r}{2} h f_N^n g_N^n, \quad (3.10)$$

where free parameters  $0 < \epsilon_l, \epsilon_r \leq 2$  scale the boundary points of the regular inner product. Naturally, if  $\epsilon_l = \epsilon_r = 1$ , Eq. (3.10) reduces to Eq. (3.9), and if  $\epsilon_l = \epsilon_r = 2$ , (3.10) reduces to (3.8).

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#### 3.2.2 Summation by parts

Extremely useful when performing energy analysis on distributed systems is *summation by parts*, which is the discrete counterpart of integration by parts. Although its application will only be apparent when actually performing an energy analysis (see fx. Sections 3.4.3 and 4.4), some definitions will be presented here for future reference.

Here, the same functions as in the previous section,  $f(x, t)$  and  $g(x, t)$  and domain  $\mathcal{D}$ , will be used. Applying a spatial derivative to  $g$ , and using Eq. (3.7), integration by parts is defined as

$$\langle f, \partial_x g \rangle_{\mathcal{D}} = -\langle \partial_x f, g \rangle_{\mathcal{D}} + fg|_0^L \quad (3.11)$$

where  $fg|_0^L$  describes the boundary terms that appeared in the process. One can observe that the spatial derivative switched functions, and is now applied to  $f$  rather than  $g$ .

In discrete time, using the same two (1D) functions as before:  $f_l^n$  and  $g_l$  and are defined for  $l \in d$  with discrete domain  $d = \{0, \dots, N\}$ . Then, using the discrete inner product in Eq. (3.8), two variants of summation by parts are defined as

$$\langle f_l^n, \delta_{x-} g_l^n \rangle_d = -\langle \delta_{x+} f_l^n, g_l^n \rangle_d + f_{N+1}^n g_N^n - f_0^n g_{-1}^n, \quad (3.12a)$$

$$\langle f_l^n, \delta_{x+} g_l^n \rangle_d = -\langle \delta_{x-} f_l^n, g_l^n \rangle_d + f_N^n g_{N+1}^n - f_{-1}^n g_0^n. \quad (3.12b)$$

A derivation of Eq. (3.12a) is given below. As in the case of integration by parts in Eq. (3.11), the process of summation by parts causes the derivative to be applied to the other function and the sign of the resulting inner product changes. Important to note, is that the sign (forward / backward) of the derivative operator has also changed. Lastly, discrete boundary terms have appeared and it can be seen that values outside of the defined domain are needed, i.e.,  $g_{N+1}^n$  and  $f_{-1}^n$ . These can be accounted for by the boundary conditions imposed on the system (see Section 2.4.2 as an example).

One could also choose to work with reduced domains after summation by parts. Domains that have one fewer point at the boundaries are defined as  $\underline{d} = \{0, \dots, N-1\}$ ,  $\bar{d} = \{1, \dots, N\}$  and  $\bar{\underline{d}} = \{1, \dots, N-1\}$ . The following identities can be shown to hold

$$\langle f_l^n, \delta_{x-} g_l^n \rangle_d = -\langle \delta_{x+} f_l^n, g_l^n \rangle_{\underline{d}} + f_N^n g_N^n - f_0^n g_{-1}^n, \quad (3.13a)$$

$$\langle f_l^n, \delta_{x+} g_l^n \rangle_d = -\langle \delta_{x-} f_l^n, g_l^n \rangle_{\bar{d}} + f_N^n g_{N+1}^n - f_0^n g_0^n, \quad (3.13b)$$

and, using the primed inner product in Eq. (3.9),

$$\langle f_l^n, \delta_{x-} g_l^n \rangle'_d = -\langle \delta_{x+} f_l^n, g_l^n \rangle_{\underline{d}} + f_N^n \mu_{x-} g_N^n - f_0^n \mu_{x-} g_0^n, \quad (3.14a)$$

$$\langle f_l^n, \delta_{x+} g_l^n \rangle'_d = -\langle \delta_{x-} f_l^n, g_l^n \rangle_{\bar{d}} + f_N^n \mu_{x+} g_N^n - f_0^n \mu_{x+} g_0^n, \quad (3.14b)$$

or the more general weighted inner product in Eq. (3.10)

$$\begin{aligned} \langle f_l^n, \delta_{x-} g_l^n \rangle_d^{\epsilon_l, \epsilon_r} &= -\langle \delta_{x+} f_l^n, g_l^n \rangle_{\underline{d}} + f_N^n g_{N-1}^n - f_0^n g_0^n \\ &\quad + \frac{\epsilon_r}{2} f_N^n (g_N^n - g_{N-1}^n) + \frac{\epsilon_l}{2} f_0^n (g_0^n - g_{-1}^n), \end{aligned} \quad (3.15a)$$

$$\begin{aligned} \langle f_l^n, \delta_{x+} g_l^n \rangle_d^{\epsilon_l, \epsilon_r} &= -\langle \delta_{x-} f_l^n, g_l^n \rangle_{\bar{d}} + f_N^n g_N^n - f_0^n g_1^n \\ &\quad + \frac{\epsilon_r}{2} f_N^n (g_{N+1}^n - g_N^n) + \frac{\epsilon_l}{2} f_0^n (g_1^n - g_0^n), \end{aligned} \quad (3.15b)$$

will it though? all of which will prove useful in energy analysis techniques later on. A derivation of (3.13a) is given below.

Finally, recalling that  $\delta_{xx} = \delta_{x+}\delta_{x-}$ , one can apply summation by parts twice to get the following identities

$$\langle f, \delta_{xx} g \rangle_d = \langle \delta_{xx} f, g \rangle_d + f_N \delta_{x+} g_N - g_N \delta_{x+} f_N - f_0 \delta_{x-} g_0 + g_0 \delta_{x-} f_0, \quad (3.16a)$$

$$\langle f, \delta_{xx} g \rangle_d = \langle \delta_{xx} f, g \rangle_{\bar{d}} + f_N \delta_{x+} g_N - g_N \delta_{x-} f_N - f_0 \delta_{x-} g_0 + g_0 \delta_{x+} f_0, \quad (3.16b)$$

$$\langle f, \delta_{xx} g \rangle'_d = \langle \delta_{xx} f, g \rangle'_d + f_N \delta_{x+} g_N - g_N \delta_{x-} f_N - f_0 \delta_{x-} g_0 + g_0 \delta_{x+} f_0. \quad (3.16c)$$

## Derivations

To see why the above identities hold true, it is useful to briefly go through a derivation. As an example, Eqs. (3.12a) and (3.13a) are derived as they have the same inner product as a starting point, but yield different results. In the following,  $d = \{0, \dots, N\}$  and  $N = 2$  are used.

Starting with Eq. (3.12a), suppressing the  $n$  superscript for brevity, and using the definition for the discrete inner product in Eq. (3.8), yields

$$\begin{aligned}
 \langle f_l, \delta_{x-} g_l \rangle_d &= \sum_{l=0}^2 h f_l \frac{1}{h} (g_l - g_{l-1}), \\
 &= f_0 g_0 - f_0 g_{-1} + f_1 g_1 - f_1 g_0 + f_2 g_2 - f_2 g_1, \\
 &= g_0(f_0 - f_1) - f_0 g_{-1} + g_1(f_1 - f_2) + g_2(f_2 - f_3) + f_3 g_2, \\
 &= -g_0(f_1 - f_0) - g_1(f_2 - f_1) - g_2(f_3 - f_2) + f_3 g_2 - f_0 g_{-1}, \\
 &= -\sum_{l=0}^2 h g_l \frac{1}{h} (f_{l+1} - f_l) + f_3 g_2 - f_0 g_{-1}, \\
 &= -\langle \delta_{x+} f_l, g_l \rangle_d + f_3 g_2 - f_0 g_{-1}.
 \end{aligned}$$

As  $N = 2$ , the result is identical to Eq. (3.12a).

Similarly, identity (3.13a) can be proven to hold:

$$\begin{aligned}
 \langle f_l, \delta_{x-} g_l \rangle_d &= \sum_{l=0}^2 h f_l \frac{1}{h} (g_l - g_{l-1}), \\
 &= f_0 g_0 - f_0 g_{-1} + f_1 g_1 - f_1 g_0 + f_2 g_2 - f_2 g_1, \\
 &= -f_0 g_{-1} + g_0(f_0 - f_1) + g_1(f_1 - f_2) + f_2 g_2, \\
 &= -g_0(f_1 - f_0) - g_1(f_2 - f_1) + f_2 g_2 - f_0 g_{-1}, \\
 &= \sum_{l=0}^1 h g_l \frac{1}{h} (f_{l+1} - f_l) + f_2 g_2 - f_0 g_{-1}, \\
 &= -\langle \delta_{x+} f_l, g_l \rangle_d + f_2 g_2 - f_0 g_{-1},
 \end{aligned}$$

where the resulting inner product has a reduced domain of  $\underline{d} = \{0, \dots, N-1\}$ . Similar processes can be used to prove the other identities presented in this section.

### 3.2.3 Product identities

Some useful identities used in this work are

$$(\delta_t \cdot u_l^n)(\delta_{tt} u_l^n) = \delta_{t+} \left( \frac{1}{2} (\delta_{t-} u_l^n)^2 \right), \quad (3.17a)$$

$$(\delta_t \cdot u_l^n) u_l^n = \delta_{t+} \left( \frac{1}{2} u_l^n e_{t-} u_l^n \right), \quad (3.17b)$$

$$(\delta_{t+} u_l^n)(\mu_{t+} u_l^n) = \delta_{t+} \left( \frac{1}{2} (u_l^n)^2 \right), \quad (3.17c)$$

$$(\delta_t \cdot u_l^n)(\mu_{t-} u_l^n) = \delta_{t+} \left( \frac{1}{2} \mu_{t-} (u_l^n)^2 \right), \quad (3.17d)$$

$$(\delta_t \cdot u_l^n)(\mu_{tt} u_l^n) = \delta_{t+} \left( \frac{1}{8} (u_l^n + e_{t-} u_l^n)^2 \right) \quad (3.17e)$$

$$u_l^n e_{t-} u_l^n = (\mu_{t-} u_l^n)^2 - \frac{k^2}{4} (\delta_{t-} u_l^n)^2 \quad (3.17f)$$

These identities can be used for spatial derivatives as well by substituting the 't' subscripts for 'x'.

When an operator is applied to a product of two grid functions, the discrete counterpart of the product rule needs to be used according to

$$\delta_{t+}(u_l^n w_l^n) = (\delta_{t+} u_l^n)(\mu_{t+} w_l^n) + (\mu_{t+} u_l^n)(\delta_{t+} w_l^n). \quad (3.18)$$

The same rule applies when the backward operator  $\delta_{t-}(u_l^n w_l^n)$  or centred operator  $\delta_t(u_l^n w_l^n)$  is used. In that case, the forward operators  $\delta_{t+}$  and  $\mu_{t+}$  in Eq. (3.18) need to be substituted for the backward or centred versions of the operators respectively.

## 3.3 Frequency domain analysis

Frequency domain analysis, also called Fourier analysis, is a way to determine various properties of a FD scheme, including conditions for stability. The process is similar to finding stability for digital filters. In essence, a FD scheme can be seen as a complex filter of which its coefficients are defined by physical parameters. This section will explain how to obtain a frequency domain representation of a scheme and will mainly follow [23], albeit in a slightly more practical manner.

### Frequency domain representation and ansatz

Frequency domain analysis of FD schemes starts by performing a *z-transform* on the scheme. The z-transform converts a discrete signal into a frequency

### 3.3. Frequency domain analysis

domain representation, and is extensively used in the field of digital signal processing (DSP) to analyse the behaviour and especially stability of digital filters. To not go too much into detail here, the interested reader is referred to the very comprehensive explanation on the z-transform given in [54, Ch. 5].

If a system is distributed in space, one can perform a spatial Fourier transform on a grid function. Frequency domain analysis in the distributed case is called *von Neumann analysis* which first appeared in [55] co-authored by John von Neumann. Later, this technique got a more general treatment in [56] and is heavily used in [23]. The discrete-time z-transform and discrete spatial Fourier transform performed on a 1D grid function are defined as [23]

$$\hat{u} = \sum_{n=-\infty}^{\infty} u_l^n z^{-n} \quad \text{and} \quad \tilde{u} = \sum_{l=-\infty}^{\infty} u_l^n e^{-jl\beta h} \quad (3.19)$$

with complex number  $z = e^{sk}$ , complex frequency  $s = j\omega + \sigma$  (more elaborated on in 3.5) and real wavenumber  $\beta$ . Frequency domain analysis in 2D will be elaborated on in Section 6.2.4.

A shortcut to performing a full frequency domain analysis is to use a test solution, or *ansatz*, and replace the grid functions by their transforms. The grid function for a 1D system can be replaced by an ansatz of the form (1D) [56]

$$u_l^n \xrightarrow{\mathcal{A}} z^n e^{jl\beta h} \quad (3.20)$$

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where " $\xrightarrow{\mathcal{A}}$ " indicates to replace the grid function with the ansatz (the shortcut to taking the full z-transform and spatial Fourier transform).

Like in the DSP realm, the power of  $z$  indicates a temporal shift, i.e.,  $z^{-1}$  is a one-sample delay. In a FDTD context, this corresponds to a time shift as seen in Section 2.2.2. For spatially distributed systems, a shift in  $l$  can be interpreted as a phase shift of a frequency with wavenumber  $\beta$ . See Table 3.1

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for the frequency domain representation of grid functions with their temporal and spatial indices shifted in different ways.

Using these definitions, the effect of various operators on a grid function can be written in their frequency domain representation. For systems distributed in space, the following trigonometric identities are extremely useful when performing the analyses [57, p. 71]:

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \Rightarrow \sin^2(x) = \frac{e^{j2x} + e^{-j2x}}{-4} + \frac{1}{2}, \quad (3.21a)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \Rightarrow \cos^2(x) = \frac{e^{j2x} + e^{-j2x}}{4} + \frac{1}{2}. \quad (3.21b)$$

Take for example

$$\delta_{xx} u_l^n = \frac{1}{h^2} (u_{l+1}^n - 2u_l^n + u_{l-1}^n) \xrightarrow{\mathcal{A}} \frac{1}{h^2} (e^{j\beta h} - 2 + e^{-j\beta h}).$$

Grid function	Ansatz	Result
$u_l^n$	$z^0 e^{j0\beta h}$	1
$u_l^{n+1}$	$z^1 e^{j0\beta h}$	$z$
$u_l^{n-1}$	$z^{-1} e^{j0\beta h}$	$z^{-1}$
$u_{l+1}^n$	$z^0 e^{j1\beta h}$	$e^{j\beta h}$
$u_{l-1}^n$	$z^0 e^{j(-1)\beta h}$	$e^{-j\beta h}$
$u_{l+2}^n$	$z^0 e^{j2\beta h}$	$e^{j2\beta h}$
$u_{l-2}^n$	$z^0 e^{j(-2)\beta h}$	$e^{-j2\beta h}$
$u_{l+1}^{n-1}$	$z^{-1} e^{j1\beta h}$	$z^{-1} e^{j\beta h}$
$u_{l-1}^{n-1}$	$z^{-1} e^{j(-1)\beta h}$	$z^{-1} e^{-j\beta h}$

**Table 3.1:** Frequency domain representation of a grid function using ansatz (3.20) with frequently appearing temporal and spatial shifts.

Then, using  $x = \beta h/2$ , identity (3.21a) can be rewritten to

$$e^{j\beta h} - 2 + e^{-j\beta h} = -4 \sin^2(\beta h/2),$$

and substituted into the above to get

$$\delta_{xx} u_l^n \xrightarrow{\mathcal{A}} -\frac{4}{h^2} \sin^2(\beta h/2).$$

Examples of various temporal FD operators applied to grid functions in their frequency domain representation are

$$\begin{aligned} \delta_{t+} u_l^n &\xrightarrow{\mathcal{A}} \frac{1}{k} (z - 1), & \delta_{t-} u_l^n &\xrightarrow{\mathcal{A}} \frac{1}{k} (1 - z^{-1}), \\ \delta_{t \cdot} u_l^n &\xrightarrow{\mathcal{A}} \frac{1}{2k} (z - z^{-1}), & \delta_{tt} u_l^n &\xrightarrow{\mathcal{A}} \frac{1}{k^2} (z - 2 + z^{-1}) \end{aligned} \quad (3.22)$$

and for spatial operators identity (3.21a) can be used to obtain

$$\delta_{xx} u_l^n \xrightarrow{\mathcal{A}} -\frac{4}{h^2} \sin^2(\beta h/2), \quad (3.23a)$$

$$\delta_{xxxx} u_l^n \xrightarrow{\mathcal{A}} \frac{16}{h^4} \sin^4(\beta h/2). \quad (3.23b)$$

### Proving stability

Similar to digital filters, the system is stable when the roots of the characteristic polynomial in  $z$  are bounded by 1 (unity)

only the denominator of the transfer function

$$|z| \leq 1. \quad (3.24)$$

### 3.3. Frequency domain analysis

In the FDTD context, the frequency domain representation of a FD scheme results in a *characteristic equation* – which is usually a second-order polynomial – in  $z$  and needs to satisfy condition (3.24) for all wave numbers  $\beta$ . It can be shown that for a polynomial of the form

$$z^2 + a^{(1)}z + a^{(2)} \quad (3.25)$$

its roots satisfy condition (3.24) when it abides the following condition [23]

$$|a^{(1)}| - 1 \leq a^{(2)} \leq 1. \quad (3.26)$$

If  $a^{(2)} = 1$ , the simpler condition

$$|a^{(1)}| \leq 2, \quad (3.27)$$

suffices.

#### 3.3.1 Mass-spring system

Recalling the FD scheme of the mass-spring system in Eq. (2.35)

$$M\delta_{tt}u^n = -Ku^n$$

a frequency domain representation can be obtained using the ansatz in (3.20) with  $l = 0$ . Using Table 3.1 and Eqs. (3.22) as a reference and substituting the definitions yields

$$\frac{M}{k^2} (z - 2 + z^{-1}) = -K.$$

Gathering the terms and moving all to the left-hand side, the characteristic equation for the mass-spring system can be obtained:

$$z - \left(2 - \frac{Kk^2}{M}\right) + z^{-1} = 0. \quad (3.28)$$

To begin to prove stability, this equation needs to be written in the form found in (3.25). Multiplying all the terms by  $z$ , and noticing that  $a^{(2)} = 1$ , one could continue with condition (3.27). However, the scheme used here is a special case where the roots of the characteristic equation can not be identical [23]. When this happens, the output of the system will grow linearly and is called “marginally unstable”. This means that  $|a^{(1)}| \neq 1$  and the condition in (3.27) becomes  $|a^{(1)}| < 2$ . Continuing with this conditions yields

$$\begin{aligned} \left| -2 + \frac{Kk^2}{M} \right| &< 2, \\ -2 < -2 + \frac{Kk^2}{M} &< 2, \\ 0 < \frac{Kk^2}{M} &< 4. \end{aligned}$$

If only non-zero values are chosen for  $K$ ,  $k$  and  $M$  they are positive (as they are already defined as being non-negative) and the first condition is always satisfied. The second condition is then easily solved for  $k$  by

$$k < 2\sqrt{\frac{M}{K}}. \quad (3.29)$$

Recalling that  $\omega_0 = \sqrt{K/M}$ , Eq (3.29) can be more compactly written as

$$k < \frac{2}{\omega_0}. \quad (3.30)$$

### 3.3.2 1D wave equation

This section will derive the stability condition for the 1D wave equation presented in Section 2.4 using von Neumann analysis.

Recalling the FD scheme in (2.42):

$$\delta_{tt} u_l^n = c^2 \delta_{xx} u_l^n,$$

its frequency domain representation can be obtained using the definitions in Eqs. (3.22) and (3.23a):

$$\frac{1}{k^2} (z - 2 + z^{-1}) = -\frac{4c^2}{h^2} \sin^2(\beta h/2). \quad (3.31)$$

Also recalling that

$$\lambda = \frac{ck}{h},$$

the characteristic equation of the 1D wave equation is

$$z + (4\lambda^2 \sin^2(\beta h/2) - 2) + z^{-1} = 0. \quad (3.32)$$

The scheme is then stable if the roots satisfy condition (3.24). As the characteristic equation is of the form in (3.25) (after multiplication with  $z$ ) with  $a^{(2)} = 1$ , stability is shown by abiding condition (3.27) for all  $\beta$  and when applied to the characteristic equation (3.32), it can be seen that

$$\begin{aligned} |4\lambda^2 \sin^2(\beta h/2) - 2| &\leq 2, \\ |2\lambda^2 \sin^2(\beta h/2) - 1| &\leq 1, \\ -1 &\leq 2\lambda^2 \sin^2(\beta h/2) - 1 \leq 1, \\ 0 &\leq 2\lambda^2 \sin^2(\beta h/2) \leq 2, \\ 0 &\leq \lambda^2 \sin^2(\beta h/2) \leq 1. \end{aligned}$$

Observing that all terms in  $\lambda^2 \sin^2(\beta h/2)$  are squared, this term will always be non-negative and therefore always satisfy the first condition. Continuing with

### 3.4. Energy analysis

the second condition, and knowing that the  $\sin^2(\beta h/2)$ -term is bounded by 1 for all  $\beta$ , yields the following stability condition:

$$\lambda \leq 1.$$

This is the CFL condition given in Eq. (2.46). To obtain the stability condition in terms of the grid spacing, the definition for  $\lambda$  is substituted and written in terms of the grid spacing

$$h \geq ck, \quad (3.33)$$

which is the stability condition given in Eq. (2.47).

### 3.3.3 Discussion

Although frequency domain analysis is very useful, it can only be applied to linear and time-invariant (LTI) systems and linear and shift-invariant (LSI) systems. These, respectively, describe systems whose properties do not change over time (LTI) and space (LSI).<sup>2</sup> Furthermore, the analysis assumes systems with infinite domains, so boundary conditions are not included. Energy analysis techniques, on the other hand, allow these types of systems, even nonlinear systems, to be analysed, as well as being able to handle boundary conditions. This will be presented below.

## 3.4 Energy analysis

Of all analysis techniques described in this chapter, energy analysis is without a doubt the most important when working with FDTD methods. First of all, from a practical point of view, it is essential for debugging implementations of FD schemes. Especially when trying to model more complex systems, programming errors are unavoidable, and energy analysis can be extremely helpful in pinpointing where the error lies. Secondly, energy analysis techniques can be used to obtain stability conditions in a much more general sense than the frequency domain analysis techniques presented in Section 3.3. Where frequency domain analysis is restricted to LTI and LSI systems with infinite domains (for distributed systems), energy analysis can be applied to nonlinear systems and include boundary conditions [23].

Gustafsson et al. in (the first edition of) [58] worked with energy to find stability conditions for FD schemes. This, they referred to as ‘the energy method’ and it effectively circumvented the need of a frequency domain representation to find stability conditions (as presented in Section 3.3). Later, energy, or more specifically ‘energy as a conserved quantity’, was used to determine stability

---

<sup>2</sup>Acoustic tubes with a spatially-varying cross-section presented in Chapter 5 are examples of non-LSI systems.

and passivity of systems. Bilbao gives an extensive overview in [23] where this has been extensively used to show stability of the FD schemes used.

One of the main goals when performing energy analysis is to find an expression for the total energy present in the system. This is referred to as the *Hamiltonian* and denoted by  $\mathfrak{H}$  in continuous time and  $\mathfrak{h}$  in discrete time. In this work, the focus of the energy analysis will be in discrete time.

In this section, four steps are presented and can be followed to perform a full energy analysis of a FD scheme and implement it afterwards. Then, the analysis will be performed on the mass-spring system and the 1D wave equation presented in Chapter 2. Finally, it will be shown how to obtain stability conditions through the techniques presented in this section.

### 3.4.1 Energy analysis: A 4-step tutorial

#### Step 1: Obtain the rate of change of the total energy $\delta_{t+}\mathfrak{h}$

The first step to energy analysis is to take the appropriate *norm* of the scheme (see Eq. (3.7)), which yields an expression for the rate of change of the energy of the system:  $\delta_{t+}\mathfrak{h}$ . Usually, this means to take the inner product of the scheme with  $(\delta_t u_i^n)$ . See Section 3.2.1 for more details on the inner product. Note that the forward time difference  $\delta_{t+}$  is used (and not the backwards or centred) because of convention and preference [Bilbao, verbally].

For the units of the resulting energy balance to add up (also see Step 3), it is useful to perform the analysis on a scheme with all physical parameters written out (so the discretised version of Eq. (2.28) rather than Eq. (2.29)).

#### Step 2: Identify different types of energy and obtain the total energy $\mathfrak{h}$ by isolating $\delta_{t+}$

The energy of a FD scheme can generally be divided into three different types: the total energy contained within the system, or Hamiltonian  $\mathfrak{h}$ , energy losses through damping  $\mathfrak{q}$  and energy input through external forces or excitations  $\mathfrak{p}$ . For distributed systems, an additional boundary term  $\mathfrak{b}$  appears, but vanishes under ‘regular’ (lossless and not energy-storing) boundary conditions. Nearly any energy balance is thus of the form

$$\delta_{t+}\mathfrak{h} = \mathfrak{b} - \mathfrak{q} - \mathfrak{p}. \quad (3.34)$$

This equation essentially says that the total energy present in the system changes due to losses and inputs. For a lossless system without externally supplied energy over the course of the simulation (so initial conditions excluded), the energy should remain unchanged over the course of the simulation,

$$\delta_{t+}\mathfrak{h} = 0 \implies \mathfrak{h}^n = \mathfrak{h}^0. \quad (3.35)$$

### 3.4. Energy analysis

As the eventual interest lies in the total energy of the system  $\mathfrak{h}$  and not its rate of change,  $\delta_{t+}$  must be isolated in the definition of  $\delta_{t+}\mathfrak{h}$ . In this step, the identities in Section 3.2.3 will come in handy, as well as summation by parts described in Section 3.2.2 for distributed systems.

The Hamiltonian itself can usually be further subdivided into kinetic energy and potential energy, denoted by the symbols  $\mathfrak{t}$  and  $\mathfrak{v}$  respectively:

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{v} \quad (3.36)$$

As a rule of thumb, the definition for kinetic energy contains ‘velocity squared’ (as in the classical-mechanics definition  $E_{\text{kin}} = \frac{1}{2}M\dot{u}$ ) and the potential energy includes the restoring forces of the system.

#### Step 3: Check the units in the expression for $\mathfrak{h}$

To know that the previous steps have been carried out correctly, it is good to check whether the units of the resulting expression for  $\mathfrak{h}$  is indeed in Joules, or  $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$ . The other quantities such as energy losses  $\mathfrak{q}$  and inputs  $\mathfrak{p}$ , should be in Joules per second or in SI units:  $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3}$ . As mentioned in Step 1, it is therefore useful to have all physical parameters written out so that the units will be correct in this step. Some information about operators and grid functions and how they ‘add’ units to the equation will be given below.

An (1D) inner product (or norm) will ‘add’ one ‘m’ unit due to the  $h$  in its definition in (3.7). A first-order temporal difference operator will ‘add’ one ‘ $\text{s}^{-1}$ -unit (because of the  $1/k$ ) and a first-order spatial difference operator will ‘add’ one ‘ $\text{m}^{-1}$ -unit ( $1/h$ ). Along these lines, a second-order time or difference operator will ‘add’ a ‘ $\text{s}^{-2}$ ’ ( $1/k^2$ ) or ‘ $\text{m}^{-2}$ -unit ( $1/h^2$ ) respectively. It is important to note that the time shift operator ( $e_{t-}$ ) does not influence the units. Finally, the appearance of a grid function  $u_l^n$  ‘adds’ whatever it describes. Usually, as  $u_l^n$  describes a displacement in m, it will ‘add’ this to the equation. If it describes anything else, it will ‘add’ that.

#### Step 4: Implement the definitions for energy and debug the FD scheme

In the end, the definition for the energy can be implemented and used as a check for whether the FD scheme has been implemented correctly. Usually, the energy of the system is calculated for every iteration in the for loop and plotted after the simulation. For a system without losses or energy inputs, the energy should be unchanged according to Eq. (3.35) and can be plotted according to

$$\mathfrak{h}_e^n = \frac{\mathfrak{h}^n - \mathfrak{h}^0}{\mathfrak{h}^0}, \quad \text{if } \mathfrak{h}^0 \neq 0, \quad (3.37)$$

where  $\mathfrak{h}_e^n$  can be seen as the normalised energy and shows the error variation. Although this equation should always return 0 (as  $\mathfrak{h}^n = \mathfrak{h}^0$ ), in a finite preci-

sion simulation, ultra slight fluctuations of the energy should be visible due to rounding errors. Plotting the Hamiltonian should show fluctuations within *machine precision*, which is usually in the range of  $10^{-15}$ . Over time, the fluctuations can add up, and possibly end up out of this range, but generally, any fluctuations less than in the  $10^{-10}$  range indicate that there is no programming error. See fx. Figures 3.1 and 3.2.

For a system with losses or energy inputs, a discrete integration, or summed form can be used (as done in fx. [59]):

$$\mathfrak{h}_e^n = \frac{\mathfrak{h}^n - \mathfrak{h}^0 + k \sum_{m=0}^{n-1} (\mathfrak{q}^m + \mathfrak{p}^m)}{\mathfrak{h}^0}, \quad \text{if } \mathfrak{h}^0 \neq 0. \quad (3.38)$$

check if the sum should indeed go until  $n - 1$  and why

### 3.4.2 Mass-spring system

Recalling the FD scheme for the simple mass-spring system in Eq. (2.34)

$$M\delta_{tt}u^n = -Ku^n$$

an energy analysis can be performed using the four steps described above.

#### Step 1: Obtain $\delta_{t+}\mathfrak{h}$

The energy balance of the simple mass-spring system presented in Section 2.3 can be obtained by first taking the product of scheme (2.34) with  $(\delta_{t+}u^n)$ :

$$\delta_{t+}\mathfrak{h} = M(\delta_{t+}u^n)(\delta_{tt}u^n) + K(\delta_{t+}u^n)(u^n) = 0. \quad (3.39)$$

Note that the inner product is not necessary as the system is not distributed.

#### Step 2: Identify energy types and isolate $\delta_{t+}$

As there are no losses or externally supplied energy present in the system, all terms are part of the Hamiltonian  $\mathfrak{h}$ . To isolate  $\delta_{t+}$  from (3.39), one can use identities (3.17a) and (3.17b) to get the following:

$$\delta_{t+}\mathfrak{h} = \delta_{t+} \left( \frac{M}{2}(\delta_{t-}u^n)^2 + \frac{K}{2}u^n e_{t-}u^n \right) = 0, \quad (3.40)$$

and the following definition for  $\mathfrak{h}$  can be obtained

$$\mathfrak{h} = \frac{M}{2}(\delta_{t-}u^n)^2 + \frac{K}{2}u^n e_{t-}u^n = 0. \quad (3.41)$$

This can be rewritten in terms of the kinetic energy  $\mathfrak{t}$  and potential energy  $\mathfrak{v}$  according to

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{v}, \quad \text{with} \quad \mathfrak{t} = \frac{M}{2}(\delta_{t-}u^n)^2, \quad \text{and} \quad \mathfrak{v} = \frac{K}{2}u^n e_{t-}u^n. \quad (3.42)$$

### 3.4. Energy analysis

#### Step 3: Check units

As mentioned above, the energy  $\mathfrak{h}$  needs to be in Joules, or  $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$ . Taking the terms in Eq. (3.42) one-by-one and writing them in their units results in

$$\begin{aligned}\mathfrak{t} &= \frac{M}{2} (\delta_{t-} u^n)^2 \xrightarrow{\text{in units}} \text{kg} \cdot (\text{s}^{-1} \cdot \text{m})^2 = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}, \\ \mathfrak{v} &= \frac{K}{2} u^n e_{t-} u^n \xrightarrow{\text{in units}} \text{N} \cdot \text{m}^{-1} \cdot \text{m} \cdot \text{m} = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2},\end{aligned}$$

and indeed have the correct units.

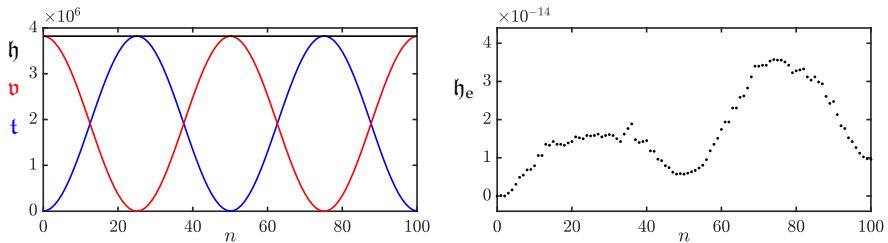
#### Step 4: Implementation

Equation (3.47) can then be implemented in the same for-loop recursion where the update is calculated.

```

1 %% Calculate the energy using Eq. (3.42)
2
3 % Kinetic energy
4 kinEnergy(n) = M / 2 * (1/k * (u - uPrev))^2;
5
6 % Potential energy
7 potEnergy(n) = K / 2 * u * uPrev;
8
9 % Total energy (Hamiltonian)
10 totEnergy(n) = kinEnergy(n) + potEnergy(n);
```

Figure 3.1 shows the normalised energy (according to Eq. (3.37)) of the mass-spring system and shows that the deviation is indeed within machine precision.



**Fig. 3.1:** The kinetic (blue), potential (red), and total (black) energy of an implementation of the mass-spring system are plotted in the left panel. The right panel shows the normalised energy (according to Eq. (3.37)). Notice that the scaling of the y-axis is  $10^{-14}$  and the energy is thus within machine precision.

### 3.4.3 1D wave equation

Energy analysis could be directly performed on the FD scheme in (2.42). However, in order for the units of the scheme to add up to energy in Joules, it is useful to write out all physical parameters. Taking the definition for the wave speed for the ideal string  $c = \sqrt{T/\rho A}$  and multiplying both sides of Eq. (2.42) by  $\rho A$  yields

$$\rho A \delta_{tt} u_l^n = T \delta_{xx} u_l^n, \quad (3.43)$$

where  $l \in d$  with discrete domain  $d \in \{0, \dots, N\}$  and number of grid points  $N + 1$ . Furthermore, Dirichlet boundary conditions as given in Eq. (2.48a) are used. A note on using Neumann boundary conditions is given at the end of this section.

#### Step 1: Obtain $\delta_{t+}\mathfrak{h}$

Taking an inner product using Eq. (3.43) with  $(\delta_t \cdot u_l^n)$  and moving all terms to the left-hand side yields the definition for the rate of change of the Hamiltonian:

$$\delta_{t+}\mathfrak{h} = \rho A \langle \delta_t \cdot u_l^n, \delta_{tt} u_l^n \rangle_d - T \langle \delta_t \cdot u_l^n, \delta_{xx} u_l^n \rangle_d = 0. \quad (3.44)$$

#### Step 2: Identify energy types and isolate $\delta_{t+}$

As in the case of the mass-spring system in the previous section, there are no losses or externally supplied energy present in the system, and all terms are part of the Hamiltonian  $\mathfrak{h}$ .

To isolate  $\delta_{t+}$  in Eq. (3.44), the terms have to be rewritten in a way that it fits the product identities in Section 3.2.3. Summation by parts as described in Section 3.2.2 can be used. Using identity (3.13a) with  $f_l^n \triangleq \delta_t \cdot u_l^n$  and  $g_l^n \triangleq \delta_{x+} u_l^n$ , the second term can be rewritten to

$$-T \langle \delta_t \cdot u_l^n, \delta_{xx} u_l^n \rangle_d = T \langle \delta_{x+}(\delta_t \cdot u_l^n), \delta_{x+} u_l^n \rangle_d - \mathfrak{b},$$

where the boundary term

$$\mathfrak{b} = T(\delta_t \cdot u_N^n)(\delta_{x+} u_N^n) - T(\delta_t \cdot u_0^n) \underbrace{(\delta_{x+} u_{-1}^n)}_{\delta_x - u_0^n},$$

and reduced domain  $\underline{d} = \{0, \dots, N - 1\}$ . As Dirichlet boundary conditions are used, the boundary term vanishes as

$$u_0^n = u_N^n = 0 \implies \delta_t \cdot u_0^n = \delta_t \cdot u_N^n = 0.$$

In other words, if the states of the system at the boundaries are zero, their velocity will also be zero. Then, using the discrete inner product in Eq. (3.8),

### 3.4. Energy analysis

Eq. (3.44) can be expanded to

$$\delta_{t+} \mathfrak{h} = \rho A \sum_{l=0}^N h(\delta_t \cdot u_l^n)(\delta_{tt} u_l^n) + T \sum_{l=0}^N h(\delta_t \cdot \delta_{x+} u_l^n)(\delta_{x+} u_l^n) \quad (3.45)$$

Then, using identities (3.17a) and (3.17b),  $\delta_{t+}$  can be isolated

$$\delta_{t+} \mathfrak{h} = \delta_{t+} \left( \frac{\rho A}{2} \|\delta_{t-} u_l^n\|_d^2 + \frac{T}{2} \langle \delta_{x+} u_l^n, e_{t-} \delta_{x+} u_l^n \rangle_d \right), \quad (3.46)$$

and the definition for the Hamiltonian and the kinetic and potential energy can be found:

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{v},$$

$$\text{with } \mathfrak{t} = \frac{\rho A}{2} \|\delta_{t-} u_l^n\|_d^2, \quad \text{and} \quad \mathfrak{v} = \frac{T}{2} \langle \delta_{x+} u_l^n, e_{t-} \delta_{x+} u_l^n \rangle_d. \quad (3.47)$$

#### Step 3: Check units

Writing out the definitions for kinetic and potential energy in Eq. (3.47) respectively, yields

$$\begin{aligned} \mathfrak{t} &= \frac{\rho A}{2} \|\delta_{t-} u_l^n\|_d^2 \xrightarrow{\text{in units}} \text{kg} \cdot \text{m}^{-3} \cdot \text{m}^2 \cdot \text{m} \cdot (\text{s}^{-1} \cdot \text{m})^2 \\ &\quad = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}, \\ \mathfrak{v} &= \frac{T}{2} \langle \delta_{x+} u_l^n, e_{t-} \delta_{x+} u_l^n \rangle_d \xrightarrow{\text{in units}} \text{N} \cdot \text{m} \cdot (\text{m}^{-1} \cdot \text{m} \cdot \text{m}^{-1} \cdot \text{m}^{-1} \cdot \text{m}) \\ &\quad = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}, \end{aligned}$$

and are indeed in Joules. Notice that an extra ‘m’ unit appears due to the norm and inner product.

#### Step 4: Implementation

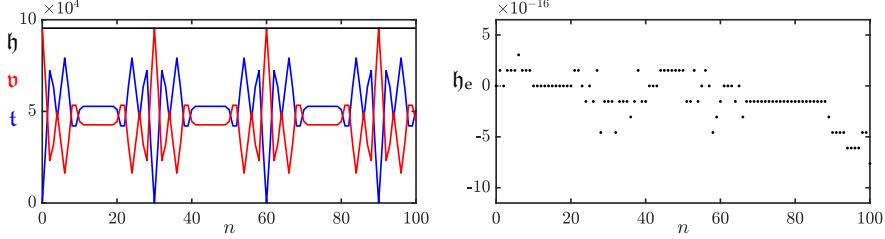
The energy balance in Eq. (3.47) can be implemented with the following code in the for-loop recursion:

```

1 %% Calculate the energy using Eq. (3.47)
2
3 % Kinetic energy
4 kinEnergy(n) = rho * A / 2 * h * sum((1/k * (u-uPrev)).^2);
5
6 % Potential energy
7 potEnergy(n) = T/(2*h) * sum(([u; 0] - [0; u]) ...
8     .* ([uPrev; 0] - [0; uPrev]));
9
10 % Total energy (Hamiltonian)
11 totEnergy(n) = kinEnergy(n) + potEnergy(n);

```

Here,  $\mathbf{u}$  is the vector  $\mathbf{u} = [u_1^n, \dots, u_{N-1}^n]^T$  (as Dirichlet boundary conditions are used) and need to be concatenated with 0 in the calculation of the potential energy as the boundaries needs to be included in the calculation, despite them being 0!<sup>13</sup> Figure 3.2 shows the plot of the normalised energy according to Eq. (3.37) and shows that the deviation of  $\mathfrak{h}^n$  is within machine precision.



**Fig. 3.2:** The kinetic (blue), potential (red), and total (black) energy of an implementation of the 1D wave equation are plotted in the left panel. The right panel shows the normalised energy (according to Eq. (3.37)) and shows that the deviation of the energy is within machine precision.

### Neumann boundary conditions

If Neumann boundary conditions – as per Eq. (2.48b) – are used instead, the primed inner product in Eq. (3.9) needs to be used in Step 1. Using the identity in (3.14a), summation by parts of the second term results in

$$-T\langle \delta_t.u_l^n, \delta_{xx}u_l^n \rangle'_d = T\langle \delta_{x+}(\delta_t.u_l^n), \delta_{x+}u_l^n \rangle_d - \mathfrak{b},$$

where the boundary term

$$\begin{aligned} \mathfrak{b} &= T(\delta_t.u_N^n)(\mu_{x-}\delta_{x+}u_N^n) - T(\delta_t.u_0^n)(\mu_{x-}\delta_{x+}u_0^n), \\ \xleftarrow{\text{Eq. (2.27b)}} \quad &= T(\delta_t.u_N^n)(\delta_{x+}u_N^n) - T(\delta_t.u_0^n)(\delta_{x+}u_0^n). \end{aligned}$$

As the Neumann boundary condition states that

$$\delta_{x+}u_0^n = \delta_{x+}u_N^n = 0$$

the boundary term vanishes and the energy balance results in

$$\begin{aligned} \mathfrak{h} &= \mathfrak{t} + \mathfrak{v}, \\ \text{with } \mathfrak{t} &= \frac{\rho A}{2} \left( \|\delta_{t-}u_l^n\|'_d \right)^2, \quad \text{and} \quad \frac{T}{2} \langle \delta_{x+}u_l^n, e_{t-}\delta_{x+}u_l^n \rangle_d. \end{aligned} \tag{3.48}$$

Using  $\mathbf{u}$  for the vector  $\mathbf{u} = [u_0^n, \dots, u_N^n]^T$ , this is then implemented as

---

<sup>13</sup>As can be seen from the definition of  $\mathfrak{v}$  in Eq. (3.47), the domain used for the inner product is  $d = \{0, \dots, N-1\}$  and  $\mathfrak{v}$  contains a forward difference in its definition requiring  $u_N^n$  as well.

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```

1 %% Calculate the energy using Eq. (3.48)
2
3 % Scaling of the boundaries through weighted inner product
4 scaling = [0.5; ones(N-1, 1); 0.5];
5
6 % Kinetic energy
7 kinEnergy(n) = rho * A / 2 * h * sum(scaling .* (1/k * (u-uPrev)).^2);
8
9 % Potential energy
10 potEnergy(n) = T/(2*h) * sum(u(2:end) - u(1:end-1) ...
11     .* (uPrev(2:end) - uPrev(1:end-1)));
12
13 % Total energy (Hamiltonian)
14 totEnergy(n) = kinEnergy(n) + potEnergy(n);

```

#### 3.4.4 Stability using energy analysis techniques

Section 3.3 showed how to obtain a stability condition of a FD scheme using a frequency domain representation. Although not operating in the frequency domain, the energy analysis techniques presented here may also be used to obtain stability conditions of FD schemes and might even be considered more powerful than the frequency domain approach, as it can also be used to analyse spatially varying and nonlinear systems!

To arrive at a stability condition, the energy must be *non-negative* ( $\mathfrak{h} \geq 0$ ) or, in some cases *positive definite* ( $\mathfrak{h} > 0$ ). Below, the mass-spring system and the 1D wave equation will be used as a test case.

long sentence

##### Mass-spring system

Section 3.3.1 mentions that the mass-spring system is a special case in that the roots of its characteristic equation can not be identical. When proving stability using energy analysis, this means that the energy of the system needs to be positive definite. It can be shown that an equation of the form

$$x^2 + y^2 + 2axy \quad (3.49)$$

is positive definite if  $|a| < 1$ .

Equation (3.49) can be used to prove stability for the mass spring system using the energy balance in Eq. (3.42). One can easily conclude that  $\mathfrak{t}$  is non-negative due to the fact that  $M > 0$  and  $(\delta_t - u^n)$  is squared. The potential energy  $\mathfrak{v}$ , however, is of indefinite sign. Expanding the operators in Eq. (3.42)

yields

$$\begin{aligned}\mathfrak{h} &= \frac{M}{2k^2} \left( (u^n)^2 - 2u^n u^{n-1} + (u^{n-1})^2 \right) + \frac{K}{2} u^n u^{n-1}, \\ &= \frac{M}{2k^2} \left( (u^n)^2 + (u^{n-1})^2 \right) + \left( \frac{K}{2} - \frac{M}{k^2} \right) u^n u^{n-1}.\end{aligned}$$

Dividing all terms by  $M/2k^2$  this equation is of the form in Eq. (3.49):

$$\mathfrak{h} = (u^n)^2 + (u^{n-1})^2 + \left( \frac{Kk^2}{M} - 2 \right) u^n u^{n-1}.$$

For  $\mathfrak{h}$  to be positive definite, the following condition must hold

$$\left| \frac{Kk^2}{2M} - 1 \right| < 1.$$

This can then be written as

$$\begin{aligned}-1 &< \frac{Kk^2}{2M} - 1 < 1 \\ 0 &< \frac{Kk^2}{2M} < 2\end{aligned}$$

where, as long as  $K$  and  $k$  are non-zero, the first inequality is always satisfied. Then the condition solved for  $k$  can easily be shown to be

$$k < 2\sqrt{\frac{M}{K}} \quad (3.50)$$

which is identical to the definition in Eq. (3.29).

## 1D wave equation

For the 1D wave equation, the energy must be proven to be non-negative. One can take the energy balance in Eq. (3.47) and conclude that  $t$  is non-negative due to the non-negativity of the parameters and  $(\delta_{t-} u_l^n)$  being squared. The potential energy, however, is of indefinite sign. One can rewrite  $\mathfrak{v}$  using identity (3.17f) as

$$\begin{aligned}\mathfrak{v} &= \frac{T}{2} \langle \delta_{x+} u_l^n, e_{t-} \delta_{x+} u_l^n \rangle_{\underline{d}}, \\ &= \frac{T}{2} \sum_{l=0}^{N-1} h(\delta_{x+} u_l^n)(e_{t-} \delta_{x+} u_l^n), \\ &= \frac{T}{2} \sum_{l=0}^{N-1} h \left( (\mu_{t-} \delta_{x+} u_l^n)^2 - \frac{k^2}{4} (\delta_{t-} \delta_{x+} u_l^n)^2 \right), \\ &= \frac{T}{2} \left( \|\mu_{t-} \delta_{x+} u_l^n\|_{\underline{d}}^2 - \frac{k^2}{4} \|\delta_{t-} \delta_{x+} u_l^n\|_{\underline{d}}^2 \right).\end{aligned}$$

### 3.5. Modal analysis

One can then use the following bound for spatial differences [23]

$$\|\delta_{x+} u_l^n\|_{\underline{d}} \leq \frac{2}{h} \|u_l^n\|'_d \leq \frac{2}{h} \|u_l^n\|_d, \quad (3.51)$$

to put a condition on  $\mathfrak{v}$

$$\begin{aligned} \mathfrak{v} &\geq \frac{T}{2} \left( \|\mu_{t-} \delta_{x+} u_l^n\|_{\underline{d}}^2 - \frac{k^2}{4} \left( \frac{2}{h} \|\delta_{t-} u_l^n\|_d \right)^2 \right), \\ &\geq \frac{T}{2} \left( \|\mu_{t-} \delta_{x+} u_l^n\|_{\underline{d}}^2 - \frac{k^2}{h^2} \|\delta_{t-} u_l^n\|_d^2 \right), \end{aligned}$$

Substituting this condition into the energy balance in Eq. (3.47) yields

$$\begin{aligned} \mathfrak{h} = \mathfrak{t} + \mathfrak{v} &\geq \frac{\rho A}{2} \|\delta_{t-} u_l^n\|_d^2 + \frac{T}{2} \left( \|\mu_{t-} \delta_{x+} u_l^n\|_{\underline{d}}^2 - \frac{k^2}{h^2} \|\delta_{t-} u_l^n\|_d^2 \right), \\ &\geq \left( \frac{\rho A}{2} - \frac{T k^2}{2 h^2} \right) \|\delta_{t-} u_l^n\|_d^2 + \frac{T}{2} \|\mu_{t-} \delta_{x+} u_l^n\|_{\underline{d}}^2. \end{aligned}$$

Recalling that  $c = \sqrt{T/\rho A}$  and  $\lambda = ck/h$ , all terms can be divided by  $\rho A$  which yields

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{v} \geq \frac{1}{2} (1 - \lambda^2) \|\delta_{t-} u_l^n\|_d^2 + \frac{c^2}{2} \|\mu_{t-} \delta_{x+} u_l^n\|_{\underline{d}}^2, \quad (3.52)$$

and is non-negative for

$$\begin{aligned} 1 - \lambda^2 &\geq 0, \\ \lambda &\leq 1. \end{aligned}$$

This is the same (CFL) condition obtained through von Neumann analysis in Section 3.3.2.

## 3.5 Modal analysis

Modes are the resonant frequencies of a system. The number of modes that a discrete system contains depends on the number of moving points. A mass-spring system thus has one resonating mode, but – as briefly touched upon in Section 2.4.3 – a FD scheme of the 1D wave equation with  $N = 30$  and Dirichlet boundary conditions will have 29 modes. Modal analysis can be used to obtain objective data on what modes a FD scheme should contain. This can then be used to determine whether this matches one's expectations or whether the output of the system matches what the analysis predicted. This section will show how to numerically obtain the modal frequencies of a FD implementation using the 1D wave equation as a test case.

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Recall the matrix form of the 1D wave equation from Eq. (3.5)

$$\frac{1}{k^2} (\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) = c^2 \mathbf{D}_{xx} \mathbf{u}^n.$$

more explanation, perhaps refer to von neumann analysis in 3.3

Following [23], a test solution of the form  $\mathbf{u}^n = z^n \phi$  can be assumed. Substituting this into the above equation yields the characteristic equation

$$(z - 2 + z^{-1})\phi = c^2 k^2 \mathbf{D}_{xx} \phi. \quad (3.53)$$

This is an eigenvalue problem (see Section B.4) where the  $p^{\text{th}}$  solution  $\phi_p$  may be interpreted as the modal shape of mode  $p$ . The corresponding modal frequencies are the solutions to the following equations:

$$\begin{aligned} z_p - 2 + z_p^{-1} &= c^2 k^2 \text{eig}_p(\mathbf{D}_{xx}), \\ z_p + \left( -2 - c^2 k^2 \text{eig}_p(\mathbf{D}_{xx}) \right) + z_p^{-1} &= 0. \end{aligned} \quad (3.54)$$

Furthermore, one can substitute a test solution  $z_p = e^{s_p k}$  with complex frequency  $s_p = j\omega_p + \sigma_p$  which contains the (angular) frequency  $\omega_p$  and damping  $\sigma_p \leq 0$  of the  $p^{\text{th}}$  mode.<sup>4</sup> As there is no damping present in the system, the test solution reduces to  $z_p = e^{j\omega_p k}$  which can be substituted into Eq (3.5) to get

$$\begin{aligned} e^{j\omega_p k} + e^{-j\omega_p k} - 2 - c^2 k^2 \text{eig}_p(\mathbf{D}_{xx}) &= 0, \\ \frac{e^{j\omega_p k} + e^{-j\omega_p k}}{-4} + \frac{1}{2} + \frac{c^2 k^2}{4} \text{eig}_p(\mathbf{D}_{xx}) &= 0. \end{aligned}$$

Finally, using the trigonometric identity in Eq. (3.21a) yields

$$\begin{aligned} \sin^2(\omega_p k / 2) + \frac{c^2 k^2}{4} \text{eig}_p(\mathbf{D}_{xx}) &= 0, \\ \sin(\omega_p k / 2) &= \frac{ck}{2} \sqrt{-\text{eig}_p(\mathbf{D}_{xx})}, \\ \omega_p &= \frac{2}{k} \sin^{-1} \left( \frac{ck}{2} \sqrt{-\text{eig}_p(\mathbf{D}_{xx})} \right), \end{aligned} \quad (3.55)$$

and can be rewritten to

$$f_p = \frac{1}{\pi k} \sin^{-1} \left( \frac{ck}{2} \sqrt{-\text{eig}_p(\mathbf{D}_{xx})} \right) \quad (3.56)$$

to get the modal frequency of the  $p^{\text{th}}$  mode in Hz.

See Figure 3.3 for a plot of the modal frequencies of an implementation of the 1D wave equation with the parameters given in Table 2.1. The figure shows

<sup>4</sup>Notice that regardless of the possible damping coefficient per mode, the eventual amplitude of each will mostly be determined by the locations of the excitation and output as discussed in Section 2.4.3.

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one great advantage of performing modal analysis on an FD scheme, over only obtaining the spectrum of its output. Although the values from the analysis do correspond to the partials shown in the frequency domain output of the 1D wave equation in Figure 2.11, the latter does not show all modes present in the system. This is due to the input and output locations of the system as discussed in Section 2.4.3. The modal analysis does obtain the frequency data regardless of the aforementioned input and output locations.

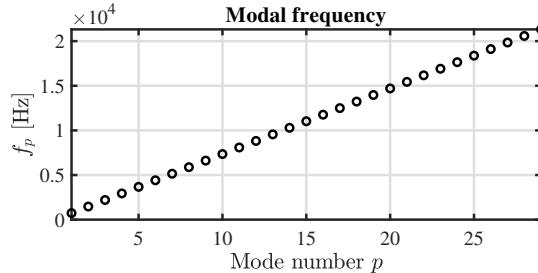


Fig. 3.3: Modal frequencies of the 1D wave equation with the parameters given in Table 2.1.

### 3.5.1 One-step form

For more complicated systems, specifically those containing damping terms, it is useful to rewrite the update in *one-step form* (also referred to as a state-space representation). The damping terms cause the coefficients of  $z$  and  $z^{-1}$  in the characteristic equation to not be identical and the trigonometric identities in (3.21) can not be used directly. Although the eigenfrequency calculation needs to be done on a larger matrix, it allows for a more general and direct way to calculate the modal frequencies and damping coefficients per mode.

If matrix  $\mathbf{A}$  has an inverse, any scheme of the form

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \mathbf{C}\mathbf{u}^{n-1}, \quad (3.57)$$

can be rewritten to

$$\underbrace{\begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{u}^n \end{bmatrix}}_{\mathbf{w}^{n+1}} = \underbrace{\begin{bmatrix} \mathbf{A}^{-1}\mathbf{B} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \mathbf{u}^n \\ \mathbf{u}^{n-1} \end{bmatrix}}_{\mathbf{w}^n} \quad (3.58)$$

which relates the unknown state of the system to the known state through matrix  $\mathbf{Q}$  which encompasses the scheme. The sizes of the identity matrix  $\mathbf{I}$  and zero matrix  $\mathbf{0}$  are the same size as  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$ .

Again, solutions of the form  $\mathbf{w}^n = z^n \phi$  can be assumed (where  $\phi$  is now less-trivially connected to the modal shapes)

$$z\phi = \mathbf{Q}\phi, \quad (3.59)$$

which can be solved for the  $p$ th eigenvalue as

$$z_p = \text{eig}_p(\mathbf{Q}). \quad (3.60)$$

As the scheme could exhibit damping, the test solution  $z_p = e^{s_p k}$  is used. Substituting this yields

$$\begin{aligned} e^{s_p k} &= \text{eig}_p(\mathbf{Q}), \\ s_p &= \frac{1}{k} \ln \left( \text{eig}_p(\mathbf{Q}) \right). \end{aligned} \quad (3.61)$$

Solutions for the frequency and damping for the  $p$ th eigenvalue can then be obtained through

$$\omega_p = \Im(s_p) \quad \text{and} \quad \sigma_p = \Re(s_p), \quad (3.62)$$

where  $\Im(\cdot)$  and  $\Re(\cdot)$  denote the “imaginary part of” and “real part of” respectively.

As the elements of  $\mathbf{Q}$  are real-valued, the solutions  $s_p$  in Eq. (3.61) come in complex conjugates (pairs of numbers of which the imaginary part has an opposite sign). For analysis, only the  $\Im(s_p) \geq 0$  should be considered as these correspond to non-negative frequencies.

## 3.6 Conclusion

This chapter presented three different analysis techniques in discrete time, that are of extreme utility when working with FD schemes. Frequency domain analysis, or von Neumann analysis in the distributed case, can be used to obtain stability conditions for LTI and LSI systems. Energy analysis techniques can also be used to prove stability and passivity, but for a larger range of systems including LTI, LSI, and nonlinear systems. Furthermore, energy analysis can be used in a practical manner to debug implementations of FD schemes and ensure that no programming errors have been made. Finally, modal analysis can be used to analyse the behaviour of a scheme in terms of its modal frequencies and modal shapes. This can be used to determine whether the auditory output matches the predictions of the analysis. Although analogous techniques in continuous time also exist, a more practical angle has been chosen for this work and only the discrete time methods have been presented. For more information about the techniques in continuous time, see [23].

All three analysis techniques will be extensively used in the rest of this document to analyse the FD schemes used in this project.

## **Part II**

# **Resonators**



# Resonators

Although the physical models described in the previous part – the simple mass-spring system and the 1D wave equation – are also considered resonators, they are *ideal* cases. In other words, these can not be found in the real world as effects such as losses or frequency dispersion are not included.

This part presents the different resonators used over the course of the project that better include these non-ideal physical processes and is structured as follows: Chapter 4 introduces the stiff string, an extension of the 1D wave equation, Chapter 5 introduces acoustic tubes, used to model brass instruments, and finally, Chapter 6 introduces 2D systems which, in this project, have been used to simulate (simplified) instrument bodies. The analysis techniques introduced in the previous part will be applied to all models and described in detail.



## Chapter 4

# The Stiff String

In earlier chapters, the case of the ideal string was presented, modelled using the 1D wave equation. This system, if the CFL condition is satisfied with equality, generates an output with harmonic partials that are integer multiples of the fundamental frequency. In the real world, however, strings exhibit a phenomenon called *frequency dispersion* due to stiffness in the material, hence the name *stiff string*. This phenomenon causes another effect known as *inharmonicity*: the “harmonic” partials get exponentially further apart the higher their frequency is. The stiffness in a string is dependent on its material properties and geometry and will be elaborated on in this chapter. The stiff string played a prominent part in the following papers: [A], [B], [C], [D] and [E].

This chapter presents the PDE of the stiff string in continuous time, and goes through the discretisation process. The analysis techniques presented in Chapter 3 will then be applied to the resulting (explicit) FD scheme and derived in detail. Finally, an example of an implicit scheme will be given and comparison to the earlier FD scheme will be made. Unless denoted otherwise, this chapter follows [23].

### 4.1 Continuous time

Consider a lossless stiff string of length  $L$  and with a circular cross-section. Its transverse displacement is described by  $u = u(x, t)$  (in m) defined for  $x \in \mathcal{D}$  with domain  $\mathcal{D} = [0, L]$  and time  $t \geq 0$ . The PDE describing its motion is

$$\rho A \partial_t^2 u = T \partial_x^2 u - EI \partial_x^4 u \quad (4.1)$$

parametrised by material density  $\rho$  (in kg/m<sup>3</sup>), cross-sectional area  $A = \pi r^2$  (in m<sup>2</sup>), radius  $r$  (in m), tension  $T$  (in N), Young’s modulus  $E$  (in Pa) and area moment of inertia  $I = \pi r^4/4$  (in m<sup>4</sup>). If  $E = 0$ , Eq (4.1) reduces to the 1D wave

equation in Eq. (2.38) where  $c = \sqrt{T/\rho A}$ , i.e., the ideal string. If instead  $T = 0$ , Eq. (4.1) reduces to the *ideal bar* equation. A more compact way to write Eq. (4.1) is

$$\partial_t^2 u = c^2 \partial_x^2 u - \kappa^2 \partial_x^4 u \quad (4.2)$$

with wave speed  $c = \sqrt{T/\rho A}$  (in m/s) and stiffness coefficient  $\kappa = \sqrt{EI/\rho A}$ .

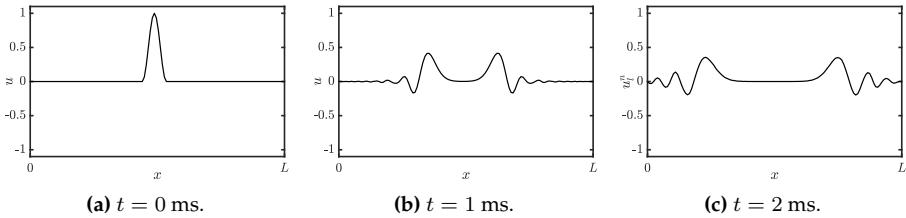
The difference between the ideal string and the stiff string is the term containing a 4<sup>th</sup>-order spatial derivative. This term adds *stiffness* to the system and causes frequency dispersion. As opposed to unwanted numerical dispersion due to numerical error (see Section 2.4.4) this type of dispersion is physical and thus something desired in the model. This phenomenon causes higher frequencies to travel faster through a medium than lower frequencies. See Figure 4.1. Furthermore, frequency dispersion is closely tied to *inharmonicity*, an effect where ‘harmonic’ partials get exponentially further apart as frequency increases. For low values of  $\kappa$ , the frequency of these partials can be expressed in terms of the fundamental frequency  $f_0 = c/2L$  (as in Eq. (2.40)) and frequency of partial  $p$  (in Hz) is defined as [23]

$$f_p = f_0 p \sqrt{1 + B p^2}, \quad (4.3)$$

with inharmonicity coefficient

$$B = \frac{\kappa^2 \pi^2}{c^2}.$$

Frequency dispersion and inharmonicity will be further discussed in Section 4.2.3.



**Fig. 4.1:** Frequency dispersion in a stiff string due to stiffness.

### 4.1.1 Adding losses

Before moving on to the discretisation of the PDE in Eq. (4.1), losses can be added to the system. In the physical world, strings lose energy through fx. air viscosity and thermoelastic effects. All frequencies lose energy and die out (damp) over time, but higher frequencies do so at a much faster rate.

#### 4.1. Continuous time

This phenomenon is called *frequency-dependent damping* and can be modelled using a mixed derivative  $\partial_t \partial_x^2$ . This way of frequency-dependent damping first appeared in [60] and has been used extensively in the literature since [24, 23]. A damped stiff string can be modelled as

$$\rho A \partial_t^2 u = T \partial_x^2 u - EI \partial_x^4 u - 2\sigma_0 \rho A \partial_t u + 2\sigma_1 \rho A \partial_t \partial_x^2 u, \quad (4.4)$$

where the non-negative loss coefficients  $\sigma_0$  (in  $s^{-1}$ ) and  $\sigma_1$  (in  $m^2/s$ ) determine the frequency-independent and frequency-dependent losses respectively. Appendix D attempts to provide some intuition on workings of these damping terms.

A more compact way to write Eq. (4.4), and is also found often in the literature [23], is to divide all terms by  $\rho A$  to get

$$\partial_t^2 u = c^2 \partial_x^2 u - \kappa^2 \partial_x^4 u - 2\sigma_0 \partial_t u + 2\sigma_1 \partial_t \partial_x^2 u, \quad (4.5)$$

where  $c = \sqrt{T/\rho A}$  is the wave speed (in m/s) as in the 1D wave equation in (2.38) and  $\kappa = \sqrt{EI/\rho A}$  is referred to as the stiffness coefficient (in  $m^2/s$ ).

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#### Boundary conditions

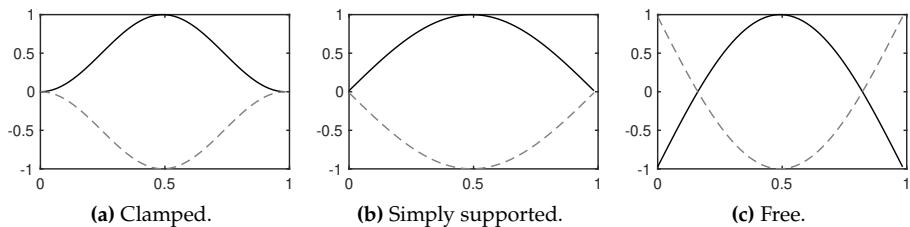
Section 2.4 presents two types of boundary conditions for the 1D wave equation in Eq. (2.39). In the case of the stiff string, these can be extended to

$$u = \partial_x u = 0 \quad (\text{clamped}) \quad (4.6a)$$

$$u = \partial_x^2 u = 0 \quad (\text{simply supported}) \quad (4.6b)$$

$$\partial_x^2 u = \partial_x^3 u = 0 \quad (\text{free}) \quad (4.6c)$$

at  $x = 0, L$ . See Figure 4.2 for plots of the first modal shape for each respective boundary condition.



**Fig. 4.2:** Plots of the first (normalised) modal shape for the three boundary conditions in Eqs. (4.6). The extremes are indicated with solid black and dashed grey lines respectively.

## 4.2 Discrete time

For the sake of compactness, Eq. (4.5) will be used in the following, rather than Eq. (4.4). Naturally, the same process can be followed for the latter, the only difference being a multiplication by  $\rho A$  of all terms.

Following Section 2.2.1 and using the FD operators presented in Section 2.2.2, Eq. (4.5) can be discretised as

$$\delta_{tt} u_l^n = c^2 \delta_{xx} u_l^n - \kappa^2 \delta_{xxxx} u_l^n - 2\sigma_0 \delta_t u_l^n + 2\sigma_1 \delta_{t-} \delta_{xx} u_l^n, \quad (4.7)$$

and is defined for domain  $l \in \{0, \dots, N\}$  and number of grid points  $N + 1$ . The  $\delta_{xxxx}$  operator is defined as the second-order spatial difference in Eq. (2.8) applied to itself:

$$\delta_{xxxx} = \delta_{xx} \delta_{xx} = \frac{1}{h^4} (e_{x+}^2 - 4e_{x+} + 6 - 4e_{x-} + e_{x-}^2). \quad (4.8)$$

A multiplication of two shift operators applied to a grid function simply means to apply each shift individually. The  $\delta_{xxxx}$  operator applied to  $u_l^n$  thus becomes

$$\delta_{xxxx} u_l^n = \frac{1}{h^4} (u_{l+2}^n - 4u_{l+1}^n + 6u_l^n - 4u_{l-1}^n + u_{l-2}^n). \quad (4.9)$$

A definition for the mixed-derivative operator can similarly be found. Recalling the definitions for  $\delta_{t-}$  in Eq. (2.3b) and  $\delta_{xx}$  Eq. (2.8), their combination results in

$$\begin{aligned} \delta_{t-} \delta_{xx} &= \frac{1}{k} (1 - e_{t-}) \frac{1}{h^2} (e_{x+} - 2 + e_{x-}), \\ &= \frac{1}{kh^2} (e_{x+} - 2 + e_{x-} - e_{t-}(e_{x+} - 2 + e_{x-})). \end{aligned} \quad (4.10)$$

To have two different shift operators multiplied together still simply means to apply each of them to the grid function individually. The  $\delta_{t-} \delta_{xx}$  operator applied to  $u_l^n$  thus yields

$$\delta_{t-} \delta_{xx} u_l^n = \frac{1}{hk^2} (u_{l+1}^n - 2u_l^n + u_{l-1}^n - u_{l+1}^{n-1} + 2u_l^{n-1} - u_{l-1}^{n-1}). \quad (4.11)$$

The reason a backwards difference is used here is to keep the system *explicit*. A scheme is explicit if the values of  $u_l^{n+1}$  can be calculated from known values at times  $n$  and  $n - 1$ . If this is not the case and values of fx.  $u_{l+1}^{n+1}$  and  $u_{l-1}^{n+1}$  are required to calculate  $u_l^{n+1}$ , the scheme is called *implicit*. An example of an implicit scheme using the centred operator for the temporal derivative in the frequency-dependent damping term instead can be found in Section 4.6.

## 4.2. Discrete time

Using the definitions above, the operators in scheme (4.7) can be expanded to get

$$\begin{aligned} \frac{1}{k^2}(u_l^{n+1} - 2u_l^n + u_l^{n-1}) &= \frac{c^2}{h^2}(u_{l+1}^n - 2u_l^n + u_{l-1}^n) \\ &\quad - \frac{\kappa^2}{h^4}(u_{l+2}^n - 4u_{l+1}^n + 6u_l^n - 4u_{l-1}^n + u_{l-2}^n) \\ &\quad - \frac{\sigma_0}{k}(u_l^{n+1} - u_l^{n-1}) \\ &\quad + \frac{2\sigma_1}{kh^2}(u_{l+1}^n - 2u_l^n + u_{l-1}^n - u_{l+1}^{n-1} + 2u_l^{n-1} + u_{l-1}^{n-1}), \end{aligned} \quad (4.12)$$

and after multiplication by  $k^2$  and collecting the terms yields

$$\begin{aligned} (1 + \sigma_0 k)u_l^{n+1} &= \left(2 - 2\lambda^2 - 6\mu^2 - \frac{4\sigma_1 k}{h^2}\right)u_l^n \\ &\quad + \left(\lambda^2 + 4\mu^2 + \frac{2\sigma_1 k}{h^2}\right)(u_{l+1}^n + u_{l-1}^n) \\ &\quad - \mu^2(u_{l+2}^n + u_{l-2}^n) + \left(-1 + \sigma_0 k + \frac{4\sigma_1 k}{h^2}\right)u_l^{n-1} \\ &\quad - \frac{2\sigma_1 k}{h^2}(u_{l+1}^{n-1} + u_{l-1}^{n-1}), \end{aligned} \quad (4.13)$$

with

$$\lambda = \frac{ck}{h} \quad \text{and} \quad \mu = \frac{\kappa k}{h^2}. \quad (4.14)$$

The update equation follows by dividing both sides by  $(1 + \sigma_0 k)$ .

The stability condition for the FD scheme in (4.7) is defined as

$$h \geq \sqrt{\frac{c^2 k^2 + 4\sigma_1 k + \sqrt{(c^2 k^2 + 4\sigma_1 k)^2 + 16\kappa^2 k^2}}{2}}, \quad (4.15)$$

and will be derived in Section 4.3 using von Neumann analysis. This condition can then be used to calculate the number of intervals  $N$  in a similar fashion as for the 1D wave equation shown in Eq. (2.53). First, Eq. (4.15) should be satisfied with equality, after which

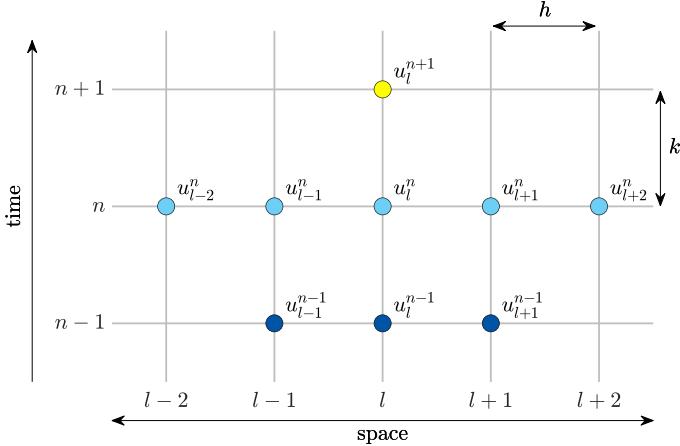
$$N := \left\lfloor \frac{L}{h} \right\rfloor, \quad \text{and} \quad h := \frac{L}{N}$$

which can then be used to calculate  $\lambda$  and  $\mu$  in (4.14).

### Stencil

As done in Section 2.4.2, a stencil for the FD scheme in Eq. (4.7) can be created. This is shown in Figure 4.3. In order to calculate  $u_l^{n+1}$ , 5 points at the current

time step are needed due to the 4<sup>th</sup>-order spatial derivative. Due to the mixed derivative in the frequency-dependent damping term, neighbouring points at the previous time step are also required.



**Fig. 4.3:** The stencil for the damped stiff string scheme in Eq. (4.7) (adapted from [A]).

### 4.2.1 Boundary conditions

Due to the 4<sup>th</sup>-order spatial derivative, two virtual grid points need to be accounted for at the boundaries of the system. Discretising the boundary conditions in (4.6) yields

$$u_l^n = \delta_{x\pm} u_l^n = 0 \quad (\text{clamped}) \quad (4.16a)$$

$$u_l^n = \delta_{xx} u_l^n = 0 \quad (\text{simply supported}) \quad (4.16b)$$

$$\delta_{xx} u_l^n = \delta_x \cdot \delta_{xx} u_l^n = 0 \quad (\text{free}) \quad (4.16c)$$

at  $l = 0, N$ . The operator in the clamped condition uses the  $\delta_{x+}$  operator at the left boundary ( $l = 0$ ) and  $\delta_{x-}$  at the right ( $l = N$ ). Note that to discretise  $\partial_x^3$  in the free boundary condition in Eq. (4.6c), the more accurate  $\delta_x \cdot \delta_{xx}$  operator has been chosen over the less accurate  $\delta_{x-} \delta_{xx}$  and  $\delta_{x+} \delta_{xx}$  operators for the left and right boundary respectively.

Below, the boundary conditions are expanded to get definitions for the virtual grid points.

#### Clamped

Expanding the operators for the clamped condition yields

$$u_0^n = u_1^n = 0 \quad \text{and} \quad u_{N-1}^n = u_N^n = 0. \quad (4.17)$$

insert figure showing virtual grid points somewhere in this section

## 4.2. Discrete time

This can be simplified by reducing the range of calculation to  $l \in \{2, \dots, N-2\}$ .

### Simply supported

As the states of the end points of a system with simply supported boundary conditions are 0 at all times, the range of calculation can be reduced to  $l \in \{1, \dots, N-1\}$ . Evaluating the update equation in Eq. (4.13) at  $l = 1$  and  $l = N-1$  shows that definitions for the virtual grid points  $u_{-1}^n$  and  $u_{N+1}^n$  are required. A definition for  $u_{-1}^n$  can be found by expanding Eq. (4.16b) at  $l = 0$ :

$$\begin{aligned} \frac{1}{h^2} (u_1^n - 2u_0^n + u_{-1}^n) &= 0, \\ \xleftarrow{u_0^n=0} u_1^n + u_{-1}^n &= 0, \\ u_{-1}^n &= -u_1^n, \end{aligned} \tag{4.18}$$

and similarly for  $u_{N+1}^n$  by expanding the condition at  $l = N$ :

$$u_{N+1}^n = -u_{N-1}^n.$$

Substituting the first definition into the expanded scheme in Eq. (4.12) at  $l = 1$  and collecting the terms thereafter yields

$$\begin{aligned} (1 + \sigma_0 k)u_1^{n+1} &= \left(2 - 2\lambda^2 - 5\mu^2 - \frac{4\sigma_1 k}{h^2}\right)u_1^n + \left(\lambda^2 + 4\mu^2 + \frac{2\sigma_1 k}{h^2}\right)u_2^n \\ &\quad - \mu^2 u_3^n + \left(-1 + \sigma_0 k + \frac{4\sigma_0 k}{h^2}\right)u_1^{n-1} - \frac{2\sigma_1 k}{h^2}u_2^{n-1}. \end{aligned} \tag{4.19}$$

Doing the same for  $l = N-1$  yields

$$\begin{aligned} (1 + \sigma_0 k)u_{N-1}^{n+1} &= \left(2 - 2\lambda^2 - 5\mu^2 - \frac{4\sigma_1 k}{h^2}\right)u_{N-1}^n + \left(\lambda^2 + 4\mu^2 + \frac{2\sigma_1 k}{h^2}\right)u_{N-2}^n \\ &\quad - \mu^2 u_{N-3}^n + \left(-1 + \sigma_0 k + \frac{4\sigma_0 k}{h^2}\right)u_{N-1}^{n-1} - \frac{2\sigma_1 k}{h^2}u_{N-2}^{n-1}. \end{aligned} \tag{4.20}$$

### Free

Finally, the free boundary condition requires all points to be calculated and the range of calculation remains  $l \in \{0, \dots, N\}$ . At each respective boundary, two virtual grid points are needed:  $u_{-1}^n$  and  $u_{-2}^n$  at the left and  $u_{N+1}^n$  and  $u_{N+2}^n$  at the right boundary respectively. The third-order spatial FD operator in Eq.

(4.16c) is defined as:

$$\begin{aligned}\delta_x \cdot \delta_{xx} &= \frac{1}{2h^3} (e_{x+} - e_{x-}) (e_{x+} - 2 + e_{x-}), \\ &= \frac{1}{2h^3} (e_{x+}^2 - 2e_{x+} + 1 - (1 - 2e_{x-} + e_{x-}^2)), \\ &= \frac{1}{2h^3} (e_{x+}^2 - 2e_{x+} + 2e_{x-} - e_{x-}^2),\end{aligned}\quad (4.21)$$

and can be used to solve for  $u_{-2}^n$  at  $l = 0$ :

$$\begin{aligned}\frac{1}{2h^3} (u_2^n - 2u_1^n + 2u_{-1}^n - u_{-2}^n) &= 0, \\ u_{-2}^n &= u_2^n - 2u_1^n + 2u_{-1}^n.\end{aligned}$$

As  $u_0^n$  is not necessarily 0 at all times, solving the first part of the boundary condition (i.e.,  $\delta_{xx} u_0^n = 0$ ) yields a different result than in the simply supported case:

$$\begin{aligned}\frac{1}{h^2} (u_1^n - 2u_0^n + u_{-1}^n) &= 0, \\ u_{-1}^n &= 2u_0^n - u_1^n.\end{aligned}$$

The same can be done at  $l = N$  to get the following definitions for the virtual grid points

$$u_{N+2}^n = u_{N-2}^n - 2u_{N-1}^n + 2u_{N+1}^n \quad \text{and} \quad u_{N+1}^n = 2u_N^n - u_{N-1}^n.$$

The update equations for the boundary points will not be given here. Instead the matrix form of the FD scheme with free boundaries will be provided below.

In practice, the simply supported boundary condition is mostly chosen as this most realistically reflects string terminations in the real world. The clamped condition could be chosen for simplicity as this does not require an alternative update at the boundaries. The free boundary condition is more often used to model a (damped) ideal bar, (Eq. (4.4) with  $T = 0$ ).

## 4.2.2 Implementation and matrix form

When using MATLAB, for a more compact implementation, it is useful to write the scheme in matrix form (see Section 3.1.2). The FD scheme of the stiff string in (4.7) can be written as

$$A\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \mathbf{C}\mathbf{u}^{n-1} \quad (4.22)$$

where

$$\begin{aligned}A &= (1 + \sigma_0 k), \quad \mathbf{B} = 2\mathbf{I} + c^2 k^2 \mathbf{D}_{xx} - \kappa^2 k^2 \mathbf{D}_{xxxx} + 2\sigma_1 k \mathbf{D}_{xx}, \\ \text{and} \quad \mathbf{C} &= -(1 - \sigma_0 k)\mathbf{I} - 2\sigma_1 k \mathbf{D}_{xx}.\end{aligned}$$

## 4.2. Discrete time

Notice that  $A$  is a scalar rather than a matrix.

The size of the state vectors and the matrix-form operators depend on the boundary conditions. For clamped conditions, the state vectors ( $\mathbf{u}^{n+1}$ ,  $\mathbf{u}^n$  and  $\mathbf{u}^{n-1}$ ) and matrices will be of size  $(N-3) \times 1$  and  $(N-3) \times (N-3)$  respectively. The  $\mathbf{D}_{xx}$  matrix will be of the form given in Eq. (3.3) and the matrix form of the  $\delta_{xxxx}$  operator is

$$\mathbf{D}_{xxxx} = \frac{1}{h^4} \begin{bmatrix} 6 & -4 & 1 & & \mathbf{0} \\ -4 & 6 & \ddots & \ddots & \\ 1 & \ddots & \ddots & \ddots & 1 \\ & \ddots & \ddots & 6 & -4 \\ \mathbf{0} & & 1 & -4 & 6 \end{bmatrix}. \quad (4.23)$$

For simply supported conditions, the state vectors and matrices will be of size  $(N-1) \times 1$  and  $(N-1) \times (N-1)$  respectively. Again,  $\mathbf{D}_{xx}$  is as defined in Eq. (3.3) and  $\mathbf{D}_{xxxx}$  can be obtained by multiplying two  $\mathbf{D}_{xx}$  matrices according to

$$\mathbf{D}_{xxxx} = \mathbf{D}_{xx} \mathbf{D}_{xx} = \frac{1}{h^4} \begin{bmatrix} 5 & -4 & 1 & & \mathbf{0} & \\ -4 & 6 & \ddots & \ddots & & \\ 1 & \ddots & \ddots & -4 & 1 & \\ & \ddots & -4 & 6 & -4 & \ddots \\ & & 1 & -4 & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & 6 & -4 \\ \mathbf{0} & & & & 1 & -4 & 5 \end{bmatrix}. \quad (4.24)$$

Finally for free boundary conditions given in Eq. (4.16c), the state vectors and matrices are  $(N+1) \times 1$  and  $(N+1) \times (N+1)$  respectively. Now, the  $\mathbf{D}_{xx}$  matrix is of the form in Eq. (3.4) instead, and

$$\mathbf{D}_{xxxx} = \frac{1}{h^4} \begin{bmatrix} 2 & -4 & 2 & & \mathbf{0} & \\ -2 & 5 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 5 & -2 \\ \mathbf{0} & & & & 2 & -4 & 2 \end{bmatrix}. \quad (4.25)$$

### 4.2.3 Parameters and output

The values of the parameters naturally determine the properties of the output sound. Where in the 1D wave equation, only the fundamental frequency  $f_0$  could be affected (through  $c$  and  $L$  in Eq. (2.40)), the stiff string has many more aspects that can be changed. See Table 4.1 for parameters most commonly used in this project.

Name	Symbol (unit)	Value
Length	$L$ (m)	1
Material density	$\rho$ (kg/m <sup>3</sup> )	7850
Radius	$r$ (m)	$5 \cdot 10^{-4}$
Tension	$T$ (N)	$100 \leq T \leq 1.5 \cdot 10^4$
Young's modulus	$E$ (Pa)	$2 \cdot 10^{11}$
Freq.-independent damping	$\sigma_0$ (s <sup>-1</sup> )	1
Freq.-dependent damping	$\sigma_1$ (m <sup>2</sup> /s)	0.005

**Table 4.1:** Parameters and their values most commonly used over the course of this project.

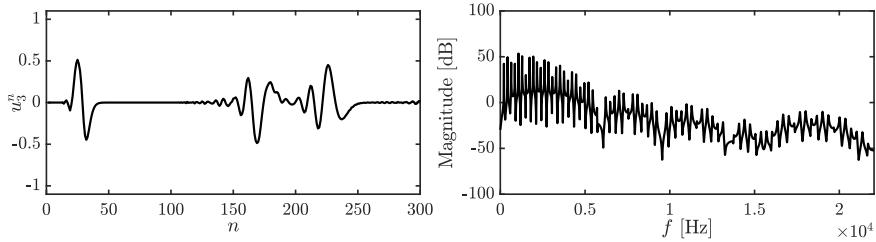
A formula exists to calculate the loss coefficients  $\sigma_0$  and  $\sigma_1$  from  $T_{60}$  values at different frequencies (see [23, Eq (7.29)]). During this project, however, these values have been tuned by ear and are usually set to be approximately those found in Table 4.1.

### Output

Figure 4.4 shows the time domain and frequency domain output (retrieved at  $l = 3$ ) of an implementation of the stiff string excited using a raised-cosine. The parameters used can be found in Table 4.1 with  $T = 1129$  N. Furthermore,  $E = 7 \cdot 10^{11}$  Pa to highlight dispersive effects. Finally, simply supported boundary conditions are chosen. From the left panel, one can observe that over time, dispersive effects show where higher-frequency components in the excitation travel faster through the medium than lower-frequency components. In the frequency domain (the right panel in Figure 4.4) this shows in the fact that the partials are not perfect integer multiples of the fundamental. Notice that the partials are closer to each other for lower frequencies and further apart as their frequency increases. Finally, the frequency-dependent damping term causes higher frequencies to have a lower amplitude than lower frequencies.

Apart from the obvious material properties such as density, stiffness and geometry, perceptual qualities of the sound are surprisingly much determined by  $\sigma_1$ , and for lower values the output can become extremely metallic.

### 4.3. von Neumann analysis and stability condition



**Fig. 4.4:** The time-domain and frequency domain output of the stiff string. The parameters are set as in Table 4.1 with  $E = 7 \cdot 10^{11}$  Pa to highlight dispersive effects and  $T = 1129$  N.

## 4.3 von Neumann analysis and stability condition

In order to obtain the stability condition for the damped stiff string, one can perform von Neumann analysis as presented in Section 3.3 on the FD scheme in Eq. (4.7).

Using the definitions found in Eq. (3.22) for the temporal operators and Eqs. (3.23a) and (3.23b) for the spatial operators, the frequency domain representation of Eq. (4.7) can be obtained:

$$\begin{aligned} \frac{1}{k^2} (z - 2 + z^{-1}) = & -\frac{4c^2}{h^2} \sin^2(\beta h/2) - \frac{16\kappa^2}{h^4} \sin^4(\beta h/2) - \frac{\sigma_0}{k} z + \frac{\sigma_0}{k} z^{-1} \\ & - \frac{8\sigma_1}{kh^2} \sin^2(\beta h/2) + \frac{8\sigma_1}{kh^2} \sin^2(\beta h/2) z^{-1}, \end{aligned}$$

and after collecting the terms, the characteristic equation is as follows

$$\begin{aligned} (1 + \sigma_0 k)z + \left( 16\mu^2 \sin^4(\beta h/2) + \left( 4\lambda^2 + \frac{8\sigma_1 k}{h^2} \right) \sin^2(\beta h/2) - 2 \right) \\ + \left( 1 - \sigma_0 k - \frac{8\sigma_1 k}{h^2} \sin^2(\beta h/2) \right) z^{-1} = 0. \end{aligned} \quad (4.26)$$

Rewriting this to the form in Eq. (3.25), and using  $\mathcal{S} = \sin^2(\beta h/2)$  for brevity, yields

$$z^2 + \left( \frac{16\mu^2 \mathcal{S}^2 + (4\lambda^2 + \frac{8\sigma_1 k}{h^2}) \mathcal{S} - 2}{1 + \sigma_0 k} \right) z + \frac{1 - \sigma_0 k - \frac{8\sigma_1 k}{h^2} \mathcal{S}}{1 + \sigma_0 k} = 0.$$

Stability of the system can then be proven using condition (3.26), and substi-

tuting the coefficients into this condition yields

$$\begin{aligned} \left| \frac{16\mu^2\mathcal{S}^2 + \left(4\lambda^2 + \frac{8\sigma_1 k}{h^2}\right)\mathcal{S} - 2}{1 + \sigma_0 k} \right| - 1 &\leq \frac{1 - \sigma_0 k - \frac{8\sigma_1 k}{h^2}\mathcal{S}}{1 + \sigma_0 k} \leq 1, \\ \left| 16\mu^2\mathcal{S}^2 + \left(4\lambda^2 + \frac{8\sigma_1 k}{h^2}\right)\mathcal{S} - 2 \right| - (1 + \sigma_0 k) &\leq 1 - \sigma_0 k - \frac{8\sigma_1 k}{h^2}\mathcal{S} \leq 1 + \sigma_0 k, \\ \left| 16\mu^2\mathcal{S}^2 + \left(4\lambda^2 + \frac{8\sigma_1 k}{h^2}\right)\mathcal{S} - 2 \right| &\leq 2 - \frac{8\sigma_1 k}{h^2}\mathcal{S} \leq 2 + 2\sigma_0 k. \end{aligned}$$

The second condition is always true due to the fact that  $\sigma_0, \sigma_1 \geq 0$ . Continuing with the first condition:

$$\begin{aligned} -2 + \frac{8\sigma_1 k}{h^2}\mathcal{S} &\leq 16\mu^2\mathcal{S}^2 + \left(4\lambda^2 + \frac{8\sigma_1 k}{h^2}\right)\mathcal{S} - 2 \leq 2 - \frac{8\sigma_1 k}{h^2}\mathcal{S}, \\ 0 &\leq 16\mu^2\mathcal{S}^2 + 4\lambda^2\mathcal{S} \leq 4 - \frac{16\sigma_1 k}{h^2}\mathcal{S}. \end{aligned}$$

As  $16\mu^2\mathcal{S}^2 + 4\lambda^2\mathcal{S}$  is non-negative, the first condition is always satisfied. Continuing with the second condition:

$$\begin{aligned} 16\mu^2\mathcal{S}^2 + \left(4\lambda^2 + \frac{16\sigma_1 k}{h^2}\right)\mathcal{S} &\leq 4, \\ 4\mu^2\mathcal{S}^2 + \left(\lambda^2 + \frac{4\sigma_1 k}{h^2}\right)\mathcal{S} &\leq 1. \end{aligned}$$

As  $\mathcal{S}$  is bounded by 1 for all values of  $\beta$ , one can set  $\mathcal{S} = 1$  [check with Stefan](#). Continuing with the substituted definitions for  $\lambda$  and  $\mu$  from Eq. (4.14) yields

$$\begin{aligned} \frac{4\kappa^2 k^2}{h^4} + \frac{c^2 k^2 + 4\sigma_1 k}{h^2} &\leq 1, \\ 4\kappa^2 k^2 + (c^2 k^2 + 4\sigma_1 k)h^2 &\leq h^4, \\ h^4 - (c^2 k^2 + 4\sigma_1 k)h^2 - 4\kappa^2 k^2 &\geq 0, \end{aligned}$$

which is a quadratic equation in  $h^2$ . Using the quadratic formula, the grid spacing  $h$  can then be shown to be bounded by

$$h \geq \sqrt{\frac{c^2 k^2 + 4\sigma_1 k + \sqrt{(c^2 k^2 + 4\sigma_1 k)^2 + 16\kappa^2 k^2}}{2}}, \quad (4.27)$$

which is the stability condition for the damped stiff string also shown in Eq. (4.15).

## 4.4 Energy analysis

As mentioned in Section 3.4, it is useful to perform the energy analysis on the scheme with all physical parameters written out. Discretising the PDE in Eq.

#### 4.4. Energy analysis

(4.4) yields

$$\rho A \delta_{tt} u_l^n = T \delta_{xx} u_l^n - EI \delta_{xxxx} u_l^n - 2\sigma_0 \rho A \delta_{t.} u_l^n + 2\sigma_1 \rho A \delta_{t-} \delta_{xx} u_l^n, \quad (4.28)$$

defined for  $l \in d$  with discrete domain  $d = \{0, \dots, N\}$ . This section will follow the 4 steps described in Section 3.4.

##### Step 1: Obtain $\delta_{t+}\mathfrak{h}$

The first step is to take the inner product (see Eq. (3.8)) of the scheme with  $(\delta_{t.} u_l^n)$  over discrete domain  $d$ :

$$\begin{aligned} \delta_{t+}\mathfrak{h} &= \rho A \langle \delta_{t.} u_l^n, \delta_{tt} u_l^n \rangle_d - T \langle \delta_{t.} u_l^n, \delta_{xx} u_l^n \rangle_d + EI \langle \delta_{t.} u_l^n, \delta_{xxxx} u_l^n \rangle_d \\ &\quad + 2\sigma_0 \rho A \langle \delta_{t.} u_l^n, \delta_{t.} u_l^n \rangle_d - 2\sigma_1 \rho A \langle \delta_{t.} u_l^n, \delta_{t-} \delta_{xx} u_l^n \rangle_d = 0. \end{aligned} \quad (4.29)$$

##### Step 2: Identify energy types and isolate $\delta_{t+}$

As there is damping present in the system, and the system is distributed, the energy balance will be of the form

$$\delta_{t+}\mathfrak{h} = \mathfrak{b} - \mathfrak{q}. \quad (4.30)$$

with boundary term  $\mathfrak{b}$  and damping term  $\mathfrak{q}$ . The latter is defined as

$$\mathfrak{q} = 2\sigma_0 \rho A \|\delta_{t.} u_l^n\|_d^2 - 2\sigma_1 \rho A \langle \delta_{t.} u_l^n, \delta_{t-} \delta_{xx} u_l^n \rangle_d, \quad (4.31)$$

virtual grid  
points needed  
for freq-dep  
damping term..

and  $\mathfrak{b}$  appears after rewriting Eq. (4.29) using summation by parts (see Section 3.2.2). Specifically, using Eq. (3.13a) for the second term and Eq. (3.16b) for the third, yields

$$\begin{aligned} \delta_{t+}\mathfrak{h} &= \rho A \langle \delta_{t.} u_l^n, \delta_{tt} u_l^n \rangle_d + T \langle \delta_{t.} \delta_{x+} u_l^n, \delta_{x+} u_l^n \rangle_d + EI \langle \delta_{t.} \delta_{xx} u_l^n, \delta_{xx} u_l^n \rangle_d \\ &= \mathfrak{b} - \mathfrak{q} \end{aligned}$$

where the boundary term becomes

$$\begin{aligned} \mathfrak{b} &= T \left( (\delta_{t.} u_N^n) (\delta_{x+} u_N^n) - (\delta_{t.} u_0^n) (\delta_{x+} u_{-1}^n) \right) \\ &\quad + EI \left( (\delta_{t.} u_N^n) (\delta_{x+} \delta_{xx} u_N^n) - (\delta_{xx} u_N^n) (\delta_{x-} \delta_{t.} u_N^n) \right) \\ &\quad - EI \left( (\delta_{t.} u_0^n) (\delta_{x-} \delta_{xx} u_0^n) - (\delta_{xx} u_0^n) (\delta_{x+} \delta_{t.} u_0^n) \right). \end{aligned}$$

For the clamped and simply supported boundary conditions in (4.16a) and (4.16b) it can easily be shown that  $\mathfrak{b} = 0$ . If free conditions as in Eq. (4.16c) are used, the boundary conditions will vanish when the primed inner product in Eq. (3.9) is used in Step 1 and identity (3.16c) is used when performing

summation by parts. Here, only the clamped / simply supported case will be considered.

Isolating  $\delta_{t+}$  to obtain the total energy  $\mathfrak{h}$  in the definition for  $\delta_{t+}\mathfrak{h}$  above, requires identities (3.17a) and (3.17b) and yields

$$\begin{aligned}\delta_{t+}\mathfrak{h} &= \delta_{t+} \left( \frac{\rho A}{2} \|\delta_{t-} u_l^n\|_d^2 + \frac{T}{2} \langle \delta_{x+} u_l^n, e_{t-} \delta_{x+} u_l^n \rangle_{\underline{d}} + \frac{EI}{2} \langle \delta_{xx} u_l^n, e_{t-} \delta_{xx} u_l^n \rangle_{\underline{d}} \right) \\ &= -\mathfrak{q}.\end{aligned}$$

From this, the definition for the Hamiltonian  $\mathfrak{h}$ , the kinetic energy  $\mathfrak{t}$  and potential energy  $\mathfrak{v}$  can be found:

$$\begin{aligned}\mathfrak{h} &= \mathfrak{t} + \mathfrak{v}, \quad \text{with } \mathfrak{t} = \frac{\rho A}{2} \|\delta_{t-} u_l^n\|_d^2, \quad \text{and} \\ \mathfrak{v} &= \frac{T}{2} \langle \delta_{x+} u_l^n, e_{t-} \delta_{x+} u_l^n \rangle_{\underline{d}} + \frac{EI}{2} \langle \delta_{xx} u_l^n, e_{t-} \delta_{xx} u_l^n \rangle_{\underline{d}}.\end{aligned}\tag{4.32}$$

### Step 3: Check units

Comparing the acquired energy balance in Eq. (4.32) to the energy balance for the 1D wave equation in Eq. (3.47), one can observe that the balances are nearly identical, the only difference being the second term in the definition for  $\mathfrak{v}$  in Eq. (4.32). Writing this term out in units, and recalling that Pa (the unit for  $E$ ) in SI units is  $\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}$ , yields

$$\begin{aligned}\frac{EI}{2} \langle \delta_{xx} u_l^n, e_{t-} \delta_{xx} u_l^n \rangle_{\underline{d}} &\xrightarrow{\text{in units}} \text{Pa} \cdot \text{m}^4 \cdot \text{m} \cdot (\text{m}^{-2} \cdot \text{m} \cdot \text{m}^{-2} \cdot \text{m}) \\ &= \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2},\end{aligned}$$

and indeed has the correct units.

As described in Section 3.4, the damping terms in  $\mathfrak{q}$  need to have units of Joules per second, or  $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3}$ . Writing the terms in Eq. (4.31) out in their units yields

$$\begin{aligned}2\sigma_0 \rho A \|\delta_{t-} u_l^n\|_d^2 &\xrightarrow{\text{in units}} \text{s}^{-1} \cdot \text{kg} \cdot \text{m}^{-3} \cdot \text{m}^2 \cdot \text{m} \cdot (\text{s}^{-1} \cdot \text{m})^2 \\ &= \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3}, \\ -2\sigma_1 \rho A \langle \delta_{t-} u_l^n, \delta_{t-} \delta_{xx} u_l^n \rangle_d &\xrightarrow{\text{in units}} \text{m}^2 \cdot \text{s}^{-1} \cdot \text{kg} \cdot \text{m}^{-3} \cdot \text{m}^2 \\ &\quad \cdot \text{m} \cdot (\text{s}^{-1} \cdot \text{m})(\text{s}^{-1} \cdot \text{m}^{-2} \cdot \text{m}), \\ &= \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3},\end{aligned}$$

and also have the correct units.

### Step 4: Implementation

An implementation of the energy calculation for the simply supported boundary condition is given in Algorithm 4.1. The damping is ignored but can be found in Appendix ???. Figure 4.5 shows that the damping present in the system causes  $\mathfrak{h}$  to decrease in the left panel. The right panel shows that the deviation of the total energy calculated using Eq. (3.38) is within machine precision.

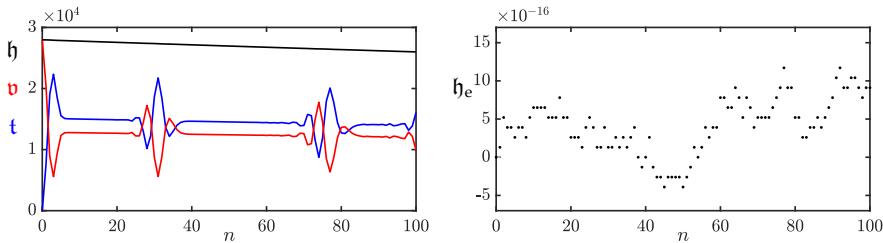
Add to appendix or refer to a gist

```

1 %%% Before the main loop: %%%
2
3 % Initialise Dx+ operator to calculate potential energy due to tension
4 % As the domain is reduced by one, the matrix needs to be of size N x N
5 Dxp = sparse(1:N, 1:N, -ones(1, N), N, N) + ...
6           sparse(1:N-1, 2:N, ones(1, N-1), N, N);
7
8 %%% In the main loop: %%%
9
10 % energy in the system
11 kinEnergy(n) = rho * A * h / 2 * sum((1/k * (u - uPrev)).^2);
12 potEnergy(n) = T / 2 * h * sum((Dxp * [0; u]) .* (Dxp * [0; uPrev]))
... + E * I * h / 2 * sum((Dxx * u) .* (Dxx * uPrev));

```

**Algorithm 4.1:** Calculating  $\mathfrak{h}$  for the simply supported boundary condition.



**Fig. 4.5:** The kinetic (blue), potential (red), and total (black) energy of an implementation of the stiff string are plotted in the left panel. The right panel shows the normalised energy (according to Eq. (3.38)) and shows that the deviation of the energy is within machine precision.

## 4.5 Modal analysis

To be able to perform a modal analysis on the FD scheme in (4.7), it must be written in one-step form – introduced in Section 3.5.1 – due to the damping present in the system. Using the matrix form of the damped stiff string in Eq. (4.22), the one-step form can be written as

$$\underbrace{\begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{u}^n \end{bmatrix}}_{\mathbf{w}^{n+1}} = \underbrace{\begin{bmatrix} \mathbf{B}/A & \mathbf{C}/A \\ \mathbf{I} & \mathbf{0} \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \mathbf{u}^n \\ \mathbf{u}^{n-1} \end{bmatrix}}_{\mathbf{w}^n}, \quad (4.33)$$

where the definitions for  $\mathbf{B}$ ,  $\mathbf{C}$  and  $A$  can be found in Section 4.2.2. In this analysis, the definitions for  $\mathbf{D}_{xx}$  and  $\mathbf{D}_{xxxx}$  for simply supported boundary conditions will be used.

Assuming test solutions of the form  $\mathbf{w}^n = z^n \phi$ , and recalling that  $z = e^{sk}$  and complex frequency  $s = j\omega + \sigma$ , yields the following eigenvalue problem (see Section B.4)

$$z\phi = \mathbf{Q}\phi, \quad (4.34)$$

which can be solved for the  $p^{\text{th}}$  complex modal frequency

$$s_p = \frac{1}{k} \ln (\text{eig}_p(\mathbf{Q})). \quad (4.35)$$

The (angular) frequency of the  $p^{\text{th}}$  mode can then be obtained using  $\Im(s_p)$  and the damping per mode as  $\Re(s_p)$ . Only selecting the non-negative frequencies obtained from  $\Im(s_p)$ , these can be plotted and are shown in Figure 4.6. The parameters used are the ones found in Table 4.1 with  $T = 1885$  N, and  $E = 2 \cdot 10^{14}$  Pa to highlight inharmonic behaviour. The left panel shows that the system is indeed inharmonic, i.e., modal frequencies increase more as the modal number increases. The right panel shows that higher modes exhibit a higher amount of damping. This is due to the frequency-dependent damping term. If  $\sigma_1 = 0$  in (4.7), it can be shown that  $\sigma_p = \sigma_0$  for every mode  $p$  (in this case  $\sigma_0 = -1$ ).

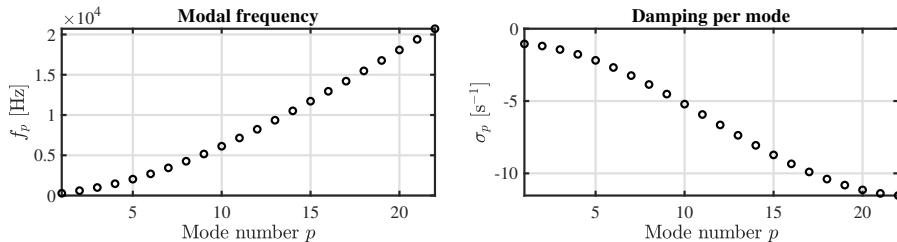


Fig. 4.6: The modal frequencies and damping per mode for the stiff string using the values in Table 4.1 and  $T = 1885$  N and  $E = 2 \cdot 10^{14}$  Pa to highlight effects of stiffness.

## 4.6 Implicit scheme

Although not used in the published work of this project, it is useful to touch upon an example of an implicit scheme. Consider a discretisation of Eq. (4.5) where the (more accurate) centred operator is used for the frequency-dependent damping term:

$$\delta_{tt} u_l^n = c^2 \delta_{xx} u_l^n - \kappa^2 \delta_{xxxx} u_l^n - 2\sigma_0 \delta_{t.} u_l^n + 2\sigma_1 \delta_{t.} \delta_{xx} u_l^n. \quad (4.36)$$

## 4.6. Implicit scheme

Using the centred operator in the mixed-spatio-temporal operator renders the system *implicit*, meaning that a definition for  $u_l^{n+1}$  can not explicitly be found from known values. The stencil in Figure 4.7 also shows this: in order to calculate  $u_l^{n+1}$ , neighbouring points at the next time step  $u_{l+1}^{n+1}$  and  $u_{l-1}^{n+1}$  are needed. The issue is that these values are unknown at the time of calculation.

Luckily, as the scheme is linear, it can be treated as a system of linear equations and solved following the technique described in Section B.3. The drawback is that this requires one matrix inversion per iteration which can be extremely costly (see Section ??). However, both von Neumann and modal analysis (below) show that using the centred instead of the backwards operator has a positive effect on the stability and the modal behaviour of the scheme.

Considering simply supported boundary conditions such that  $l \in \{1, \dots, N-1\}$ , the system will have  $N-1$  unknowns ( $u_l^{n+1}$  for  $l \in \{1, \dots, N-1\}$ ) that can be calculated using  $N-1$  (update) equations. Writing this in matrix form using column vector  $\mathbf{u}^n = [u_1^n, u_2^n, \dots, u_{N-1}^n]$  yields

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \mathbf{C}\mathbf{u}^{n-1} \quad (4.37)$$

where

$$\begin{aligned} \mathbf{A} &= (1 + \sigma_0 k)\mathbf{I} - \sigma_1 k \mathbf{D}_{xx}, & \mathbf{B} &= c^2 k^2 \mathbf{D}_{xx} - \kappa^2 k^2 \mathbf{D}_{xxxx} \\ \text{and } \mathbf{C} &= -(1 - \sigma_0 k)\mathbf{I} - \sigma_1 k \mathbf{D}_{xx}. \end{aligned}$$

Equation (4.37) can be considered a system of linear equations (see Section B.3) and the state at the next time step  $\mathbf{u}^{n+1}$  can then be retrieved using a matrix inversion (see B.2)

$$\mathbf{u}^{n+1} = \mathbf{A}^{-1} (\mathbf{B}\mathbf{u}^n + \mathbf{C}\mathbf{u}^{n-1}). \quad (4.38)$$

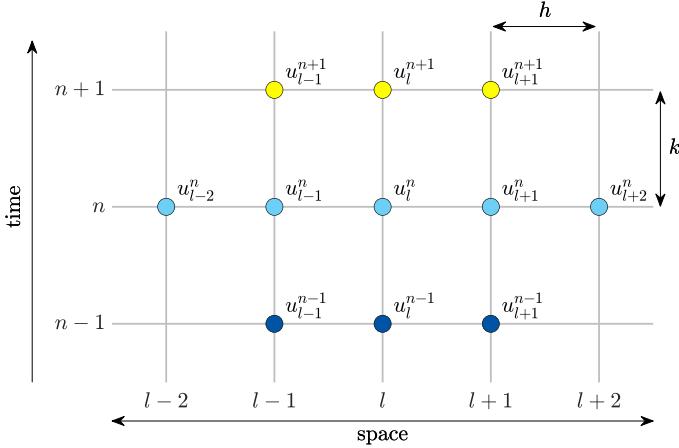
### 4.6.1 von Neumann analysis

Using the same process as in Section 4.3, the definitions in Section 3.3 can be used to obtain a frequency domain representation of the FD scheme in Eq. (4.36):

$$\begin{aligned} \frac{1}{k^2} (z - 2 + z^{-1}) &= -\frac{4c^2}{h^2} \sin^2(\beta h/2) - \frac{16\kappa^2}{h^4} \sin^4(\beta h/2) - \frac{\sigma_0}{k} z + \frac{\sigma_0}{k} z^{-1} \\ &\quad - \frac{4\sigma_1}{kh^2} \sin^2(\beta h/2) z + \frac{4\sigma_1}{kh^2} \sin^2(\beta h/2) z^{-1}, \end{aligned} \quad (4.39)$$

and collecting the terms, yields the following characteristic equation:

$$\begin{aligned} \left(1 + \sigma_0 k + \frac{4\sigma_1 k}{h^2} \sin^2(\beta h/2)\right) z + (16\mu^2 \sin^4(\beta h/2) + 4\lambda^2 \sin^2(\beta h/2) - 2) \\ + \left(1 - \sigma_0 k - \frac{4\sigma_1 k}{h^2} \sin^2(\beta h/2)\right) z^{-1} = 0. \end{aligned} \quad (4.40)$$



**Fig. 4.7:** The stencil for the damped stiff string scheme in (4.36).

Rewriting this to the form found in Eq. (3.25) and, again, using  $\mathcal{S} = \sin^2(\beta h/2)$  yields:

$$z^2 + \frac{16\mu^2\mathcal{S}^2 + 4\lambda^2\mathcal{S} - 2}{1 + \sigma_0 k + \frac{4\sigma_1 k}{h^2}\mathcal{S}} z + \frac{1 - \sigma_0 k - \frac{4\sigma_1 k}{h^2}\mathcal{S}}{1 + \sigma_0 k + \frac{4\sigma_1 k}{h^2}\mathcal{S}} = 0.$$

Stability of the system can then be proven using condition (3.26), and after substitution of the coefficients yields

$$\left| \frac{16\mu^2\mathcal{S}^2 + 4\lambda^2\mathcal{S} - 2}{1 + \sigma_0 k + \frac{4\sigma_1 k}{h^2}\mathcal{S}} \right| - 1 \leq \frac{1 - \sigma_0 k - \frac{4\sigma_1 k}{h^2}\mathcal{S}}{1 + \sigma_0 k + \frac{4\sigma_1 k}{h^2}\mathcal{S}} \leq 1,$$

$$\begin{aligned} |16\mu^2\mathcal{S}^2 + 4\lambda^2\mathcal{S} - 2| - \left( 1 + \sigma_0 k + \frac{4\sigma_1 k}{h^2}\mathcal{S} \right) &\leq 1 - \sigma_0 k - \frac{4\sigma_1 k}{h^2}\mathcal{S} \\ &\leq 1 + \sigma_0 k + \frac{4\sigma_1 k}{h^2}\mathcal{S}, \end{aligned}$$

$$|16\mu^2\mathcal{S}^2 + 4\lambda^2\mathcal{S} - 2| \leq 2 \leq 2 + 2\sigma_0 k + \frac{8\sigma_1 k}{h^2}\mathcal{S}.$$

Because  $\sigma_0, \sigma_1, k, \mathcal{S}$  and  $h$  are all non-negative, the last condition is always satisfied. Continuing with the first condition:

$$\begin{aligned} -2 &\leq 16\mu^2\mathcal{S}^2 + 4\lambda^2\mathcal{S} - 2 \leq 2, \\ 0 &\leq 16\mu^2\mathcal{S}^2 + 4\lambda^2\mathcal{S} \leq 4. \end{aligned}$$

#### 4.6. Implicit scheme

Again, the first condition is always satisfied due to the non-negativity of all coefficients. Continuing with the second condition yields

$$4\mu^2\mathcal{S}^2 + \lambda^2\mathcal{S} \leq 1,$$

and knowing that  $\mathcal{S}$  is bounded by 1 for all  $\beta$ , the process can be finalised:

$$\begin{aligned} 4\mu^2 + \lambda^2 &\leq 1, \\ \frac{4\kappa^2 k^2}{h^4} + \frac{c^2 k^2}{h^2} &\leq 1, \\ h^4 - c^2 k^2 h^2 - 4\kappa^2 k^2 &\geq 0, \end{aligned}$$

and yields the following stability condition:

$$h \geq \sqrt{\frac{c^2 k^2 + \sqrt{c^4 k^4 + 16\kappa^2 k^2}}{2}}. \quad (4.41)$$

Comparing this to the stability condition for the explicit scheme in Eq. (4.15), one can observe that the terms containing  $\sigma_1$  have vanished. It can thus be concluded that if the centred (rather than the backwards) difference is used to discretise the temporal derivative in the frequency-dependent damping term,  $\sigma_1$  no longer influences the stability of the scheme and the condition is more relaxed. What this means in terms of behaviour of the scheme will be elaborated on in the following section.

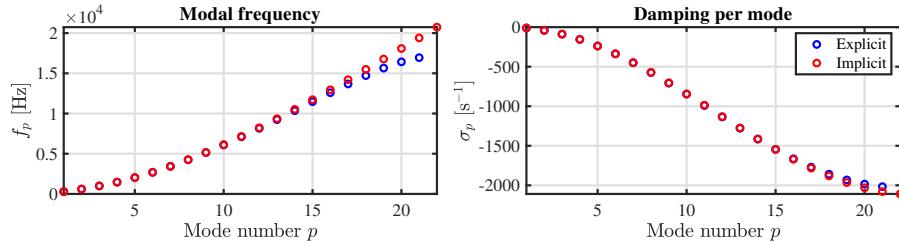
#### 4.6.2 Modal analysis

As the matrix form of the implicit FD scheme in Eq. (4.37) matches the form in Eq. (3.57), one can perform a modal analysis by writing the scheme in one-step form as explained in Section 3.5.1. The results of the analysis are shown in Figure 4.8. To highlight the difference between using the backwards and centred difference for the frequency-dependent damping term,  $\sigma_1$  has been set to 1, which is much higher than one would normally use.

One can observe from Figure 4.8 that especially higher-frequency modes in the explicit scheme are affected by  $\sigma_1$ . In the continuous case, the modal frequencies should only be affected by values for  $c$  and  $\kappa$  as per Eq. (4.3) and the damping should not influence the frequencies of the partials, as one could expect. However, as  $\sigma_1$  increases,  $h$  increases due to Eq. (4.15), causing  $\lambda$  and  $\mu$  to decrease. This introduces numerical dispersion as explained in Section 2.4.4, and the higher the value of  $\sigma_1$ , the more numerical dispersion is introduced.

As the stability condition for the implicit scheme in Eq. (4.41) does not contain  $\sigma_1$ , this value will not affect  $\lambda$  and  $\mu$  and will thus not affect the modal frequencies. As can be observed from the figure, it even allows for one more grid point to be included in the simulation. It can be concluded that because the

frequency-dependent damping term no longer affects the stability condition for the implicit scheme, a more accurate simulation can be obtained with fewer numerically dispersive effects.



**Fig. 4.8:** A comparison between the modal frequencies and damping per mode of the explicit (blue) and implicit (red) scheme. Here,  $T = 1885$  N,  $E = 2 \cdot 10^{14}$  Pa and  $\sigma_1 = 1$  m $^2$ /s to highlight differences between the two schemes.

### 4.6.3 Conclusion

This section presented an implicit discretisation of the stiff string where the centred operator has been used to discretise the temporal derivative in the frequency-dependent damping term. By means of stability analysis and modal analysis several advantages that the implicit scheme has over its explicit counterpart (presented in Section 4.2) have been shown.

As these advantages only show for higher values of  $\sigma_1$ , much higher than the ones used in this project, it has been chosen to use the explicit scheme for all further implementation. The decrease in accuracy is negligible for lower values of  $\sigma_1$  and the calculation of the scheme becomes orders of magnitude more computationally expensive if the implicit scheme is used.

# Chapter 5

## Acoustic Tubes

The dynamics of woodwind and brass instruments is based on wave propagation in acoustic tubes. Although the physical processes that generate the sound are fundamentally different from those in strings, the underlying models have many similarities. The main difference between acoustic tubes and the (ideal) strings, is that tubes have a varying cross-sectional area, causing wave dispersion and greatly influencing the modal frequencies and modal shapes present in the system.

In this work, planar wave propagation is assumed (rather than spherical), such that the behaviour of the systems can be approximated using 1D systems. Although higher-dimensional models might better capture some physical effects (see e.g. [61]), 1D systems already show good agreement between model and measurement [62]. Moreover, looking towards real-time implementation of these models, the choice to simplify to 1D has been made due to the low relative computational cost.

This chapter first presents Webster's equation, which extends the 1D wave equation presented in Section 2.4 by introducing a spatially varying cross-section. Although not used for the contributions in Part ??, Webster's equation forms a good basis for the second part of this chapter, which decomposes Webster's equation into a system of two coupled first-order PDEs. This has been used to model the trombone in paper [H].

### 5.1 Webster's equation

For an (axially symmetric) acoustic tube of length  $L$  (in m) where the wavelengths of the frequencies at interest are much larger than the radius of the tube, one can simplify the system to be one-dimensional [25]. For low-amplitude vibrations, the air propagation in this tube can be described using Webster's

equation [63]

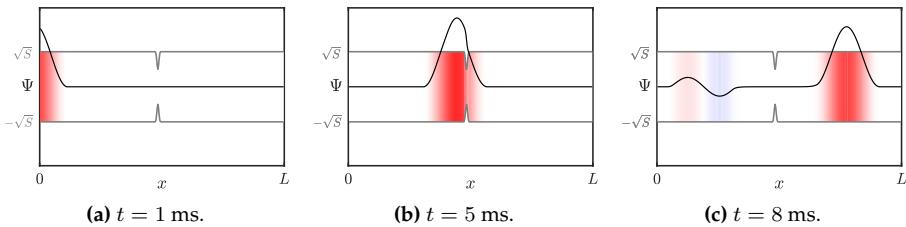
$$S\partial_t^2\Psi = c^2\partial_x(S\partial_x\Psi), \quad (5.1)$$

with *acoustic potential*  $\Psi = \Psi(x, t)$  (in  $\text{m}^2/\text{s}$ ), the cross-sectional area along the tube, or bore profile  $S = S(x)$  (in  $\text{m}^2$ ) and the speed of sound in air  $c$  (in  $\text{m/s}$ ). The state variable  $\Psi$  is defined for  $t \geq 0$  and  $x \in \mathcal{D}$  where domain  $\mathcal{D} = [0, L]$ . If  $S(x)$  is constant, Eq. (5.1) reduces to the 1D wave equation in Eq. (2.38). This shows that for cylindrical acoustic tube, the fundamental frequency is not affected by the cross-sectional area, but solely relies on length  $L$  and wave speed  $c$  according to Eq. (2.40). The acoustic potential can be related to pressure  $p = p(x, t)$  (in Pa) and particle velocity  $v = v(x, t)$  (in  $\text{m/s}$ ) according to [25]

$$p = \rho_0\partial_t\Psi, \quad \text{and} \quad v = -\partial_x\Psi, \quad (5.2)$$

with air density  $\rho_0$  (in  $\text{kg/m}^3$ ).

The interesting thing about the presence of a variable cross-section, is that it causes dispersive or scattering behaviour, especially at locations of high (spatial) variation of  $S$ . See Figure 5.1. Contrary to frequency dispersion as happens in a stiff string (see Chapter 4), all frequencies travel at the same speed, but some components of the wave get reflected due to the geometry of the tube.



**Fig. 5.1:** Wave propagation and dispersion in an acoustic tube of varying cross-section (shown in grey) modelled by Webster's equation in Eq. (5.1). Positive acoustic potential  $\Psi$  is shown in red and negative in blue, which is also plotted for clarity.

### Boundary conditions

The choices for boundary conditions in an acoustic tube are open and closed, defined as [25]

$$\partial_t\Psi(0, t) = 0, \quad \partial_t\Psi(L, t) = 0, \quad (\text{Dirichlet, open}), \quad (5.3a)$$

$$\partial_x\Psi(0, t) = 0, \quad \partial_x\Psi(L, t) = 0, \quad (\text{Neumann, closed}). \quad (5.3b)$$

This might be slightly counter-intuitive as when compared to the 1D wave equation, as “closed” might imply the “fixed” or Dirichlet boundary condition.

### 5.1. Webster's equation

The opposite can be intuitively shown by imagining a wave front with a positive acoustic potential (and thus positive pressure according to Eq. (5.2)) moving through a tube and hitting a closed end. What reflects is also a wave front with a positive acoustic potential, i.e., the sign of the potential does not flip. This also happens using the free or Neumann condition for the 1D wave equation (see Figure 2.9). Here, the following boundaries are chosen

$$\partial_x \Psi(0, t) = 0, \quad \text{and} \quad \partial_t \Psi(L, t) = 0, \quad (5.4)$$

i.e. closed at the left end and open at the right end.

#### 5.1.1 Discrete time

The state variable is discretised to the grid function  $\Psi_l^n$  and is defined for  $n \in \mathbb{N}^0$  and  $l = \{0, \dots, N\}$ , where  $N$  is the number of intervals between the grid points. As the cross-section is distributed in space,  $S(x)$  needs to be discretised to a grid function as well, albeit only in space (as it is not time-varying). Following [25], it is useful to introduce *interleaved grid points* at  $l - 1/2$  and  $l + 1/2$  for  $S$  and are defined as

$$S_{l-1/2} = \mu_{x-} S(x = lh), \quad \text{and} \quad S_{l+1/2} = \mu_{x+} S(x = lh), \quad (5.5)$$

and approximate a ‘true’ (possibly measured) bore profile  $S(x)$  sampled at  $x = lh$  with grid spacing  $h$  (see Figure 5.2). Using these definitions, one can discretise Eq. (5.1) to the following FD scheme [23]<sup>1</sup>

$$\bar{S}_l \delta_{tt} \Psi_l^n = c^2 \delta_{x-} (S_{l+1/2} (\delta_{x+} \Psi_l^n)), \quad (5.6)$$

where

$$\bar{S}_l = \mu_{x+} S_{l-1/2} = \mu_{x-} S_{l+1/2} = \mu_{xx} S(x = lh), \quad (5.7)$$

the choice of which will become apparent in Section 5.1.6. The right-hand side of the scheme contains an operator applied to two grid functions ( $S$  and  $\Psi$ ) multiplied onto each other. In order to expand this, the product rule must be used. Recalling Eq. (3.18) and applying this to backwards spatial operators instead yields

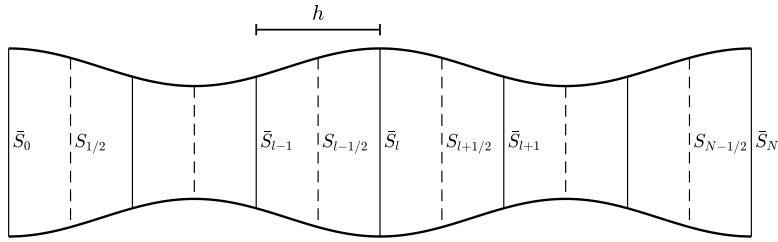
$$\delta_{x-} (u_l^n w_l^n) = (\delta_{x-} u_l^n) (\mu_{x-} w_l^n) + (\mu_{x-} u_l^n) (\delta_{x-} w_l^n). \quad (5.8)$$

Using the product rule, the right-hand side of Eq. (5.6) can be expanded to

$$\bar{S} \delta_{tt} \Psi_l^n = c^2 [(\delta_{x-} S_{l+1/2}) (\mu_{x-} (\delta_{x+} \Psi_l^n)) + (\mu_{x-} S_{l+1/2}) (\delta_{x-} (\delta_{x+} \Psi_l^n))],$$

---

<sup>1</sup>Notice that in [23], Webster’s equation is  $\bar{S}_l \delta_{tt} \Psi_l^n = c^2 \delta_{x+} (S_{l-1/2} (\delta_{x-} \Psi_l^n))$  but is identical to Eq. (5.6). This discretisation has been chosen for a more straightforward energy analysis in Section 5.1.5.



**Fig. 5.2:** Approximations to  $S(x)$  used in the FD scheme implementing Webster's equation. Dashed lines indicate the interleaved grid on which  $S$  is sampled (Eq. (5.5)) and solid lines indicate  $\bar{S}$  which are averages of these (Eq. (5.7)).

and solving for  $\Psi_l^{n+1}$  yields the following update equation (see Appendix E.1):

$$\Psi_l^{n+1} = 2(1 - \lambda^2)\Psi_l^n - \Psi_l^{n-1} + \frac{\lambda^2 S_{l+1/2}}{\bar{S}_l} \Psi_{l+1}^n + \frac{\lambda^2 S_{l-1/2}}{\bar{S}_l} \Psi_{l-1}^n, \quad (5.9)$$

with

$$\lambda = \frac{ck}{h}, \quad (5.10)$$

and similar to the 1D wave equation in Section 2.4.2 needs to abide

$$\lambda \leq 1, \quad (5.11)$$

in order for the scheme to be stable. See Section 5.1.6 for a derivation. The number of grid points  $N$  can then be calculated in the same way as for the 1D wave equation in Eq. (2.53).

Notice that at the boundaries, Eq. (5.9) requires values of  $S$  (through its definition in Eq. (5.7)) outside of the defined domain, i.e.,  $S_{N+1/2}$  and  $S_{-1/2}$ . To solve this, one can set  $\bar{S}_0 = S(0)$  and  $\bar{S}_N = S(L)$  from which  $S_{-1/2}$  and  $S_{N+1/2}$  can be calculated according to

$$\bar{S}_0 = \frac{1}{2}(S_{1/2} + S_{-1/2}) \Rightarrow S_{-1/2} = 2\bar{S}_0 - S_{1/2}, \quad (5.12a)$$

$$\bar{S}_N = \frac{1}{2}(S_{N+1/2} + S_{N-1/2}) \Rightarrow S_{N+1/2} = 2\bar{S}_N - S_{N-1/2}. \quad (5.12b)$$

Although these values will not be needed when discretising the boundary conditions in Eq. (5.3), they will be useful at a later point.

### Boundary conditions

One can discretise the continuous boundary conditions in Eq. (5.4) (closed at  $x = 0$ , open at  $x = L$ ) using centred difference operators for higher accuracy

$$\delta_x \cdot \Psi_0^n = 0 \Rightarrow \Psi_{-1}^n = \Psi_1^n, \quad (\text{Neumann, closed}), \quad (5.13a)$$

$$\delta_t \cdot \Psi_N^n = 0 \Rightarrow \Psi_N^n = 0, \quad (\text{Dirichlet, open}). \quad (5.13b)$$

### 5.1. Webster's equation

At the left boundary, Eq. (5.9) can be expanded to:

$$\begin{aligned} \Psi_0^{n+1} &= 2(1 - \lambda^2)\Psi_0^n - \Psi_0^{n-1} + \frac{\lambda^2 S_{1/2}}{\bar{S}_0} \Psi_1^n + \frac{\lambda^2 S_{-1/2}}{\bar{S}_0} \Psi_{-1}^n, \\ \xleftarrow{\text{Eq. (5.13a)}} \quad \Psi_0^{n+1} &= 2(1 - \lambda^2)\Psi_0^n - \Psi_0^{n-1} + \frac{\lambda^2(S_{1/2} + S_{-1/2})}{\bar{S}_0} \Psi_1^n, \end{aligned}$$

and as  $\bar{S}_0 = \frac{1}{2}(S_{1/2} + S_{-1/2})$  through Eq. (5.7), this can be solved to

$$\Psi_0^{n+1} = 2(1 - \lambda^2)\Psi_0^n - \Psi_0^{n-1} + 2\lambda^2\Psi_1^n. \quad (5.14)$$

One can implement the right boundary condition by simply reducing the range of operation to  $l = \{0, \dots, N-1\}$ , as  $\Psi_N^n = 0$  according to Eq. (5.13b). A more realistic boundary condition for the open end is presented in the following.

### 5.1.2 Radiation

One of the ways that an acoustic tube loses energy, is through radiation. The right boundary condition presented in Eq. (5.4) can be changed to be radiating according to [23]

$$\partial_x \Psi(L, t) = -a_1 \partial_t \Psi(L, t) - a_2 \Psi(L, t), \quad (5.15)$$

where, for a tube terminating on an infinite plane [64]

$$a_1 = \frac{1}{2(0.8216)^2 c}, \quad \text{and} \quad a_2 = \frac{L}{0.8216 \sqrt{S_0 S(1)/\pi}}, \quad (5.16)$$

which determine the amount of loss and inertia at the radiating boundary respectively.

The radiating boundary in Eq. (5.15) can then be discretised to [23]

$$\delta_x \cdot \Psi_N^n = -a_1 \delta_t \cdot \Psi_N^n - a_2 \mu_t \cdot \Psi_N^n, \quad (5.17)$$

which can be expanded and solved for  $\Psi_{N+1}^n$  according to

$$\Psi_{N+1}^n = h \left( -\frac{a_1}{k} (\Psi_N^{n+1} - \Psi_N^{n-1}) - a_2 (\Psi_N^{n+1} + \Psi_N^{n-1}) \right) + \Psi_{N-1}^n. \quad (5.18)$$

Substitution into Eq. (5.9) at the  $l = N$  yields the following update equation

$$\Psi_N^{n+1} = \frac{2(1 - \lambda^2)\Psi_N^n - \Psi_N^{n-1} + \alpha_- \Psi_N^{n-1} + 2\lambda^2\Psi_{N-1}^n}{(1 + \alpha_+)}, \quad (5.19)$$

where

$$\alpha_{\pm} = h \left( \frac{a_1}{k} \pm a_2 \right) \frac{\lambda^2 S_{N+1/2}}{\bar{S}_N}. \quad (5.20)$$

One can observe that  $S_{N+1/2}$  is needed which is outside the defined domain. As mentioned before, setting  $\bar{S}_N = S(L)$ , one can calculate  $S_{N+1/2}$  using Eq. (5.12b) to solve the issue.

### 5.1.3 Excitation

maybe refer to  
chapter/sec-  
tion instead

Although excitations will be discussed more in-depth in Part III, a simple way to excite Webster's equation will be presented here.

Following [25], one can create an input signal  $v_{\text{in}} = v_{\text{in}}(t)$  that interacts with the particle velocity of the tube. As this relates to the acoustic potential as in Eq. (5.2), one can change the boundary condition of the left boundary to

$$\partial_x \Psi(0, t) = -v_{\text{in}}. \quad (5.21)$$

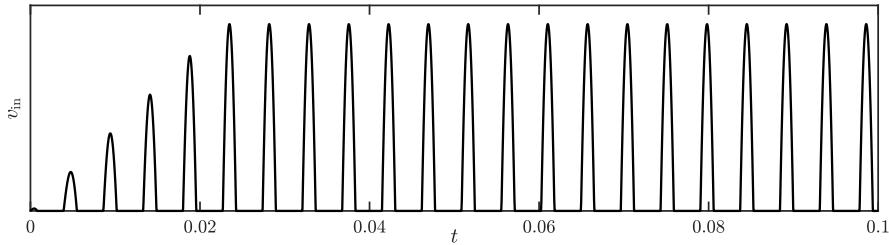
Discretising this using the centred spatial operator yields

$$\delta_x \cdot \Psi_0^n = -v_{\text{in}}^n \Rightarrow \Psi_{-1}^n = 2hv_{\text{in}}^n + \Psi_1^n, \quad (5.22)$$

and can be substituted into the update equation in Eq. (5.9) at  $l = 0$  to get

$$\begin{aligned} \Psi_0^{n+1} &= 2(1 - \lambda^2)\Psi_0^n - \Psi_0^{n-1} \frac{\lambda^2 S_{1/2}}{\bar{S}_0} \Psi_1^n + \frac{\lambda^2 S_{-1/2}}{\bar{S}_0} (2hv_{\text{in}}^n + \Psi_1^n), \\ \Psi_0^{n+1} &= 2(1 - \lambda^2)\Psi_0^n - \Psi_0^{n-1} + 2\lambda^2 \Psi_1^n + \frac{2h\lambda^2 S_{-1/2}}{\bar{S}_0} v_{\text{in}}^n. \end{aligned} \quad (5.23)$$

The input signal is arbitrary, but looking towards lip excitation, and following [23], one can set the input to a pulse train as shown in Figure 5.3. More details can be found in Chapter 7.



**Fig. 5.3:** A pulse train with a frequency of 213 Hz. This is used to excite the implementation of Webster's equation in Section 5.1.4.

### 5.1.4 Matrix form and output

One can write scheme (5.6) in matrix form by saving the state in a vector  $\Psi^n = [\Psi_0^n, \dots, \Psi_N^n]^T$  and creating a  $\mathbf{D}_{xx}$  matrix that includes the effect of the

### 5.1. Webster's equation

cross-sectional area  $S$ . Assuming Neumann boundary conditions yields

$$\mathbf{D}_{xx} = \frac{1}{h^2} \begin{bmatrix} -2 & 2 & & & & \mathbf{0} \\ \frac{S_{1/2}}{S_1} & -2 & \frac{S_{3/2}}{S_1} & & & \\ \ddots & \ddots & \ddots & & & \\ & \frac{S_{l-1/2}}{S_l} & -2 & \frac{S_{l+1/2}}{S_l} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{S_{N-3/2}}{S_{N-1}} & -2 & \frac{S_{N-1/2}}{S_{N-1}} \\ \mathbf{0} & & & & 2 & -2 \end{bmatrix}. \quad (5.24)$$

Notice that there are no appearances of  $S$  at the boundaries as these vanish due to the boundary conditions as in Eq. (5.14). Using  $\mathbf{I}_N$  as the  $N \times N$  identity matrix, one can write scheme (5.6) in matrix form as

$$\mathbf{A}\Psi^{n+1} = \mathbf{B}\Psi^n + \mathbf{C}\Psi^{n-1} + \mathbf{v}^n \quad (5.25)$$

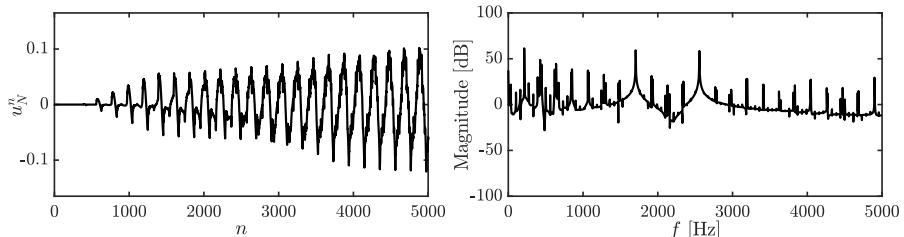
where

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & 1 + \alpha_+ \end{bmatrix}, \quad \mathbf{B} = 2\mathbf{I} + c^2 k^2 \mathbf{D}_{xx}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} -\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & -1 + \alpha_- \end{bmatrix},$$

and the  $(N + 1) \times 1$  input vector  $\mathbf{v}^n$  consists of zeros except for the first index:

$$\mathbf{v}_i^n = \begin{cases} \frac{2h\lambda^2 S_{-1/2}}{S_0} v_{in}^n, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (5.26)$$

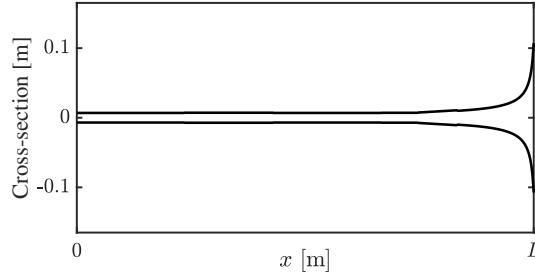
Notice how the radiation is included by changing the last entry of matrices  $\mathbf{A}$  and  $\mathbf{C}$ . The output of an implementation of Webster's equation is shown in Figure 5.4. The parameters used for the scheme, the input signal and the geometry used to obtain the output can be found in Table 5.1, Figure 5.3, and Figure 5.5 respectively.



**Fig. 5.4:** The output of Webster's equation at  $\Psi_N^n$  using the parameters in Table 5.1 and geometry in Figure 5.5.

Name	Symbol (unit)	Value
Length	$L$ (m)	$\approx 3$
Wave speed	$c$ (m/s)	343
Cross-sectional area	$S(x)$	See e.g. paper [H]

**Table 5.1:** Parameters for the implementation of Webster's equation. The length is slightly below 3 m to yield  $\lambda = 1$  in Eq. (5.11).



**Fig. 5.5:** The geometry used for the implementation. See paper [H] for more details.

### 5.1.5 Energy analysis

The energy analysis of Webster's equation with a radiating boundary might seem straightforward. However, due to the varying cross-sectional area, the energy balance deserves a more detailed treatment, especially at the boundaries. For this analysis, (centred) Neumann boundary conditions are used for both boundaries and the input is ignored. This section follows the steps presented in Section 3.4.

#### Step 1: Obtain $\delta_{t+}\mathfrak{h}$

Usually, to ensure vanishing boundary terms when using centred Neumann boundary conditions, the primed inner product in Eq. (3.9) is chosen. However, as the system has a spatially varying cross-section, the more general *weighted inner product* in Eq. (3.10) has to be chosen instead.

Taking an inner product weighted by free parameters  $0 < \epsilon_l, \epsilon_r \leq 2$  at the left and right boundary respectively, of scheme (5.6) with  $(\delta_t, \Psi_l^n)$  over discrete domain  $d$  yields

$$\delta_{t+}\mathfrak{h} = \langle \delta_t, \Psi_l^n, \bar{S}_l \delta_{tt} \Psi_l^n \rangle_d^{\epsilon_l, \epsilon_r} - c^2 \langle \delta_t, \Psi_l^n, \delta_{x-} (S_{l+1/2} (\delta_{x+} \Psi_l^n)) \rangle_d^{\epsilon_l, \epsilon_r} = 0. \quad (5.27)$$

### 5.1. Webster's equation

#### Step 2: Identify energy types and isolate $\delta_{t+}$

As the right boundary is set to be radiating according to Eq. (5.17), the energy balance will eventually be of the following form:

$$\delta_{t+}(\mathfrak{h} + \mathfrak{h}_b) = \mathfrak{b} - \mathfrak{q}_b, \quad (5.28)$$

where  $\mathfrak{h}_b$  is the energy stored by the radiating boundary through the inertia term,  $\mathfrak{q}_b$  describes the the energy losses through radiation and  $\mathfrak{b}$  is the general boundary term.

Starting at Eq. (5.27), the last term can – using identity (3.15a) – be rewritten to

$$c^2 \langle S_{l+1/2} \delta_{t+} \delta_{x+} \Psi_l^n, (\delta_{x+} \Psi_l^n) \rangle_d + \mathfrak{b}_r - \mathfrak{b}_l,$$

where

$$\mathfrak{b}_r = c^2 (\delta_{t+} \Psi_N^n) \left( \frac{\epsilon_r}{2} S_{N+1/2} (\delta_{x+} \Psi_N^n) + \left(1 - \frac{\epsilon_r}{2}\right) S_{N-1/2} (\delta_{x-} \Psi_N^n) \right), \quad (5.29a)$$

$$\mathfrak{b}_l = c^2 (\delta_{t+} \Psi_0^n) \left( \frac{\epsilon_l}{2} S_{-1/2} (\delta_{x-} \Psi_0^n) + \left(1 - \frac{\epsilon_l}{2}\right) S_{1/2} (\delta_{x+} \Psi_0^n) \right), \quad (5.29b)$$

are the right and left boundary term respectively (notice that  $\mathfrak{b}_l$  is subtracted). One can immediately observe that the boundary terms vanish if Dirichlet boundary conditions would be used.

Then, using identities (3.17a) and (3.17b) yields

$$\delta_{t+} \mathfrak{h} = \mathfrak{b}_r - \mathfrak{b}_l$$

where

$$\begin{aligned} \mathfrak{h} &= \mathfrak{t} + \mathfrak{v}, \quad \text{with} \quad \mathfrak{t} = \frac{1}{2} \left( \|\sqrt{\bar{S}_l} \delta_{t-} \Psi_l^n\|_d^{\epsilon_l, \epsilon_r} \right)^2 \quad \text{and} \\ \mathfrak{v} &= \frac{c^2}{2} \langle S_{l+1/2} \delta_{x+} \Psi_l^n, e_{t-} \delta_{x+} \Psi_l^n \rangle_d \end{aligned} \quad (5.30)$$

Notice that  $\bar{S}_l$  is included in the norm by using its square-root.

The next step is to find definitions for  $\epsilon_l$  and  $\epsilon_r$  such that the boundary terms vanish if radiation were to be ignored. In other words, the boundary terms need to be rewritten such that

$$\delta_x \cdot \Psi_0^n = 0 \Rightarrow \mathfrak{b}_l = 0$$

$$\delta_x \cdot \Psi_N^n = 0 \Rightarrow \mathfrak{b}_r = 0$$

for the left and right boundary respectively. It can be shown that, for the special cases of  $\epsilon_r = S_{N-1/2}/\mu_{xx} S_N$  and  $\epsilon_l = S_{1/2}/\mu_{xx} S_0$  the boundary terms vanish in this case

$$\mathfrak{b}_r = c^2 (\delta_{t+} \Psi_N^n) S_{N-1/2} (2 - \epsilon_r) (\delta_x \cdot \Psi_N^n), \quad (5.31a)$$

$$\mathfrak{b}_l = c^2 (\delta_{t+} \Psi_0^n) S_{1/2} (2 - \epsilon_l) (\delta_x \cdot \Psi_0^n). \quad (5.31b)$$

See Appendix E.2 for a derivation of this.

To add the energy stored and dissipated by the radiating boundary, its definition in Eq. (5.17) can be substituted into the right boundary term  $b_r$  in Eq. (5.31a) as

$$\begin{aligned} b_r &= c^2(\delta_{t-}\Psi_N^n)S_{N-1/2}(2-\epsilon_r)(-a_1\delta_{t-}\Psi_N^n - a_2\mu_{t-}\Psi_N^n), \\ &= c^2S_{N-1/2}(2-\epsilon_r)\left(-a_1(\delta_{t-}\Psi_N^n)^2 - a_2(\delta_{t-}\Psi_N^n)(\mu_{t-}\Psi_N^n)\right). \end{aligned}$$

This can then be decomposed in  $\mathfrak{h}_b$  and  $q_b$  used in Eq. (5.27).

Using identity (3.17d) yields the definitions for  $\mathfrak{h}_b$  and  $q_b$  in Eq. (5.28)

$$\mathfrak{h}_b = \frac{c^2S_{N-1/2}(2-\epsilon_r)a_2}{2}\mu_{t-}(\Psi_N^n)^2, \quad \text{and} \quad q_b = c^2S_{N-1/2}(2-\epsilon_r)a_1(\delta_{t-}\Psi_N^n)^2. \quad (5.32)$$

Finally,  $b = b_l$  and can be shown to vanish for both Dirichlet and (centred) Neumann conditions.

### Step 3: Check units

It seems like in order for the units to make sense, one must write Eq. (5.1) as

$$\frac{S\rho}{c^2}\partial_t^2\Psi = \frac{B}{c^2}\partial_x(S(\partial_x\Psi)), \quad (5.33)$$

where  $B$  is the bulk modulus of air (in N/m<sup>2</sup>)

### Step 4: Implementation

Figure 5.6 shows the energetic output of Webster's equation with a radiating boundary at  $x = L$ . To highlight the effect of the radiation on the energy the parameters are set to  $L = 1$  and  $S(x) = 0.01$  for all  $x \in \mathcal{D}$ . The system is excited with a raised cosine close to the left boundary, and when the excitation reaches the radiating boundary, the total energy in the system decreases due to the losses. The energy stored by the boundary  $\mathfrak{h}_b$  is also shown and indeed increases when the wave reaches the boundary.

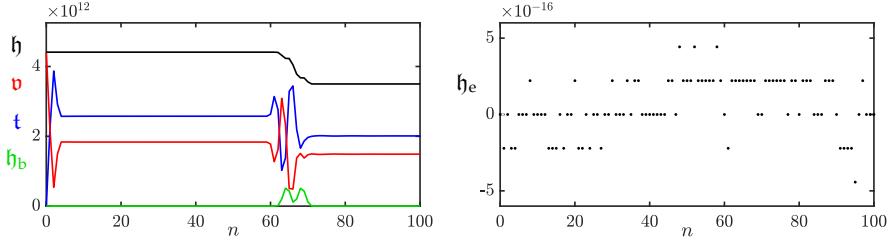
## 5.1.6 Stability through energy analysis

Frequency domain analysis as presented in Section 3.3, or more specifically, von Neumann analysis, can not be performed on Webster's equation due to the varying cross-section of the system [23]. Instead, stability conditions can be obtained through energy analysis explained in Section 3.4.4.

Consider the following scheme

$$[S]_l\delta_{tt}\Psi_l^n = c^2\delta_{x-}(S_{l+1/2}(\delta_{x+}\Psi_l^n)), \quad (5.34)$$

### 5.1. Webster's equation



**Fig. 5.6:** The kinetic (blue), potential (red), and total (black) energy as well as the energy stored by the radiation condition (green) of an implementation of Webster's equation are plotted in the left panel. Notice that the energy decreases between  $n = 60$  and  $n = 70$  as the excitation reached the boundary where damping is included. The right panel shows the normalised energy (according to Eq. (3.38)) and shows that the deviation of the energy is within machine precision.

where  $[S]_l$  is a still undetermined second-order approximation to the true geometry of the acoustic tube and will be shown to be  $\bar{S}_l$  below. As done for the 1D wave equation in Section 3.4.4, the potential energy  $\mathfrak{v}$  in Eq. (5.30) can be rewritten using identity (3.17f) as

$$\mathfrak{v} = \frac{c^2}{2} \left( \|\sqrt{S_{l+1/2}}\mu_{t-}\delta_{x+}\Psi_l^n\|_d^2 - \frac{k^2}{4} \|\sqrt{S_{l+1/2}}\delta_{t-}\delta_{x+}\Psi_l^n\|_d^2 \right).$$

For spatially varying systems, one can use the following extension of the bound given in Eq. (3.51) [23]

$$\|\sqrt{\phi_l}\delta_{x+}u_l^n\|_d \leq \frac{2}{h} \|\sqrt{\mu_{x-}\phi_l}u_l^n\|_d, \quad (5.35)$$

where spatially varying function  $\phi_l > 0$  is defined over the same domain as  $u$ . The following condition can then be put on  $\mathfrak{v}$

$$\begin{aligned} \mathfrak{v} &\geq \frac{c^2}{2} \left( \|\sqrt{S_{l+1/2}}\mu_{t-}\delta_{x+}\Psi_l^n\|_d^2 - \frac{k^2}{4} \left( \frac{2}{h} \|\sqrt{\mu_{x-}S_{l+1/2}}\delta_{t-}\Psi_l^n\|_d \right)^2 \right), \\ &\geq \frac{c^2}{2} \left( \|\sqrt{S_{l+1/2}}\mu_{t-}\delta_{x+}\Psi_l^n\|_d^2 - \frac{k^2}{h^2} \|\sqrt{\mu_{xx}S_l}\delta_{t-}\Psi_l^n\|_d^2 \right) \\ &\geq \frac{c^2}{2} \left( \|\sqrt{S_{l+1/2}}\mu_{t-}\delta_{x+}\Psi_l^n\|_d^2 - \frac{k^2}{h^2} \left( \|\sqrt{\mu_{xx}S_l}\delta_{t-}\Psi_l^n\|_d^{\epsilon_l, \epsilon_r} \right)^2 \right), \end{aligned}$$

where last step is possible because  $0 < \epsilon_l, \epsilon_r \leq 2$ . Substituting this into the energy balance in Eq. (5.30) yields

$$\begin{aligned} \mathfrak{h} = \mathfrak{t} + \mathfrak{v} &\geq \frac{1}{2} \left( \|\sqrt{[S]_l}\delta_{t-}\Psi_l^n\|_d^{\epsilon_l, \epsilon_r} \right)^2 \\ &+ \frac{c^2}{2} \left( \|\sqrt{S_{l+1/2}}\mu_{t-}\delta_{x+}\Psi_l^n\|_d^2 - \frac{k^2}{h^2} \left( \|\sqrt{\mu_{xx}S_l}\delta_{t-}\Psi_l^n\|_d^{\epsilon_l, \epsilon_r} \right)^2 \right), \end{aligned}$$

and as  $\|\sqrt{S_{l+1/2}}\mu_{t-}\delta_{x+}\Psi_l^n\|_d^2$  is non-negative, the following is also true

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{v} \geq \frac{1}{2} \left( \|\sqrt{[S]_l}\delta_{t-}\Psi_l^n\|_d^{\epsilon_l, \epsilon_r} \right)^2 - \frac{\lambda^2}{2} \left( \|\sqrt{\mu_{xx}S_l}\delta_{t-}\Psi_l^n\|_d^{\epsilon_l, \epsilon_r} \right)^2.$$

This can be written as

$$\mathfrak{h} \geq \frac{1}{2} \sum_d \left( \sqrt{[S]_l} - \lambda^2 \sqrt{\mu_{xx}S_l} \right) (\delta_{t-}\Psi_l^n)^2 \quad (5.36)$$

which is non-negative if

$$\min \left( \sqrt{[S]_l} - \lambda^2 \sqrt{\mu_{xx}S_l} \right) \geq 0,$$

$$\lambda \leq \min \left( \sqrt{\frac{[S]_l}{\mu_{xx}S_l}} \right).$$

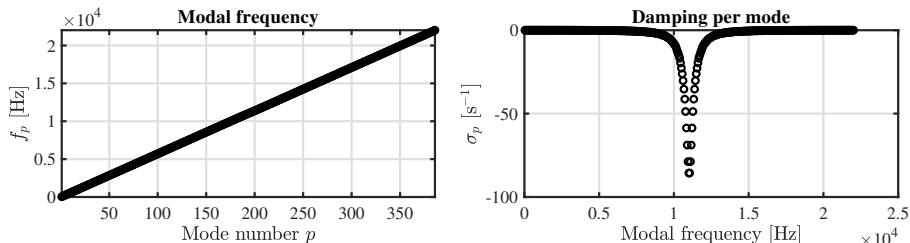
For the special choice of  $[S]_l = \mu_{xx}S_l$ , this condition reduces to

$$\lambda \leq 1, \quad (5.37)$$

also given in (5.11). This choice of  $[S]_l$  is equal to  $\bar{S}_l$  through Eq. (5.7), hence its choice in Eq. (5.6).

### 5.1.7 Modal analysis

The variable cross-section causes the modes of the system to vary in interesting ways. If the tube is perfectly cylindrical and  $S(x)$  is thus constant, Webster's equation reduces to the 1D wave equation and the modal frequencies are integer multiples of the fundamental frequency. For low cross-sectional variations, the modes generally follow a linear pattern. See Figure 5.7. The damping per mode, however, follows a different pattern, where higher damping occurs around  $f_s/4$ , and is due to the comb-filtering effect that the radiation has on the system. [is this true?](#)



**Fig. 5.7:** The result of a modal analysis of Webster's equation with parameters and geometry given in 5.1.4. Notice the higher damping for modes around  $f_s/4$ .

## 5.2 First-order system

Until now, only PDEs have been presented that second-order in time, i.e., that are dependent on the acceleration of the state variable. This section presents a system of two coupled first-order PDEs which are instead dependent on the velocity. State-of-the-art research on brass instruments in the context of FDTD methods also uses this coupled system (see e.g.[65, 59]) and has been used in this project to model the trombone in paper [H].

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### 5.2.1 Continuous time

Using the same variables as before for cross-sectional area  $S = S(x)$ , wave speed  $c$ , and air density  $\rho_0$ , a system of PDEs that describes air propagation in an acoustic tube can be defined as follows:

$$\frac{S}{\rho_0 c^2} \partial_t p = -\partial_x(Sv), \quad (5.38a)$$

$$\rho_0 \partial_t v = -\partial_x p, \quad (5.38b)$$

where pressure  $p = p(x, t)$  (Pa) and particle velocity  $v = v(x, t)$  (m/s) are defined for  $x \in \mathcal{D}$  with domain  $\mathcal{D} = [0, L]$  and tube length  $L$  (in m). These state variables are related to the acoustic potential  $\Psi$  as shown in Eq. (5.2) as

$$p = \rho_0 \partial_t \Psi, \quad v = -\partial_x \Psi. \quad (5.39)$$

Indeed it can be shown by substituting these definitions into Eq. (5.38a), Webster's equation can be obtained:

$$\frac{S}{\rho_0 c^2} \partial_t(\rho_0 \partial_t \Psi) = -\partial_x(S(-\partial_x \Psi)) \implies S \partial_t^2 \Psi = c^2 \partial_x(S \partial_x \Psi).$$

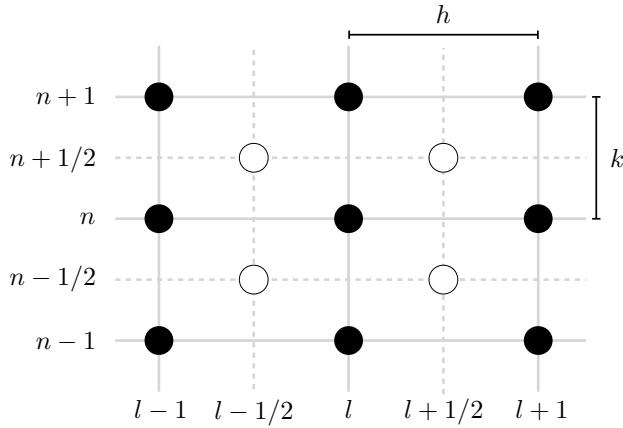
#### Boundary conditions

For the first-order PDE system in Eq. (5.38), the boundary conditions are defined as follows:

$$p(0, t) = 0, \quad p(L, t) = 0, \quad (\text{Dirichlet, open}), \quad (5.40a)$$

$$S(0)v(0) = 0, \quad S(L)v(L) = 0, \quad (\text{Neumann, closed}), \quad (5.40b)$$

which, through Eq. (5.39), relate to the boundary conditions of Webster's equation in Eq. (5.3).



**Fig. 5.8:** The interleaved grid used for the system of FD schemes in Eq. (5.41). Grid points on the regular grid (in black) are used for pressure  $p$ , while points on the interleaved grid (in white) are used for particle velocity  $v$ .

## 5.2.2 Discrete time

It is useful to place either  $p$  or  $v$  on an interleaved grid (see Figure 5.8). Following [59],  $v$  is placed on this interleaved grid both in space and time. Accordingly, system (5.38) is discretised as

$$\frac{\bar{S}_l}{\rho_0 c^2} \delta_{t+} p_l^n = -\delta_x (S_{l+1/2} v_{l+1/2}^{n+1/2}), \quad (5.41a)$$

$$\rho_0 \delta_{t-} v_{l+1/2}^{n+1/2} = -\delta_x + p_l^n, \quad (5.41b)$$

after which the update equations become

$$p_l^{n+1} = p_l^n - \frac{\rho_0 c \lambda}{\bar{S}_l} (S_{l+1/2} v_{l+1/2}^{n+1/2} - S_{l-1/2} v_{l-1/2}^{n+1/2}), \quad (5.42a)$$

$$v_{l+1/2}^{n+1/2} = v_{l+1/2}^{n-1/2} - \frac{\lambda}{\rho_0 c} (p_{l+1}^n - p_l^n), \quad (5.42b)$$

where (again)  $\lambda = ck/h \leq 1$  for stability. The pressure is defined for  $l = \{0, \dots, N\}$  and the velocity for  $l = \{0, \dots, N-1\}$  where  $N$  is the number of intervals between the grid points on the pressure grid. Notice that the range of calculation for the particle velocity is one fewer grid point than the that of the pressure.

An advantage of using an interleaved grid like this, is that the forward and backward first-order are second-order accurate, and can be shown through a Taylor series expansion as done in 2.2.2 (also see [59]).

### Boundary conditions

The boundary conditions in Eq. (5.40a) can be discretised as follows

$$p_0^n = 0, \quad p_N^n = 0, \quad (\text{Dirichlet, open}), \quad (5.43\text{a})$$

$$\mu_{x-}(S_{1/2}v_{1/2}^n) = 0, \quad \mu_{x+}(S_{N-1/2}v_{N-1/2}^n) = 0, \quad (\text{Neumann, closed}). \quad (5.43\text{b})$$

### 5.2.3 Matrix form

System (5.41) can be written in matrix form, by saving the states of  $p_l^n$  and  $v_{l+1/2}^{n+1}$  in vectors as

$$\mathbf{p}^n = [p_0^n, \dots, p_N^n]^T, \quad \text{and} \quad \mathbf{v}^{n+1/2} = [v_{1/2}^{n+1/2}, \dots, v_{N-1/2}^{n+1/2}]^T \quad (5.44)$$

which are of sizes  $(N + 1) \times 1$  and  $N \times 1$  respectively. One may then write the scheme in matrix form as

$$\mathbf{v}^{n+1/2} = \mathbf{v}^{n-1/2} + \mathbf{B}_p \mathbf{p}^n \quad (5.45\text{a})$$

$$\mathbf{p}^{n+1} = \mathbf{p}^n + \mathbf{B}_v \mathbf{v}^{n+1/2} \quad (5.45\text{b})$$

where

$$\mathbf{B}_v = \frac{\rho_0 c}{\lambda} \begin{bmatrix} -\frac{2S_{1/2}}{S_0} & & & & \mathbf{0} \\ \frac{S_{1/2}}{S_1} & -\frac{S_{3/2}}{S_1} & & & \\ & \ddots & \ddots & & \\ & & \frac{S_{N-3/2}}{S_{N-1}} & -\frac{S_{N-1/2}}{S_{N-1}} & \\ \mathbf{0} & & & \frac{2S_{N-1/2}}{S_N} & \end{bmatrix}. \quad (5.46)$$

is of size  $(N + 1) \times N$ .

$$\mathbf{B}_p = \frac{\lambda}{\rho c} \begin{bmatrix} 1 & -1 & & & \mathbf{0} & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 & \\ \mathbf{0} & & & & 1 & -1 \end{bmatrix}. \quad (5.47)$$

is of size  $N \times (N + 1)$ .

Alternatively, one can write the scheme in one-step form by concatenating the states of the pressure and particle velocity into one vector. The matrix form will then be

$$\underbrace{\begin{bmatrix} \mathbf{p}^{n+1} \\ \mathbf{v}^{n+1/2} \end{bmatrix}}_{\mathbf{u}^{n+1}} = \mathbf{B} \underbrace{\begin{bmatrix} \mathbf{p}^n \\ \mathbf{v}^{n-1/2} \end{bmatrix}}_{\mathbf{u}^n} \quad (5.48)$$

where

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_{N+1} + \mathbf{B}_v \mathbf{B}_p & \mathbf{B}_v \\ \mathbf{B}_p & \mathbf{I}_N \end{bmatrix}, \quad (5.49)$$

is of size  $(2N + 1) \times (2N + 1)$  and may be directly used as matrix  $\mathbf{Q}$  for the modal analysis using a one-step form described in Section 3.5.1.

Finally the concatenated vector is defined as,

$$\mathbf{u}^n = [p_0^n, \dots, p_N^n, v_{1/2}^{n-1/2}, \dots, v_{N-1/2}^{n-1/2}]^T \quad (5.50)$$

and will be of size  $(2N + 1) \times 1$ .

## 5.2.4 Energy analysis

This section presents an energy analysis of the first-order system presented above using the techniques presented in Section 3.4. For the bulk of the analysis, [59] is followed.

### Step 1: Obtain $\delta_{t+}\mathfrak{h}$

To obtain the correct energy balance, an inner product of Eq. (5.41a) with  $\mu_{t+}p_l^n$  needs to be taken over discrete domain  $d = \{0, \dots, N\}$ . Using the primed inner product in Eq. (3.9) and after taking all terms to the left-hand side yields<sup>2</sup>

$$\delta_{t+}\mathfrak{h} = \frac{1}{\rho_0 c^2} \langle \mu_{t+}p_l^n, \bar{S}\delta_{t+}p_l^n \rangle'_d + \langle \mu_{t+}p_l^n, \delta_{x-}(S_{l+1/2}v_{l+1/2}^{n+1/2}) \rangle'_d = 0. \quad (5.51)$$

### Step 2: Identify energy types and isolate $\delta_{t+}$

For the rest of the analysis, the following superscripts and subscripts will be assumed unless denoted otherwise:  $n$  and  $l$  for  $p$ ,  $l$  for  $\bar{S}$ ,  $l + 1/2$  for  $S$  and  $l + 1/2$  and  $n + 1/2$  for  $v$ . After performing summation by parts of the last term using identity (3.13a), Eq. (5.51) becomes

$$\delta_{t+}\mathfrak{h} = \frac{1}{\rho_0 c^2} \langle \mu_{t+}p, \bar{S}\delta_{t+}p \rangle'_d - \langle \mu_{t+}\delta_{x+}p, Sv \rangle_d = -\mathfrak{b} \quad (5.52)$$

where the boundary term is

$$\mathfrak{b} = \mathfrak{b}_r + \mathfrak{b}_l, \quad \text{with}$$

$$\mathfrak{b}_r = (\mu_{t+}p_N)\mu_{x+}(S_{N-1/2}v_{N-1/2}) \quad \text{and} \quad (5.53)$$

$$\mathfrak{b}_l = -(\mu_{t+}p_0)\mu_{x-}(S_{1/2}v_{1/2}), \quad (5.54)$$

---

<sup>2</sup>The primed rather than the weighted inner product can be used here as the eventual boundary terms can be shown to vanish when using the boundary conditions in Eq. (5.43a).

## 5.2. First-order system

and can be shown to vanish under the boundary conditions shown in Eq. (5.43a). Then, Eq. (5.41b) can be substituted into Eq. (5.52) to get

$$\delta_{t+} \mathfrak{h} = \frac{1}{\rho_0 c^2} \langle \mu_{t+} p, \bar{S} \delta_{t+} p \rangle'_d - \langle \mu_{t+} (-\rho_0 \delta_{t-} v), S v \rangle_d = 0 \quad (5.55)$$

$$= \frac{1}{\rho_0 c^2} \langle \mu_{t+} p, \bar{S} \delta_{t+} p \rangle'_d + \rho_0 \langle \delta_{t-} v, S v \rangle_d = 0. \quad (5.56)$$

Finally, one can use identities (3.17c) and (3.17b) for the first and second term respectively to get

$$\begin{aligned} \mathfrak{h} &= \mathfrak{t} + \mathfrak{v} \quad \text{where} \\ \mathfrak{t} &= \frac{\rho_0}{2} \langle S v, e_{t-} v \rangle_d \quad \text{and} \quad \mathfrak{v} = \frac{1}{2\rho_0 c^2} \left( \|\sqrt{\bar{S}} p\|'_d \right)^2. \end{aligned} \quad (5.57)$$

### Step 3: Check units

Writing the terms in Eq. (5.57) in their units yields

$$\begin{aligned} \frac{\rho_0}{2} \langle S v, e_{t-} v \rangle_d &\xrightarrow{\text{in units}} \text{kg} \cdot \text{m}^{-3} \cdot \text{m} \cdot (\text{m}^2 \cdot \text{m} \cdot \text{s}^{-1} \cdot \text{m} \cdot \text{s}^{-1}), \\ &= \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}, \\ \frac{1}{2\rho_0 c^2} \left( \|\sqrt{\bar{S}} p\|'_d \right)^2 &\xrightarrow{\text{in units}} (\text{kg} \cdot \text{m}^{-3} \cdot \text{m}^2 \cdot \text{s}^{-2})^{-1} \cdot (\text{m} \cdot \text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2})^2, \\ &= \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}, \end{aligned}$$

and indeed have the correct units.

### Step 4: Implementation

Figure 5.9 shows the energetic output of an implementation of the first order system in Eq. (5.41), and shows that the energy is within machine precision.

## 5.2.5 Adding radiation

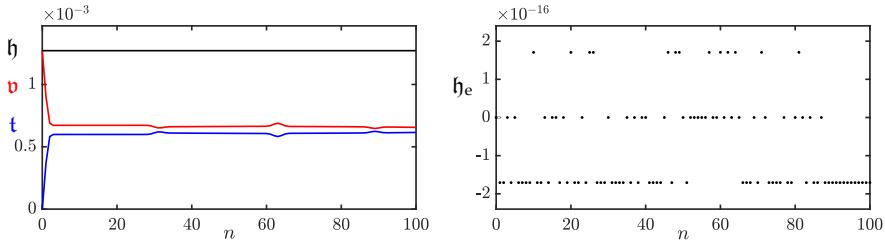
Following [59], radiation can be added to the schemes using a circuit representation of the Levine and Schwinger radiation model (See Figure 5.10). This section will provide a derivation I'll actually move most of this to an appendix...

The system can be described as

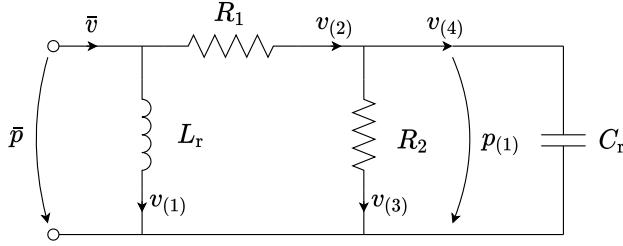
$$\bar{v} = \mu_{t+} v_{(1)} + \frac{1}{R_2} \mu_{t+} p_{(1)} + C_r \delta_{t+} p_{(1)}, \quad (5.58a)$$

$$\bar{p} = L_r \delta_{t+} v_{(1)}, \quad (5.58b)$$

$$\bar{p} = \left( 1 + \frac{R_1}{R_2} \right) \mu_{t+} p_{(1)} + R_1 C_r \delta_{t+} p_{(1)}, \quad (5.58c)$$



**Fig. 5.9:** The kinetic (blue), potential (red), and total (black) energy of an implementation of the first-order system in Eq. (5.41) are plotted in the left panel. The right panel shows the normalised energy (according to Eq. (3.37)) and shows that the deviation of the energy is within machine precision.



**Fig. 5.10:** The circuit representation of the Levine and Schwinger radiation model.

where  $\bar{p}^{n+1/2}$  and  $\bar{v}^{n+1/2}$  are placed on the interleaved temporal grid and are related to the tube by

$$\bar{p} = \mu_{t+} p_N^n, \quad \bar{S}_N \bar{v} = \mu_{x-} \left( S_{N+1/2} v_{N+1/2}^{n+1/2} \right). \quad (5.59)$$

This can be applied to the right boundary of the tube by evaluating Eq. (5.42a) at  $l = N$

$$p_N^{n+1} = p_N^n - \frac{\rho_0 c \lambda}{\bar{S}_N} \left( S_{N+1/2} v_{N+1/2}^{n+1/2} - S_{N-1/2} v_{N-1/2}^{n+1/2} \right), \quad (5.60)$$

and, rewriting this to

$$\begin{aligned} p_N^{n+1} &= p_N^n - \frac{\rho_0 c \lambda}{\bar{S}_N} \left( 2\mu_{x-} \left( S_{N+1/2} v_{N+1/2}^{n+1/2} \right) - 2S_{N-1/2} v_{N-1/2}^{n+1/2} \right), \\ p_N^{n+1} &= p_N^n - \frac{2\rho_0 c \lambda}{\bar{S}_N} \left( \bar{S}_N \bar{v} - S_{N-1/2} v_{N-1/2}^{n+1/2} \right). \end{aligned} \quad (5.61)$$

A definition for  $\bar{v}$  can then be found by first expanding Eq. (5.58a) to

$$\bar{v} = \frac{1}{2} \left( v_{(1)}^{n+1} + v_{(1)}^n \right) + \left( \frac{1}{2R_2} + \frac{C_r}{k} \right) p_{(1)}^{n+1} + \left( \frac{1}{2R_2} - \frac{C_r}{k} \right) p_{(1)}^n. \quad (5.62)$$

## 5.2. First-order system

The goal is to make this definition solely dependent on the known values  $v_{(1)}^n$ ,  $p_{(1)}^n$  and  $p_N^n$  and the unknown  $p_N^{n+1}$  (as this can be obtained using Eq. (5.61)). Equation (5.58b) can be expanded to

$$v_{(1)}^{n+1} = \frac{k}{L_r} \bar{p} + v_{(1)}^n, \quad (5.63)$$

and Eq. (5.58c) to

$$\begin{aligned} \bar{p} &= \left(1 + \frac{R_1}{R_2}\right) \mu_t + p_{(1)} + R_1 C_r \delta_{t+p_{(1)}}, \\ \bar{p} &= \frac{1}{2} \left(1 + \frac{R_1}{R_2}\right) (p_{(1)}^{n+1} + p_{(1)}^n) + \frac{R_1 C_r}{k} (p_{(1)}^{n+1} - p_{(1)}^n), \\ \left(\frac{1}{2} + \frac{R_1}{2R_2} + \frac{R_1 C_r}{k}\right) p_{(1)}^{n+1} &= \bar{p} + \left(\frac{R_1 C_r}{k} - \frac{1}{2} - \frac{R_1}{2R_2}\right) p_{(1)}^n \end{aligned}$$

and finally solved for  $p_{(1)}^{n+1}$  as

$$p_{(1)}^{n+1} = \underbrace{\left(\frac{2R_2 k}{2R_1 R_2 C_r + k(R_1 + R_2)}\right)}_{\zeta_1} \bar{p} + \underbrace{\left(\frac{2R_1 R_2 C_r - k(R_1 + R_2)}{2R_1 R_2 C_r + k(R_1 + R_2)}\right)}_{\zeta_2} p_{(1)}^n. \quad (5.64)$$

Equations (5.63) and (5.64) can then be substituted into Eq. (5.62) and using the definition of  $\bar{p}$  from Eq. (5.59) yields

$$\begin{aligned} \bar{v} &= \frac{1}{2} \left( \frac{k}{L_r} (\mu_t + p_N^n) + 2v_{(1)}^n \right) + \left( \frac{1}{2R_2} + \frac{C_r}{k} \right) \zeta_1 \mu_t + p_N^n \\ &\quad + \left( \frac{1}{2R_2} + \frac{C_r}{k} \right) \zeta_2 p_{(1)}^n + \left( \frac{1}{2R_2} - \frac{C_r}{k} \right) p_{(1)}^n, \\ \bar{v} &= \underbrace{\left( \frac{k}{2L_r} + \frac{\zeta_1}{2R_2} + \frac{C_r \zeta_1}{k} \right)}_{\zeta_3} \mu_t + p_N^n + v_{(1)}^n + \underbrace{\left( \frac{\zeta_2 + 1}{2R_2} + \frac{C_r \zeta_2 - C_r}{k} \right)}_{\zeta_4} p_{(1)}^n. \quad (5.65) \end{aligned}$$

Finally, substituting this definition for  $\bar{v}$  into Eq. (5.61) yields

$$\begin{aligned} p_N^{n+1} &= p_N^n - \frac{2\rho_0 c \lambda}{\bar{S}_N} \left( \bar{S}_N \left[ \zeta_3 \left( \frac{p_N^{n+1} + p_N^n}{2} \right) + v_{(1)}^n + \zeta_4 p_{(1)}^n \right] - S_{N-1/2} v_{N-1/2}^{n+1/2} \right) \\ p_N^{n+1} &= p_N^n - \rho_0 c \lambda \left( \zeta_3 (p_N^{n+1} + p_N^n) + 2(v_{(1)}^n + \zeta_4 p_{(1)}^n) - \frac{2S_{N-1/2} v_{N-1/2}^{n+1/2}}{\bar{S}_N} \right) \end{aligned}$$

and yields a definition for  $p_N^{n+1}$  based on known values of the system

$$p_N^{n+1} = \frac{1 - \rho_0 c \lambda \zeta_3}{1 + \rho_0 c \lambda \zeta_3} p_N^n - \frac{2\rho_0 c \lambda}{1 + \rho_0 c \lambda \zeta_3} \left( v_{(1)}^n + \zeta_4 p_{(1)}^n - \frac{S_{N-1/2} v_{N-1/2}^{n+1/2}}{\bar{S}_N} \right). \quad (5.66)$$

### 5.2.6 Energy

**not done yet...** Recalling the condition at the right boundary from (5.53)

$$\mathfrak{b}_r = (\mu_{t+} p_N) \underbrace{\mu_{x+} (S_{N-1/2} v_{N-1/2})}_{\mu_{x-} S_{N+1/2} v_{N+1/2}}, \quad (5.67)$$

using Eq. (5.59) one can rewrite this to

$$\mathfrak{b}_r = \bar{p} \bar{S}_N \bar{v}. \quad (5.68)$$

then we can

# Chapter 6

## 2D Systems

The previous chapters considered systems distributed over one spatial dimension. As not all musical instruments or instrument-components can be simplified to this, higher dimensional systems need to be taken into consideration, such as 2D systems. 2D PDEs can be used to model drum membranes, plate reverbs or simplified instrument bodies.

Apart from being slightly more complex than 1D models, the main issue with 2D systems is that their implementations are orders of magnitude heavier to compute than 1D schemes. This chapter will therefore also provide details on how to best implement these schemes in MATLAB. Implementation in C++ will be detailed in Chapter ??.

This chapter starts by providing some additional information about 2D grid functions and operators. Then, the 2D wave equation is presented, which is used to extend the analysis techniques presented in Chapter 3 to 2D.<sup>1</sup> Afterwards, two 2D models used in this work, the thin plate and the stiff membrane will be described in a similar fashion. The systems modelled in this work are simplified to be rectangular and are defined on a Cartesian coordinate system. Section 6.5 briefly elaborates on radial coordinate systems and their shortcomings. Unless denoted otherwise, the theory in this chapter follows [23].

### 6.1 PDEs and FD schemes in 2D

Consider a rectangular 2D system with side lengths  $L_x$  and  $L_y$  (both in m) and its state described by  $u = u(x, y, t)$ . The system is defined for  $t \geq 0$  and  $(x, y) \in \mathcal{D}$  where domain  $\mathcal{D} \in [0, L_x] \times [0, L_y]$  is two-dimensional.

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<sup>1</sup>The abbreviation 2D will also be used for ‘two dimensions’.

Similar to the 1D case explained in Section 2.2.1, the state variable can be discretised to a 2D grid function according to  $u(x, y, t) \cong u_{l,m}^n$  with space  $x = lh$  and  $y = mh$  and time  $t = nk$  with  $k = 1/f_s$ . The temporal index  $n \in \mathbb{N}^0$  and the spatial indices  $l \in \{0, \dots, N_x\}$  and  $m \in \{0, \dots, N_y\}$  index the grid function in space in the  $x$  and  $y$  directions respectively. Here,  $N_x$  is the number of intervals between grid points in the  $x$  direction and  $N_y$  in the  $y$  direction. For simplicity, the grid spacing  $h$  is set to be the same in both the  $x$  and  $y$  directions in this work.

### Additional operators

In continuous time, an additional operator referred to as the *Laplacian*, can be defined as

$$\Delta = \partial_x^2 + \partial_y^2, \quad (6.1)$$

which describes a second-order spatial derivative in 2D. A 4<sup>th</sup>-order spatial derivative in 2D, used to model stiffness like in the stiff string in Chapter 4 is called the *biharmonic* operator, and is defined as the Laplacian applied to itself:

$$\Delta\Delta = \partial_x^4 + 2\partial_x^2\partial_y^2 + \partial_y^4. \quad (6.2)$$

In discrete time, the same temporal and spatial shift operators as defined in Section 2.2.2 can be applied to grid function  $u_{l,m}^n$  the latter of which only affects the spatial index  $l$ . Additional operators affecting spatial index  $m$  for the  $y$  direction are

$$e_{y+}u_{l,m}^n = u_{l,m+1}^n, \quad \text{and} \quad e_{y-}u_{l,m}^n = u_{l,m-1}^n. \quad (6.3)$$

Using these shift operators, a discrete approximation of the Laplacian in Eq. (6.1) can be made<sup>2</sup>

$$\Delta \cong \delta_\Delta \triangleq \frac{1}{h^2} (e_{x+} + e_{x-} + e_{y+} + e_{y-} - 4), \quad (6.4)$$

and when applied to a grid function yields

$$\Delta u \cong \delta_\Delta u_{l,m}^n = \frac{1}{h^2} (u_{l+1,m}^n + u_{l-1,m}^n + u_{l,m+1}^n + u_{l,m-1}^n - 4u_{l,m}^n). \quad (6.5)$$

See Figure 6.1a for the stencil of the discrete Laplacian. Similarly, an approximation of the biharmonic operator in Eq. (6.2) can be made as

$$\begin{aligned} \Delta\Delta \cong \delta_\Delta \delta_\Delta \triangleq \frac{1}{h^4} & \left[ (e_{x+}^2 + e_{x-}^2 + e_{y+}^2 + e_{y-}^2) \right. \\ & + 2(e_{x+}e_{y+} + e_{x+}e_{y-} + e_{x-}e_{y+} + e_{x-}e_{y-}) \\ & \left. - 8(e_{x+} + e_{x-} + e_{y+} + e_{y-}) + 20 \right], \end{aligned} \quad (6.6)$$

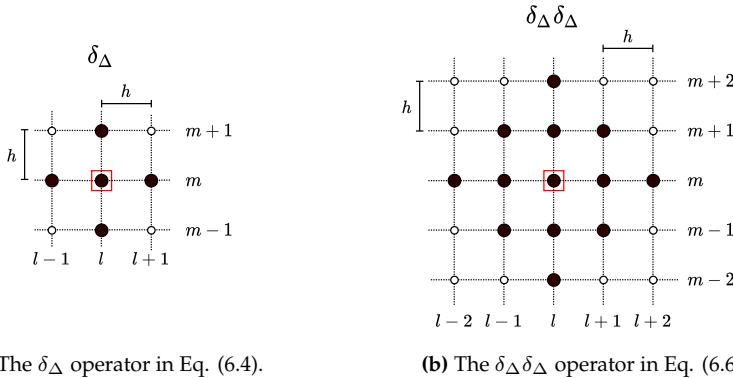
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<sup>2</sup>Notice that the  $\delta_\Delta$  operator is identical to  $\delta_{\Delta\Box}$  in [23], but will not be used here as Eq. (6.4) is the only discretisation to the  $\Delta$  operator used in this work.

and when applied to a grid function yields

$$\begin{aligned}\Delta\Delta u \approx \delta_\Delta\delta_\Delta u_{l,m}^n = \frac{1}{h^4} & \left[ (u_{l+2,m}^n + u_{l-2,m}^n + u_{l,m+2}^n + u_{l,m-2}^n) \right. \\ & + 2(u_{l+1,m+1}^n + u_{l-1,m+1}^n + u_{l+1,m-1}^n + u_{l-1,m-1}^n) \\ & \left. - 8(u_{l+1,m}^n + u_{l-1,m}^n + u_{l,m+1}^n + u_{l,m-1}^n) + 20u_{l,m}^n \right].\end{aligned}$$

See Figure 6.1b for the stencil of the discrete biharmonic operator.



**Fig. 6.1:** The stencils of the 2D spatial FD operators in Eqs. (6.4) and (6.6) respectively. The red square denotes what grid point the operator is applied to. The stencils follow the same layout as Figure 2.3, but the vertical axis denotes a second spatial dimension rather than time.

## 6.2 The 2D wave equation

The 2D wave equation is the simplest 2D model in the context of musical acoustics and using the operators presented above it is a fairly straightforward extension to the 1D wave equation. Similar to how the 1D wave equation is used to model an ideal string, the 2D wave equation can be used to model an ideal membrane.

The first appearance of an implementation of the 2D wave equation in a musical context was due to van Duyne and Smith who used digital waveguides, or more specifically a waveguide mesh, to discretise it [66]. The implementation is identical to the FD scheme that will be presented here. *though it is more general here due to  $\lambda^2$  instead of hard-coded 0.5*

This section will present the 2D wave equation in continuous time and its discretisation afterwards. The resulting FD scheme is then used as a test-case to extend the various analysis techniques presented in Chapter 3 to 2D.

### 6.2.1 Continuous time

Consider a system modelling the 2D wave equation with side lengths  $L_x$  and  $L_y$  (both in m) and its state described by  $u = u(x, y, t)$ . The system is defined over  $(x, y) \in \mathcal{D}$  with domain  $\mathcal{D} = [0, L_x] \times [0, L_y]$  and its motion is described by the following PDE:

$$\partial_t^2 u = c^2 \Delta u, \quad (6.7)$$

with wave speed  $c$  (in m/s) and the Laplacian operator as defined in Eq. (6.1). If the 2D wave equation is used to model an ideal membrane, the wave speed is defined as  $c = \sqrt{T/\rho H}$  (in m/s), with tension per unit length (applied to the boundary)  $T$  (in N/m), material density  $\rho$  (in kg/m<sup>3</sup>) and thickness  $H$  (in m).

check whether  
this wording is  
right..

#### Boundary conditions

Similar to the 1D wave equation, two alternatives for boundary conditions are

$$\left. \begin{array}{l} u(0, y, t) = u(L_x, y, t) = 0 \\ u(x, 0, t) = u(x, L_y, t) = 0 \end{array} \right\} \quad (\text{Dirichlet, fixed}), \quad (6.8a)$$

$$\left. \begin{array}{l} \partial_x u(0, y, t) = \partial_x u(L_x, y, t) = 0 \\ \partial_y u(x, 0, t) = \partial_y u(x, L_y, t) = 0 \end{array} \right\} \quad (\text{Neumann, free}), \quad (6.8b)$$

where  $\forall$  means 'for all values of'.

### 6.2.2 Discrete time

Using the definition for the approximation of the Laplacian in Eq. (6.4), the 2D wave equation PDE in Eq. (6.7) can be discretised to

$$\delta_{tt} u_{l,m}^n = c^2 \delta_\Delta u_{l,m}^n, \quad (6.9)$$

with  $l \in \{0, \dots, N_x\}$  and  $m \in \{0, \dots, N_y\}$  where  $N_x$  and  $N_y$  are the number of intervals between grid points in the  $x$  and  $y$  direction respectively. The operators can then be expanded (see Eq. (6.5)) and solving for  $u_{l,m}^n$  yields the following update equation

$$u_{l,m}^{n+1} = 2u_{l,m}^n - u_{l,m}^{n-1} + \lambda^2 (u_{l+1,m}^n + u_{l-1,m}^n + u_{l,m+1}^n + u_{l,m-1}^n - 4u_{l,m}^n), \quad (6.10)$$

where the Courant number

$$\lambda = \frac{ck}{h}, \quad (6.11)$$

and needs to abide

$$\lambda \leq \frac{1}{\sqrt{2}} \quad (6.12)$$

for the scheme to be stable. See Section 6.2.4 for a derivation of this. Writing this condition in terms of the grid spacing places the following limit on  $h$ :

$$h \geq \sqrt{2}ck. \quad (6.13)$$

### Discrete boundary conditions

The boundary conditions in Eqs. (6.8) can be discretised to

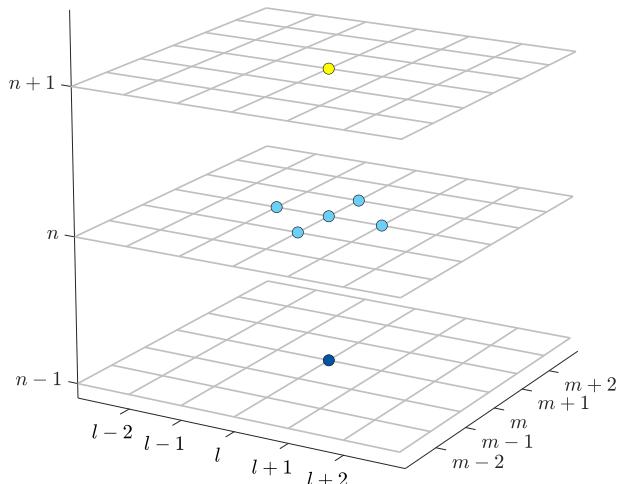
$$\left. \begin{array}{l} u_{0,m}^n = u_{N_x,m}^n = 0 \quad \forall m, \\ u_{l,0}^n = u_{l,N_y}^n = 0 \quad \forall l, \end{array} \right\} \text{(Dirichlet, fixed),} \quad (6.14a)$$

$$\left. \begin{array}{l} \delta_x \cdot u_{0,m}^n = \delta_x \cdot u_{N_x,m}^n = 0 \quad \forall m, \\ \delta_y \cdot u_{l,0}^n = \delta_y \cdot u_{l,N_y}^n = 0 \quad \forall l, \end{array} \right\} \text{(Neumann, free).} \quad (6.14b)$$

If the Dirichlet boundary conditions are used (for all sides), the domain of calculation can simply be reduced to  $l \in \{1, \dots, N_x - 1\}$  and  $m \in \{1, \dots, N_y - 1\}$ .

### Stencil

Figure 6.2 shows the stencil of the 2D wave equation FD scheme in Eq. (6.9). The grid points use the same colour-coding as previous stencils (see e.g. Figure 2.10).



**Fig. 6.2:** The stencil for the 2D wave equation FD scheme in Eq. (6.9).

### 6.2.3 Matrix form and output

Similar to how the number of intervals between grid points are calculated 1D in Eq. (2.53), these can be calculated using the following operations:

$$h := \sqrt{2}ck, \quad N_x := \left\lfloor \frac{L_x}{h} \right\rfloor, \quad N_y := \left\lfloor \frac{L_y}{h} \right\rfloor, \quad h := \min \left( \frac{L_x}{N_x}, \frac{L_y}{N_y} \right), \quad \lambda := \frac{ck}{h}, \quad (6.15)$$

where the ‘min’ operator selects the smallest value of  $L_x/N_x$  and  $L_y/N_y$  to stay as close to the stability condition as possible. To implement the update equation in Eq. (6.10), one could save the states of the system in matrices (as opposed to vectors in the 1D case such as done in Section 4.2.2) and directly work with these. Using Dirichlet boundary conditions the  $(N_x - 1) \times (N_y - 1)$  state matrix at time index  $n$  would be

$$\mathbf{U}^n = \begin{bmatrix} u_{1,1}^n & \cdots & u_{1,N_x-1}^n \\ \vdots & \ddots & \vdots \\ u_{N_y-1,1}^n & \cdots & u_{N_y-1,N_x-1}^n \end{bmatrix}, \quad (6.16)$$

and could be used to make a ‘for-loop implementation’ of the update equation. This would indeed be the strategy if one would implement the scheme in e.g. C++ (see Chapter ??). For a more compact and faster implementation in MATLAB, however, one could ‘stack’ or ‘flatten’ the state matrices to vectors and update the scheme using matrix-vector multiplication as done for, e.g., the stiff string in Section 4.2.2. Again using Dirichlet boundary conditions, the stacked state vector will be structured as

$$\mathbf{u}^n = [(\mathbf{u}_1^n)^T, \dots, (\mathbf{u}_{N_x-1}^n)^T]^T, \quad \text{with } \mathbf{u}_l^n = [u_{l,1}^n, \dots, u_{l,N_y-1}^n]^T, \quad (6.17)$$

and has a size of  $(N_x - 1) \cdot (N_y - 1) \times 1$ . See Figure 6.3 for a visualisation of the matrix-stacking process.

To obtain a matrix form of the  $\delta_\Delta$  operator that can be applied to this stacked state vector, the *kronecker product* and *kronecker sum* must be introduced [67]. The kronecker product between two arbitrarily-sized matrices (using their dimensions as a subscript) is

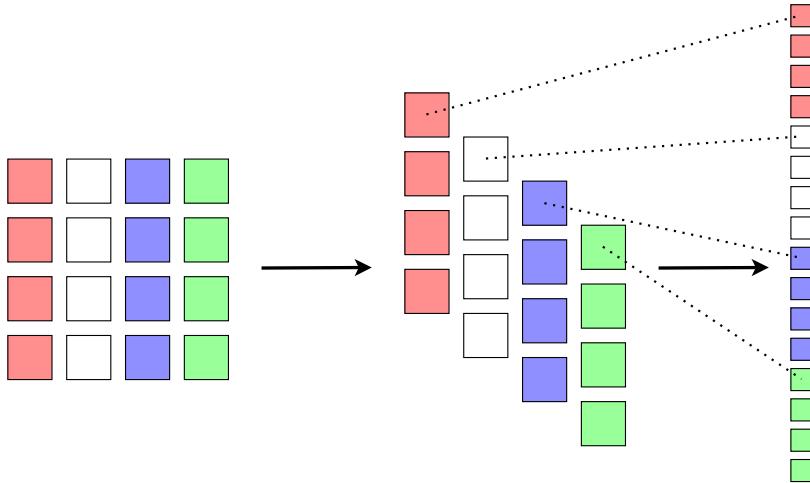
$$\mathbf{A}_{M \times N} \otimes \mathbf{B}_{K \times L} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1N}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{M1}\mathbf{B} & \cdots & a_{MN}\mathbf{B} \end{bmatrix}_{MK \times NL}. \quad (6.18)$$

The kronecker sum between two square matrices is

$$\mathbf{A}_{M \times M} \oplus \mathbf{B}_{N \times N} = \mathbf{I}_N \otimes \mathbf{A} + \mathbf{B} \otimes \mathbf{I}_M, \quad (6.19)$$

where  $\mathbf{I}_P$  is the identity matrix of size  $P \times P$ .

## 6.2. The 2D wave equation



**Fig. 6.3:** Stacking, or ‘flattening’ a  $4 \times 4$  matrix to a 16-element vector.

For Dirichlet boundary conditions, the  $\mathbf{D}_{xx}$  matrix of size  $(N_x - 1) \times (N_x - 1)$  and the  $\mathbf{D}_{yy}$  matrix of size  $(N_y - 1) \times (N_y - 1)$  can be defined (similar to Eq. (3.3)) as

$$\mathbf{D}_{xx} = \frac{1}{h^2} \underbrace{\begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{bmatrix}}_{(N_x-1) \times (N_x-1)} \quad \text{and} \quad \mathbf{D}_{yy} = \frac{1}{h^2} \underbrace{\begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{bmatrix}}_{(N_y-1) \times (N_y-1)}. \quad (6.20)$$

Following [47], the matrix form of the  $\delta_\Delta$  operator can then be defined as the kronecker sum of  $\mathbf{D}_{yy}$  and  $\mathbf{D}_{xx}$  yielding

$$\mathbf{D}_\Delta = \mathbf{D}_{yy} \oplus \mathbf{D}_{xx} = \begin{bmatrix} \ddots & & 0 \\ & \mathbf{D}_{yy} & \\ & \mathbf{D}_{yy} & \ddots \\ 0 & & \mathbf{D}_{yy} & \ddots \end{bmatrix} + \frac{1}{h^2} \begin{bmatrix} \ddots & \ddots & & 0 \\ & -2\mathbf{I} & \mathbf{I} & \\ & \mathbf{I} & -2\mathbf{I} & \mathbf{I} \\ 0 & & \mathbf{I} & -2\mathbf{I} & \ddots \end{bmatrix}, \quad (6.21)$$

where the identity matrix  $\mathbf{I} = \mathbf{I}_{N_x-1}$ . The  $\mathbf{D}_\Delta$  matrix is square and of size  $(N_x - 1) \cdot (N_y - 1) \times (N_x - 1) \cdot (N_y - 1)$ .

Using the above, the FD scheme in Eq. (6.9) can then be compactly written

in matrix form as

$$\mathbf{u}^{n+1} = (\mathbf{I} + c^2 k^2 \mathbf{D}_\Delta) \mathbf{u}^n - \mathbf{u}^{n-1}, \quad (6.22)$$

where the identity matrix is of the same size as  $\mathbf{D}_\Delta$ . See Appendix C.3 for a MATLAB implementation of the 2D wave equation.<sup>3</sup>

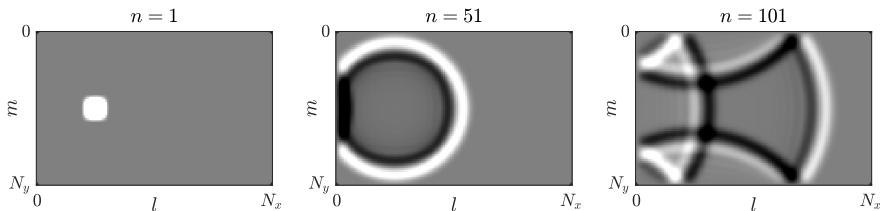
If one would like to visualise the system state as a 2D grid, one can revert the stacked vector back to a matrix by using the `reshape` function in MATLAB:

```
uMatrix = reshape(u, Ny-1, Nx-1);
```

A 2D raised-cosine excitation can be implemented in the same way by reshaping an excitation matrix to a vector (see lines 40–52 in Appendix C.3).

## Output

Figure 6.4 shows the wave propagation of an implementation of the 2D wave equation with Dirichlet boundary conditions. Parameter values are  $L_x = 1.5$  m,  $L_y = 1$  m and  $c = 360$  m/s. Waves reflect at the boundaries at an increasing rate. This is also shown in Figure 6.5, where the output – taken at  $(x, y) = (0.15, 0.85)$  – in time domain shows an increase in oscillations over time due to these reflections. The right panel shows that the output contains many partials that are close together, i.e., the output is highly inharmonic. As opposed to the output of the 1D wave equation shown in Figure 2.11, where the partials are integer multiples of the fundamental frequency, the 2D wave equation exhibits aperiodic behaviour due to the aforementioned reflections causing this inharmonicity.



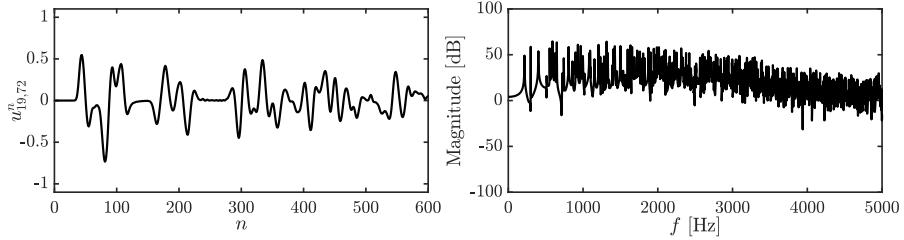
**Fig. 6.4:** Wave propagation of an implementation of the 2D wave equation with  $L_x = 1.5$  m,  $L_y = 1$  m and  $c = 360$  m/s. The system is excited with a 2D raised cosine at  $(0.25L_x, 0.5L_y)$ .

### 6.2.4 Frequency domain analysis in 2D

Section 3.3 showed how to perform frequency domain analysis to obtain stability conditions for an FD scheme. This section shows extensions to this in 2D and follows [23, Ch. 10].

<sup>3</sup>As the matrices are extremely sparse (many 0-entries), it is useful to utilise MATLAB's optimisation for sparse matrices using the `sparse()` function. One can use `speye()` for sparse identity matrices.

## 6.2. The 2D wave equation



**Fig. 6.5:** The output of the 2D wave equation at  $(x, y) = (0.15, 0.85)$  corresponding to Figure 6.4. The partials are extremely close together (notice that only frequency up to 5000 Hz are shown) and is related to the aperiodic nature of the system behaviour.

In 2D, the ansatz in Eq. (3.20) can be extended to

$$u_{l,m}^n \xrightarrow{\mathcal{A}} z^n e^{jh(l\beta_x + m\beta_y)} \quad (6.23)$$

where  $\beta_x$  and  $\beta_y$  are components of a 2D wavenumber  $\beta$  in the  $x$  and  $y$  directions respectively. Frequency domain representations of temporal operators shown in Eq. (3.22) do not change in the 2D case. Using

$$p_x = \sin^2(\beta_x h/2) \quad \text{and} \quad p_y = \sin^2(\beta_y h/2) \quad (6.24)$$

for brevity, the following frequency domain representation of spatial operators can be obtained through the ansatz in Eq. (6.23)

$$\delta_{xx} u_{l,m}^n \xrightarrow{\mathcal{A}} -\frac{4}{h^2} p_x u_{l,m}^n \quad \text{and} \quad \delta_{yy} u_{l,m}^n \xrightarrow{\mathcal{A}} -\frac{4}{h^2} p_y u_{l,m}^n, \quad (6.25)$$

from which it follows that

$$\delta_{\Delta} u_{l,m}^n \xrightarrow{\mathcal{A}} -\frac{4}{h^2} (p_x + p_y) u_{l,m}^n, \quad (6.26)$$

$$\delta_{\Delta} \delta_{\Delta} u_{l,m}^n \xrightarrow{\mathcal{A}} \frac{16}{h^4} (p_x + p_y)^2 u_{l,m}^n. \quad (6.27)$$

Using these definitions, a frequency domain interpretation of the 2D wave equation FD scheme in Eq. (6.9) can be obtained

$$\frac{1}{k^2} (z - 2 + z^{-1}) = -\frac{4c^2}{h^2} (p_x + p_y).$$

Recalling  $\lambda$  in Eq. (6.11), this can be rewritten to the following characteristic equation

$$z + (4\lambda^2(p_x + p_y) - 2) + z^{-1} = 0. \quad (6.28)$$

As (after multiplication by  $z$ ) the characteristic equation is of the form in Eq. (3.25) and  $a^{(2)} = 1$ , its roots are bounded by condition (3.27)

$$|4\lambda^2(p_x + p_y) - 2| \leq 2.$$

Further derivation yields

$$\begin{aligned} -2 &\leq 4\lambda^2(p_x + p_y) - 2 \leq 2, \\ 0 &\leq 4\lambda^2(p_x + p_y) \leq 4, \end{aligned}$$

and as middle term is non-negative the first condition is always satisfied, yields

$$\lambda^2(p_x + p_y) \leq 1.$$

Finally, as  $p_x$  and  $p_y$  are bounded by 1 for all wavenumbers  $\beta_x$  and  $\beta_y$  respectively, the following condition must hold

$$\begin{aligned} 2\lambda^2 &\leq 1, \\ \lambda &\leq \frac{1}{\sqrt{2}} \end{aligned} \tag{6.29}$$

which is the stability condition given in Eq. (6.12).

### 6.2.5 Energy analysis in 2D

Energy analysis for the 1D case is introduced in Section 3.4. Extensions for the analysis in 2D will be given here.

Analogous to the 1D inner product presented in Section 3.2.1, one can define a 2D inner product. For two functions  $f = f(x, y, t)$  and  $g(x, y, t)$  defined for a 2D domain  $\mathcal{D}$  their inner product over this domain is defined as

$$\langle f, g \rangle_{\mathcal{D}} = \iint_{\mathcal{D}} f g dx dy. \tag{6.30}$$

Like in the 1D case, these functions do not have to be a function of time, but they are here, for coherence.

For two (grid) functions  $f_{l,m}^n$  and  $g_{l,m}^n$  defined over a discrete domain  $d \in \{0, \dots, N_x\} \times \{0, \dots, N_y\}$  their discrete inner product is defined as

$$\langle f_{l,m}^n, g_{l,m}^n \rangle_d = \sum_{l=0}^{N_x} \sum_{m=0}^{N_y} h^2 f_{l,m}^n g_{l,m}^n. \tag{6.31}$$

Notice that the multiplication with the grid spacing is squared due to the inner product over a 2D domain (and is the discrete counterpart of  $dx dy$ ). Useful for energy analysis are the following reduced 2D domains

$$\underline{d}_x = \{0, \dots, N_x - 1\} \times \{0, \dots, N_y\}, \tag{6.32a}$$

$$\overline{d}_x = \{1, \dots, N_x - 1\} \times \{0, \dots, N_y\}, \tag{6.32b}$$

$$\underline{d}_y = \{0, \dots, N_x\} \times \{0, \dots, N_y - 1\}, \tag{6.32c}$$

$$\overline{d}_y = \{0, \dots, N_x\} \times \{1, \dots, N_y - 1\} \tag{6.32d}$$

$$\overline{\underline{d}} = \{1, \dots, N_x - 1\} \times \{1, \dots, N_y - 1\} \tag{6.32e}$$

Below, the steps to perform energy analysis presented in Section 3.4 will be followed:

**Step 1: Obtain  $\delta_{t+\mathfrak{h}}$**

Using the definition of wave speed for the ideal membrane, i.e.,  $c = \sqrt{T/\rho H}$ , the FD scheme in Eq. (6.9) can be multiplied by  $\rho H$  and a 2D inner product (see Eq. (6.31)) with  $(\delta_t \cdot u_{l,m}^n)$  over discrete domain  $d$  can be taken to yield a definition for  $\delta_{t+\mathfrak{h}}$ :

$$\delta_{t+\mathfrak{h}} = \rho H \langle \delta_t \cdot u_{l,m}^n, \delta_{tt} u_{l,m}^n \rangle_d - T \langle \delta_t \cdot u_{l,m}^n, \delta_\Delta u_{l,m}^n \rangle_d = 0,$$

which can be rewritten to

$$\delta_{t+\mathfrak{h}} = \rho H \langle \delta_t \cdot u_{l,m}^n, \delta_{tt} u_{l,m}^n \rangle_d - T (\langle \delta_t \cdot u_{l,m}^n, \delta_{xx} u_{l,m}^n \rangle_d + \langle \delta_t \cdot u_{l,m}^n, \delta_{yy} u_{l,m}^n \rangle_d) = 0.$$

**Step 2: Identify energy types and isolate  $\delta_{t+}$**

Summation by parts as described in Section 3.2.2 can also be applied to  $\delta_{yy}$  and the following energy balance follows

$$\delta_{t+\mathfrak{h}} = \mathfrak{b},$$

where

$$\begin{aligned} \mathfrak{h} &= \mathfrak{t} + \mathfrak{v} \quad \text{with} \quad \mathfrak{t} = \frac{\rho H}{2} \|\delta_{t-} u_{l,m}^n\|_d^2 \quad \text{and} \\ \mathfrak{v} &= \frac{T}{2} \left( \langle \delta_{x+} u_{l,m}^n, e_{t-} \delta_{x+} u_{l,m}^n \rangle_{\underline{d}_x} + \langle \delta_{y+} u_{l,m}^n, e_{t-} \delta_{y+} u_{l,m}^n \rangle_{\underline{d}_y} \right). \end{aligned} \quad (6.33)$$

Here, the reduced domains  $\underline{d}_x$  and  $\underline{d}_y$  are as defined in Eqs. (6.32). The boundary term is

$$\begin{aligned} \mathfrak{b} &= \frac{T}{2} \left[ \langle \delta_t \cdot u_{N_x,m}^n, \delta_{x+} u_{N_x,m}^n \rangle_{(N_x,y)} - \langle \delta_t \cdot u_{0,m}^n, \delta_{x-} u_{0,m}^n \rangle_{(0,y)} \right. \\ &\quad \left. + \langle \delta_t \cdot u_{l,N_y}^n, \delta_{y+} u_{l,N_y}^n \rangle_{(x,N_y)} - \langle \delta_t \cdot u_{l,0}^n, \delta_{y-} u_{l,0}^n \rangle_{(x,0)} \right], \end{aligned}$$

where  $(l, y) = \{l\} \times \{0, \dots, N_y\}$  and  $(x, m) = \{0, \dots, N_x\} \times \{m\}$  are slices of domain  $d$ . The boundary term can be shown to vanish under Dirichlet boundary conditions in Eq. (6.14a). Neumann conditions will not be considered here.

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SWEEP: check what equations have numbers when performing energy analysis (and stability for that matter)

### Step 3: Check units

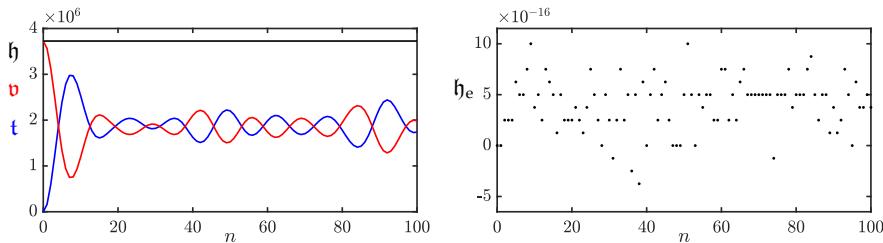
As the addition of the two inner products in the definition for  $\mathfrak{v}$  in Eq. (6.33) does not affect the units, only one term is used to check the units. Recalling that, as opposed to the 1D case, the symbol  $T$  is tension per unit length and thus in N/m, one can write the terms in Eq. (6.33) in their units:

$$\begin{aligned} \frac{\rho H}{2} \|\delta_{t-} u_{l,m}^n\|_d^2 &\xrightarrow{\text{in units}} \text{kg} \cdot \text{m}^{-3} \cdot \text{m} \cdot \text{m}^2 \cdot (\text{s}^{-1} \cdot \text{m})^2 \\ &= \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \\ \frac{T}{2} \langle \delta_{x+} u_{l,m}^n, e_{t-} \delta_{x+} u_{l,m}^n \rangle_{dx} &\xrightarrow{\text{in units}} \text{N} \cdot \text{m}^{-1} \cdot \text{m}^2 \cdot (\text{m}^{-1} \cdot \text{m}) \cdot (\text{m}^{-1} \cdot \text{m}) \\ &= \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \end{aligned}$$

which have the correct units.

### Step 4: Implementation

Figure 6.6 shows the energetic output of an implementation of the 2D wave equation and shows that the energy deviation is within machine precision.



**Fig. 6.6:** The kinetic (blue), potential (red), and total (black) energy of an implementation of the 2D wave equation are plotted in the left panel. The right panel shows the normalised energy (according to Eq. (3.37)) and shows that the deviation of the energy is within machine precision.

### 6.2.6 Modal analysis in 2D

Given that the state vector is stacked as described Section 6.2.3 and the update equation is written in matrix form as in Eq. (6.22), performing a modal analysis on a 2D system does not differ from a 1D system and follows the same process presented in Section 3.5.

Inserting a test solution of  $u^n = z^n \phi$  into the matrix form of the 2D wave equation in Eq. (6.22) yields the following characteristic equation

$$(z - 2 + z^{-1}) \phi = c^2 k^2 \mathbf{D}_\Delta \phi. \quad (6.34)$$

### 6.3. The thin plate

The  $p^{\text{th}}$  modal frequency can then be obtained by finding the roots of

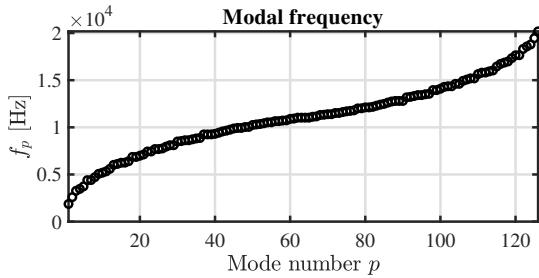
$$z_p + \left( -2 - c^2 k^2 \text{eig}_p(\mathbf{D}_\Delta) \right) + z_p^{-1} = 0, \quad (6.35)$$

which, using test solution  $z_p = e^{j\omega_p k}$  for (angular) frequency  $\omega_p$ , can be shown to be

$$f_p = \frac{1}{\pi k} \sin^{-1} \left( \frac{ck}{2} \sqrt{-\text{eig}_p(\mathbf{D}_\Delta)} \right). \quad (6.36)$$

Notice the similarity to the equation for the modal frequencies of the 1D wave equation in Eq. (3.56). Again, the number of modes is equal to the number of moving grid points in the system.

See Figure 6.7 for the result of a modal analysis of the 2D wave equation. One can observe that the modes do not follow a linear pattern as opposed to those of the 1D wave equation shown in Figure 3.3. This confirms the inharmonic behaviour of the 2D wave equation discussed 6.2.3.



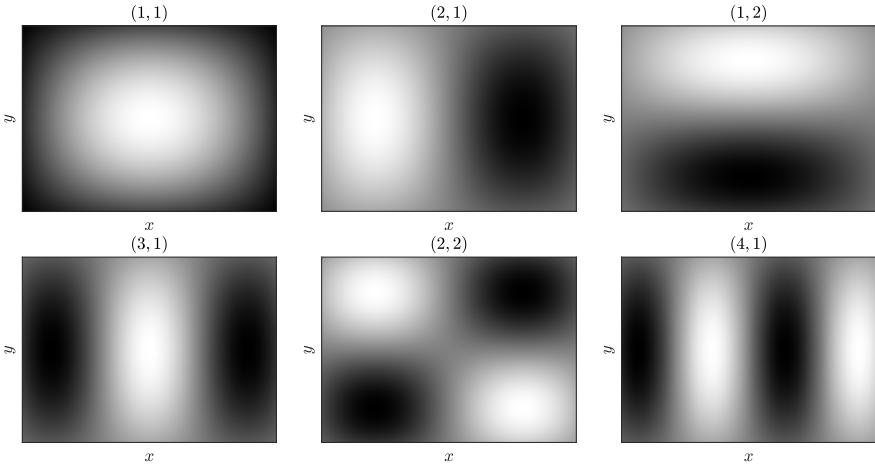
**Fig. 6.7:** Modal frequencies of the 2D wave equation with  $L_x = 1.5$  m,  $L_y = 1$  m and  $c \approx 3118$  m/s, such that  $\lambda = 1/\sqrt{2}$  according to Eq. (6.12).

### Modal shapes

Using the line of code in Appendix B.4 and the `reshape` function, the modal shapes of the system can also be obtained. Figure 6.8 shows the six lowest-frequency modes of the 2D wave equation with  $L_x = 1.5$  m and  $L_y = 1$  m. The mode number  $(x, y)$  corresponds to the modal number in the  $x$  and  $y$  direction.

## 6.3 The thin plate

The thin plate, also known as the Kirchhoff model [68], differs from the 2D wave equation in that its restoring force is solely due to stiffness rather than tension. Like for the stiffness term in the stiff string (see Chapter 4), this causes frequency dispersion, and exhibits interesting timbres.



**Fig. 6.8:** The first (lowest-frequency) six modal shapes of 2D wave equation with  $L_x = 1.5$  m and  $L_y = 1$  m.

The plate model is quite versatile and can be used to model a plate reverberation [69] as well as simplified instrument bodies, as done in [A], [B], [D] and [E]. This section presents the thin plate PDE and FD scheme, after which it will be subjected to the various analysis techniques extended to 2D in the previous section.

### 6.3.1 Continuous time

Consider a rectangular thin plate with side lengths  $L_x$  and  $L_y$  (both in m) and its transverse displacement described by  $u = u(x, y, t)$  (in m). The system is defined for  $(x, y) \in \mathcal{D}$  where 2D domain  $\mathcal{D} = [0, L_x] \times [0, L_y]$ . Using the biharmonic operator introduced in Eq. (6.2), the PDE for the thin plate, can be defined as [68]

$$\rho H \partial_t^2 u = -D \Delta \Delta u, \quad (6.37)$$

where  $D = EH^3/12(1-\nu^2)$  is a stiffness coefficient (in  $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$ ) parametrised by Young's Modulus  $E$  (in Pa), thickness  $H$  (in m) and the dimensionless Poisson's ratio  $\nu$ . Although Eq. (6.37) does not hold for thick plates and only accounts for low-amplitude vibration (as it is linear), these properties can be assumed in musical instrument simulations making this model sufficient in this work.

Adding losses to Eq. (6.37) yields

$$\rho H \partial_t^2 u = -D \Delta \Delta u - 2\sigma_0 \rho H \partial_t u + 2\sigma_1 \rho H \partial_t \Delta u \quad (6.38)$$

where, as in the case of the stiff string in Eq. (4.4),  $\sigma_0$  and  $\sigma_1$  are the frequency

### 6.3. The thin plate

independent (in  $\text{s}^{-1}$ ) and frequency dependent damping coefficient (in  $\text{m}^2/\text{s}$ ) respectively.

check hyphens after 'frequency'

#### Boundary conditions

Similar to the stiff string, clamped and simply supported boundary conditions can be defined as

$$\left. \begin{array}{l} u = \partial_x u = 0, \quad \text{if } y = \{0, L_y\}, \quad \forall x \\ u = \partial_y u = 0, \quad \text{if } x = \{0, L_x\}, \quad \forall y \end{array} \right\} \quad (\text{Clamped}), \quad (6.39\text{a})$$

$$\left. \begin{array}{l} u = \partial_x^2 u = 0, \quad \text{if } y = \{0, L_y\}, \quad \forall x \\ u = \partial_y^2 u = 0, \quad \text{if } x = \{0, L_x\}, \quad \forall y \end{array} \right\} \quad (\text{Simply supported}). \quad (6.39\text{b})$$

Naturally, a free condition can be added too, but is much less trivial. As it will not be used in this work, it will not be given here, and the interested reader is instead referred to [23, Ch. 12].

#### 6.3.2 Discrete time

Equation (6.38) can be discretised to the following FD scheme:

$$\rho H \delta_{tt} u_{l,m}^n = -D \delta_\Delta \delta_\Delta u_{l,m}^n - 2\sigma_0 \rho H \delta_{t.} u_{l,m}^n + 2\sigma_1 \rho H \delta_{t-} \delta_\Delta u_{l,m}^n \quad (6.40)$$

where  $l \in \{0, \dots, N_x\}$  and  $m \in \{0, \dots, N_y\}$ . Like for the stiff string FD scheme in Eq. (4.7), the backwards difference operator is used for the frequency-dependent damping term to yield an explicit scheme. A more compact way to write this scheme is after a division by  $\rho H$  which yields

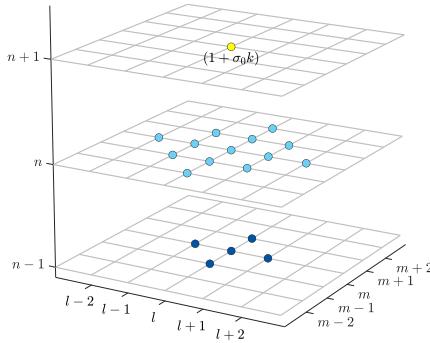
$$\delta_{tt} u_{l,m}^n = -\kappa^2 \delta_\Delta \delta_\Delta u_{l,m}^n - 2\sigma_0 \delta_{t.} u_{l,m}^n + 2\sigma_1 \delta_{t-} \delta_\Delta u_{l,m}^n \quad (6.41)$$

with

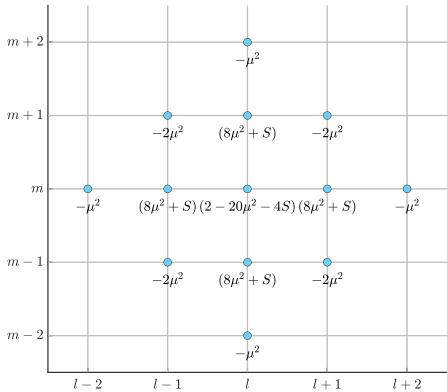
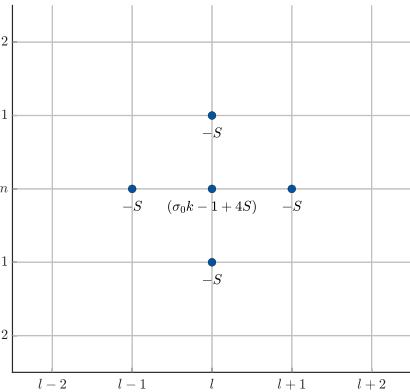
$$\kappa = \sqrt{\frac{D}{\rho H}}. \quad (6.42)$$

Using the expansion of the discrete biharmonic operator in Eq. (6.6), Eq. (6.41) can be expanded and solved for  $u_{l,m}^{n+1}$  according to

$$\begin{aligned} u_{l,m}^{n+1} &= (2 - 20\mu^2 - 4S)u_{l,m}^n \\ &\quad + (8\mu^2 + S)(u_{l+1,m}^n + u_{l-1,m}^n + u_{l,m+1}^n + u_{l,m-1}^n) \\ &\quad - 2\mu^2(u_{l+1,m+1}^n + u_{l-1,m+1}^n + u_{l+1,m-1}^n + u_{l-1,m-1}^n) \\ &\quad - \mu^2(u_{l+2,m}^n + u_{l-2,m}^n + u_{l,m+2}^n + u_{l,m-2}^n), \\ &\quad + (\sigma_0 k - 1 + 4S)u_{l,m}^{n-1} \\ &\quad - S(u_{l+1,m}^{n-1} + u_{l-1,m}^{n-1} + u_{l,m+1}^{n-1} + u_{l,m-1}^{n-1}) \end{aligned} \quad (6.43)$$



(a) Full stencil.


 (b) Stencil of  $u_{l,m}^n$ .

 (c) Stencil of  $u_{l,m}^{n-1}$ .

**Fig. 6.9:** The stencil of the plate with coefficients corresponding to those in update equation (6.43). (a) A full overview of the stencil. (b) The current time-step  $n$  highlighted. (c) The previous time-step  $n - 1$  highlighted.

where

$$\mu = \frac{\kappa k}{h^2} \quad (6.44)$$

and  $S = 2\sigma_1 k / h^2$  for compactness. See Figure 6.9 for the stencil of this scheme. The stability condition of the scheme can be shown to be

$$h \geq 2 \sqrt{k \left( \sigma_1 + \sqrt{\kappa^2 + \sigma_1^2} \right)}, \quad (6.45)$$

and will be derived in Section 6.3.4.

### 6.3. The thin plate

#### Discrete boundary conditions

The boundary conditions shown in Eq. (6.39) can be discretised to

$$\left. \begin{array}{ll} u_{l,m}^n = \delta_{x+} u_{l,m}^n = 0 & \text{if } m = 0 \\ u_{l,m}^n = \delta_{x-} u_{l,m}^n = 0 & \text{if } m = N_y \\ u_{l,m}^n = \delta_{y+} u_{l,m}^n = 0 & \text{if } l = 0 \\ u_{l,m}^n = \delta_{y-} u_{l,m}^n = 0 & \text{if } l = N_x \end{array} \right\} \quad \begin{array}{l} \forall l \\ \forall l \\ \forall m \\ \forall m \end{array} \quad (\text{Clamped}), \quad (6.46a)$$

$$\left. \begin{array}{ll} u_{l,m}^n = \delta_{xx} u_{l,m}^n = 0 & \text{if } m = \{0, N_y\} \\ u_{l,m}^n = \delta_{yy} u_{l,m}^n = 0 & \text{if } l = \{0, N_x\} \end{array} \right\} \quad \begin{array}{l} \forall l \\ \forall m \end{array} \quad (\text{Simply supported}). \quad (6.46b)$$

The clamped condition can be implemented by simply reducing the discrete range of operation to  $l = \{2, \dots, N_x - 2\}$  and  $m = \{2, \dots, N_y - 2\}$ . For the simply supported case, the range of operation reduces to  $l = \{1, \dots, N_x - 1\}$  and  $m = \{1, \dots, N_y - 1\}$ , and similar to the simply supported stiff string described in Section 4.2.1, the virtual grid points needed for this condition become

$$\begin{aligned} u_{-1,m}^n &= -u_{1,m}^n & \text{and} & \quad u_{N_x+1,m}^n = -u_{N_x-1,m}^n \quad \forall m, \\ u_{l,-1}^n &= -u_{l,-1}^n & \text{and} & \quad u_{l,N_y+1}^n = -u_{l,N_y-1}^n \quad \forall l. \end{aligned}$$

#### 6.3.3 Matrix form and output

Similar to the implementation of the 2D wave equation in Section 6.2.3, one can use a stacked state vector. If simply supported boundary conditions are used, one can easily obtain a matrix form of the  $\delta_\Delta \delta_\Delta$  operator by multiplying two  $\mathbf{D}_\Delta$  matrices presented in Eq. (6.21) to get  $\mathbf{D}_{\Delta\Delta} = \mathbf{D}_\Delta \mathbf{D}_\Delta$ .

Using a stacked form of the state as described in Eq. (6.17) the scheme in Eq. (6.41) in matrix form is

$$A\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \mathbf{C}\mathbf{u}^{n-1}, \quad (6.47)$$

where

$$\begin{aligned} A &= (1 + \sigma_0 k), & \mathbf{B} &= 2\mathbf{I} - \kappa^2 k^2 \mathbf{D}_{\Delta\Delta} + 2\sigma_1 k \mathbf{D}_\Delta, \\ \text{and} \quad \mathbf{C} &= -(1 - \sigma_0 k)\mathbf{I} - 2\sigma_1 k \mathbf{D}_\Delta, \end{aligned}$$

and the identity matrix  $\mathbf{I}$  is of the same size as  $\mathbf{D}_{\Delta\Delta}$  and  $\mathbf{D}_\Delta$ .

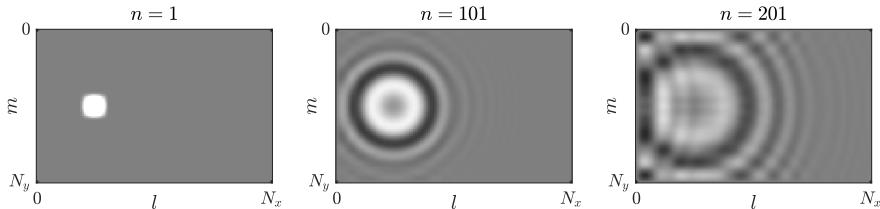
As a starting point for implementation, possible parameters are given in Table 6.1. Figure 6.10 shows the wave propagation of a thin plate excited with a 2D raised cosine at  $(x, y) = (0.25L_x, 0.5L_y)$  and uses these parameters.

When compared to Figure 6.4, dispersive effects – where higher-frequency components travel faster than lower-frequency ones – due to stiffness are apparent. Figure 6.11 shows the time domain and frequency domain output of

Name	Symbol (unit)	Value
Side length $x$	$L_x$ (m)	1.5
Side length $y$	$L_y$ (m)	1
Material density	$\rho$ (kg/m <sup>3</sup> )	7850
Thickness	$H$ (m)	$5 \cdot 10^{-3}$
Young's modulus	$E$ (Pa)	$2 \cdot 10^{11}$
Poisson's ratio	$\nu$ (-)	0.3
Freq.-independent damping	$\sigma_0$ (s <sup>-1</sup> )	1
Freq.-dependent damping	$\sigma_1$ (m <sup>2</sup> /s)	0.005

**Table 6.1:** Parameters for the thin plate and possible values to use as a starting point for the simulation.

the thin plate at  $(x, y) = (0.15L_x, 0.85L_y)$ . Compared to the output of the 2D wave equation in Figure 6.5, there are several interesting differences. The amplitude is much lower, waves are closer together and the first wave arrives much earlier, all due to dispersive effects.



**Fig. 6.10:** Wave propagation of a thin plate with simply supported boundary conditions and parameters as shown in Table 6.1. The system is excited with a 2D raised cosine at  $(x, y) = (0.25L_x, 0.5L_y)$  and dispersive effects are apparent.

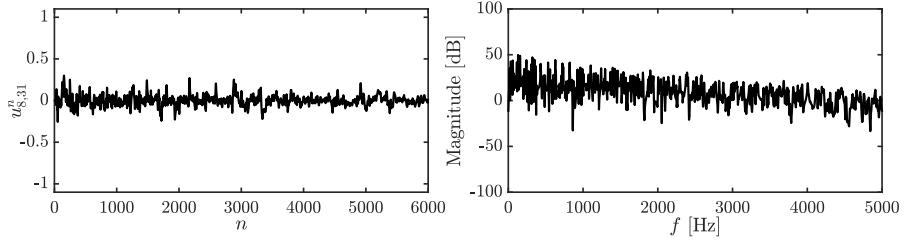
### 6.3.4 Frequency domain analysis

This section follows the process presented in Section 3.3 with the extensions to 2D shown in 6.2.4.

Using Eqs. (6.26) and (6.27) one can obtain a frequency domain representation of the FD scheme in Eq. (6.41) and obtain the following characteristic equation

$$(1 + \sigma_0 k)z + \left( 16\mu^2(p_x + p_y)^2 + \frac{8\sigma_1 k}{h^2}(p_x + p_y) - 2 \right) \\ + \left( 1 - \sigma_0 k - \frac{8\sigma_1 k}{h^2}(p_x + p_y) \right) z^{-1} = 0. \quad (6.48)$$

### 6.3. The thin plate



**Fig. 6.11:** The output of the thin plate at  $(x, y) = (0.15, 0.85)$  corresponding to Figure 6.10.

This can, similar to the damped stiff string in Section 4.3, be solved to

$$4\mu^2(p_x + p_y)^2 + \frac{4\sigma_1 k}{h^2}(p_x + p_y) \leq 1.$$

Recalling the definitions  $p_x$  and  $p_y$  in Eq. (6.24), and given the fact that these are bounded by 1, the following can be written

$$\begin{aligned} 4\mu^2(1+1)^2 + \frac{4\sigma_1 k}{h^2}(1+1) &\leq 1 \\ 16\mu^2 + \frac{8\sigma_1 k}{h^2} &\leq 1. \end{aligned}$$

Finally, recalling the definition for  $\kappa$  in Eq. (6.42) solving for  $h$  yields

$$\begin{aligned} 1 &\geq \frac{16\kappa^2 k^2}{h^4} + \frac{8\sigma_1 k}{h^2}, \\ h^4 - 8\sigma_1 k h^2 - 16\kappa^2 k^2 &\geq 0, \\ h &\geq \sqrt{\frac{8\sigma_1 k + \sqrt{(8\sigma_1 k)^2 + 64\kappa^2 k^2}}{2}}, \\ h &\geq \sqrt{\frac{8\sigma_1 k + 8\sqrt{\sigma_1^2 k^2 + \kappa^2 k^2}}{2}}, \\ h &\geq 2\sqrt{k \left( \sigma_1 + \sqrt{\sigma_1^2 + \kappa^2} \right)}, \end{aligned} \tag{6.49}$$

which is the stability condition given in Eq. (6.45).

#### 6.3.5 Energy analysis

Using the steps described in Section 3.4 with the extensions to 2D presented in Section 6.2.5, one can obtain the total energy of the FD scheme in Eq. (6.41).

**Step 1: Obtain  $\delta_{t+}\mathfrak{h}$** 

To obtain the rate of change of the total energy, one can take an inner product of the scheme in Eq. (6.41) with  $(\delta_t u_{l,m}^n)$  over discrete (2D) domain  $d = \{0, \dots, N_x\} \times \{0, \dots, N_y\}$  to get

$$\begin{aligned}\delta_{t+}\mathfrak{h} &= \rho H \langle \delta_t u_{l,m}^n, \delta_{tt} u_{l,m}^n \rangle_d + D \langle \delta_t u_{l,m}^n, \delta_\Delta \delta_\Delta u_{l,m}^n \rangle_d \\ &\quad + 2\sigma_0 \rho H \langle \delta_t u_{l,m}^n, \delta_t u_{l,m}^n \rangle_d - 2\sigma_1 \rho H \langle \delta_t u_{l,m}^n, \delta_{t-} \delta_\Delta u_{l,m}^n \rangle_d = 0.\end{aligned}\quad (6.50)$$

**Step 2: Identify energy types and isolate  $\delta_{t+}$** 

Due to the damping present in the system and because the system is distributed in space, the energy balance will be of the following form

$$\delta_{t+}\mathfrak{h} = \mathfrak{b} - \mathfrak{q},$$

with boundary term  $\mathfrak{b}$  and damping term

$$\mathfrak{q} = 2\sigma_0 \rho H \|\delta_t u_{l,m}^n\|_d^2 - 2\sigma_1 \rho H \langle \delta_t u_{l,m}^n, \delta_{t-} \delta_\Delta u_{l,m}^n \rangle_d. \quad (6.51)$$

Expanding the stiffness term in Eq. (6.50) to

$$\begin{aligned}& D \langle \delta_t u_{l,m}^n, (\delta_{xx} + \delta_{yy}) \delta_\Delta u_{l,m}^n \rangle_d \\ \iff & D \left( \langle \delta_t u_{l,m}^n, \delta_{xx} \delta_\Delta u_{l,m}^n \rangle_d + \langle \delta_t u_{l,m}^n, \delta_{yy} \delta_\Delta u_{l,m}^n \rangle_d \right),\end{aligned}$$

one can perform summation by parts twice using Eq. (3.16b) for both terms to get

$$D \left( \langle \delta_t \delta_{xx} u_{l,m}^n, \delta_\Delta u_{l,m}^n \rangle_{\underline{\underline{d}_x}} + \langle \delta_t \delta_{yy} u_{l,m}^n, \delta_\Delta u_{l,m}^n \rangle_{\underline{\underline{d}_y}} \right) + \mathfrak{b}.$$

The definitions for the reduced domains can be found in Eqs. (6.32). Finally, as the boundaries are always 0 due to the boundary conditions in Eq. (6.46),  $\underline{\underline{d}_x}$  and  $\underline{\underline{d}_y}$  can be further reduced to  $\underline{\underline{d}}$  and the terms can be combined as [check with stefan](#)

$$D \langle \delta_t \delta_\Delta u_{l,m}^n, \delta_\Delta u_{l,m}^n \rangle_{\underline{\underline{d}}} + \mathfrak{b},$$

and using identities (3.17a) and (3.17b) a definition for the total energy can be found:

$$\begin{aligned}\mathfrak{h} &= \mathfrak{t} + \mathfrak{v}, \quad \text{with} \quad \mathfrak{t} = \frac{\rho H}{2} \|\delta_{t-} u_{l,m}^n\|_d^2 \quad \text{and} \\ \mathfrak{v} &= \frac{D}{2} \langle \delta_\Delta u_{l,m}^n, e_{t-} \delta_\Delta u_{l,m}^n \rangle_{\underline{\underline{d}}}.\end{aligned}\quad (6.52)$$

The definition of the boundary term  $\mathfrak{b}$  will not be given here, but can be shown to vanish under the boundary conditions given in Eq. (6.46) [23].

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## 6.4. The stiff membrane

### Step 3: Check units

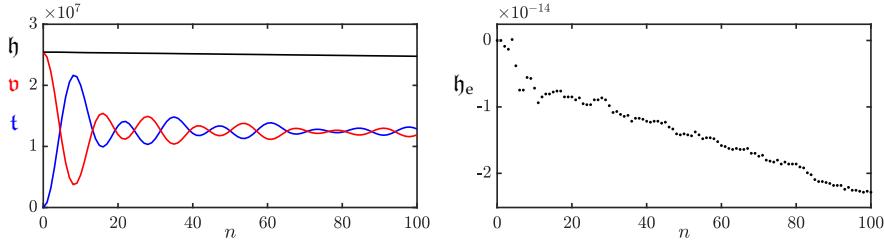
As  $t$  is identical to its definition in Eq. (6.33), only the units for  $v$  will be checked here. Recalling that  $D = EH^3/12(1 - \nu^2)$ , which in units is  $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$ , yields

$$\begin{aligned} \frac{D}{2} \langle \delta_{\Delta} u_{l,m}^n, e_t - \delta_{\Delta} u_{l,m}^n \rangle_{\bar{d}} &\xrightarrow{\text{in units}} \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{m}^2 \cdot (\text{m}^{-2} \cdot \text{m}) \cdot (\text{m}^{-2} \cdot \text{m}) \\ &= \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \end{aligned}$$

and shows that  $v$  indeed has the correct units.

### Step 4: Implementation

Figure 6.12 shows the energetic output of an implementation of the thin plate and shows that the energy is conserved.



**Fig. 6.12:** The kinetic (blue), potential (red), and total (black) energy of an implementation of the thin plate are plotted in the left panel. Notice that the damping present in the system causes  $h$  to decrease. The right panel shows the normalised energy (according to Eq. (3.38)) and shows that the deviation of the energy is within machine precision.

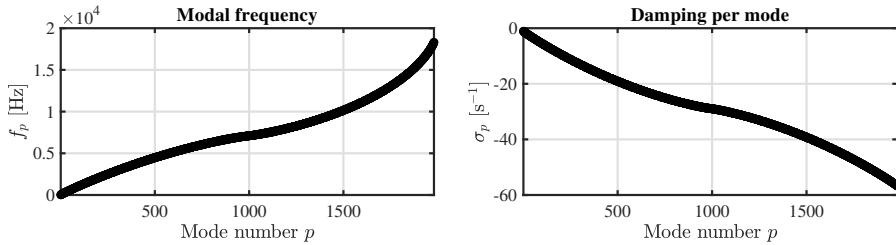
### 6.3.6 Modal analysis

Using the matrix form in Eq. (6.47), a modal analysis of the system can be performed using a one-step form described in Section 3.5.1.

Figure 6.13 shows the results of the analysis with parameter values as listed in Table 6.1. Although the modal frequencies follow a similar pattern to those of the 2D wave equation in Figure 6.8, the pattern is slightly more exponential like the stiff string in Figure 4.6.  
not really sure what to say here honestly

## 6.4 The stiff membrane

The term *stiff membrane* first appeared in [22] and is essentially a 2D version of true? the stiff string. It can be used to model membranes with dispersive effects or



**Fig. 6.13:** The result of a modal analysis of the thin plate using the parameters in Table 6.1. Notice that the damping is plotted against modal frequency rather than mode number.

provide tension control for thin plates. In this work, the stiff membrane has only been used in paper [F] to model a drum membrane.

Similar to previous sections, this section will provide the continuous-time and discrete-time equations of the model. Only a frequency domain analysis will be given, as energy and modal analyses too similar to those previously presented.

#### 6.4.1 Continuous time

The PDE for a stiff membrane can be obtained as a combination of the 2D wave equation in Eq. (6.7) and the thin plate in Eq. (6.37). Adding losses as in Eq. (6.38) yields the following PDE

$$\rho H \partial_t^2 u = T \Delta u - D \Delta \Delta u - 2\sigma_0 \rho H \partial_t u + 2\sigma_1 \rho H \partial_t \Delta u, \quad (6.53)$$

where the parameters are identical to those in Eqs. (6.7) and (6.38).

#### 6.4.2 Discrete time

Using familiar operators, Eq. (6.53) can be discretised to

$$\rho H \delta_{tt} u_{l,m}^n = T \delta_\Delta u_{l,m}^n - D \delta_\Delta \delta_\Delta u_{l,m}^n - 2\sigma_0 \rho H \delta_{t-} u_{l,m}^n + 2\sigma_1 \rho H \delta_{t-} \delta_\Delta u_{l,m}^n, \quad (6.54)$$

or, using a more compact form after division by  $\rho H$ , to

$$\delta_{tt} u_{l,m}^n = c^2 \delta_\Delta u_{l,m}^n - \kappa^2 \delta_\Delta \delta_\Delta u_{l,m}^n - 2\sigma_0 \delta_{t-} u_{l,m}^n + 2\sigma_1 \delta_{t-} \delta_\Delta u_{l,m}^n, \quad (6.55)$$

where  $c = \sqrt{T/\rho H}$  and  $\kappa = \sqrt{D/\rho H}$ .

The update equation can then be obtained using the expansions of the

#### 6.4. The stiff membrane

Laplacian and biharmonic operators in Eqs. (6.4) and (6.6) respectively to get

$$\begin{aligned} u_{l,m}^{n+1} = & (2 - 4\lambda^2 - 20\mu^2 - 4S)u_{l,m}^n \\ & + (\lambda^2 + 8\mu^2 + S)(u_{l+1,m}^n + u_{l-1,m}^n + u_{l,m+1}^n + u_{l,m-1}^n) \\ & - 2\mu^2(u_{l+1,m+1}^n + u_{l-1,m+1}^n + u_{l+1,m-1}^n + u_{l-1,m-1}^n) \\ & - \mu^2(u_{l+2,m}^n + u_{l-2,m}^n + u_{l,m+2}^n + u_{l,m-2}^n), \\ & + (\sigma_0 k - 1 + 4S)u_{l,m}^{n-1} \\ & - S(u_{l+1,m}^{n-1} + u_{l-1,m}^{n-1} + u_{l,m+1}^{n-1} + u_{l,m-1}^{n-1}) \end{aligned} \quad (6.56)$$

where

$$\lambda = \frac{ck}{h} \quad \text{and} \quad \mu = \frac{\kappa k}{h^2} \quad (6.57)$$

and again,  $S = 2\sigma_1 k/h^2$  for compactness. The stability condition for this scheme will be shown in Section 6.4.4.

#### 6.4.3 Implementation

Writing Eq. (6.55) in matrix form yields

$$A\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \mathbf{C}\mathbf{u}^{n-1}, \quad (6.58)$$

with

$$\begin{aligned} A = & (1 + \sigma_0 k), \quad \mathbf{B} = 2\mathbf{I} + c^2 k^2 \mathbf{D}_\Delta - \kappa^2 k^2 \mathbf{D}_{\Delta\Delta} + 2\sigma_1 k \mathbf{D}_\Delta, \\ \text{and} \quad \mathbf{C} = & -(1 - \sigma_0 k)\mathbf{I} - 2\sigma_1 k \mathbf{D}_\Delta. \end{aligned}$$

Notice that the only difference with Eq. (6.47) is the addition of the wave speed term in the definition of the  $\mathbf{B}$  matrix.

#### 6.4.4 Frequency domain analysis

Following familiar techniques from Sections 3.5.1 and 6.2.4, the characteristic equation of the FD scheme in Eq. (6.55) can be obtained:

$$\begin{aligned} (1 + \sigma_0 k)z + & \left( 4\lambda^2(p_x + p_y) + 16\mu^2(p_x + p_y)^2 + \frac{8\sigma_1 k}{h^2}(p_x + p_y) - 2 \right) \\ & + \left( 1 - \sigma_0 k - \frac{8\sigma_1 k}{h^2}(p_x + p_y) \right) z^{-1} = 0. \end{aligned} \quad (6.59)$$

Similar to the stiff string in Section 4.3 and the thin plate in Section 6.3.4, this can be solved to

$$\lambda^2(p_x + p_y) + 4\mu^2(p_x + p_y)^2 + \frac{4\sigma_1 k}{h^2}(p_x + p_y) \leq 1,$$

and recalling that  $p_x$  and  $p_y$  are bounded by 1 yields

$$\begin{aligned}\lambda^2(1+1) + 4\mu^2(1+1)^2 + \frac{4\sigma_1 k}{h^2}(1+1) &\leq 1, \\ 2\lambda^2 + 16\mu^2 + \frac{8\sigma_1 k}{h^2} &\leq 1.\end{aligned}$$

Recalling the definitions for  $\lambda$  and  $\mu$  from 6.57, one can solve for  $h$

$$\begin{aligned}\frac{2c^2k^2}{h^2} + \frac{16\kappa^2k^2}{h^4} + \frac{8\sigma_1 k}{h^2} &\leq 1, \\ h^4 - (2c^2k^2 + 8\sigma_1 k)h^2 - 16\kappa^2k^2 &\geq 0, \\ h \geq \sqrt{\frac{2c^2k^2 + 8\sigma_1 k + \sqrt{(2c^2k^2 + 8\sigma_1 k)^2 + 64\kappa^2k^2}}{2}}, \\ h \geq \sqrt{c^2k^2 + 4\sigma_1 k + \frac{1}{2}\sqrt{4(c^2k^2 + 4\sigma_1 k)^2 + 64\kappa^2k^2}}, \\ h \geq \sqrt{c^2k^2 + 4\sigma_1 k + \sqrt{(c^2k^2 + 4\sigma_1 k)^2 + 16\kappa^2k^2}},\end{aligned}\tag{6.60}$$

and is the stability condition for the stiff membrane.

## 6.5 Radial coordinates

This chapter presented various models using a Cartesian coordinate system. Circular or elliptical systems, such as membranes or gongs, could be modelled using a radial coordinate system [23, Ch. 10]. However, using explicit schemes causes these systems to exhibit high amounts much numerical dispersion and reduction of bandwidth [23, Ch. 11]. For better behaviour, one could resort to an implicit scheme, but comes with the drawbacks mentioned in Section 4.6. A better alternative is to retain the cartesian coordinate system and set quasi-circular boundary conditions according to a staircase approximation as done in [S4] (see fx. [47, 59]).

# **Part III**

# **Exciters**



# Exciters

Several resonators have been introduced in part II, different mechanisms to excite them will be introduced here. First, different examples of

and have a great effect on the eventual timbre of the sound. Chapter 7 presents various physically inspired excitations some of which made a brief appearance in Part II. Chapter 8 introduces a static and a dynamic friction model that can be used for bowing resonators. Additionally, this chapter presents the contribution made in [C]: the elasto-plastic friction model applied to FDTD stiff strings. Finally, Chapter 9 presents the lip reed as a way to excite brass instruments.



# Chapter 7

## Physically-Inspired Excitations

This short chapter introduces several simple ways to excite the different resonators presented in Part II.

First, 1D systems will be considered...

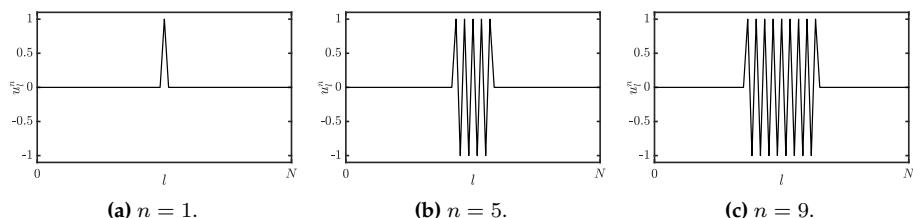
### 7.1 Initial conditions

The easiest way to excite a system is to set its initial conditions to non-zero values. This has been done several times before using a raised cosine

To not give the system an initial velocity, both  $u^0$  and  $u^1$  will have to be initialised with the same value

#### 7.1.1 Impulse

The simplest way of exciting a system is to add an impulse to spatial Nyquist



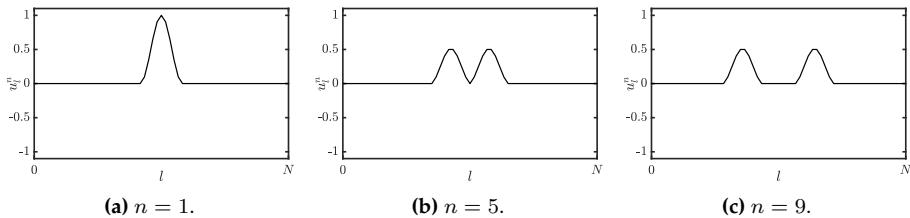
**Fig. 7.1:** The 1D wave equation initialised with an impulse at  $l = 0.5N$ .

### 7.1.2 Raised cosine

As the impulse triggers very high-frequency behaviour

Also referred to as a pluck – due to the displacing and letting go – but a more physically reasonable pluck is presented in Section 7.1.3

Simplest way is hann.

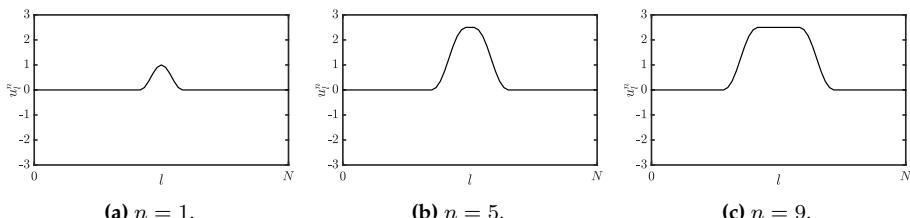


**Fig. 7.2:** The 1D wave equation initialised with a raised cosine at  $l = 0.5N$ .

### Strike

If  $u_l^0$  is not set to be the same value as  $u_l^1$  at the start of the simulation, one can model a strike.

As can be observed from Figure 7.3, the amplitude of the displacement will be higher and, in the case of the 1D wave equation excited with a raised cosine, can be calculated to have a maximum amplitude of half the sum of the excitation.



**Fig. 7.3:** The 1D wave equation initialised with a strike at  $l = 0.5N$ . Notice the scaling of the y-axis compared to the other figures.

### 7.1.3 Pluck

- Cut-off raised cosine
- Triangle (for string)

### 7.1.4 Noise

Noise input

Stress-testing

## 7.2 Time-varying excitations

If one would like to excite the system, not at the start, but later on in the simulation,

$$\delta_{tt} u_t^n = c^2 \delta_{xx} u_t^n + F(t) \quad (7.1)$$

where  $F$  is a scaled force with units of m/s if  $c = \sqrt{T/\rho A}$ , then  $F = f/\rho A$ ,  $f$  being a force in Newtons.

### 7.2.1 Alternative pluck

SMC figure

### 7.2.2 Hammer

### 7.2.3 Pulse train

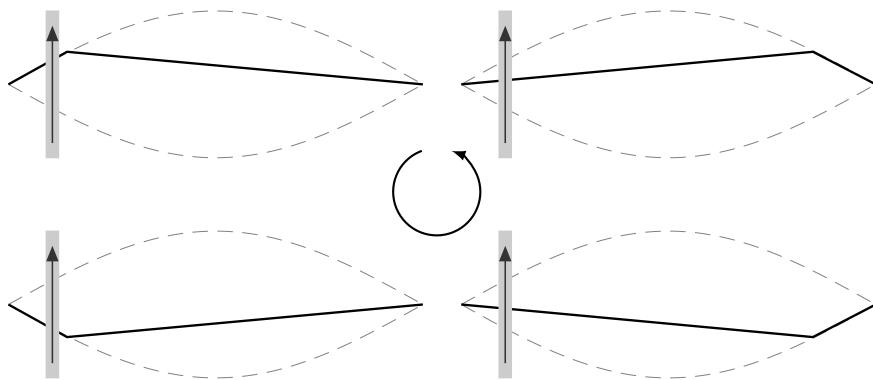
For brass

## Chapter 7. Physically-Inspired Excitations

# Chapter 8

## The Bow

The bow is a very interesting excitation mechanism from a simulation perspective. The bow excites the resonator at hand with a force due to friction, which introduces a nonlinear element that is dependent on the relative velocity between the bow and the string. The string ‘sticks’ to the bow after which it ‘slips’ when the restoring force of the string is too great and overcomes the friction force. This ‘stick-slip’ behaviour, first coined by Bowden and Leben in 1939 [70] causes the string to move in a characteristic triangular motion where the corner of the triangle moves back and forth along the string (see Figure 8.1). Herman Helmholtz was the first to discover this behaviour, which later got named *Helmholtz motion* in his honour [71].<sup>1</sup>



**Fig. 8.1:** Helmholtz motion. If the bow moves up on the left side of the string, the ‘Helmholtz corner’ travels anti-clockwise.

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<sup>1</sup>Also see <https://www.youtube.com/watch?v=6JeyiM0YNo4>

The Helmholtz motion gives bowed string instruments, such as the violin and cello, their characteristic sound. [helmholtz output figure](#)

### Brief history of bowed-string simulation

The first nonlinear systems in the context of musical instrument simulations, including the bowed string, were presented by McIntyre, et al. in 1983 [14]. The first real-time implementation of the bowed string was due to Smith in 1986 and used digital waveguides for the string and a look-up table for the friction model [72]. Simultaneously, Florens et al. presented a real-time implementation of the bowed string, but instead, the string was modelled using mass-spring systems and the friction model used a static friction model [73] (see Section 8.3). One of the most complex friction models applied in a musical context to-date is the elasto-plastic friction model due to Dupont [74], which Serafin et al. [75, 76] applied to a digital waveguide implementation of the string.

The first appearance of FDTD methods in bowed string simulations was in a publication by Pitteroff and Woodhouse in [77]. Later, Maestre et al. in [30] used FDTD methods to implement a thermal friction model proposed by Woodhouse in [78]. In both cases, the string was implemented using digital waveguides. Desvages implemented a bowed string model using a static friction model and a two-polarisation FDTD model for the string in [49, 48], but did not implement this in real-time. This chapter presents the first contribution of this dissertation published in papers [A] and [C]. Firstly, paper [A] shows the first real-time implementation of a bowed stiff string fully modelled using FDTD methods. Secondly, [C] presents the first (real-time) implementation of the elasto-plastic friction model applied to a FD scheme in a musical context.

Before introducing various friction models, a brief introduction on interpolation and spreading operators will be given, which will be necessary to work with this type of excitation.

## 8.1 Interpolation and spreading operators

As should be clear by now, FD schemes are an approximation to continuous space (and time) using a finite number of points. Working with the system between grid points is not impossible, but requires an extra step. If one would like to listen to a location between specified grid points, one can use interpolation. For this end, an *interpolation operator*  $I(x_i)$  can be introduced and can be applied to a grid function [23]. This operator is a function of  $x_i$ , the (continuous) location of interest and can be defined in various levels of accuracy. Here, a 1D system  $u(x, t)$  is assumed where  $x \in \mathcal{D}$  for spatial domain  $\mathcal{D}$ .

## 8.1. Interpolation and spreading operators

An interpolation operator can be applied to a grid function  $u_l^n$  and when expanded performs the following operation:

$$I_{l,o}(x_i)u_l^n = \sum_{l \in d} I_{l,o}(x_i) \cdot u_l^n, \quad (8.1)$$

where  $o$  is the order of the operator, and discrete domain  $d$  needs to be the same for  $I_{l,o}(x_i)$  and  $u_l^n$ . The simplest interpolation operator is of '0<sup>th</sup>'-order and is defined as

$$I_{l,0}(x_i) = \begin{cases} 1, & \text{if } l = l_i, \\ 0, & \text{otherwise,} \end{cases} \quad (8.2)$$

where the grid location of interest is defined as  $l_i = \lfloor x_i/h \rfloor$  (see Figure 8.2a). Instead of actually performing an interpolation operation,  $I_0$  simply floors its input to the grid location below. A slightly more accurate way to perform 0<sup>th</sup>-order interpolation is to *round*  $x_i/h$  to the nearest neighbour, rather than using the flooring operation.

First-order or linear interpolation also uses the fractional part of the flooring operation according to  $\alpha_i = x_i/h - l_i$  and is defined as

$$I_{l,1}(x_i) = \begin{cases} (1 - \alpha_i), & \text{if } l = l_i, \\ \alpha_i, & \text{if } l = l_i + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (8.3)$$

See Figure 8.2b.

The highest order interpolator used in this project is the Lagrange cubic interpolator:

$$I_{l,3}(x_i) = \begin{cases} -\alpha_i(\alpha_i - 1)(\alpha_i - 2)/6, & l = l_i - 1, \\ (\alpha_i - 1)(\alpha_i + 1)(\alpha_i - 2)/2, & l = l_i, \\ -\alpha_i(\alpha_i + 1)(\alpha_i - 2)/2, & l = l_i + 1, \\ \alpha_i(\alpha_i + 1)(\alpha_i - 1)/6, & l = l_i + 2, \\ 0, & \text{otherwise.} \end{cases} \quad (8.4)$$

See Figure 8.2c. Notice that the sum of all values of  $I(x_i)$ , regardless of the order of interpolation or the value of  $\alpha_i$ , add up to 1.

One could potentially create higher-order interpolation operators, but as one is restricted to a finite domain, the flexibility of the implementation will become less. Notice that if  $\alpha_i = 0$ , the higher-order interpolators reduce to the 0<sup>th</sup>-order one in Eq. (8.2).

Apart from interpolation operators, one may define *spreading operators* which can be interpreted as an inverse interpolation operation. A spreading operator  $J(x_i)$  is used to interact with a distributed FD scheme in the form

of an excitation or other interactions such as connections or collisions between multiple schemes (also see Part IV).

The spreading operators can be defined in the same way as the interpolation operators described above, yielding a 0<sup>th</sup>-order spreading operator

$$J_{l,0}(x_i) = \frac{1}{h} \begin{cases} 1, & \text{if } l = l_i, \\ 0, & \text{otherwise,} \end{cases} \quad (8.5)$$

a linear spreading operator

$$J_{l,1}(x_i) = \frac{1}{h} \begin{cases} (1 - \alpha_i), & \text{if } l = l_i, \\ \alpha_i, & \text{if } l = l_i + 1, \\ 0 & \text{otherwise,} \end{cases} \quad (8.6)$$

and a Lagrange cubic spreading operator

$$J_{l,3}(x_i) = \frac{1}{h} \begin{cases} -\alpha_i(\alpha_i - 1)(\alpha_i - 2)/6, & l = l_i - 1, \\ (\alpha_i - 1)(\alpha_i + 1)(\alpha_i - 2)/2, & l = l_i, \\ -\alpha_i(\alpha_i + 1)(\alpha_i - 2)/2, & l = l_i + 1, \\ \alpha_i(\alpha_i + 1)(\alpha_i - 1)/6, & l = l_i + 2, \\ 0, & \text{otherwise.} \end{cases} \quad (8.7)$$

Notice the scaling by  $1/h$  which will be more elaborated on in Chapter 10. As is the case with the interpolation operators, higher-order spreading operators reduce to Eq. (8.5) if  $\alpha_i = 0$ .

Spreading operators approximate the spatial Dirac delta function  $\delta(x - x_i)$  (in  $m^{-1}$ ), a test function defined as

$$\delta(x) = \begin{cases} \infty, & x = 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) dx = 1, \quad (8.8)$$

used in continuous time to locate an external force to a location  $x_i$  along a system distributed over space  $x$ . Note that the definition in (8.8) will not be used directly. Instead, it can be approximated using the spreading operators presented in this section.

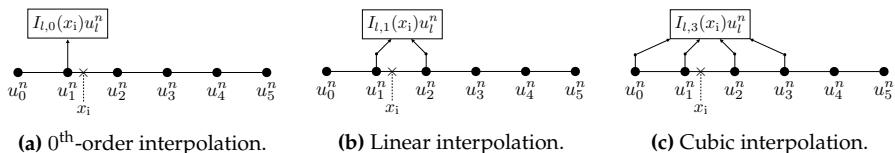


Fig. 8.2: Interpolation with varying orders of accuracy.

## 8.2. The Newton-Raphson method

The following identity is extremely useful when solving systems including interpolation and spreading operators of the same order  $o$

$$\langle f, J_o(x_i) \rangle_d = I_o(x_i) f, \quad (8.9)$$

for any (grid) function  $f$  and discrete domain  $d$ . From this, it follows that taking the norm of a spreading operator  $J_o(x_i)$  over a given domain is identical to applying to its ‘dual’ interpolation operator (of the same order  $o$  and same input  $x_i$ ):

$$\langle J_o(x_i), J_o(x_i) \rangle_d = \|J_o(x_i)\|_d^2 = I_o(x_i) J_o(x_i). \quad (8.10)$$

See Section 3.2.1 for more details on the inner product and the norm.

### Other distributions

This section presented interpolation and spreading operators that interact with the state of a FD scheme at a single location  $x_i$  and either interpolates or distributes over a range of points. Although multiple grid points might be used for these operations, the interpolation or spreading is not *distributed*. Physical excitors such as mallets or bows have a non-zero width and thus interact with a larger part of the system. One could make an arbitrary distribution function  $E$  with elements  $e_l$  (in 1D) where  $l \in d$  for discrete domain  $d$  of the system at hand. The *distribution* and spreading operators become

$$I_l = \frac{e_l}{\sum_d e_l} \quad \text{and} \quad J_l = \frac{1}{h} I_l. \quad (8.11)$$

Although any values for  $E$  would work, to retain correct scaling, the sum of  $E$  needs to be normalised to 1 as shown above.

## 8.2 The Newton-Raphson method

Before moving on to more complex nonlinear excitation mechanisms, it is useful to go over the process of how to solve some of these mechanisms using an iterative root-finding method called the *Newton-Raphson* method (or Newton-Raphson for short).

If a FD scheme not be solved explicitly, due to a nonlinear dependence on a variable for example, Newton-Raphson can be used. For a continuous and differentiable function  $f(x) = 0$  its root can be approached using the following iteration

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad (8.12)$$

with iteration number  $i$  and the tick is used to denote a derivation with respect to  $x$ . This iteration will then be carried out until the difference between the

values of two consecutive iterations is smaller than a given threshold:

$$|x_{i+1} - x_i| < \epsilon, \quad (8.13)$$

where  $\epsilon$  is small, but its exact value depends on the situation at hand. To prevent Newton's method from iterating endlessly (which can happen in some cases), one can put a cap on the number of iterations allowed.

Preferably, the starting point of the iteration,  $x_0$ , should be close to the value of where the root is expected to be. This is especially the case for a higher-ordered function with multiple roots (non-uniqueness) or many local variations.

Algorithm 8.1 shows an example of an implementation of Newton-Raphson using  $f(x) = e^x - 1 \Rightarrow f'(x) = e^x$  and Figure 8.3 visualises the iterative algorithm.

---

```

1 % An example of the Newton Raphson method using f(x) = exp(x) - 1
2
3 x = 1;           % starting point
4 eps = 1e-4;      % threshold
5
6 % if the threshold has not been crossed before this number of
7 % iterations, do not iterate more
8 maxIterations = 100;
9
10 % loop until a maximum number of iterations
11 for i = 1:maxIterations
12
13     % calculate next iteration (Eq. (8.12))
14     xNext = x - (exp(x) - 1) / (exp(x));
15
16     % threshold check (Eq. (8.13))
17     if abs(xNext - x) < eps
18         break; % break out of the for loop
19     end
20
21     % update the value of x
22     x = xNext;
23 end
24 disp("The root of f(x) is at x = " + xNext)

```

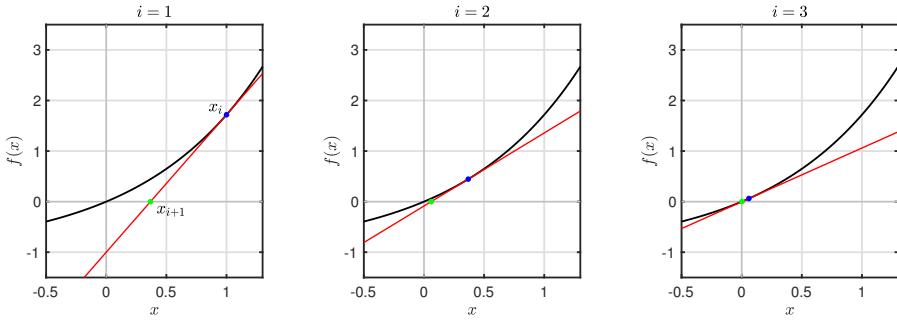
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**Algorithm 8.1:** Example of an implementation of the Newton-Raphson method using  $f(x) = e^x - 1$ .

### 8.2.1 Multivariate Newton-Raphson

For  $M$  functions  $f_m$  dependent on the same number of independent variables  $x_m$  with  $m = \{1, \dots, M\}$ , Newton-Raphson can be extended to the following

## 8.2. The Newton-Raphson method



**Fig. 8.3:** The Newton-Raphson method. The  $x$ -value of the root of the tangent line at  $f(x_i)$  is used to evaluate the next iteration.

citation?

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \underbrace{\begin{bmatrix} \frac{\partial f_1(\mathbf{x}_i)}{\partial x_1} & \dots & \frac{\partial f_1(\mathbf{x}_i)}{\partial x_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_M(\mathbf{x}_i)}{\partial x_1} & \dots & \frac{\partial f_M(\mathbf{x}_i)}{\partial x_M} \end{bmatrix}}_{\mathbf{J}(\mathbf{x})}^{-1} \begin{bmatrix} f_1(\mathbf{x}_i) \\ \vdots \\ f_M(\mathbf{x}_i) \end{bmatrix}, \quad (8.14)$$

where the independent variables are collected in a column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix}, \quad (8.15)$$

and the iteration number is again denoted by  $i$ . The matrix in Eq. (8.14) is referred to as the *Jacobian matrix*  $\mathbf{J}$  and contains the derivatives of all functions with respect to each individual independent variable.

As an example, consider the following system of equations<sup>2</sup>:

$$f_1(\mathbf{x}) = 3x_1 - \cos(x_2 x_3) - 3/2 = 0, \quad (8.16a)$$

$$f_2(\mathbf{x}) = 4x_1^2 - 625x_2^2 + 2x_3 - 1 = 0, \quad (8.16b)$$

$$f_3(\mathbf{x}) = 20x_3 + e^{-x_1 x_2} + 9 = 0. \quad (8.16c)$$

The Jacobian matrix will be

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} 3 & x_3 \sin(x_2 x_3) & x_2 \sin(x_2 x_3) \\ 8x_1 & -1250x_2 & 2 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix},$$

and its roots can be found by iteratively calculating Eq. (8.14).

---

<sup>2</sup>Taken from <http://fourier.eng.hmc.edu/e176/lectures/NM/node21.html>

### 8.3 Static friction models

A friction model is a nonlinear function that is (at least) dependent on the relative velocity  $v_{\text{rel}}$  between the bow and the string. This function scales how much the bow force affects the bowed object. In static friction models, the friction force is defined as a function of this relative velocity only. The first mathematical description of friction was proposed by Coulomb in 1773 [79] to which static friction, or *stiction*, was added by Morin in 1833 [80] and viscous friction, or velocity-dependent friction, by Reynolds in 1886 [81]. In 1902, Stribeck found a smooth transition between the static and the coulomb part of the friction curve now referred to as the Stribeck effect [82]. The latter is still the standard for static friction models today.

In this project, only the following static friction model has been used [23]

$$\Phi(v_{\text{rel}}) = \sqrt{2av_{\text{rel}}} e^{-av_{\text{rel}}^2 + 1/2}. \quad (8.17)$$

See Figure 8.4. Consider a string, its transverse displacement described by  $u(x, t)$  defined for  $x \in \mathcal{D}$  (see Chapter 4). The relative velocity between the string at bow position  $x_B = x_B(t) \in \mathcal{D}$  and the bow is described as

$$v_{\text{rel}} = \partial_t u(x_B, t) - v_B(t) \quad (8.18)$$

(in m/s) with bow velocity  $v_B = v_B(t)$  (in m/s).

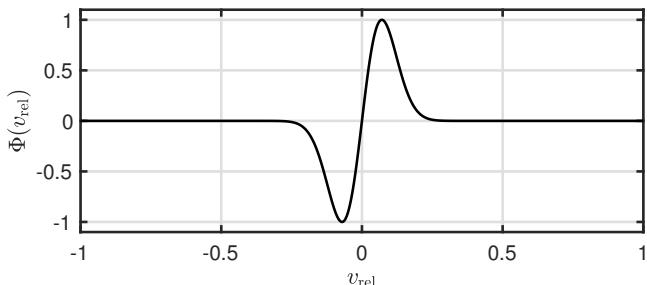


Fig. 8.4: The friction model in (8.17) with  $a = 100$ .

Many of the friction models contain a discontinuity where the relative velocity  $v_{\text{rel}} = 0$  due to a multiplication with  $\text{sgn}(v_{\text{rel}})$  in their definition. Equation (8.17) approximates discontinuous bow models using a continuous and differentiable function. This makes it easier to work with when trying to implement the system as will be explained next.

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### 8.3.1 The bowed stiff string

Recalling the PDE of the stiff string in Eq. (4.4)

$$\rho A \partial_t^2 u = T \partial_x^2 u - EI \partial_x^4 u - 2\sigma_0 \rho A \partial_t u + 2\sigma_1 \rho A \partial_t \partial_x^2 u, \quad (8.19)$$

one can add the bow force to the equation according to

$$\rho A \partial_t^2 u = T \partial_x^2 u - EI \partial_x^4 u - 2\sigma_0 \rho A \partial_t u + 2\sigma_1 \rho A \partial_t \partial_x^2 u - \delta(x - x_B) f_B \Phi(v_{\text{rel}}) \quad (8.20)$$

where spatial Dirac delta function  $\delta(x - x_B)$  (in  $\text{m}^{-1}$ ) (see Section 8.1) positions the bow along the string and  $f_B = f_B(t) \geq 0$  is the bow force (in N).<sup>3</sup>

#### Intuition

From Eq. (8.20) it can be seen that the bow force gets scaled by the friction model  $\Phi(v_{\text{rel}})$  shown in Figure 8.4. The figure shows that if  $v_{\text{rel}}$  is too large (either positively or negatively) the bow term in (8.20) becomes 0. If, on the other hand,  $v_{\text{rel}}$  is closer to 0, the bow will have an effect on the string. This can be interpreted in terms of static and dynamic friction<sup>4</sup>. A stationary object requires more force to be moved than a moving object, i.e., the static friction coefficient is always higher than the dynamic friction coefficient. This is essentially what the friction model tries to simulate.

#### Discrete time

Dividing all terms in Eq. (8.21) by  $\rho A$  and discretising the system yields

$$\delta_{tt} u_l^n = c^2 \delta_{xx} u_l^n - \kappa^2 \delta_{xxxx} u_l^n - 2\sigma_0 \delta_t u_l^n + 2\sigma_1 \delta_t - \delta_{xx} u_l^n - J_l(x_B^n) F_B^n \Phi(v_{\text{rel}}^n), \quad (8.21)$$

with  $F_B^n = f_B^n / \rho A$  and spreading operator  $J_l(x_B)$  (in  $\text{m}^{-1}$ ) as described in Section 8.1, the order of which remains undetermined for now. Notice that, as the bow position, bow velocity and bow force are time-dependent, these have received a superscript  $n$ . These parameters are called control parameters and will be supplied by the performer in the eventual implementation.

The relative velocity in Eq. (8.18) is discretised using a centred difference operator according to

$$v_{\text{rel}}^n = I_l(x_B^n) \delta_t u_l^n - v_B^n. \quad (8.22)$$

The main issue with Eq. (8.22) is that, due to the centred difference operator, the FD scheme is now nonlinearly dependent on  $u_l^{n+1}$ . To solve Eq. (8.21), an iterative solver is required, such as Newton-Raphson described in Section 8.2. This process could be circumvented by using a backward difference operator in Eq. (8.22), but this will affect accuracy of the bow model.

---

<sup>3</sup>If the spatial Dirac delta function were omitted, the bow force would be applied to the entire string domain rather than only the bow location  $x_B$ .

<sup>4</sup>'Static' and 'dynamic' friction are unrelated to 'static' and 'dynamic' friction models.

### Solution

To find a solution for  $u_l^{n+1}$  at the bow location, an inner product of the scheme in Eq. (8.21) with spreading operator  $J_l(x_B^n)$  must be taken over the discrete domain of the string  $d$  which isolates the scheme at the bowing location. Performing this operation and using identity (8.9) yields

$$I_l(x_B^n) \delta_{tt} u_l^n = c^2 I_l(x_B^n) \delta_{xx} u_l^n - \kappa^2 I_l(x_B^n) \delta_{xxxx} u_l^n - 2\sigma_0 I_l(x_B^n) \delta_t u_l^n + 2\sigma_1 I_l(x_B^n) \delta_{t-} \delta_{xx} u_l^n - \|J_l(x_B^n)\|_d^2 F_B^n \Phi(v_{\text{rel}}^n). \quad (8.23)$$

One can rewrite Eq. (8.22) to

$$I_l(x_B^n) \delta_t u_l^n = v_{\text{rel}}^n + v_B^n, \quad (8.24)$$

and using identity (2.27a), Eq. (8.23) can be rewritten and assigned to a function  $g(v_{\text{rel}}^n)$

$$g(v_{\text{rel}}^n) = \left( \frac{2}{k} + 2\sigma_0 \right) v_{\text{rel}}^n + \|J_l(x_B^n)\|_d^2 F_B^n \Phi(v_{\text{rel}}^n) + b^n = 0, \quad (8.25)$$

where the terms not dependent on  $v_{\text{rel}}$  are collected in

$$\begin{aligned} b^n = & -\frac{2}{k} I_l(x_B^n) \delta_{t-} u_l^n - c^2 I_l(x_B^n) \delta_{xx} u_l^n + \kappa^2 I_l(x_B^n) \delta_{xxxx} u_l^n \\ & + \left( \frac{2}{k} + 2\sigma_0 \right) v_B^n - 2\sigma_1 I_l(x_B^n) \delta_{t-} \delta_{xx} u_l^n. \end{aligned}$$

One can then perform the Newton-Raphson method detailed in Section 8.2 to iteratively solve for  $v_{\text{rel}}$

$$(v_{\text{rel}}^n)_{i+1} = (v_{\text{rel}}^n)_i - \frac{g((v_{\text{rel}}^n)_i)}{g'((v_{\text{rel}}^n)_i)}, \quad (8.26)$$

where

$$g'(v_{\text{rel}}^n) = \frac{2}{k} + 2\sigma_0 + \|J_l(x_B^n)\|_d^2 F_B^n \Phi'(v_{\text{rel}}^n), \quad (8.27)$$

and

$$\Phi'(v_{\text{rel}}^n) = \sqrt{2a} (1 - 2a(v_{\text{rel}}^n)^2) e^{-a(v_{\text{rel}}^n)^2 + 1/2}.$$

### Implementation

**still need to write this bit** The Newton-Raphson iteration needs to be performed for every sample

$$\epsilon = 10^{-4}$$

Limit of iterations is set to 100

$$I_l(x_B^n) \delta_{xx} u_l^n = (1 - \alpha) \delta_{xx} u_{l_B}^n + \alpha \delta_{xx} u_{l_B+1}^n \quad (8.28)$$

### 8.3.2 Energy analysis

Following the energy analysis of the stiff string presented in Section 4.4, taking an inner product of Eq. (8.21) (after multiplication by  $\rho A$ ) with  $(\delta_t u_l^n)$  over discrete domain  $d$  one arrives at the following

$$\begin{aligned} \delta_{t+}\mathfrak{h} + \mathfrak{q} &= -\langle (\delta_t u_l^n), J_l(x_B^n) f_B^n \Phi(v_{\text{rel}}^n) \rangle_d \\ \xleftarrow{\text{Eq. (8.9)}} &= -I_l(x_B^n)(\delta_t u_l^n) f_B^n \Phi(v_{\text{rel}}^n) \\ \xleftarrow{\text{Eq. (8.24)}} &= \underbrace{-f_B^n \Phi(v_{\text{rel}}^n) v_{\text{rel}}^n}_{\text{loss}} - \underbrace{f_B^n \Phi(v_{\text{rel}}^n) v_B^n}_{\text{power}}, \end{aligned}$$

where  $\mathfrak{h}$  and  $\mathfrak{q}$  are as defined in Eqs. (4.32) and (4.31) respectively. As  $\text{sgn}(\Phi(v_{\text{rel}}^n)) = \text{sgn}(v_{\text{rel}}^n)$  through Eq (8.17), one can observe that the first term on the right-hand side always has a negative effect on the rate of change of the total energy. This term can therefore be interpreted as the loss of power through the bow (as indicated). The last term is of indeterminate sign and can thus be interpreted as the power supplied by the bow.

The final energy balance can thus be written as

$$\delta_{t+}\mathfrak{h} = -\mathfrak{q} - \mathfrak{q}_B - \mathfrak{p} \quad (8.29)$$

where

$$\mathfrak{q}_B = f_B^n \Phi(v_{\text{rel}}^n) v_{\text{rel}}^n \quad \text{and} \quad \mathfrak{p} = f_B^n \Phi(v_{\text{rel}}^n) v_B^n.$$

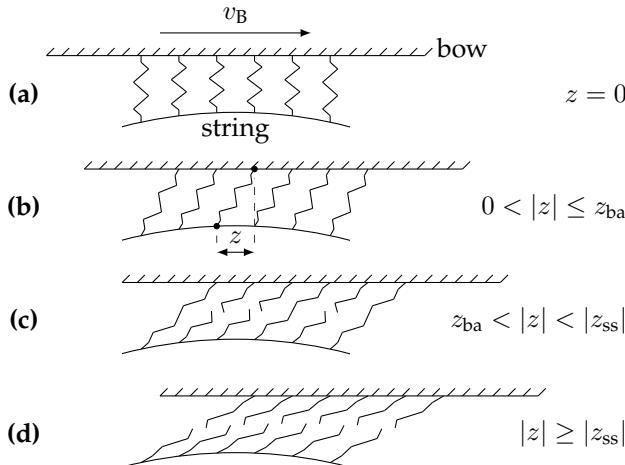
## 8.4 Dynamic friction models

As opposed to less complex static friction models, dynamic friction models relate the relative velocity to the friction force using a differential equation. Dynamic friction models exhibit a phenomenon called *hysteresis*, which is the dependence of a system on its history.

The first dynamic friction model was due to Dahl [83] and captured hysteresis effects. The Stribeck effect was, however, not taken into account. The LuGre model (named after the collaboration between Lund and Grenoble) was then proposed by Canudas de Wit et al. in [84, 85] and extended the Dahl model to take the Stribeck effect into account. The model assumes a large ensemble of bristles between the two sliding surfaces, each of which contributes a tiny amount to the total friction force. The drawback of this model is that it exhibits drift for extremely small external forces. In [74], Dupont et al. extended the LuGre model by allowing for a purely elastic regime that solves the drift issue. This model is referred to as the *elasto-plastic* friction model and is used in this project.

### 8.4.1 The elasto-plastic friction model

In a musical context, the elasto-plastic friction has been investigated in-depth by Serafin et al. in [75, 76, 86]. Like the LuGre model, the elasto-plastic friction model assumes that the friction between the bow and the string is caused by a large quantity of bristles, all contributing a fraction of the total amount of friction. See Figure 8.5.



**Fig. 8.5:** A visualisation of the microscopic displacements of the bristles between the bow and the string that the elasto-plastic friction model assumes. The bow moves right with a velocity  $v_B$ . (a) The initial state is where the average bristle displacement  $z = 0$ . (b) The bow moves right relative to the string and the purely elastic, or ‘presliding’ regime is entered (stick). (c) After  $|z|$  gets larger than break-away displacement  $z_{ba}$ , more and more bristles start to ‘break’. This is defined as the elasto-plastic regime. (d) After  $|z| \geq z_{ss}$  all bristles have ‘broken’, the steady state (slip) is reached and the purely plastic regime is entered. (Adapted from paper [C].)

Unless denoted otherwise, this section follows the original model by Dupont et al. in [74], but with the appropriate corrections added as presented in paper [C]. As opposed to the static friction model described in the previous section, the friction force  $f$  (in N) is now dependent on the average bristle displacement  $z = z(t)$  (in m) on top of the relative velocity  $v = v(t)$  (in m/s). The force is defined as

$$f(v, z) = s_0 z + s_1 \dot{z} + s_2 v + s_3 w, \quad (8.30)$$

with bristle stiffness  $s_0 \geq 0$  (in N/m), bristle damping  $s_1 \geq 0$  (in kg/s), viscous friction  $s_2 \geq 0$  (in kg/s) and, as presented in [76], a dimensionless noise coefficient  $s_3$  multiplied onto a pseudorandom function  $w = w(t)$  (in N) generating values between  $-1$  and  $1$ . Moreover, for a string defined over domain  $\mathcal{D}$ , the relative velocity between the string at bowing location  $x_B =$

#### 8.4. Dynamic friction models

$x_B(t) \in \mathcal{D}$  and the bow is (similar to Eq. (8.18))

$$v = \partial_t u(x_B, t) - v_B, \quad (8.31)$$

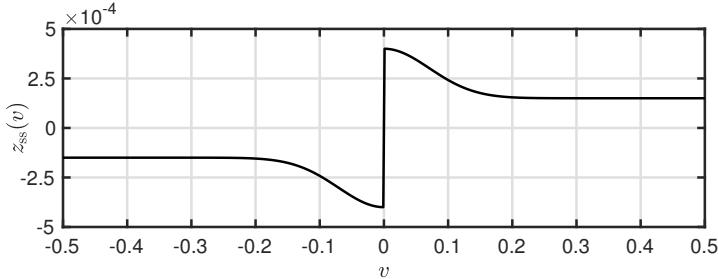
with bow velocity  $v_B = v_B(t)$  (in m/s). Lastly,  $\dot{z}$  is the rate of change of the bristle displacement (in m/s) and is related to  $v$  according to

$$\dot{z} = r(v, z) = v \left[ 1 - \alpha(v, z) \frac{z}{z_{ss}(v)} \right]. \quad (8.32)$$

Here,  $z_{ss}$  is the steady-state function

$$z_{ss}(v) = \frac{\operatorname{sgn}(v)}{s_0} \left[ f_C + (f_S - f_C) e^{-(v/v_S)^2} \right], \quad (8.33)$$

where the  $v_S$  is the Stribeck velocity (in m/s). Furthermore, using the normal force  $f_N = f_N(t)$  (in N), the Coulomb force and stiction force can be calculated according to  $f_C = f_N \mu_C$  and  $f_S = f_N \mu_S$  respectively (both in N). In these definitions  $\mu_C$  and  $\mu_S$  are the dimensionless dynamic and static friction coefficients respectively. A plot of the steady state function can be found in Figure 8.6.



**Fig. 8.6:** The steady-state function  $z_{ss}(v)$  plotted against relative velocity  $v$  with  $s_0 = 10^4$ ,  $\mu_C = 0.3$ ,  $\mu_S = 0.8$ ,  $v_S = 0.1$  and  $f_N = 5$ .

Finally,  $\alpha(v, z)$  in Eq. (8.32) is an adhesion map between the bow and the string and is defined as

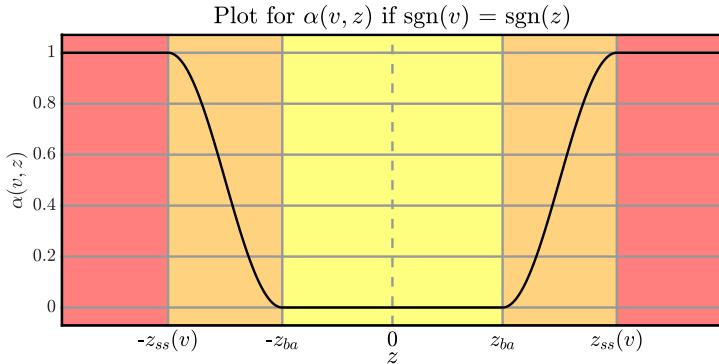
$$\alpha(v, z) = \begin{cases} 0 & |z| \leq z_{ba} \\ \alpha_m(v, z) & z_{ba} < |z| < |z_{ss}(v)| \\ 1 & |z| \geq |z_{ss}(v)| \end{cases} \quad \begin{array}{l} \text{if } \operatorname{sgn}(v) = \operatorname{sgn}(z) \\ \text{if } \operatorname{sgn}(v) \neq \operatorname{sgn}(z), \end{array} \quad (8.34)$$

where the transition between the elastic and plastic behaviour is defined as

$$\alpha_m = \frac{1}{2} \left[ 1 + \operatorname{sgn}(z) \sin \left( \pi \frac{z - \operatorname{sgn}(z) \frac{1}{2} (|z_{ss}(v)| + z_{ba})}{|z_{ss}(v)| - z_{ba}} \right) \right], \quad (8.35)$$

with break-away displacement  $z_{ba} = z_{ba}(t) = 0.7 f_C / s_0$  determines the value of  $z$  before bristles start to break. The adhesion map is visualised in Figure 8.7 and relates to Figure 8.5 as described in its caption.

so is  $z_{ba} = z_{ba}(t)$



**Fig. 8.7:** A plot of the adhesion map  $\alpha(v, z)$  in Eq. (8.34) plotted against  $z$  when  $\text{sgn}(v) = \text{sgn}(z)$ . The different coloured regions correspond to Figure 8.5 according to: yellow - a) & b), orange - c) and red - d). (Adapted from paper [C].)

### Discrete time

Equation (8.30) can be discretised to

$$f(v^n, z^n) = s_0 z^n + s_1 r^n + s_2 v^n + s_3 w^n \quad (8.36)$$

where discrete relative velocity in Eq. (8.31)

$$v^n = I(x_B^n) \delta_t u_l^n - v_B^n, \quad (8.37)$$

and

$$r^n = r(v^n, z^n) = v^n \left[ 1 - \alpha(v^n, z^n) \frac{z^n}{z_{ss}(v^n)} \right] \quad (8.38)$$

is the discrete counterpart of (8.32). The discrete adhesion map is identical to the continuous definition given in Eqs. (8.34) and (8.35), but with superscripts  $n$  added for appearances of  $v$  and  $z$ .

#### 8.4.2 Applied to a FDTD stiff string

**The first appearance contribution of the PhD project in this dissertation** The first appearance of a contribution of this work is the elasto-plastic friction model applied to a stiff string implemented using FDTD methods. This has been presented in paper [C] and will be extended in this section by providing more details on the implementation.

In the same way as done with the static friction model in Section 8.3, one can add the friction force to the stiff string PDE in Eq. (4.4) and discretise the system as follows:

$$\delta_{tt} u_l^n = c^2 \delta_{xx} u_l^n - \kappa^2 \delta_{xxx} u_l^n - 2\sigma_0 \delta_t u_l^n + 2\sigma_1 \delta_{t-} \delta_{xx} u_l^n - J_l(x_B^n) \frac{f(v^n, z^n)}{\rho A}. \quad (8.39)$$

will it though?  
And should I  
state the stabil-  
ity stuff here?

#### 8.4. Dynamic friction models

Following the same procedure as for the static friction model in Section 8.3, one takes an inner product with  $J_l(x_B^n)$  over discrete string domain  $d$  and using identities (8.9) and (2.27a), one can rewrite this similar to the static friction model in Eq. (8.25) as

$$g_1(v^n, z^n) = \left( \frac{2}{k} + 2\sigma_0 \right) v^n + \|J_l(x_B^n)\|_d^2 \frac{f(v^n, z^n)}{\rho A} + b^n = 0, \quad (8.40)$$

where

$$\begin{aligned} b^n = & -\frac{2}{k} I_l(x_B^n) \delta_{t-} u_l^n - c^2 I_l(x_B^n) \delta_{xx} u_l^n + \kappa^2 I_l(x_B^n) \delta_{xxxx} u_l^n \\ & + \left( \frac{2}{k} + 2\sigma_0 \right) v_B^n - 2\sigma_1 I_l(x_B^n) \delta_{t-} \delta_{xx} u_l^n. \end{aligned}$$

As  $g_1$  contains two unknown variables  $v^n$  and  $z^n$  that need to be solved for, the multivariate Newton-Raphson method presented in 8.2.1 must be performed. To be able to do this, an extra function must be included.

As  $r$  describes  $\dot{z}$  in Eq. (8.32), one can take another approach to approximate  $\dot{z}$  using the trapezoid rule [23]

$$a^n = (\mu_{t-})^{-1} \delta_{t-} z^n \implies a^n = \frac{2}{k} (z^n - z^{n-1}) + a^{n-1}. \quad (8.41)$$

As both  $a^n$  and  $r^n$  approximate  $\dot{z}$  these can be used to create the second function necessary to solve the full system

$$g_2(v^n, z^n) = r^n - a^n = 0. \quad (8.42)$$

Performing the multivariate Newton-Raphson method described in yields

$$\begin{bmatrix} v^n \\ z^n \end{bmatrix}_{i+1} = \begin{bmatrix} v^n \\ z^n \end{bmatrix}_i - \begin{bmatrix} \frac{\partial g_1}{\partial v} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial v} & \frac{\partial g_2}{\partial z} \end{bmatrix}^{-1} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \quad (8.43)$$

The derivatives can be shown to be... [should I give those here..? Could have an appendix with derivations haha](#)

#### 8.4.3 Implementation and output

[still need to write](#)

Plots of output  
and hysteresis  
loops

#### 8.4.4 Stability through energy analysis

As the elasto-plastic bow model is a differential equation in itself, its approximation will need to abide a stability condition as well. As the system at hand is nonlinear, frequency domain analysis as described in Section 3.3 can not

be performed. Energy analysis, on the other hand, can be used here to determine the necessary stability condition for this model. This section follows the concepts introduced in Section 3.4.4 to obtain a stability conditions for the elasto-plastic friction model. A similar process for finding stability for the LuGre model has been done by Olsson in [87, p. 55] and the derivation below is inspired by his.

First, all terms of Eq. (8.39) are multiplied by  $\rho A$  to get the appropriate units for the analysis. Then, the inner product with  $\delta_t \cdot u_l^n$  over the string domain  $\mathcal{D}$  is taken to get

$$\delta_{t+} h_s + q_s = -p_B \quad (8.44)$$

where the definitions for the discrete Hamiltonian  $h_s$  and the damping term  $q_s$  for the string can be found in Section 4.4. The input power introduced by the bow is defined as (writing  $f(v^n, z^n) = f^n$ )

$$p_B = \langle (\delta_t \cdot u_l^n), J(x_B^n) f^n \rangle_{\mathcal{D}}$$

which, using identity (8.9) can be written as

$$p_B = I_l(x_B^n) \delta_t \cdot u_l^n f^n.$$

Finally, using Eq. (8.37) yields

$$p_B = f^n v^n + f^n v_B^n. \quad (8.45)$$

The term  $f^n v^n$  is the important one as  $f^n v_B^n$  is a driving term, and is zero when the external bow velocity is zero. This means that this does not affect the internal stability of the system. In the following, the superscript  $n$  is suppressed for clarity.

Substituting Eq. (8.36) into  $fv$  and ignoring the noise term  $s_3 w^n$  for now, yields

$$p_B = fv = \sigma_0 zv + \sigma_1 rv + \sigma_2 v^2. \quad (8.46)$$

The definition for  $r^n$  in Eq. (8.38) may be rewritten as

$$\begin{aligned} r &= v \left[ 1 - \alpha \frac{z}{z_{ss}(v)} \right], \\ r &= v - \frac{v \alpha z}{z_{ss}(v)}, \\ v &= r + \frac{v \alpha z}{z_{ss}(v)}, \end{aligned}$$

and (following Olsson) may be substituted in Eq. (8.46) as

$$p_B = s_0 z \left( r + \frac{v \alpha z}{z_{ss}(v)} \right) + s_1 r \left( r + \frac{v \alpha z}{z_{ss}(v)} \right) + s_2 v^2, \quad (8.47)$$

#### 8.4. Dynamic friction models

or

$$\mathfrak{p}_B = s_0 z r + s_1 \left( r + \frac{v\alpha z}{2z_{ss}(v)} \right)^2 + \frac{v\alpha z^2}{z_{ss}(v)} \left( s_0 - \frac{s_1 v \alpha}{4z_{ss}(v)} \right) + s_2 v^2. \quad (8.48)$$

The power introduced by the bow can then be subdivided into the total energy in the bristles and their damping. As  $r$  approximates  $\dot{z}$  the first term, one can rewrite this to

NEED TO LOOK AT THE BELOW

$$\delta_{t+} \mathfrak{h}_{\text{brist}} = s_0 z^n (\delta_{t-} z^n) + \mathfrak{q}_{\text{brist}} \geq 0$$

and needs to be non-negative for passivity. Using identity (3.17b)

$$\begin{aligned} \mathfrak{h}_B &= \frac{s_0}{2} z^n e_{t-} z^n, \\ &= \frac{s_0}{2} \left( (\mu_{t-} z^n)^2 - \frac{k^2}{4} (\delta_{t-} z^n)^2 \right) \end{aligned}$$

and

$$\mathfrak{q}_{\text{brist}} = s_1 \left( r + \frac{v\alpha z}{2z_{ss}(v)} \right)^2 + \frac{v\alpha z^2}{z_{ss}(v)} \left( s_0 - \frac{s_1 v \alpha}{4z_{ss}(v)} \right) + s_2 v^2. \quad (8.49)$$

Finally, for the system to be passive,  $\mathfrak{q}$  must be non-negative. As all coefficients are non-negative and  $\text{sgn}(v) = \text{sgn}(z_{ss}(v))$  (through a multiplication by  $\text{sgn}(v)$  in the definition of in Eq. (8.33)) the following must hold

$$s_0 - \frac{s_1 v \alpha}{4z_{ss}(v)} \geq 0,$$

where both terms are non-negative. Finally, as  $\alpha$  is bounded by 1 the following condition must hold for stability

$$s_1 \leq \frac{4s_0 z_{ss}(v)}{v}. \quad (8.50)$$

This is the same stability condition as Olsson presents in [87]. So as long as one knows the limit of the velocity of the system, the coefficient  $s_1$  can be set accordingly.

## Chapter 8. The Bow

# Chapter 9

## Lip Reed

Lip reed model  
Coupling to Tube

### 9.1 Mass-spring systems revisited: Damping

Before moving on to the lip reed system, a small extension to the mass-spring system given in Section 2.3 will be given here.

Damping can be easily be added to Eq. (2.28) according to

$$M\ddot{u} = -Ku - Ru \quad (9.1)$$

with damping coefficient  $R$  (in kg/s). The damping term is analogous to the frequency-independent damping in the damped stiff string in Eq. (4.4). Equation (9.1) can then be discretised to the following FD scheme:

$$M\delta_{tt}u^n = -Ku^n - R\delta_t.u^n. \quad (9.2)$$

Expanding and solving for  $u^{n+1}$  yields the following update equation (before division with the term multiplied onto  $u^{n+1}$ ):

$$\left(1 + \frac{Rk}{2M}\right)u^{n+1} = 2u^n - u^{n-1} - \frac{Kk^2}{M}u^n + \frac{Rk}{2M}u^{n-1}. \quad (9.3)$$

#### 9.1.1 Energy analysis

Following Section 3.4, one can obtain the energy of Eq. (9.2) through a multiplication of the scheme by  $(\delta_t.u^n)$  to get

$$M(\delta_{tt}u^n)(\delta_t.u^n) = -K(\delta_t.u^n)u^n - R(\delta_t.u^n)^2. \quad (9.4)$$

As there is damping present in the system, the energy balance will be of the form

$$\delta_{t+} \mathfrak{h} = -\mathfrak{q}.$$

Using identities (3.17a) and (3.17b),  $\mathfrak{h}$  and  $\mathfrak{q}$  can be obtained from Eq. (9.4)

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{v}, \quad \text{with} \quad \mathfrak{t} = \frac{M}{2}(\delta_t u^n)^2, \quad \text{and} \quad \mathfrak{v} = \frac{K}{2}u^n e_t u^n. \quad (9.5)$$

and

$$\mathfrak{q} = R(\delta_t u^n)^2. \quad (9.6)$$

## 9.2 Continuous time

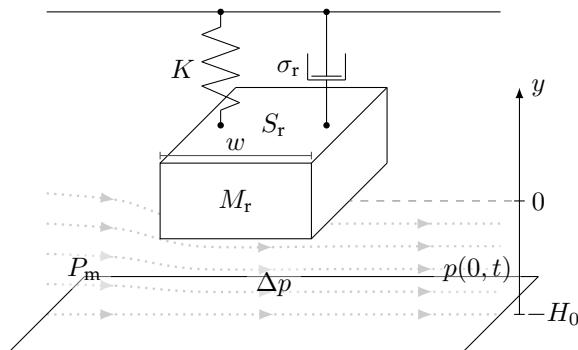
Section 5.1.3 already covered A more physical approach, that is bidirectional, is to model a lip as a mass-spring system that interacts with the left boundary of the tube (see Figure 9.1). Following [59] we get

$$M_r \frac{d^2y}{dt^2} = -M_r \omega_0^2 y - M_r \sigma_r \frac{dy}{dt} + S_r \Delta p, \quad (9.7)$$

with displacement of the lip reed from equilibrium  $y = y(t)$ , mass of the lip reed  $M_r$  (kg) natural angular frequency of the lip reed  $\omega_0 = \sqrt{K/M_r}$  (rad/s), spring stiffness of the lip  $K$  (N/m), loss parameter  $\sigma_r$  ( $s^{-1}$ ), effective surface area of the lip  $S_r$  ( $m^2$ ) and

$$\Delta p = P_m - p(0, t) \quad (9.8)$$

is the difference between the pressure in the mouth  $P_m$  (kPa) and the pressure in the mouth piece  $p(0, t)$  (kPa). This pressure difference causes a volume flow



**Fig. 9.1:** Lip-reed system with the equilibrium at 0 and the distance from the lower lip  $H_0$ . (Adapted from paper [H].)

### 9.3. Discrete time

velocity following the Bernoulli equation

$$U_B = w[y + H_0]_+ \operatorname{sgn}(\Delta p) \sqrt{\frac{2|\Delta p|}{\rho_0}}, \quad (9.9)$$

with effective lip-reed width  $w$  (m), static equilibrium separation  $H_0$  (m) and  $[x]_+ = 0.5(x+|x|)$  describes the “positive part of”. Notice that when  $y+H_0 \leq 0$ , the lips are closed and the volume velocity  $U_B$  is 0. Another volume flow is generated by the lip reed itself according to

$$U_r = S_r \frac{dy}{dt}. \quad (9.10)$$

Assuming that the volume flow velocity is conserved we define the total air volume entering the system as

$$S(0)v(0, t) = U_B(t) + U_r(t). \quad (9.11)$$

## 9.3 Discrete time

Placing  $y$ ,  $\Delta p$ , and thereby  $U_B$  and  $U_r$  on the interleaved temporal grid (but on the non-interleaved spatial grid), we discretise the equations above to get the following system

$$\begin{cases} M_r \delta_{tt} y^{n+1/2} \\ \Delta p^{n+1/2} \end{cases} = -M_r \omega_0^2 \mu_t y^{n+1/2} - M_r \sigma_r \delta_t y^{n+1/2} + S_r \Delta p^{n+1/2} \quad (9.12a)$$

$$\begin{cases} \Delta p^{n+1/2} \\ U_B^{n+1/2} \end{cases} = P_m - \mu_t + p_0^n \quad (9.12b)$$

$$\begin{cases} U_B^{n+1/2} \\ U_r^{n+1/2} \end{cases} = w[y^{n+1/2} + H_0]_+ \operatorname{sgn}(\Delta p^{n+1/2}) \sqrt{\frac{2|\Delta p^{n+1/2}|}{\rho_0}} \quad (9.12c)$$

$$\begin{cases} U_r^{n+1/2} \\ \mu_{x-}(S_{1/2} v_{1/2}^{n+1/2}) \end{cases} = S_r \delta_t y^{n+1/2} \quad (9.12d)$$

$$\begin{cases} \mu_{x-}(S_{1/2} v_{1/2}^{n+1/2}) \\ U_B^{n+1/2} + U_r^{n+1/2} \end{cases} = U_B^{n+1/2} + U_r^{n+1/2} \quad (9.12e)$$

In the following we will suppress the superscript  $n+1/2$  for the aforementioned variables. Expanding and solving (9.12a) for  $y^{n+3/2}$  yields

$$\begin{aligned} \left(1 + \frac{\omega_0^2 k^2}{2} + \frac{\sigma_r k}{2}\right) y^{n+3/2} &= 2y^{n+1/2} - \left(1 + \frac{\omega_0^2 k^2}{2} - \frac{\sigma_r k}{2}\right) y^{n-1/2} + \frac{S_r k^2}{M_r} \Delta p \\ \alpha_r y^{n+3/2} &= 4y^{n+1/2} + \beta_r y^{n-1/2} + \xi_r \Delta p \end{aligned} \quad (9.13)$$

where

$$\alpha_r = 2 + \omega_0^2 k^2 + \sigma_r k, \quad \beta_r = \sigma_r k - 2 - \omega_0^2 k^2, \quad \text{and} \quad \xi_r = \frac{2S_r k^2}{M_r}. \quad (9.14)$$

### 9.3.1 Obtaining $\Delta p$

With all other parameters user-defined, the only unknown in our system is now  $\Delta p$ . Using the following identities

$$\delta_{tt} = \frac{2}{k}(\delta_{t\cdot} - \delta_{t-}), \quad \text{and} \quad \mu_{t\cdot} = k\delta_{t\cdot} + e_{t-} \quad (9.15)$$

where  $e_{t-}$  is a backwards time-shift of one sample (so  $e_{t-}y^{n+1/2} = y^{n-1/2}$ ) we can rewrite (9.12a) to

$$\begin{aligned} \frac{2}{k}(\delta_{t\cdot} - \delta_{t-})y &= -\omega_0^2(k\delta_{t\cdot} + e_{t-})y - \sigma_r\delta_{t\cdot}y + \frac{S_r}{M_r}\Delta p \\ a_1\delta_{t\cdot}y - a_2\Delta p - a_3^n &= 0, \end{aligned} \quad (9.16)$$

where

$$a_1 = \frac{2}{k} + \omega_0^2k + \sigma_r \geq 0, \quad a_2 = \frac{S_r}{M_r} \geq 0, \quad \text{and} \quad a_3^n = \left( \frac{2}{k}\delta_{t\cdot} - \omega_0^2e_{t-} \right) y. \quad (9.17)$$

Note that because  $a_1$  and  $a_2$  are calculated solely from non-negative parameters we can apply the condition that these are greater than or equal to 0. The same will be done for other coefficients below. We can substitute Eq. (9.12d) into Eq. (9.16)

$$\frac{a_1}{S_r}U_r - a_2\Delta p - a_3^n = 0 \quad (9.18)$$

and consequently (9.12e) to get

$$\frac{a_1}{S_r} \left( \mu_{x-}(S_{1/2}v_{1/2}) - U_B \right) - a_2\Delta p - a_3^n = 0 \quad (9.19)$$

To get a definition for  $\mu_{x-}(S_{1/2}v_{1/2})$ , we include the expression for Webster's equation at  $l = 0$

$$\frac{\bar{S}_0}{\rho_0 c^2} \delta_{t+} p_0^n = -\delta_{x-}(S_{1/2}v_{1/2}), \quad (9.20)$$

which, using the following identity (derived from Eq. (2.7d) from [23])

$$\delta_{x\pm} = \pm \frac{2}{h}(\mu_{x\pm} - 1), \quad (9.21)$$

can be rewritten to

$$\frac{\bar{S}_0}{\rho_0 c^2} \delta_{t+} p_0^n = \frac{2}{h} \left( \mu_{x-}(S_{1/2}v_{1/2}) - S_{1/2}v_{1/2} \right). \quad (9.22)$$

Then using the same identity (9.21) but for  $\delta_{t+}$  we get

$$\frac{2\bar{S}_0}{\rho_0 c^2 k} (\mu_{t+} p_0^n - p_0^n) = \frac{2}{h} \left( \mu_{x-}(S_{1/2}v_{1/2}) - S_{1/2}v_{1/2} \right). \quad (9.23)$$

### 9.3. Discrete time

Using Eq. (9.12b) we can rewrite this to

$$\begin{aligned} \frac{\bar{S}_0 h}{\rho_0 c^2 k} (P_m - \Delta p - p_0^n) &= \frac{2}{h} \left( \mu_{x-} (S_{1/2} v_{1/2}) - S_{1/2} v_{1/2} \right). \\ \mu_{x-} (S_{1/2} v_{1/2}) &= b_1^n - b_2 \Delta p \end{aligned} \quad (9.24)$$

where

$$b_1^n = S_{1/2} v_{1/2} + \frac{\bar{S}_0 h}{\rho_0 c^2 k} (P_m - p_0^n), \quad \text{and} \quad b_2 = \frac{\bar{S}_0 h}{\rho_0 c^2 k} \geq 0. \quad (9.25)$$

We can then substitute Eqs. (9.24) and (9.12c) into Eq. (9.19) to get

$$\begin{aligned} \frac{a_1}{S_r} \left( b_1^n - b_2 \Delta p - w[y + H_0]_+ \operatorname{sgn}(\Delta p) \sqrt{\frac{2|\Delta p|}{\rho_0}} \right) - a_2 \Delta p - a_3^n &= 0, \\ -w[y + H_0]_+ \operatorname{sgn}(\Delta p) \sqrt{\frac{2|\Delta p|}{\rho_0}} - b_2 \Delta p - \frac{a_2 S_r}{a_1} \Delta p + b_1^n - \frac{a_3^n S_r}{a_1} &= 0, \\ -c_1^n \operatorname{sgn}(\Delta p) \sqrt{|\Delta p|} - c_2 \Delta p + c_3^n &= 0 \end{aligned} \quad (9.26)$$

where

$$c_1^n = w[y + H_0]_+ \sqrt{\frac{2}{\rho_0}} \geq 0, \quad c_2 = b_2 + \frac{a_2 S_r}{a_1} \geq 0, \quad \text{and} \quad c_3^n = b_1^n - \frac{a_3^n S_r}{a_1}. \quad (9.27)$$

We can then divide Eq. (9.26) by  $-\operatorname{sgn}(\Delta p)$  to get a quadratic equation in  $\sqrt{|\Delta p|}$

$$c_2 |\Delta p| + c_1^n \sqrt{|\Delta p|} - \frac{c_3^n}{\operatorname{sgn}(\Delta p)} = 0. \quad (9.28)$$

Now, as we know that  $c_1^n, c_2 \geq 0$ , for any real solutions to exist the following must be true

$$\operatorname{sgn}(c_3^n) = \operatorname{sgn}(\Delta p) \implies \frac{c_3^n}{\operatorname{sgn}(\Delta p)} = |c_3^n|. \quad (9.29)$$

Now we can solve for  $\sqrt{|\Delta p|}$ :

$$\sqrt{|\Delta p|} = \frac{-c_1^n \pm \sqrt{(c_1^n)^2 + 4c_2|c_3^n|}}{2c_2}. \quad (9.30)$$

As  $\sqrt{(c_1^n)^2 + 4c_2|c_3^n|} \geq c_1^n$ , we can only guarantee that the solution is positive if we take the positive solution the square root. Furthermore, using Eq. (9.29) we can solve for the pressure difference

$$\Delta p = \operatorname{sgn}(c_3^n) \left( \frac{-c_1^n + \sqrt{(c_1^n)^2 + 4c_2|c_3^n|}}{2c_2} \right)^2. \quad (9.31)$$

This we can then apply to the update of the lip reed in Eq. (9.12a).

### 9.3.2 Coupling to the tube

To couple the reed to the tube, we take Eq. (5.42a) at  $l = 0$  and rewrite it to

$$p_0^{n+1} = p_0^n - \frac{\rho_0 c \lambda}{S_0} \left( -2\mu_{x-}(S_{1/2}v_{1/2}) + 2S_{1/2}v_{1/2} \right), \quad (9.32)$$

and substitute Eq. (9.12e) to get

$$p_0^{n+1} = p_0^n - \frac{\rho_0 c \lambda}{S_0} \left( -2(U_B + U_r) + 2S_{1/2}v_{1/2} \right). \quad (9.33)$$

## 9.4 Energy analysis

We start by multiplying Eq. (9.12a) by  $\delta_t.y$  (superscript  $n + 1/2$  is again suppressed):

$$\begin{aligned} & \xrightarrow{\text{Eq. (9.12b)}} & M_r \delta_t.y \delta_{tt}y + M_r \omega_0^2 \delta_t.y \mu_t.y + M_r \sigma_r(\delta_t.y)^2 - S_r \delta_t.y \Delta p \\ & \xrightleftharpoons{\text{Eqs. (9.12d) \& (9.12e)}} & M_r \delta_t.y \delta_{tt}y + M_r \omega_0^2 \delta_t.y \mu_t.y + M_r \sigma_r(\delta_t.y)^2 - (\mu_{x-}(S_{1/2}v_{1/2}) - U_B) \Delta p \\ & \xrightleftharpoons{\text{Eq. (9.12b)}} & M_r \delta_t.y \delta_{tt}y + M_r \omega_0^2 \delta_t.y \mu_t.y + M_r \sigma_r(\delta_t.y)^2 + U_B \Delta p - \mu_{x-}(S_{1/2}v_{1/2})(P_m - \mu_{t+p_0}) \end{aligned}$$

Recalling that the energy of the tube  $\delta_{t+}\mathfrak{h}_t = \mathfrak{b}_r + \mathfrak{b}_l$  and  $\mathfrak{b}_l = -(\mu_{t+p_0})\mu_{x-}(S_{1/2}v_{1/2})$  we get, assuming that  $\mathfrak{b}_r = 0$

$$\xrightleftharpoons{\text{Eq. (5.54)}} M_r \delta_t.y \delta_{tt}y + M_r \omega_0^2 \delta_t.y \mu_t.y + M_r \sigma_r(\delta_t.y)^2 + U_B \Delta p - \mu_{x-}(S_{1/2}v_{1/2})P_m + \delta_{t+}\mathfrak{h}_t$$

Then we arrive at the following energy balance

$$\delta_{t+}(\mathfrak{h}_t + \mathfrak{h}_r) + \mathfrak{Q}_r + \mathfrak{p}_r = 0 \quad (9.34)$$

where

$$\mathfrak{h}_r = \frac{M_r}{2} ((\delta_t.y)^2 + \omega_0^2 \mu_{t-}(y^2)) \geq 0 \quad (9.35)$$

$$\mathfrak{Q}_r = M_r \sigma_r(\delta_t.y)^2 + U_B \Delta p \geq 0 \quad (9.36)$$

$$\mathfrak{p}_r = -(U_B + U_r)P_m \quad (9.37)$$

# **Part IV**

# **Interactions**



# Interactions

The models described in Part II can already sound quite convincing on their own. However, most musical instruments are composed of a combination of these individual resonators and need to be combined to approximate fully functional (virtual) instruments. The following chapters will describe different ways of interaction between individual systems. Chapter 10 describes various ways to connect different systems and Chapter 11 describes collision interactions between models.

$\varepsilon$   
Newtons third law (action reaction)



# Chapter 10

## Connections

Part II introduced various resonators in isolation. Most traditional musical instruments are made up of several of these systems which all interact with one another

Although briefly mentioned in the previous chapter in Section 11.3,

One can use connections to simulate a point-like damping finger on a string to create different pitches.

Pointlike  $\delta(x - x_c)$  or distributed  $E_c$

When connecting systems, notation becomes extremely important. Subscripts will be extensively used throughout this chapter to denote what variables belong to what system. Although this results in something of a notational jungle, it is better to be explicit and avoid confusion.

[51] presents a way to explicitly solve an arbitrary system of connections by writing everything in an extremely compact matrix form...

Before moving on to connected systems, interpolation and spreading operators in 2D will be presented.

### 10.1 Interpolation and spreading in 2D

One can extend the interpolation and spreading operators presented in Section 8.1 to 2D by adding an additional argument to the operators [23]. Using  $l_i = \lfloor x_i/h \rfloor$  and  $m_i = \lfloor y_i/h \rfloor$ , a 0<sup>th</sup>-order interpolation operator  $I_0(x_i, y_i) = I_{(l,m),0}(x_i, y_i)$  is defined as

$$I_0(x_i, y_i) = \begin{cases} 1, & \text{if } l = l_i \text{ and } m = m_i, \\ 0, & \text{otherwise.} \end{cases} \quad (10.1)$$

Notice that the same value for the grid spacing  $h$  is used for both the  $x$  and  $y$  direction.

Using the fractional part of the flooring operations  $\alpha_x = x_i/h - l_i$  and  $\alpha_y = y_i/h - m_i$ , a 2D linear interpolator  $I_1(x_i, y_i) = I_{(l,m),1}(x_i, y_i)$  can then be composed of two 1D linear interpolators as

$$I_1(x_i, y_i) = \begin{cases} (1 - \alpha_x)(1 - \alpha_y) & \text{if } l = l_i \text{ and } m = m_i, \\ (1 - \alpha_x)\alpha_y & \text{if } l = l_i \text{ and } m = m_i + 1, \\ \alpha_x(1 - \alpha_y) & \text{if } l = l_i + 1 \text{ and } m = m_i, \\ \alpha_x\alpha_y & \text{if } l = l_i + 1 \text{ and } m = m_i + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (10.2)$$

Spreading operators are defined in the same way as in Section 8.1. A 0<sup>th</sup>-order spreading operator  $J_0(x_i, y_i) = J_{(l,m),0}(x_i, y_i)$  can be defined as

$$J_0(x_i, y_i) = \frac{1}{h^2} \begin{cases} 1, & \text{if } l = l_i \text{ and } m = m_i, \\ 0, & \text{otherwise,} \end{cases} \quad (10.3)$$

as well as a linear spreading operator  $J_1(x_i, y_i) = J_{(l,m),1}(x_i, y_i)$  as

$$J_1(x_i, y_i) = \frac{1}{h^2} \begin{cases} (1 - \alpha_x)(1 - \alpha_y) & \text{if } l = l_i \text{ and } m = m_i, \\ (1 - \alpha_x)\alpha_y & \text{if } l = l_i \text{ and } m = m_i + 1, \\ \alpha_x(1 - \alpha_y) & \text{if } l = l_i + 1 \text{ and } m = m_i, \\ \alpha_x\alpha_y & \text{if } l = l_i + 1 \text{ and } m = m_i + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (10.4)$$

Notice that the scaling is by  $1/h^2$  (due to the 2D system) rather than  $1/h$  in the 1D case. Some intuition on this will be given below.

As in the 1D case, the spreading operator approximates a spatial Dirac delta function which in 2D is defined as

$$\delta(x, y) = \begin{cases} \infty, & x = y = 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) dx dy = 1. \quad (10.5)$$

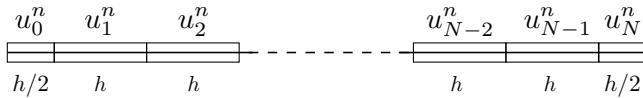
where  $\delta(x, y)$  has units of  $m^{-2}$ . Again, as described in Section 8.1, this definition will be approximated by spreading operators, rather than be used directly.

### Alternative interpretation of grid points

Section 2.2.1 gives an introduction to how a continuous 1D system is subdivided into grid points in space (see Figure 2.2) in a process called discretisation. An alternative way to see grid points after discretisation is shown in Figure 10.1. Rather than grid ‘points’ with a spacing  $h$  between them, a continuous system is divided into grid ‘sections’ of width  $h$ . This interpretation allows the ‘weight’ of a grid point to be calculated from its material properties and

geometry. Notice that boundaries have a width of  $h/2$  such that the total length  $L = Nh$  m.

As an example, the weight of one grid point (or now rather grid section) of a string can be calculated as  $\rho Ah$ . The weight of one grid point of a 2D plane can be calculated as  $\rho Hh^2$ . As these grid points interact with each other, the forces resulting from this interaction will scaled by their respective weight per grid point as will be shown in Section 10.5. This interpretation hopefully provides a better intuition for the interactions between components shown in this chapter.



**Fig. 10.1:** Alternative interpretation of the discretisation of  $u(x, t)$ . The continuous system is divided into  $N - 1$  sections of width  $h$  plus 2 sections of width  $h/2$  at the boundaries. Through this interpretation, the ‘weight’ of a grid point can be calculated from its physical parameters.

## 10.2 Connected ideal strings

As a test case for the following sections, consider two ideal strings of length  $L_u$  and  $L_w$  (both in m) their transverse displacement denoted as  $u = u(x, t)$  and  $w = w(\chi, t)$  (both in m) (see Section 2.4).<sup>1</sup> The systems are defined for  $x \in \mathcal{D}_u$  with domain  $\mathcal{D}_u = [0, L_u]$  and  $\chi \in \mathcal{D}_w$  with domain  $\mathcal{D}_w = [0, L_w]$  respectively. Notice that  $\chi$  is used as the spatial coordinate for  $w$  to denote that the two systems use different coordinate systems. Connecting these systems at  $x_c \in \mathcal{D}_u$  and  $\chi_c \in \mathcal{D}_w$  yields the following system of PDEs:

$$\rho_u A_u \partial_t^2 u = T_u \partial_x^2 u - \delta(x - x_c) f, \quad (10.6a)$$

$$\rho_w A_w \partial_t^2 w = T_w \partial_\chi^2 w + \delta(\chi - \chi_c) f, \quad (10.6b)$$

where subscripts  $u$  and  $w$  denote whether a variable belongs to system  $u$  or  $w$  respectively. Notice that the  $\partial_\chi$  in Eq. (10.6b) denotes a partial derivative with respect to  $\chi$  and is an identical operation to  $\partial_x$ , but on a different coordinate system. Furthermore,  $f = f(t)$  is the connection force (in N) which should be equal and opposite for the connected systems (hence the inverse signs). The definition for  $f$  depends on the connection type, and two alternatives will be given shortly. Finally, the spatial Dirac delta function  $\delta$  is defined as in Eq. (8.8), and localises the connection force along the systems.

---

<sup>1</sup>Recall that the ideal string is the 1D wave equation with  $c = \sqrt{T/\rho A}$ .

### 10.2.1 Discrete time

One can then discretise the state variables  $u$  and  $w$  to grid functions  $u_l^n$  and  $w_m^n$  using  $x = lh_u$  and  $\chi = mh_w$  where  $l \in \{0, \dots, N_u\}$  and  $m \in \{0, \dots, N_w\}$ .<sup>2</sup> Also see Section 2.2.1. Furthermore,  $h_u$  and  $h_w$  are the values of the grid spacing (both in m) and  $N_u + 1$  and  $N_w + 1$  are the number of grid points for  $u_l^n$  and  $w_m^n$  respectively. Dividing Eqs. (10.6a) and (10.6b) by  $\rho_u A_u$  and  $\rho_w A_w$  respectively yields

$$\delta_{tt} u_l^n = c_u^2 \delta_{xx} u_l^n - J_{l,u}(x_c) \frac{f^n}{\rho_u A_u}, \quad (10.7a)$$

$$\delta_{tt} w_m^n = c_w^2 \delta_{\chi\chi} w_m^n + J_{m,w}(\chi_c) \frac{f^n}{\rho_w A_w}, \quad (10.7b)$$

where  $c_u = \sqrt{T_u / \rho_u A_u}$  and  $c_w = \sqrt{T_w / \rho_w A_w}$ . The spreading operators  $J_{l,u}(x_c) = J_{l,o_u,u}(x_c)$  and  $J_{l,w}(\chi_c) = J_{l,o_w,w}(\chi_c)$  are as defined in Section 8.1, and their orders  $o_u$  and  $o_w$  are left unspecified.

The next step would be to solve for connection force  $f^n$ . First, one needs to isolate the schemes in system (10.7) at their respective connection locations  $x_c$  and  $\chi_c$ . This is done by taking an inner product of each scheme in system (10.7) with their respective spreading operator  $J_{l,u}$  and  $J_{l,w}$  over discrete domains  $d_u = \{0, \dots, N_u\}$  and  $d_w = \{0, \dots, N_w\}$  respectively. Using identity (8.9) one can write

$$I_{l,u}(x_c) \delta_{tt} u_l^n = c_u^2 I_{l,u}(x_c) \delta_{xx} u_l^n - \|J_{l,u}(x_c)\|_{d_u}^2 \frac{f^n}{\rho_u A_u}, \quad (10.8a)$$

$$I_{m,w}(\chi_c) \delta_{tt} w_m^n = c_w^2 I_{m,w}(\chi_c) \delta_{\chi\chi} w_m^n + \|J_{m,w}(\chi_c)\|_{d_w}^2 \frac{f^n}{\rho_w A_w}, \quad (10.8b)$$

Here, interpolation operators  $I_{l,u}(x_c) = I_{l,o_u,u}(x_c)$  and  $I_{m,w}(\chi_c) = I_{l,o_w,w}(\chi_c)$  are as defined in Section 8.1. Notice that the order of these operators need to match their ‘dual’ spreading operator, but the orders  $o_u$  and  $o_w$  may differ.

The definition of the force depends on the connection type. Below, two alternatives will be presented: the rigid connection and the spring connection.

## 10.3 Rigid connection

The simplest connection-type is the *rigid connection*. This connection type states that the displacement of two connected points should always be equal, and thus the distance between them should be 0 at all times. For the rigid connection, the following is true

$$u(x_c, t) = w(\chi_c, t), \quad (10.9)$$

---

<sup>2</sup>Here,  $m$  is used for the spatial index of  $w_m^n$  to avoid double subscripts  $l_u$  and  $l_w$ .

### 10.3. Rigid connection

which in discrete time becomes

$$I_{l,u}(x_c)u_l^n = I_{m,w}(\chi_c)w_m^n. \quad (10.10)$$

If Eq. (10.10) is true, the following must also hold:

$$I_{l,u}(x_c)\delta_{tt}u_l^n = I_{m,w}(\chi_c)\delta_{tt}w_m^n. \quad (10.11)$$

In other words, if the displacement of two objects is equal, their acceleration must also be. This definition can then immediately be used to solve for  $f^n$  and the right-hand sides in system (10.8) can be substituted in Eq. (10.11) to get

$$c_u^2 I_{l,u}(x_c) \delta_{xx} u_l^n - \|J_{l,u}(x_c)\|_{d_u}^2 \frac{f^n}{\rho_u A_u} = c_w^2 I_{m,w}(\chi_c) \delta_{\chi\chi} w_m^n + \|J_{m,w}(\chi_c)\|_{d_w}^2 \frac{f^n}{\rho_w A_w},$$

which can be explicitly solved for  $f^n$  according to

$$f^n = \frac{c_u^2 I_{l,u}(x_c) \delta_{xx} u_l^n - c_w^2 I_{m,w}(\chi_c) \delta_{\chi\chi} w_m^n}{\frac{\|J_{l,u}(x_c)\|_{d_u}^2}{\rho_u A_u} + \frac{\|J_{m,w}(\chi_c)\|_{d_w}^2}{\rho_w A_w}}. \quad (10.12)$$

This value can then be used in the update equation obtained after expanding system (10.7) as

$$u_l^{n+1} = (2 - 2\lambda_u^2) u_l^n + \lambda_u^2 (u_{l+1}^n + u_{l-1}^n) - u_l^{n-1} - J_{l,u}(x_c) \frac{k^2 f^n}{\rho_u A_u} \quad (10.13a)$$

$$w_m^{n+1} = (2 - 2\lambda_w^2) w_m^n + \lambda_w^2 (w_{m+1}^n + w_{m-1}^n) - w_m^{n-1} + J_{m,w}(\chi_c) \frac{k^2 f^n}{\rho_w A_w} \quad (10.13b)$$

where  $\lambda_u = c_u k / h_u \leq 1$  and  $\lambda_w = c_w k / h_w \leq 1$  are the Courant numbers for each individual scheme (see Section 2.4).

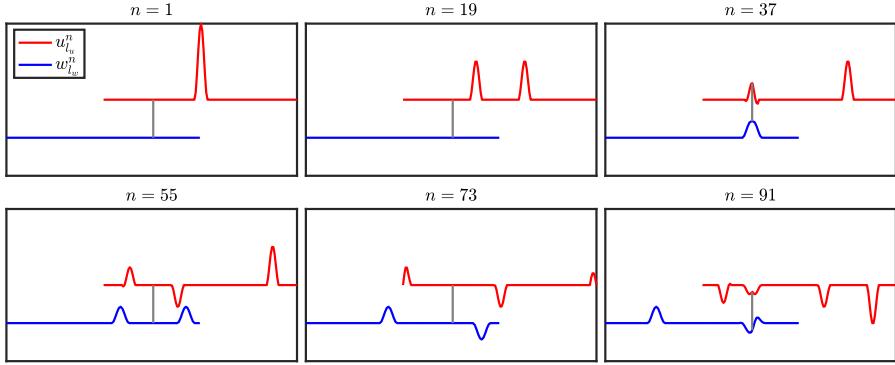
Figure 10.2 shows an implementation of system (10.7) with  $x_c = 0.25$  m and  $\chi_c = 0.75$  m. The wave equations have the same mass per unit length, i.e.,  $\rho_u A_u = \rho_w A_w$ , and the same length  $L_u = L_w = 1$  m, but operate at different wave speeds  $c_u = 300$  m/s and  $c_w = 400$  m/s.

### Notation simplification

In the above equations, the orders of the spreading and interpolation operators have been left unspecified to retain generality. If one would like to connect two systems at specified grid points without needing to worry about interpolation, the notation can greatly simplified.

Recalling that  $I_{l,u}(x_c) = I_{l,o_u,u}(x_c)$  and  $I_{m,w}(\chi_c) = I_{l,o_w,w}(\chi_c)$ , one can set the interpolation orders to 0, i.e.,  $o_u = o_w = 0$ , to yield the following short-hand notations

$$I_{l,0,u}(x_c)u_l^n = u_{l_c}^n, \quad \text{and} \quad I_{m,0,w}(\chi_c)w_m^n = w_{m_c}^n, \quad (10.14)$$



**Fig. 10.2:** The wave propagation of two connected 1D wave equations with the same mass per unit length. The systems are offset for clarity, but the relative displacement at the connection location is 0.

where  $l_c = \lfloor x_c/h_u \rfloor$  and  $m_c = \lfloor \chi_c/h_w \rfloor$ , and

$$\|J_{l,0,u}(x_c)\|_{d_u}^2 = \frac{1}{h_u}, \quad \text{and} \quad \|J_{m,0,w}(\chi_c)\|_{d_w}^2 = \frac{1}{h_w}. \quad (10.15)$$

This simplifies Eqs. (10.8) to

$$\delta_{tt} u_{l_c}^n = c_u^2 \delta_{xx} u_{l_c}^n - \frac{f^n}{\rho_u A_u h_u}, \quad (10.16a)$$

$$\delta_{tt} w_{m_c}^n = c_w^2 \delta_{\chi\chi} w_{m_c}^n + \frac{f^n}{\rho_w A_w h_w}, \quad (10.16b)$$

which, after rewriting Eq. (10.11) to

$$\delta_{tt} u_{l_c}^n = \delta_{tt} w_{m_c}^n, \quad (10.17)$$

one can solve for  $f^n$ , yielding the following simplified form of Eq. (10.12)

$$f^n = \frac{c_u^2 \delta_{xx} u_{l_c}^n - c_w^2 \delta_{\chi\chi} w_{m_c}^n}{\frac{1}{\rho_u A_u h_u} + \frac{1}{\rho_w A_w h_w}}. \quad (10.18)$$

### 10.3.1 Energy Analysis

This section follows the energy analysis techniques shown in Section 3.4 (though not explicitly following Steps 1-4). As the analysis has previously been performed on the 1D wave equation, this part of the analysis will not be detailed here.

Starting with the FD scheme in Eq. (10.7a), one can take an inner product of the scheme (after a multiplication with  $\rho_u A_u$ ) with  $(\delta_t u_l^n)$  over discrete domain  $d_u$  to get

$$\delta_{t+} \mathfrak{h}_u = \langle \delta_t u_l^n, -J_{l,u}(x_c) f^n \rangle_{d_u}, \quad (10.19)$$

### 10.3. Rigid connection

where  $\mathfrak{h}_u$  is the total energy in system  $u$  and is as defined in Eq. (3.47). The same can be done for Eq. (10.7b) (after a multiplication with  $\rho_w A_w$ ) by taking an inner product with  $(\delta_t \cdot w_m^n)$  over discrete domain  $d_w$  to get

$$\delta_{t+} \mathfrak{h}_w = \langle \delta_t \cdot w_m^n, J_{m,w}(\chi_c) f^n \rangle_{d_w}, \quad (10.20)$$

where  $\mathfrak{h}_w$  is the total energy in system  $w$ . As the total energy in the system is an addition of  $\mathfrak{h}_u$  and  $\mathfrak{h}_w$ , and using identity (8.9) for the right hand sides of Eqs. (10.19) and (10.20), one can write

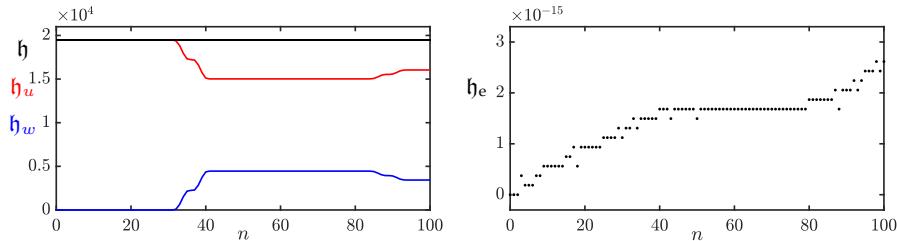
$$\delta_{t+} (\mathfrak{h}_u + \mathfrak{h}_w) = -I_{l,u}(x_c) \delta_t \cdot u_l^n f^n + I_{m,w}(\chi_c) \delta_t \cdot w_m^n f^n. \quad (10.21)$$

Finally, due to the rigid connection in Eq. (10.10),  $I_{l,u}(x_c) \delta_t \cdot u_l^n = I_{m,w}(\chi_c) \delta_t \cdot w_m^n$  (if the displacements are equal, their velocities must also be) and the right hand side vanishes:

$$\delta_{t+} (\mathfrak{h}_u + \mathfrak{h}_w) = 0.$$

This shows that the rigid connection does not affect the total energy in the system and thus does not affect the stability of the scheme.

Figure 10.3 shows the energy of an implementation of the 1D wave system in (10.7). The



**Fig. 10.3:** The energy of  $u$  (blue), the energy of  $w$  (red), and the total (black) energy of the system of connected 1D wave equations in (10.7). The energy corresponds to Figure 10.2. The right panel shows the normalised energy (according to Eq. (3.37)) shows that the deviation of the energy is within machine precision.

#### 10.3.2 Modal analysis and matrix form

might want to remove this entire section.. In order to perform a modal analysis on the system, one must use the matrix form of the system in Eq. (10.23).

As the schemes are connected, the matrix form is slightly more involved. To start, the definition of the force in Eq. (10.12) must be substituted into the

system, which, after expansion of the left hand side of the system, becomes

$$u_l^{n+1} = 2u_l^n - u_l^{n-1} + c^2 k^2 \delta_{xx} u_l^n - J_{l,u}(x_c) \frac{k^2}{\rho_u A_u} \left( \frac{c_u^2 I_{l,u}(x_c) \delta_{xx} u_l^n - c_w^2 I_{m,w}(\chi_c) \delta_{\chi\chi} w_m^n}{\frac{\|J_{l,u}(x_c)\|_{d_u}^2}{\rho_u A_u} + \frac{\|J_{m,w}(\chi_c)\|_{d_w}^2}{\rho_w A_w}} \right), \quad (10.22a)$$

$$w_m^{n+1} = 2w_m^n - w_m^{n-1} + c^2 k^2 \delta_{xx} w_m^n + J_{m,w}(\chi_c) \frac{k^2}{\rho_w A_w} \left( \frac{c_w^2 I_{l,u}(x_c) \delta_{xx} u_l^n - c_w^2 I_{m,w}(\chi_c) \delta_{\chi\chi} w_m^n}{\frac{\|J_{l,u}(x_c)\|_{d_u}^2}{\rho_u A_u} + \frac{\|J_{m,w}(\chi_c)\|_{d_w}^2}{\rho_w A_w}} \right). \quad (10.22b)$$

Using Dirichlet boundary conditions for both ideal strings, their values can be stored in the following vectors

$$\mathbf{u}^n = [u_1^n, \dots, u_{N_u-1}^n]^T, \quad \text{and} \quad \mathbf{w}^n = [w_1^n, \dots, w_{N_w-1}^n]^T.$$

These vectors can then be concatenated to one larger state vector and after the terms in Eqs. (10.22) are grouped by the grid functions at various time indices, one obtains the following compact matrix form of system (10.7):

$$\begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{w}^{n+1} \end{bmatrix} = \mathbf{B} \begin{bmatrix} \mathbf{u}^n \\ \mathbf{w}^n \end{bmatrix} - \begin{bmatrix} \mathbf{u}^{n-1} \\ \mathbf{w}^{n-1} \end{bmatrix} \quad (10.23)$$

where

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_u & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_w \end{bmatrix} + \begin{bmatrix} -\mathbf{j}_u \\ \mathbf{j}_w \end{bmatrix} [\mathbf{f}_u \quad -\mathbf{f}_w].$$

The first matrix in the definition of  $\mathbf{B}$  contains the operations of the 1D wave equation (also see Eq. (3.5)),

$$\mathbf{B}_u = 2\mathbf{I}_{N_u-1} + c_u^2 k^2 (\mathbf{D}_{xx})_u \quad \text{and} \quad \mathbf{B}_w = 2\mathbf{I}_{N_w-1} + c_w^2 k^2 (\mathbf{D}_{xx})_w \quad (10.24)$$

where matrices  $(\mathbf{D}_{xx})_u$  and  $(\mathbf{D}_{xx})_w$  are as defined in Eq. (3.3) and are of the appropriate sizes. The vector multiplication in the definition of  $\mathbf{B}$  results in a matrix, and adds the effect of the connection force to the system. Here,  $\mathbf{j}_u$  and  $\mathbf{j}_w$  are column vectors of size  $(N_u - 1) \times 1$  and  $(N_w - 1) \times 1$  containing the values of the spreading operators  $J_{l,u}(x_c)$  and  $J_{m,w}(\chi_c)$  respectively. Finally,

$$\mathbf{f}_u = \frac{k^2}{\rho_u A_u} \left( \frac{c_u^2 \mathbf{i}_u (\mathbf{D}_{xx})_u}{\frac{\mathbf{i}_u \mathbf{j}_u}{\rho_u A_u} + \frac{\mathbf{i}_w \mathbf{j}_w}{\rho_w A_w}} \right), \quad \mathbf{f}_w = \frac{k^2}{\rho_w A_w} \left( \frac{c_w^2 k^2 \mathbf{i}_w (\mathbf{D}_{xx})_w}{\frac{\mathbf{i}_u \mathbf{j}_u}{\rho_u A_u} + \frac{\mathbf{i}_w \mathbf{j}_w}{\rho_w A_w}} \right),$$

where  $\mathbf{i}_u$  and  $\mathbf{i}_w$  are row vectors of size  $1 \times (N_u - 1)$  and  $1 \times (N_w - 1)$  containing the values of the interpolation operators  $I_{l,u}(x_c)$  and  $I_{m,w}(\chi_c)$  respectively. Finally,  $\mathbf{i}_u \mathbf{j}_u$  and  $\mathbf{i}_w \mathbf{j}_w$  are matrix-vector forms of  $\|J_{l,u}(x_c)\|_d^2$  and  $\|J_{m,w}(\chi_c)\|_d^2$  respectively (see Eq. (8.10)) and reduce to a scalar. [check with stefan if this is a correct notation for this](#)

## 10.4. Spring connection

Equation (10.23) can then easily be rewritten in one-step form and used for modal analysis.<sup>3</sup> Following Section 3.5.1, one can rewrite Eq. (10.23) to

$$\underbrace{\begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{w}^{n+1} \\ \mathbf{u}^n \\ \mathbf{w}^n \end{bmatrix}}_{\mathbf{v}^{n+1}} = \underbrace{\begin{bmatrix} \mathbf{B} & -\mathbf{I} \\ \mathbf{I}, & \mathbf{0} \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \mathbf{u}^n \\ \mathbf{w}^n \\ \mathbf{u}^{n-1} \\ \mathbf{w}^{n-1} \end{bmatrix}}_{\mathbf{v}^n}, \quad (10.25)$$

one can insert a test solution  $\mathbf{v}^n = z^n \phi$  to get

$$z\phi = Q\phi,$$

and solved for the  $p^{\text{th}}$  eigenvalue by following the process in Section 3.5.1.

figure?

## 10.4 Spring connection

An alternative connection type is the *spring connection*. As in the rigid case, forces are still equal and opposite, but spring connections allow the relative displacement between the two connected elements to be non-zero. This relative displacement is used to determine the connection force. Interestingly, nonlinearities can be added in a fairly straightforward manner, after which the system can still be solved explicitly. The most complex springs used in this project have a linear and a nonlinear component, as well as a damping term. For ease of explanation, this section will only use a linear spring. A damped nonlinear spring will appear in Section 10.5.

wording

The force between two components connected by a linear spring can be defined as

$$f = f(t) = K\eta, \quad (10.26)$$

where  $K \geq 0$  is the spring constant (in N/m) and

$$\eta = \eta(t) = u(x_c, t) - w(\chi_c, t) \quad (10.27)$$

is the relative displacement between the two systems at their respective connection locations (in m).

In discrete time, Eq. (10.26) becomes

$$f^n = K\mu_t \cdot \eta^n, \quad (10.28)$$

where

$$\eta^n = I_{l,u}(x_c)u_l^n - I_{m,w}(\chi_c)w_m^n. \quad (10.29)$$

---

<sup>3</sup>Note that due to the lack of damping in the system, the system can be analysed without using a one-step form. To retain generality, the latter is used here.

Here, the centred averaging operator is used for stability (see Section 10.4.2), but when substituted into Eq. (10.8) seems to make the system implicit. However, one can find an explicit solution, even for an arbitrary amount of connections [51]. These systems are therefore referred to as being *semi-implicit*, and an explicit solution will be shown below.

### 10.4.1 Explicit solution

When compared to the rigid connection in Section 10.3, solving for  $f^n$  requires an extra step. After isolating the schemes at their respective connection locations – resulting in Eqs. (10.8) – one needs to expand the scheme and solve for the states at  $n + 1$ :

$$I_{l,u}(x_c)u_l^{n+1} = u^* - \|J_{l,u}(x_c)\|_{d_u}^2 \frac{k^2 f^n}{\rho_u A_u}, \quad (10.30a)$$

$$I_{m,w}(\chi_c)w_m^{n+1} = w^* + \|J_{m,w}(\chi_c)\|_{d_w}^2 \frac{k^2 f^n}{\rho_w A_w}, \quad (10.30b)$$

where

$$u^* = I_{l,u}(x_c)(2u_l^n - u_l^{n-1}) + c_u^2 k^2 I_{l,u}(x_c) \delta_{xx} u_l^n,$$

and

$$w^* = I_{m,w}(\chi_c)(2w_m^n - w_m^{n-1}) + c_w^2 k^2 I_{m,w}(\chi_c) \delta_{xx} w_m^n,$$

are the update equations of the system at their respective connection locations without the term containing the connection force. Evaluating Eq. (10.29) at  $n + 1$  yields

$$\eta^{n+1} = I_{l,u}(x_c)u_l^{n+1} - I_{m,w}(\chi_c)w_m^{n+1},$$

into which Eqs. (10.30) can be substituted

$$\eta^{n+1} = u^* - \|J_{l,u}(x_c)\|_{d_u}^2 \frac{k^2 f^n}{\rho_u A_u} - \left( w^* + \|J_{m,w}(\chi_c)\|_{d_w}^2 \frac{k^2 f^n}{\rho_w A_w} \right). \quad (10.31)$$

A second definition for  $\eta^{n+1}$  can be obtained after expanding Eq. (10.28):

$$\eta^{n+1} = \frac{2f^n}{K} - \eta^{n-1} \quad (10.32)$$

and can be substituted into Eq. (10.31) to get

$$\frac{2f^n}{K} - \eta^{n-1} = u^* - \|J_{l,u}(x_c)\|_{d_u}^2 \frac{k^2 f^n}{\rho_u A_u} - \left( w^* + \|J_{m,w}(\chi_c)\|_{d_w}^2 \frac{k^2 f^n}{\rho_w A_w} \right). \quad (10.33)$$

Finally, one can group the terms for  $f^n$

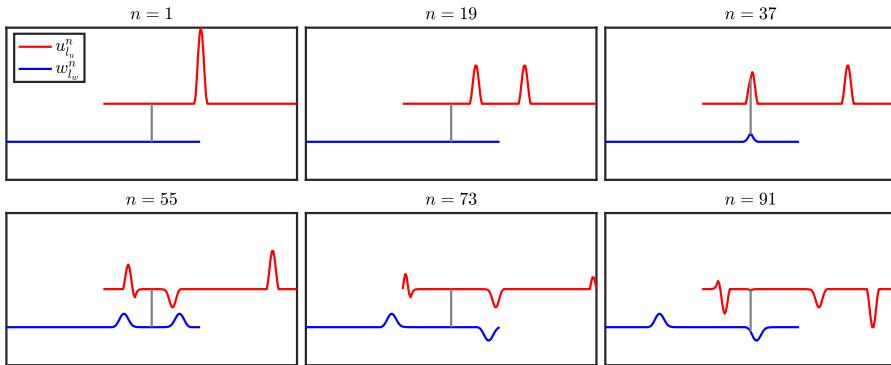
$$\left( \frac{2}{K} + \frac{\|J_{l,u}(x_c)\|_{d_u}^2 k^2}{\rho_u A_u} + \frac{\|J_{m,w}(\chi_c)\|_{d_w}^2 k^2}{\rho_w A_w} \right) f^n = u^* - w^* + \eta^{n-1}$$

## 10.4. Spring connection

and solve for the force, solely based on known values of the system

$$f^n = \frac{u^* - w^* + \eta^{n-1}}{\frac{2}{K} + \frac{\|J_{l,u}(x_c)\|_{d_u}^2 k^2}{\rho_u A_u} + \frac{\|J_{m,w}(\chi_c)\|_{d_w}^2 k^2}{\rho_w A_w}}. \quad (10.34)$$

Figure 10.4 shows the wave propagation of the system connected with a spring with spring constant  $K = 5 \cdot 10^4$  N/m. The same parameters and excitation are used as for the rigid connection in Section 10.3. When compared to the wave propagation of the system with a rigid connection in Figure 10.2, one can observe that, the transfer of energy from  $u$  to  $w$  is slower than in the rigid case due to the extension of the spring.



**Fig. 10.4:** The wave propagation of two ideal strings with the same mass per unit length connected using a spring with  $K = 5 \cdot 10^4$  N/m.

### 10.4.2 Energy analysis

As springs store energy themselves, it needs to be shown that stability is maintained. That the discretisation of the spring force chosen in Eq. (10.28) is inherently stable will be shown below.

Following the same process as Section 10.3.1, one can analyse system (10.7), and arrive at Eq. (10.21):

$$\delta_{t+}(\mathfrak{h}_u + \mathfrak{h}_w) = -I_{l,u}(x_c)\delta_t.u_l^n f^n + I_{m,w}(\chi_c)\delta_t.w_m^n f^n,$$

which can be rewritten to

$$\delta_{t+}(\mathfrak{h}_u + \mathfrak{h}_w) = -\delta_t. (I_{l,u}(x_c)u_l^n - I_{m,w}(\chi_c)w_m^n) f^n.$$

One can then substitute the definitions for  $f^n$  and  $\eta^n$  from Eqs. (10.28) and (10.29) to get

$$\delta_{t+}(\mathfrak{h}_u + \mathfrak{h}_w) = -K(\delta_t.\eta^n)(\mu_t.\eta^n), \quad (10.35)$$

which, using identity (3.17d), can be rewritten to

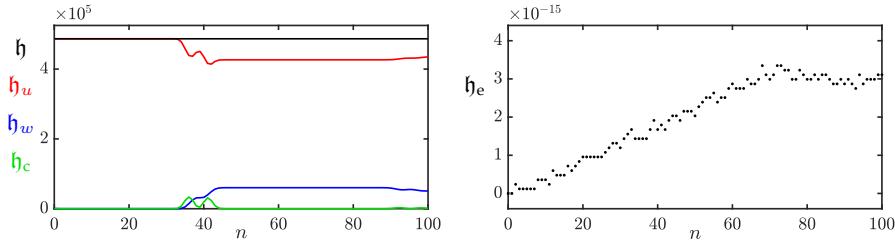
$$\delta_{t+}(\mathfrak{h}_u + \mathfrak{h}_w + \mathfrak{h}_c) = 0, \quad (10.36)$$

where

$$\mathfrak{h}_c = \frac{K}{2} (\mu_{t-}(\eta^n)^2) \quad (10.37)$$

is the energy stored by the connection. As this definition is non-negative it is inherently stable. The spring stiffness  $K$  could potentially be infinitely large, which would effectively reduce the spring connection to a rigid connection.

Figure 10.5 shows the energy of an implementation of system (10.7) connected with a spring with stiffness  $K = 5 \cdot 10^4$  and other parameters are the same as for the rigid connection in Section 10.3. One can observe that when compared to the energetic behaviour of the system with a rigid connection in Figure 10.3, less energy is transferred from  $u$  to  $w$ , and some energy is stored in the spring shown in green.



**Fig. 10.5:** The energy of  $u$  (blue), the energy of  $w$  (red), the energy of the spring (green) and the total energy (black) of the system of connected 1D wave equations in (10.7). The energy corresponds to the system in Figure 10.4. The right panel shows the normalised energy (according to Eq. (3.37)) shows that the deviation of the energy is within machine precision.

### Unstable discretisation

To show why an averaging operator is used in Eq. (10.28), consider a more straightforward discretisation of Eq. (10.26) without the averaging operator, such that

$$f^n = K\eta^n.$$

Performing an energy analysis of the system would yield

$$\delta_{t+}(\mathfrak{h}_u + \mathfrak{h}_w) = -K(\delta_{t+}\eta^n)(\eta^n),$$

(instead of Eq. (10.35)) and using identity (3.17b), this can be rewritten to

$$\delta_{t+}(\mathfrak{h}_u + \mathfrak{h}_w + \mathfrak{h}_c) = 0,$$

## 10.5. String-plate connection

where

$$\mathfrak{h}_c = \frac{K}{2} (\eta^n e_t - \eta^n).$$

As this is not necessarily non-negative, the connection places a larger restriction on the stability of the system at the connection location. In other words,  $\lambda \leq 1$  for the ideal strings does not ensure stability at the connection location. Also see [23, pp. 190–192].

## 10.5 String-plate connection

As an example of a more complicated connected system used in papers [A] and [B], consider a stiff string connected to a plate using a nonlinear damped spring. This could be interpreted as a simplified form of how for a stringed instrument the string would be connected to the body. In the following, subscripts ‘s’ and ‘p’ are used to denote a string or plate parameter respectively.

### 10.5.1 Continuous time

Consider a damped stiff string of length  $L$  (in m), its transverse displacement described by  $u = u(\chi, t)$  (in m) defined for  $t \geq 0$  and  $\chi \in \mathcal{D}_s$  where domain  $\mathcal{D}_s = [0, L]$ . Its PDE is described as (see Eq. (4.4))

$$\rho_s A \partial_t^2 u = T \partial_{\chi\chi} u - E_s I \partial_{\chi\chi\chi} u - 2\sigma_{0,s} \rho_s A \partial_t u + 2\sigma_{1,s} \rho_s A \partial_t \partial_{\chi\chi} u, \quad (10.38)$$

where parameters are as in Eq. (4.4).

The transverse displacement of a damped rectangular thin plate of side lengths  $L_x$  and  $L_y$  (both in m) can be described as  $w = w(x, y, t)$  (in m), which is defined for  $t \geq 0$  and  $(x, y) \in \mathcal{D}_p$  where domain  $\mathcal{D}_p = [0, L_x] \times [0, L_y]$ . Its PDE is defined as (see Eq. (6.38))

$$\rho_p H \partial_t^2 w = -D \Delta \Delta w - 2\sigma_{0,p} \rho_p H \partial_t w + 2\sigma_{1,p} \rho_p H \partial_t \partial_x^2 w, \quad (10.39)$$

where parameters are as in Eq. (6.38).

One can connect the above PDEs by adding a localised connection force. After a division by  $\rho_s A$  and  $\rho_p H$  respectively the connected string-plate system becomes

$$\partial_t^2 u = c^2 \partial_{\chi\chi} u - \kappa_s^2 \partial_{\chi\chi\chi} u - 2\sigma_{0,s} \partial_t u + 2\sigma_{1,s} \partial_t \partial_{\chi\chi} u - \delta(\chi - \chi_c) \frac{f}{\rho_s A}, \quad (10.40a)$$

$$\partial_t^2 w = -\kappa_p^2 \Delta \Delta w - 2\sigma_{0,p} \partial_t w + 2\sigma_{1,p} \partial_t \partial_x^2 w + \delta(x - x_c, y - y_c) \frac{f}{\rho_p H}, \quad (10.40b)$$

where  $\delta(\chi - \chi_c)$  (in  $\text{m}^{-1}$ ) and  $\delta(x - x_c, y - y_c)$  (in  $\text{m}^{-2}$ ) are the 1D and 2D spatial Dirac delta functions as defined in Eqs. (8.8) and (10.5) respectively and locate

the connection force at  $\chi_c \in \mathcal{D}_s$  (in m) along the string and  $(x_c, y_c) \in \mathcal{D}_p$  (in (m, m)) on the plate.

The force between the two components (in N) is set to be a nonlinear damped spring defined as

$$f = f(t) = K_1 \eta + K_3 \eta^3 + R \dot{\eta}, \quad (10.41)$$

with linear and nonlinear spring coefficients  $K_1$  (in N/m) and  $K_3$  (in N/m<sup>3</sup>) and damping coefficient  $R$  (in kg/s). Furthermore, the distance (in m) between the string and the plate at their respective connection locations is defined as

$$\eta = \eta(t) = u(\chi_c, t) - w(x_c, y_c, t). \quad (10.42)$$

### 10.5.2 Discrete time

To discretise  $u(\chi, t)$ , one can use grid function  $u_q^n$  where  $n \in \mathbb{N}^0$  and  $q \in \{0, \dots, N\}$  with number of grid points  $N+1$  (see Section 2.2.1). Next,  $w(x, y, t)$  can be discretised using grid function  $w_{l,m}^n$  with  $l \in \{0, \dots, N_x\}$  and  $m \in \{0, \dots, N_y\}$  where  $N_x+1$  and  $N_y+1$  are the number of grid points in the  $x$  and  $y$  direction respectively (see Section 6.1).

Using these grid functions, system (10.40) can then be discretised as

$$\begin{aligned} \delta_{tt} u_q^n &= c^2 \delta_{\chi\chi} u_q^n - \kappa_s^2 \delta_{\chi\chi\chi\chi} u_q^n - 2\sigma_{0,s} \delta_t u_q^n + 2\sigma_{1,s} \delta_{t-} \delta_{\chi\chi} u_q^n \\ &\quad - J_q(\chi_c) \frac{f^n}{\rho_s A}, \end{aligned} \quad (10.43)$$

$$\begin{aligned} \delta_{tt} w_{l,m}^n &= -\kappa_p^2 \delta_{\Delta} \delta_{\Delta} w_{l,m}^n - 2\sigma_{0,p} \delta_t w_{l,m}^n + 2\sigma_{1,p} \delta_{t-} \delta_{xx} w_{l,m}^n \\ &\quad + J_{l,m}(x_c, y_c) \frac{f^n}{\rho_p H}, \end{aligned} \quad (10.44)$$

where  $J_q(\chi_c) = J_{q,o_s}(\chi_c)$  is a 1D spreading operator of order  $o_s$  as defined in Section 8.1 and  $J_{l,m}(x_c, y_c) = J_{(l,m), o_p}(x_c, y_c)$  is a 2D spreading operator of order and  $o_p$  as defined in Section 10.1.

The definition of the force in Eq. (10.41) can be discretised as<sup>4</sup>

$$f^n = K_1 \mu_{tt} \eta^n + K_3 (\eta^n)^2 \mu_t \cdot \eta^n + R \delta_t \cdot \eta^n, \quad (10.45)$$

where

$$\eta^n = I_q(x_c) u_q^n - I_{l,m}(x_c, y_c) w_m^n. \quad (10.46)$$

Here,  $I_q(\chi_c) = I_{q,o_s}(\chi_c)$  and  $I_{l,m}(x_c, y_c) = I_{(l,m), o_p}(x_c, y_c)$  are interpolation operators of the same order as  $J_q(\chi_c)$  and  $J_{l,m}(x_c)$  respectively.

---

<sup>4</sup>The second-order averaging operator has been chosen here as a test case, could also be a centred first-order averaging operator as used in Eq.(10.28).

### 10.5.3 Solving for $f$

Following the same process as in Section 10.4.1, system (10.43) needs be isolated at the connection locations. This is done by taking an inner product of the schemes in (10.43) with their respective spreading operators over discrete domains  $d_u = \{0, \dots, N\}$  and  $d_w = \{0, \dots, N_x\} \times \{0, \dots, N_y\}$  respectively. Taking these inner products, expanding the  $\delta_{tt}$  and  $\delta_t$  operators and solving for the states at  $n + 1$  yields

$$I_q(\chi_c)u_q^{n+1} = u^* - \|J_q(\chi_c)\|_{d_u}^2 \frac{f^n}{\rho_s A(1 + \sigma_{0,s})}, \quad (10.47)$$

$$I_{l,m}(x_c)w_{l,m}^n = w^* + \|J_{l,m}(x_c, y_c)\|_{d_w}^2 \frac{f^n}{\rho_p H(1 + \sigma_{0,p}k)}, \quad (10.48)$$

where

$$\begin{aligned} u^* = & \left( I_q(\chi_c)(2u_q^n - u_q^{n-1}) + c^2 k^2 I_q(\chi_c) \delta_{\chi\chi} u_q^n - \kappa_s^2 k^2 I_q(\chi_c) \delta_{\chi\chi\chi\chi} u_q^n \right. \\ & \left. + \sigma_{0,s} k I_q(\chi_c) u_q^{n-1} + 2\sigma_{1,s} k^2 I_q(\chi_c) \delta_{t-\delta_{\chi\chi}} u_q^n \right) / (1 + \sigma_{0,s}k) \end{aligned}$$

and

$$w^* = \frac{-\kappa_p^2 I_{l,m}(x_c) \delta_{\Delta} \delta_{\Delta} w_{l,m}^n - \sigma_{0,p} k I_{l,m}(x_c) w_{l,m}^{n-1} + 2\sigma_{1,p} I_{l,m}(x_c) \delta_{t-\delta_{xx}} w_{l,m}^n}{1 + \sigma_{0,p}k},$$

are the update equations of the schemes at their respective connection locations without the force term.

Evaluating Eq. (10.46) at  $n + 1$  and substituting Eqs. (10.47) yields

$$\eta^{n+1} = u^* - \|J_q(\chi_c)\|_{d_u}^2 \frac{f^n}{\rho_s A(1 + \sigma_{0,s}k)} - \left( w^* + \|J_{l,m}(x_c, y_c)\|_{d_w}^2 \frac{f^n}{\rho_p H(1 + \sigma_{0,p}k)} \right). \quad (10.49)$$

Then expanding Eq. (10.45) to

$$\eta^n = \underbrace{\left( \frac{K_1}{4} + \frac{K_3(\eta^n)^2}{2} + \frac{R}{2k} \right)}_{r_+^n} \eta^{n+1} + \frac{K_1}{2} \eta^n + \underbrace{\left( \frac{K_1}{4} + \frac{K_3(\eta^n)^2}{2} - \frac{R}{2k} \right)}_{r_-^n} \eta^{n-1},$$

and solving for  $\eta^{n+1}$  yields

$$\eta^{n+1} = \frac{f^n}{r_+^n} - \frac{K_1}{2r_+^n} \eta^n - \frac{r_-^n}{r_+^n} \eta^{n-1}. \quad (10.50)$$

Substituting this into Eq. (10.49), one can find a definition for the connecting force

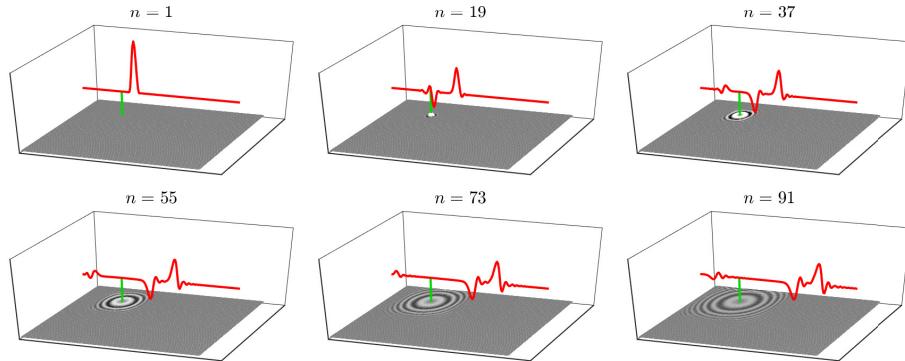
$$f^n = \frac{u^* - w^* + \frac{K_1}{2r_+^n} \eta^n + \frac{r_-^n}{r_+^n} \eta^{n-1}}{\frac{1}{r_+^n} + \frac{\|J_q(\chi_c)\|_{d_u}^2}{\rho_s A(1 + \sigma_{0,s}k)} + \frac{\|J_{l,m}(x_c, y_c)\|_{d_w}^2}{\rho_p H(1 + \sigma_{0,p}k)}}.$$

### 10.5.4 Implementation

This section shows an example of an implementation of the string-plate system. The MATLAB code of the implementation can be found online<sup>5</sup> The parameters for the stiff string can be found in Table 4.1 (with  $L = 1.5$  m and  $T = 555$  N) and for the thin plate in Table 6.1 (with  $H = 5 \cdot 10^{-4}$ ). Both systems have simply supported boundary conditions. Additional parameters used for the connection are

$$K_1 = 10^4 \text{ N/m}, \quad K_3 = 10^7 \text{ N/m}^3, \quad \text{and} \quad R = 10 \text{ kg/s}.$$

Figure 10.6 shows a visualisation of the string plate system excited with a raised cosine.



**Fig. 10.6:** A visualisation of string-plate system. The string is shown in red, the plate in gray and the connection in green.

When implementing connections (or any other interactions for that matter), one mostly does the following:

1. Calculate intermediate scheme, i.e., without force terms:  $\mathbf{u}^I$  and  $\mathbf{w}^I$
2. Calculate  $u^*$  and  $w^*$ :  $\mathbf{i}_u \mathbf{u}^I$  and  $\mathbf{i}_w \mathbf{w}^I$
3. Calculate force  $f^n$
4. Add force term to intermediate scheme  $\mathbf{u}^{n+1} = \mathbf{u}^I - \mathbf{j}_u f^n$  and  $\mathbf{w}^{n+1} = \mathbf{w}^I + \mathbf{j}_w f^n$

not done

---

<sup>5</sup><https://gist.github.com/SilvinWillemsen/91d10f6279af7fd0da7fb65fbd7647da>

### 10.5.5 Energy analysis

still needs stuff

$$\delta_{t+}(\mathfrak{h}_s + \mathfrak{h}_p) + q_s + q_p = -I_q(\chi_c)(\delta_{t.}u_q^n)f^n + I_{l,m}(x_c)(\delta_{t.}w_{l,m}^n)f^n \quad (10.51)$$

and can be rewritten to

$$\delta_{t+}(\mathfrak{h}_s + \mathfrak{h}_p) + q_s + q_p = -\delta_{t.}(I_q(\chi_c)u_q^n - I_{l,m}(x_c)w_{l,m}^n)f^n$$

which, when substituting the definitions for  $f^n$  and  $\eta^n$  from Eqs. (10.45) and (10.46) respectively, yields

$$\delta_{t+}(\mathfrak{h}_s + \mathfrak{h}_p) + q_s + q_p = -(\delta_{t.}\eta^n)(K_1\mu_{tt}\eta^n + K_3(\eta^n)^2\mu_{t.}\eta^n + R\delta_{t.}\eta^n).$$

Due to the nonlinear dependency on  $\eta^n$  one must isolate  $\delta_{t+}$  from the nonlinear term manually, according to

$$\begin{aligned} & K_3(\eta^n)^2(\delta_{t.}\eta^n)(\mu_{t.}\eta^n), \\ &= \frac{K_3(\eta^2)}{2k}(\eta^{n+1} - \eta^{n-1})\frac{1}{2}(\eta^{n+1} + \eta^{n-1}), \\ &= \frac{K_3(\eta^n)^2}{4k}((\eta^{n+1})^2 - (\eta^{n-1})^2), \\ &= \frac{K_3}{4k}((\eta^{n+1}\eta^n)^2 - (\eta^n\eta^{n-1})^2), \\ &= \delta_{t+} \left( \frac{K_3}{4}(\eta^n\eta^{n-1})^2 \right). \end{aligned}$$

Finally, using identity (3.17e) for the linear term, the following balance follows

$$\delta_{t+}(\mathfrak{h}_s + \mathfrak{h}_p + \mathfrak{h}_c) = -q_s - q_p - q_c, \quad (10.52)$$

where the energy stored by the spring connection is

$$\mathfrak{h}_c = \frac{K_1}{8}(\eta^n + \eta^{n-1})^2 + \frac{K_3}{4}(\eta^n\eta^{n-1})^2$$

and the damping term of the connection is

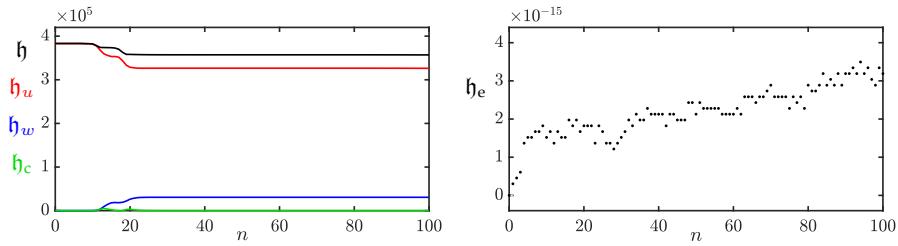
$$q_c = R(\delta_{t.}\eta^n)^2.$$

#### Relative location of objects

Notice that it is important to keep in mind the relative location of the connected objects i.e., whether one object is ‘above’ or ‘below’ an other. If component  $a$  is located above component  $b$ , and their relative displacement is defined as  $\eta = a - b$ , then a positive  $\eta$  is going to have a negative effect on  $a$  and a positive effect on  $b$  and vice-versa. This is important for the signs when adding the force terms to the schemes.

look at this  
compared to  
when I talk  
about it in  
chapter 11

Forces should be equal and opposite.



**Fig. 10.7:** The energy of the string (red), the plate (blue), the spring (green) and the total energy (black) of the connected string-plate system (10.43). The energy corresponds to the system in Figure 10.6. The right panel shows the normalised energy (according to Eq. (3.38)) shows that the deviation of the energy is within machine precision.

# Chapter 11

## Collisions

Many musical instruments rely on collisions in some way. Examples are the collision between a hammer and a piano string, a guitar pick and the string, and even the lips of a trumpet player.

In this work, the collision models used rely on penalising methods. For perfectly rigid colliding objects, the colliding objects are supposed to interpenetrate and collision is interpreted as a *penalty*. The eventual force acting on the colliding objects is then dependent on the level of penetration. For deformable objects, such as the hammer felt tip of a piano, the penalty is dependent on the level of deformation. These collision models were first used in a musical context by e.g. [88, 89].

The discretisations proposed in [88, 89] rely on implicit nonlinear schemes which require an iterative method, such as Newton-Raphson presented in Section 8.2, to obtain their solution. The exact number of iterations required per time step, especially in interactive applications, is usually unknown. This could be detrimental to real-time applications, as the number of iterations and consequently the extra number of computations needed could be very large in a particular situation. Furthermore, and perhaps more importantly, existence and uniqueness of the solution might not be available.

In [90] (co-authored by the PhD student [O3]), Ducceschi et al. propose a method based on quadratisation of the collision potential energy, that circumvents the need of an iterative method to solve nonlinear collisions. Energy quadratisation for explicit schemes first appeared in the context of Port-Hamiltonian systems and was due to Lopes et al. in [91]. The introduction of an additional state variable, which is what Ducceschi's work is based on, was introduced in [92, 93]. Publications [D] and [E] follow an earlier iteration of the non-iterative collision algorithm from [94, 95] which exhibited spurious oscillations that Ducceschi et al. resolve in [90]. The latter will be used this work and presented in this chapter.

This chapter will first provide a definition for the collision potential as well as its quadratisation, used as the basis for the explicit method. Then, the method will be applied to a simple mass-barrier collision and finally, a mass-spring – string collision which can be used to model a finger-fretted string.

### Collision potential

Collisions can be modelled using a nonlinear *collision potential*, which can be defined as

$$\phi(\eta) = \frac{K}{\alpha_c + 1} [\eta]_+^{\alpha_c + 1}, \quad (11.1)$$

with collision stiffness  $K \geq 0$  (in N/m $^{\alpha_c}$ ) and dimensionless nonlinear collision coefficient  $\alpha_c \geq 1$ . Here  $\eta = \eta(t)$  describes the relative displacement between the two colliding bodies (in m). The  $[\cdot]_+$  operator, defined as

$$[\cdot]_+ = \frac{\cdot + |\cdot|}{2} \quad (11.2)$$

describes the ‘positive part of’ and when applied to  $\eta$  in Eq. (11.1) causes the potential  $\phi$  to only be non-zero when the two colliding bodies are in contact. See Figure 11.1.

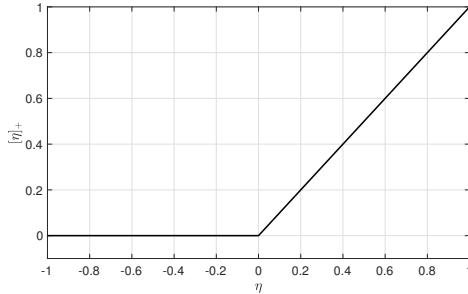


Fig. 11.1: A plot of  $[\eta]_+$ .

The derivative of Eq. (11.1) with respect to  $\eta$  is defined as

$$\phi'(\eta) = K[\eta]_+^{\alpha_c} \quad (11.3)$$

and can then be used in the PDE at hand.

The issue with this form of the collision potential, is that an iterative method, such as Newton-Raphson presented in Section 8.2 needs to be used in order to solve the system [90].

### Quadratic form

In [90], the authors propose to rewrite the potential in Eq. (11.3) in a quadratic form. Using the chain rule and  $\psi = \psi(\eta)$ , Eq. (11.3) can be rewritten as

$$\phi'(\eta) = \psi\psi' \quad \text{where} \quad \psi = \sqrt{2\phi} \quad \text{and} \quad \psi' = \frac{\dot{\psi}}{\dot{\eta}}, \quad (11.4)$$

where a dot denotes a single derivative with respect to time.

This form of the potential can be discretised to a FD scheme that can be solved explicitly. This process will be shown below, using an example of the simple mass-barrier collision.

## 11.1 The mass – rigid barrier collision

As a test case, the simplest collision case – a mass colliding with a rigid barrier – is presented here. Consider a mass at location  $u = u(t)$  (in m) colliding with a barrier at location  $b$  (in m).

If the barrier is placed above the mass, the force it exerts on the mass will be negative and its system would be described as

$$M\ddot{u} = -\psi\psi', \quad (11.5)$$

with mass  $M$  (in kg) and  $\psi = \psi(\eta)$  and  $\psi'$  are as defined in Eq. (11.4) with  $\eta = \eta(t) = u(t) - b$ .

Looking towards the discretisation the mass-barrier collision, one could rewrite Eq. (11.5) to the following system of equations

$$M\ddot{u} = -\psi g, \quad (11.6a)$$

$$\dot{\psi} = g\dot{\eta}, \quad (11.6b)$$

$$\eta(t) = u(t) - b, \quad (11.6c)$$

where  $g = \psi'$ .

### Location of objects

As explained in Chapter 10, it is important to consider whether an object is located ‘above’ or ‘below’ the other,

or, which has a more positive or negative displacement than the other object. A mass with a displacement of 0.01 m will thus be ‘above’ a barrier with a displacement of -0.05 m. Along these lines, a positive force acting on an element will accelerate it upwards and a negative force will accelerate it downwards.

The relative location of the two colliding objects will affect two things in (11.6). Firstly, the location of the object determines the direction of the collision force, i.e., the sign of the right-hand side in system (11.6a). In this case, the barrier is placed above the mass, and will exert a downwards (negative) force on the mass. If the barrier was placed below the mass, the opposite would have applied. Secondly, the definition of  $\eta$  in (11.6c) is affected by the relative location of the objects. The collision potential in Eq (11.1) is only non-zero when  $\eta$  is positive. If the barrier is placed above the mass,  $u(t) - b$  will be positive on collision. It is thus important to remember that  $\eta$  should be defined as the element above subtracted from the element below.

This might need to be moved to the previous chapter. At least highlight the difference between a “pushing” collision and a “pulling” connection

### 11.1.1 Discrete time

Before discretising system (11.6) in full, the discrete approximation to the collision potential will be elaborated on. Following [90],  $\psi$  is placed on an interleaved temporal grid using

$$\psi^{n-1/2} = \mu_{t-}\psi^n, \quad (11.7)$$

where the interleaved temporal grid is used here as it results in energy conservation in discrete time as will be shown below. Approximations to  $\psi$  and  $g$  in Eq. (11.6) can then be made as

$$\psi \approx \mu_{t+}\psi^{n-1/2} \quad (11.8)$$

and

$$g \approx g^n = \frac{\delta_{t+}\psi^{n-1/2}}{\delta_t.\eta^n}, \quad (11.9)$$

respectively. Notice that applying a first-order difference operator to a grid function on an interleaved grid is second-order accurate.<sup>1</sup> The result of the approximation to in Eq. (11.9) allows  $\psi$  to be treated as an independent time series:

$$\delta_{t+}\psi^{n-1/2} = g^n \delta_t.\eta^n. \quad (11.10)$$

With the above approximations in place, system (11.6) can be discretised and yields the following system of equations:

$$M\delta_{tt}u^n = - \left( \mu_{t+}\psi^{n-1/2} \right) g^n, \quad (11.11a)$$

$$\delta_{t+}\psi^{n-1/2} = g^n \delta_t.\eta^n, \quad (11.11b)$$

$$\eta^n = u^n - b. \quad (11.11c)$$

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<sup>1</sup>  $\delta_{t+}\psi^{n-1/2} \stackrel{\text{Eq. (11.9)}}{=} \delta_{t+}\mu_{t-}\psi^n \stackrel{\text{Eq. (2.27b)}}{=} \delta_t.\psi^n$  which is second-order accurate (see Section 2.2.2).

### An explicit definition for $g^n$

To be able to calculate  $\psi^{n+1/2}$  and  $u^{n+1}$  in system (11.11) explicitly, a definition for  $g^n$  only based on known values must be found. As  $g^n \approx \psi'$  as per Eq. (11.9), the derivative can be computed analytically according to

$$g^n = \psi' \Big|_{\eta=\eta^n} \stackrel{\text{Eq. (11.4)}}{=} \frac{\phi'}{\sqrt{2\phi}} \Big|_{\eta=\eta^n}. \quad (11.12)$$

Recalling (11.3) and (11.1), this can conveniently be rewritten to

$$g^n = \frac{K[\eta^n]_+^\alpha}{\sqrt{\frac{2K}{\alpha+1}[\eta^n]_+^{\alpha+1}}} = K\sqrt{\frac{\alpha+1}{2K}}[\eta^n]_+^\alpha [\eta^n]_+^{-\frac{(\alpha+1)}{2}} = \sqrt{\frac{K(\alpha+1)}{2}}[\eta^n]_+^{\frac{\alpha-1}{2}}. \quad (11.13)$$

For implementation purposes, one can expand the  $[\cdot]_+$  operator as the following (equivalent) condition:

$$g^n = \begin{cases} \sqrt{\frac{K_c(\alpha_c + 1)}{2}} \cdot (\eta^n)^{\frac{\alpha_c - 1}{2}} & \text{if } \eta^n \geq 0 \\ 0, & \text{if } \eta^n < 0 \end{cases} \quad (11.14a)$$

$$(11.14b)$$

This implementation is the one presented in [94], but exhibited spurious oscillations and ‘sticking’ behaviour. This is due to the possibility of negative forces for positive penetrations due to the discontinuity in the definition for  $g^n$  at  $\eta^n = 0$ .

In [90], the definition for  $g^n$  in (11.14) is extended, starting out by using an implicit equation for  $g^n$  by directly discretising Eq. (11.9)

$$g_{\text{imp}}^n = 2 \frac{\psi^{n+1/2} - \psi^{n-1/2}}{\eta^{n+1} - \eta^{n-1}}. \quad (11.15)$$

If there is, however, no collision at  $n + 1/2$ ,  $\psi^{n+1/2} = 0$  and Eq. (11.15) reduces to

$$g_{\text{imp}}^n = -2 \frac{\psi^{n-1/2}}{\eta^{n+1} - \eta^{n-1}}.$$

Furthermore, due to the fact that there is no collision,  $\eta^{n+1}$  can be calculated according to  $\eta^{n+1} = \eta^* = u^* - b$  where  $u^*$  is the value of  $u^{n+1}$  calculated using the scheme in Eq. (11.11a) without the collision force (as done in Chapter 10). Expanding Eq. (11.11a) without the collision force yields

$$\frac{M}{k^2} (u^* - 2u^n + u^{n-1}) = 0 \implies u^* = 2u^n - u^{n-1}.$$

Thus, if there is no collision,  $g_{\text{imp}}^n$  can now be explicitly calculated from known

values and be used in the definition for  $g^n$  in Eq. (11.14) according to [90]

$$g^n = \begin{cases} \kappa \sqrt{\frac{K_c(\alpha_c + 1)}{2}} \cdot (\eta^n)^{\frac{\alpha_c - 1}{2}} & \text{if } \eta^n \geq 0, \\ -2 \frac{\psi^{n-1/2}}{\eta^* - \eta^{n-1}} & \text{if } \eta^n < 0 \text{ and } \eta^* \neq \eta^{n-1}, \\ 0, & \text{if } \eta^n < 0 \text{ and } \eta^* = \eta^{n-1}. \end{cases} \quad (11.16a)$$

$$(11.16b) \quad (11.16c)$$

Here,  $\kappa = 1$  if  $\psi^{n-1/2} \geq 0$ , otherwise  $\kappa = -1$  and aims to resolve the ‘sticking’ behaviour by forcing an outwardly-directed force at all times. As was done in paper [H], (11.16c) has been added to the definition of  $g$  from [90] to prevent a division by 0 in (11.16b).

This definition for  $g^n$  does not exhibit the spurious oscillations that the old definition in Eq. (11.14) did, and can still be explicitly calculated from known values of the system.

### 11.1.2 Solving the system

To implement the system in Eq. (11.11), the equations need to be slightly rewritten. Using identity (2.27c),  $\mu_{t+}\psi^{n-1/2}$  can be rewritten to

$$\mu_{t+}\psi^{n-1/2} = \frac{k}{2}\delta_{t+}\psi^{n-1/2} + \psi^{n-1/2}.$$

Then, substituting (11.11b) into this yields

$$\mu_{t+}\psi^{n-1/2} = \frac{k}{2}g^n\delta_t\cdot\eta^n + \psi^{n-1/2},$$

and inserting this into (11.11a) yields

$$M\delta_{tt}u^n = -\left(\frac{k}{2}g^n\delta_t\cdot\eta^n + \psi^{n-1/2}\right)g^n. \quad (11.17)$$

As the position of barrier  $b$  is static, the following is true:

$$\frac{d\eta}{dt} = \frac{d}{dt}(u - b) \implies \delta_t\cdot\eta^n = \delta_t.u^n, \quad (11.18)$$

i.e., the time derivative of  $\eta$  equals the time derivative of  $u$ .<sup>2</sup> Now (11.17) can be solved explicitly as  $u^{n+1}$  is the only unknown in the system

$$\left(\frac{M}{k^2} + \frac{(g^n)^2}{4}\right)u^{n+1} = \frac{M}{k^2}(2u^n - u^{n-1}) + \frac{(g^n)^2}{4}u^{n-1} - \psi^{n-1/2}g^n, \quad (11.19)$$

and can be solved by a simple division.

---

<sup>2</sup>Note that if the barrier was placed underneath the mass, making (11.11c)  $\eta^n = b - u^n$ , this would result in  $\delta_t\cdot\eta^n = -\delta_t.u^n$ .

## 11.2. Mass-spring – string collision

Finally,  $u^{n+1}$  can be used to calculate  $\eta^{n+1}$  can be calculated using (11.11c) at  $n + 1$ :

$$\eta^{n+1} = u^{n+1} - b, \quad (11.20)$$

which is used to calculate  $\psi^{n+1/2}$  by expanding and rewriting (11.11b) to

$$\psi^{n+1/2} = \psi^{n-1/2} + \frac{\eta^{n+1} - \eta^{n-1}}{2}. \quad (11.21)$$

### 11.1.3 Energy analysis

To prove that the collision term does not add any additional energy into the system (retaining passivity) and that it does not add additional constraints on the stability of the system, the energy analysis techniques presented in Section 3.4 can be used. Notice that for brevity, the steps presented Section 3.4 will not explicitly be followed.

Multiplying Eq. (11.11a) by  $(\delta_t.u^n)$  yields

$$\delta_{t+}\mathfrak{h} = - \left( \mu_{t+}\psi^{n-1/2} \right) g^n(\delta_t.u^n)$$

where

$$\mathfrak{h} = \frac{M}{2}(\delta_{t-}u^n)^2 \quad (11.22)$$

(see Eq. (3.42)). Expanding  $g^n$  yields

$$\begin{aligned} \delta_{t+}\mathfrak{h} &= - \left( \mu_{t+}\psi^{n-1/2} \right) \frac{\delta_{t+}\psi^{n-1/2}}{\delta_t.\eta^n} (\delta_t.u^n) \\ &\stackrel{\text{Eq. (11.18)}}{=} - \left( \mu_{t+}\psi^{n-1/2} \right) \delta_{t+}\psi^{n-1/2}, \end{aligned}$$

which, using identity (3.17c) can be rewritten to

$$\delta_{t+}(\mathfrak{h} + \mathfrak{h}_c) = 0, \quad (11.23)$$

with collision energy

$$\mathfrak{h}_c = \frac{(\psi^{n-1/2})^2}{2}. \quad (11.24)$$

Recall that in order for a scheme to be passive, its energy must be non-negative. The fact that  $\psi$  is squared proves passivity for the system in (11.11).

## 11.2 Mass-spring – string collision

The mass-spring – string collision is slightly trickier than the mass – rigid barrier collision, as there are two moving components rather than one. This

system is chosen as an example as it has the interesting use-case of fretting a string to change the pitch, modelling the fretting finger as a mass.

Consider a lossless stiff string of length  $L$ , its transverse displacement described by  $u = u(x, t)$  (in m) and defined over  $x \in \mathcal{D}$  with domain  $\mathcal{D} = [0, L]$ . The mass with displacement  $w = w(t)$  (in m) will model the fretting finger. The PDE for the stiff string and its parameter definitions can be found in Eq. (4.1) and for the mass-spring system in Eq. (2.28). Placing the string above the mass, the following system emerges:

$$\rho A \partial_t^2 u = T \partial_x^2 u - EI \partial_x^4 u + \delta(x - x_m) \psi g \quad (11.25a)$$

$$M \ddot{w} = -Kw - \psi g \quad (11.25b)$$

$$\dot{\psi} = g \dot{\eta}, \quad (11.25c)$$

$$\eta(t) = w(t) - u(x_m, t), \quad (11.25d)$$

where spatial Dirac delta function  $\delta(x - x_m)$  localises the mass (finger) along the string at location  $x_m \in \mathcal{D}$ . Furthermore  $\psi = \psi(\eta)$  and  $g = \psi'$  and are as defined in Eq. (11.4).

Discretising system (11.25), with the collision discretised according to the process explained in Section 11.1, yields

$$\rho A \delta_{tt} u_l^n = T \delta_{xx} u_l^n - EI \delta_{xxxx} u_l^n + J_l(x_m) (\mu_{t+} \psi^{n-1/2}) g^n, \quad (11.26a)$$

$$M \delta_{tt} w^n = -Kw^n - (\mu_{t+} \psi^{n-1/2}) g^n, \quad (11.26b)$$

$$\delta_{t+} \psi^{n-1/2} = g^n \delta_t \eta^n, \quad (11.26c)$$

$$\eta^n = w^n - I_l(x_m) u_l^n, \quad (11.26d)$$

where spreading and interpolation operators  $I_l(x_m)$  and  $J_l(x_m)$  are as defined in Section 8.1 (the order is not specified here). Following the same process as in Section 11.1.2, Eqs. (11.26a) and (11.26b) can be rewritten to

$$\rho A \delta_{tt} u_l^n = T \delta_{xx} u_l^n - EI \delta_{xxxx} u_l^n + J_l(x_m) \left( \frac{k}{2} g^n \delta_t \eta^n + \psi^{n-1/2} \right) g^n, \quad (11.27a)$$

$$M \delta_{tt} w^n = -Kw^n - \left( \frac{k}{2} g^n \delta_t \eta^n + \psi^{n-1/2} \right) g^n, \quad (11.27b)$$

which can be used as a starting point for solving the system.

### 11.2.1 Solving the system

As the colliding objects are both moving, Eq. (11.18) is not valid anymore and another strategy needs to be used. One can start to solve the system by isolating the string at the collision location  $x_m$  by taking an inner product of Eq. (11.27a) with  $J_l(x_m)$ . Using identity (8.9) and dividing all terms by  $\rho A$

## 11.2. Mass-spring – string collision

yields

$$\begin{aligned}\delta_{tt} I_l(x_m) u_l^n &= c^2 I_l(x_m) \delta_{xx} u_l^n - \kappa^2 I_l(x_m) \delta_{xxxx} u_l^n \\ &\quad + \frac{\|J_l(x_m)\|_D^2}{\rho A} \left( \frac{k}{2} g^n \delta_t \eta^n + \psi^{n-1/2} g^n \right) g^n.\end{aligned}$$

with  $c = \sqrt{T/\rho A}$  and  $\kappa = \sqrt{EI/\rho A}$ . Expanding the temporal FD operators yields

$$I_l(x_m) u_l^{n+1} = u^\star + \underbrace{\frac{\|J_l(x_m)\|_D^2 k^2}{\rho A}}_{\mathfrak{J}_l} \left( \frac{(g^n)^2}{4} (\eta^{n+1} - \eta^{n-1}) + \psi^{n-1/2} g^n \right),$$

where

$$u^\star = I_l(x_m)(2u_l^n - u_l^{n-1}) + c^2 k^2 I_l(x_m) \delta_{xx} u_l^n - \kappa^2 k^2 I_l(x_m) \delta_{xxxx} u_l^n$$

is the result of the update equation of the string at  $x_m$  without the collision term. Then, Eq. (11.26d) at  $n+1$ , which is  $\eta^{n+1} = w^{n+1} - I(x_m) u_l^{n+1}$ , can be substituted, which results in

$$\begin{aligned}\left(1 + \mathfrak{J}_l \frac{(g^n)^2}{4}\right) I_l(x_m) u_l^{n+1} - \mathfrak{J}_l \frac{(g^n)^2}{4} w^{n+1} &= u^\star \\ &\quad + \mathfrak{J}_l \left( -\frac{(g^n)^2}{4} \eta^{n-1} + \psi^{n-1/2} g^n \right).\end{aligned}\tag{11.28}$$

Performing this same process on the FD scheme of the mass in Eq. (11.27b) yields

$$\begin{aligned}-\frac{(g^n)^2 k^2}{4M} I_l(x_m) u_l^{n+1} + \left(1 + \frac{(g^n)^2 k^2}{4M}\right) w^{n+1} &= w^\star \\ &\quad - \frac{k^2}{M} \left( -\frac{(g^n)^2}{4} \eta^{n-1} + \psi^{n-1/2} g^n \right),\end{aligned}\tag{11.29}$$

where

$$w^\star = 2w^n - w^{n-1} - \frac{Kk^2}{M} w^n$$

is again the result of the update equation of the mass without the collision term. Equations (11.28) and (11.29) can be treated as a system of linear equations (see Section B.3) with unknowns  $I_l(x_m) u_l^{n+1}$  and  $w^{n+1}$ . Writing the aforementioned equations in matrix form yields

$$\begin{bmatrix} I_l(x_m) u_l^{n+1} \\ w^{n+1} \end{bmatrix} = \mathbf{A}^{-1} \mathbf{v}\tag{11.30}$$

where

$$\mathbf{A} = \begin{bmatrix} \left(1 + \mathfrak{J}_l \frac{(g^n)^2}{4}\right) & -\mathfrak{J}_l \frac{(g^n)^2}{4} \\ -\frac{(g^n)^2 k^2}{4M} & \left(1 + \frac{(g^n)^2 k^2}{4M}\right) \end{bmatrix} \quad \text{and}$$

$$\mathbf{v} = \begin{bmatrix} u^* + \mathfrak{J}_l \left(-\frac{(g^n)^2}{4} \eta^{n-1} + \psi^{n-1/2} g^n\right) \\ w^* - \frac{k^2}{M} \left(-\frac{(g^n)^2}{4} \eta^{n-1} + \psi^{n-1/2} g^n\right) \end{bmatrix}.$$

From this,  $\eta^{n+1}$  can be calculated, and can consequently be applied to the string and mass in system (11.27).

### 11.2.2 Energy analysis

This section follows Section 3.4 without explicitly following the steps for brevity.

One can obtain the energy of the stiff string FD scheme in Eq. (11.26a) by taking the inner product of scheme by  $\delta_{t+} u_l^n$  over domain  $\mathcal{D}$  to obtain

$$\delta_{t+} \mathfrak{h}_s = \left\langle (\delta_{t+} u_l^n), J_l(x_m) \left(\mu_{t+} \psi^{n-1/2}\right) g^n \right\rangle_{\mathcal{D}} \quad (11.31)$$

where (see Eq. (4.32))

$$\mathfrak{h}_s = \mathfrak{t}_s + \mathfrak{v}_s, \quad \text{with} \quad \mathfrak{t}_s = \frac{\rho A}{2} \|\delta_{t-} u_l^n\|_d^2, \quad \text{and}$$

$$\mathfrak{v}_s = \frac{T}{2} \langle \delta_{x+} u_l^n, e_{t-} \delta_{x+} u_l^n \rangle_d + \frac{EI}{2} \langle \delta_{xx} u_l^n, e_{t-} \delta_{xx} u_l^n \rangle_{\bar{d}}.$$

Energy analysis for the mass in Eq. (11.26b) can be done by simply multiplying the scheme by  $\delta_{t+} w^n$  to get

$$\delta_{t+} \mathfrak{h}_m = -(\delta_{t+} w^n) \left(\mu_{t+} \psi^{n-1/2}\right) g^n, \quad (11.32)$$

where (see Eq. (3.42))

$$\mathfrak{h}_m = \mathfrak{t}_m + \mathfrak{v}_m, \quad \text{with} \quad \mathfrak{t}_m = \frac{M}{2} (\delta_{t-} w^n)^2, \quad \text{and} \quad \mathfrak{v}_m = \frac{K}{2} w^n e_{t-} w^n.$$

The total energy in the system is the addition of Eqs. (11.31) and (11.32), which, using identity (8.9) for the former, can be written as:

$$\begin{aligned} \delta_{t+} (\mathfrak{h}_s + \mathfrak{h}_m) &= \left(I_l(x_m) (\delta_{t+} u_l^n) - (\delta_{t+} w^n)\right) \left(\mu_{t+} \psi^{n-1/2}\right) g^n, \\ &= \underbrace{\delta_{t+} (I_l(x_m) u_l^n - w^n)}_{-\delta_{t+} \eta^n} \left(\mu_{t+} \psi^{n-1/2}\right) g^n. \end{aligned}$$

### 11.3. Two-sided collision: A connection

Then, expanding  $g^n$  according to Eq. (11.9) yields

$$\begin{aligned}\delta_{t+}(\mathfrak{h}_s + \mathfrak{h}_m) &= -\delta_t \cdot \eta^n \left( \mu_{t+} \psi^{n-1/2} \right) \frac{\delta_{t+} \psi^{n-1/2}}{\delta_t \cdot \eta^n} \\ &= - \left( \mu_{t+} \psi^{n-1/2} \right) \delta_{t+} \psi^{n-1/2}\end{aligned}$$

and finally, using identity (3.17c), can be rewritten as

$$\delta_{t+}(\mathfrak{h}_s + \mathfrak{h}_m + \mathfrak{h}_c) = 0, \quad (11.33)$$

with collision energy

$$\mathfrak{h}_c = \frac{(\psi^{n-1/2})^2}{2}.$$

Again, the fact that  $\psi$  is squared here, means that  $\mathfrak{h}_c$  is non-negative, and proves passivity of the system.

## 11.3 Two-sided collision: A connection

Using the methods presented in this chapter, one could devise a two-sided collision and alter the collision potential in Eq. (11.1) to [95]

$$\phi(\eta) = \frac{K}{\alpha_c + 1} |\eta|^{\alpha_c + 1}, \quad (11.34)$$

and taking its derivative with respect to  $\eta$  yields

$$\phi'(\eta) = \text{sgn}(\eta) K |\eta|^{\alpha_c}. \quad (11.35)$$

One can observe that, as opposed to the (one-sided) potential presented in Eq. (11.1), the collision force will be non-zero, for both a positive and negative  $\eta$ . This two-sided collision can be used as a connection – as an alternative to the connections presented in Chapter 10 – and has been used in papers [D] and [E] in combination with Eq. (11.1) to model the mechanics of the tromba marina. See Chapter ?? for more details.

## Chapter 11. Collisions

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ences or not

**Part V**

**Papers**



# Paper A

## Real-Time Control of Large-Scale Modular Physical Models using the Sensel Morph

Silvin Willemse, Nikolaj Andersson, Stefania Serafin  
and Stefan Bilbao

The paper has been published in the  
*Proceedings of the 16th Sound and Music Computing (SMC) Conference*, pp.  
275–280, 2019.



# Paper B

## Physical Models and Real-Time Control with the Sensel Morph

Silvin Willemse, Stefan Bilbao, Nikolaj Andersson  
and Stefania Serafin

The paper has been published in the  
*Proceedings of the 16th Sound and Music Computing (SMC) Conference*, pp.  
95–96, 2019.



# Paper C

## Real-Time Implementation of an Elasto-Plastic Friction Model applied to Stiff Strings using Finite-Difference Schemes

Silvin Willemse, Stefan Bilbao and Stefania Serafin

The paper has been published in the  
*Proceedings of the 22nd International Conference on Digital Audio Effects*  
(DAFx-19), pp. 40–46, 2019.



# Paper D

## Real-time Implementation of a Physical Model of the Tromba Marina

Silvin Willemsen, Stefania Serafin, Stefan Bilbao and Michele  
Ducceschi

The paper has been published in the  
*Proceedings of the 17th Sound and Music Computing (SMC) Conference*, pp.  
161–168, 2020.



# Paper E

## Resurrecting the Tromba Marina: A Bowed Virtual Reality Instrument using Haptic Feedback and Accurate Physical Modelling

Silvin Willemse, Razvan Paisa and Stefania Serafin

The paper has been published in the  
*Proceedings of the 17th Sound and Music Computing (SMC) Conference*, pp.  
300–307, 2020.



# Paper F

## DigiDrum: A Haptic-based Virtual Reality Musical Instrument and a Case Study

Silvin Willemse, Anca-Simona Horvath and Mauro Nascimben

The paper has been published in the  
*Proceedings of the 17th Sound and Music Computing (SMC) Conference*, pp.  
292–299, 2020.



# Paper G

## Dynamic Grids for Finite-Difference Schemes in Musical Instrument Simulations

Silvin Willemse, Stefan Bilbao, Michele Ducceschi and Stefania Serafin

The paper has been published in the  
*Proceedings of the 23rd International Conference on Digital Audio Effects*  
(DAFx2020in21), 2021.



# Paper H

## A Physical Model of the Trombone using Dynamic Grids for Finite-Difference Schemes

Silvin Willemse, Stefan Bilbao, Michele Ducceschi and Stefania Serafin

The paper has been published in the  
*Proceedings of the 23rd International Conference on Digital Audio Effects*  
(DAFx2020in21), 2021.

Paper H.

# **Part VI**

# **Appendix**



# Appendix A

## Paper Errata

Here, some errors in the published papers will be listed:

### Elasto-plastic [C]

- The authors in reference [15] are a bit shuffled.

### Real-Time Tromba [D]

- The minus sign in Eq. (28) (and thus Eqs. (31) and (35)) should be a plus sign.
- $\sigma_{1,s}$  in Eq. (21) should obviously be  $\sigma_{1,p}$
- the unit of the spatial Dirac delta function  $\delta$  should be  $m^{-1}$

### DigiDrum [F]

- $\sigma_0$  and  $\sigma_1$  should be multiplied by  $\rho H$  in order for the stability condition to hold.
- stability condition is wrong. Should be:

$$h \geq \sqrt{c^2 k^2 + 4\sigma_1 k + \sqrt{(c^2 k^2 + 4\sigma_1 k)^2 + 16\kappa^2 k^2}} \quad (\text{A.1})$$

- Unit for membrane tension is N/m.

## Appendix A. Paper Errata

# Appendix B

# Matrices

This appendix aims to provide some fundamental knowledge on matrices and linear algebra used throughout this thesis.

A matrix is a rectangular array with numerical elements and its dimensions are denoted using “*row* × *column*”. A  $3 \times 5$  matrix, for example, thus has 3 rows and 5 columns (see Figure B.1a). Along those lines, a *row vector* is a matrix with 1 row and more than 1 column and a *column vector* is a matrix with 1 column and more than 1 row.

In this document, matrices and vectors are written using bold symbols. A matrix is denoted by a capital letter – such as  $\mathbf{A}$  – whereas vectors are decapitalised – such as  $\mathbf{u}$ . An element in a matrix is denoted with a non-bold, decapitalised variable, where the subscripts indicate the indices of the row and column. For example, the element in the 2<sup>nd</sup> row and the 4<sup>th</sup> column of a matrix  $\mathbf{A}$  is denoted as  $a_{24}$ . An element in a vector only has one subscript, regardless of whether it is a row or a column vector.

## B.1 Operations

Multiplying and dividing a matrix by a scalar (a single number) is valid and happens on an element-by-element basis. For a  $2 \times 2$  matrix  $\mathbf{A}$  and scalar  $p$  the following operations hold

$$p\mathbf{A} = \mathbf{A}p = \begin{bmatrix} p \cdot a_{11} & p \cdot a_{12} \\ p \cdot a_{21} & p \cdot a_{22} \end{bmatrix}, \quad \text{and} \quad \mathbf{A}/p = \begin{bmatrix} a_{11}/p & a_{12}/p \\ a_{21}/p & a_{22}/p \end{bmatrix}.$$

Notice that although a matrix can be divided by a scalar, a scalar can not necessarily be divided by a matrix. See Section B.2 for more information.

### Matrix transpose

A matrix or vector can be *transposed*, and is indicated with the  $T$  operator. Transposing a matrix  $\mathbf{A}$  is denoted by  $\mathbf{A}^T$ . This means that the elements in the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column of the original matrix become the elements in the  $j^{\text{th}}$  row and the  $i^{\text{th}}$  column of the transposed matrix. Essentially the row and column indices of the elements inside the matrix get switched according to

$$a_{ij} = a_{ji}. \quad (\text{B.1})$$

Also see Figure B.1. For a row vector, the transpose operation simply changes it to a column vector and vice versa. Another way of seeing a transpose is that all the elements get flipped over the *main diagonal* of the matrix. The main diagonal comprises the elements  $a_{ij}$  where  $i = j$  and a transpose does not affect the location of these elements.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix} \quad \mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \\ a_{15} & a_{25} & a_{35} \end{bmatrix}$$

(a) A  $3 \times 5$  matrix  $\mathbf{A}$ .

(b) A transposed matrix  $\mathbf{A}^T$  of size  $5 \times 3$ .

**Fig. B.1:** A matrix  $\mathbf{A}$  and its transpose  $\mathbf{A}^T$ . The elements get flipped along the main diagonal of the matrix according to Eq. (B.1).

### Matrix multiplication

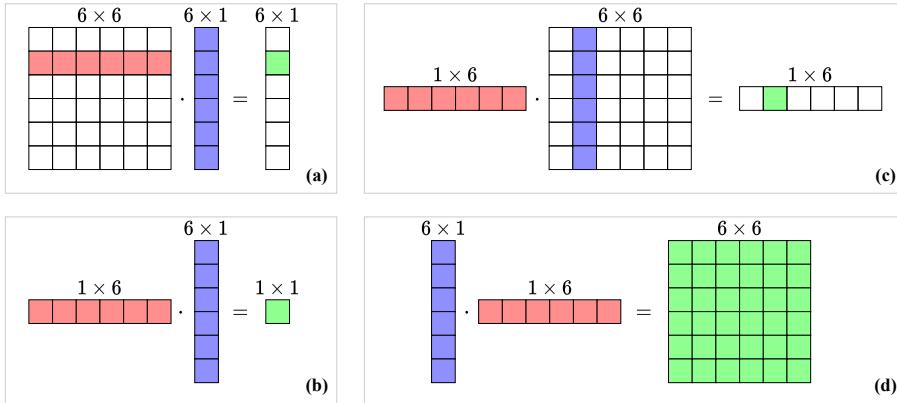
Matrix multiplication (this includes matrix-vector multiplication) is different from regular multiplication in that it needs to abide several extra rules. The multiplication of two matrices  $\mathbf{A}$  and  $\mathbf{B}$  to a resulting matrix  $\mathbf{C}$  is defined as

$$c_{ij} = \sum_{k=1}^K a_{ik} b_{kj}, \quad (\text{B.2})$$

where  $K$  is both the number columns of matrix  $\mathbf{A}$  and the number of rows in matrix  $\mathbf{B}$ . It thus follows that, order for matrix multiplication to be valid, the number of columns of the first matrix needs to be equal to the number of rows in the second matrix. The result will then be a matrix with a number of rows equal to that of the first matrix and a number of columns equal to that of the second matrix. See Figure B.2 for reference.

As an example, consider the  $L \times M$  matrix  $\mathbf{A}$  and a  $M \times N$  matrix  $\mathbf{B}$  with  $L \neq N$ . The multiplication  $\mathbf{AB}$  is defined as the number of columns of matrix  $\mathbf{A}$  ( $M$ ) is equal to the number of rows of matrix  $\mathbf{B}$  (also  $M$ ). The result,  $\mathbf{C}$ , is a  $L \times N$  matrix. The multiplication  $\mathbf{BA}$  is undefined as the number of columns of the first matrix does not match the number of rows in the second matrix. A valid multiplication of two matrices written in their dimensions is

$$\overbrace{(L \times M)}^{\mathbf{A}} \cdot \overbrace{(M \times N)}^{\mathbf{B}} = \overbrace{(L \times N)}^{\mathbf{C}}. \quad (\text{B.3})$$



**Fig. B.2:** Visualisation of valid matrix multiplications (see Eq. (B.2)). The “inner” dimensions (columns of the left matrix and rows of the right) must match and result in a matrix with a size of “outer” dimensions (rows of the left matrix and columns of the right).

## B.2 Matrix inverse

If a matrix has the same number of rows as columns, it is called a *square matrix*. Square matrices have special properties, one of which is that it (usually) can be *inverted*. A square matrix  $\mathbf{A}$  is invertable if there exists a matrix  $\mathbf{B}$  such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}. \quad (\text{B.4})$$

This matrix  $\mathbf{B}$  is then called the *inverse* of  $\mathbf{A}$  and can be written as  $\mathbf{A}^{-1}$ . Not all square matrices have an inverse, in which case it is called *singular*. Rather than going through manually inverting a matrix, or determining whether it is singular, the following function in MATLAB will provide the inverse of a matrix  $\mathbf{A}$ :

```
A_inverted = inv(A);
```

The inverse of a *diagonal matrix* (a matrix with non-zero elements on its main diagonal and the rest zeros) is obtained by replacing the diagonal elements by their reciprocal. So for a diagonal  $3 \times 3$  matrix, the following holds:

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{12}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}.$$

### B.3 Systems of linear equations

Matrices can be conveniently used to solve *systems of linear equations*, a set of linear equations containing the same set of variables.

For example, take the system of linear equations

$$\begin{aligned} x + z &= 6, \\ z - 3y &= 7, \\ 2x + y + 3z &= 15, \end{aligned}$$

with independent variables  $x, y$  and  $z$ . The goal is to find a solution for these variables that satisfy all three equations. This system could be solved by hand using algebraic methods, but alternatively, the system can be written in matrix form:

$$\mathbf{A}\mathbf{u} = \mathbf{w}. \quad (\text{B.5})$$

Here, column vector  $\mathbf{u}$  contains the independent variables  $x, y$ , and  $z$ , matrix  $\mathbf{A}$  contains the coefficients multiplied onto these variables and  $\mathbf{w}$  contains the right-hand side, i.e., the coefficients not multiplied onto any of the variables:

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 1 \\ 2 & 1 & 3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 6 \\ 7 \\ 15 \end{bmatrix}}_{\mathbf{w}}$$

This can be solved for  $\mathbf{u}$  by taking the inverse of  $\mathbf{A}$  (see Section B.2) and multiplying this onto  $\mathbf{w}$

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{w}. \quad (\text{B.6})$$

Generally, if  $X$  unknowns are described by  $X$  equations, the unknowns can be solved for using this method.

Solving a system of linear equations can be implemented in MATLAB by using the code given in Section B.2 and multiplying this onto a vector  $\mathbf{w}$

$$\mathbf{u} = \text{inv}(\mathbf{A}) * \mathbf{w};$$

or more compactly, by using the ' $\backslash$ ' operator:

$$\mathbf{u} = \mathbf{A} \backslash \mathbf{w};$$

## B.4 Eigenvalue problems

A square matrix  $\mathbf{A}$  is characterised by its *eigenvalues* and corresponding *eigenvectors*. In a FDTD context, these are usually associated with the modes of a system, where the eigenvalues relate to the modal frequencies and the eigenvectors to the modal shapes. Section 3.5 provides more information on this.

To find these characteristic values for a  $p \times p$  matrix  $\mathbf{A}$ , an equation of the following form must be solved

$$\mathbf{A}\phi = \lambda\phi. \quad (\text{B.7})$$

This is called is an *eigenvalue problem* and has  $p$  solutions (corresponding to the dimensions of  $\mathbf{A}$ ). These are the  $p^{\text{th}}$  eigenvector  $\phi_p$  and the corresponding eigenvalue  $\lambda_p$  which is calculated using

$$\lambda_p = \text{eig}_p(\mathbf{A}), \quad (\text{B.8})$$

where  $\text{eig}_p(\cdot)$  denotes the  $p^{\text{th}}$  eigenvalue of. Instead of delving too deep into eigenvalue problems and the process of how to solve them, an easy way to obtain the solutions using MATLAB is provided here:

```
[phi, lambda] = eig(A, 'vector');
```

The  $p^{\text{th}}$  eigenvector appears in the  $p^{\text{th}}$  column of  $p \times p$  matrix  $\phi$  and the correspondsing eigenvalues are given in a  $p \times 1$  column vector  $\lambda$ .

## Appendix B. Matrices

# Appendix C

## Code Snippets

### C.1 Mass-spring system (Section 2.3)

```
1 %% Initialise variables
2 fs = 44100; % sample rate [Hz]
3 k = 1 / fs; % time step [s]
4 lengthSound = fs; % length of the simulation (1 second) [samples]
5
6 f0 = 440; % fundamental frequency [Hz]
7 omega0 = 2 * pi * f0; % angular (fundamental) frequency [Hz]
8 M = 1; % mass [kg]
9 K = omega0^2 * M; % spring constant [N/m]
10
11 %% initial conditions (u0 = 1, d/dt u0 = 0)
12 u = 1;
13 uPrev = 1;
14
15 % initialise output vector
16 out = zeros(lengthSound, 1);
17
18 %% Simulation loop
19 for n = 1:lengthSound
20
21     % Update equation Eq. (2.35)
22     uNext = (2 - K * k^2 / M) * u - uPrev;
23
24     out(n) = u;
25
26     % Update system states
27     uPrev = u;
28     u = uNext;
29 end
```

## C.2 1D wave equation (Section 2.4)

```

1 %% Initialise variables
2 fs = 44100; % Sample rate [Hz]
3 k = 1 / fs; % Time step [s]
4 lengthSound = fs; % Length of the simulation (1 second) [samples]
5
6 c = 300; % Wave speed [m/s]
7 L = 1; % Length [m]
8 h = c * k; % Grid spacing [m] (from CFL condition)
9 N = floor(L/h); % Number of intervals between grid points
10 h = L / N; % Recalculation of grid spacing based on integer N
11
12 lambdaSq = c^2 * k^2 / h^2; % Courant number squared
13
14 % Boundary conditions ([D]irichlet or [N]eumann)
15 bcLeft = "D";
16 bcRight = "D";
17
18 %% Initialise state vectors (one more grid point than the number of
19 % intervals)
20 uNext = zeros(N+1, 1);
21 u = zeros(N+1, 1);
22
23 %% Initial conditions (raised cosine)
24 loc = round(0.8 * N); % Center location
25 halfWidth = round(N/10); % Half-width of raised cosine
26 width = 2 * halfWidth; % Full width
27 rcX = 0:width; % x-locations for raised cosine
28
29 rc = 0.5 - 0.5 * cos(2 * pi * rcX / width); % raised cosine
30 u(loc-halfWidth : loc+halfWidth) = rc; % initialise current state
31
32 % Set initial velocity to zero
33 uPrev = u;
34
35 % Range of calculation
36 range = 2:N;
37
38 % Output location
39 outLoc = round(0.3 * N);
40
41 %% Simulation loop
42 for n = 1:lengthSound
43
44     % Update equation Eq. (2.44)
45     uNext(range) = (2 - 2 * lambdaSq) * u(range) ...
46                     + lambdaSq * (u(range+1) + u(range-1)) - uPrev(range);
47
48     % boundary updates Eq. (2.49)
49     if bcLeft == "N"
50         uNext(1) = (2 - 2 * lambdaSq) * u(1) - uPrev(1) ...
51                     + 2 * lambdaSq * u(2);

```

### C.3. 2D wave equation (Section 6.2)

```

51 end
52
53 % Eq. (2.50)
54 if bcRight == "N"
55     uNext(N+1) = (2 - 2 * lambdaSq) * u(N+1) - uPrev(N+1) ...
56     + 2 * lambdaSq * u(N);
57 end
58
59 out(n) = u(outLoc);
60
61 % Update system states
62 uPrev = u;
63 u = uNext;
64 end

```

## C.3 2D wave equation (Section 6.2)

```

1
2 %% Initialise variables
3 fs = 44100;           % Sample rate [Hz]
4 k = 1 / fs;           % Time step [s]
5 lengthSound = fs;     % Length of the simulation (1 second) [samples]
6
7 rho = 7850;           % Material density [kg/m^3]
8 H = 0.0005;            % Thickness [m]
9 T = 1000000;          % Tension per unit length [N/m]
10 c = sqrt(T / (rho * H)); % Wave speed [m/s]
11
12 Lx = 1;               % Length in x direction [m]
13 Ly = 2;               % Length in y direction [m]
14
15 h = sqrt(2) * c * k; % Grid spacing [m]
16 Nx = floor(Lx/h);    % Number of intervals in x direction
17 Ny = floor(Ly/h);    % Number of intervals in y direction
18 h = min(Lx/Nx, Ly/Ny); % Recalculation of grid spacing
19
20 lambdaSq = c^2 * k^2 / h^2; % Courant number squared
21 h = max(Lx/Nx, Ly/Ny); % Recalculation of grid spacing
22
23 %% Create scheme matrices with Dirichlet boundary conditions
24 Nxu = Nx - 1;
25 Nyu = Ny - 1;
26 Dxx = toeplitz([-2, 1, zeros(1, Nxu-2)]);
27 Dyy = toeplitz([-2, 1, zeros(1, Nyu-2)]);
28
29 % Kronecker sum
30 D = kron(speye(Nxu), Dyy) + kron(Dxx, speye(Nyu));
31 D = D / h^2;
32

```

## Appendix C. Code Snippets

```
33 % Total number of grid points
34 Nu = Nxu * Nyu;
35
36 %% Initialise state vectors (one more grid point than the number of
   intervals)
37 uNext = zeros(Nu, 1);
38 u = zeros(Nu, 1);
39
40 %% Initial conditions (2D raised cosine)
41 halfWidth = floor(min(Nx, Ny) / 5);
42 width = 2 * halfWidth + 1;
43 xLoc = floor(0.3 * Nx);
44 yLoc = floor(0.6 * Ny);
45 xRange = xLoc-halfWidth : xLoc+halfWidth;
46 yRange = yLoc-halfWidth : yLoc+halfWidth;
47
48 rcMat = zeros(Nyu, Nxu);
49 rcMat(yRange, xRange) = hann(width) * hann(width)';
50
51 % initialise current state
52 u = reshape(rcMat, Nu, 1);
53
54 % Set initial velocity to zero
55 uPrev = u;
56
57 % Output location
58 xOut = 0.45;
59 yOut = 0.25;
60 outLoc = round((xOut + yOut * Nyu) * Nxu);
61 out = zeros(lengthSound, 1);
62
63 %% Simulation loop
64 for n = 1:lengthSound
65
66     %% Update equation Eq. (6.22)
67     uNext = (2 * eye(Nu) + c^2 * k^2 * D) * u - uPrev;
68
69     % Update system states
70     uPrev = u;
71     u = uNext;
72
73 end
```

## Appendix D

# Intuition for the Damping Terms in the Stiff String PDE

This appendix will delve deeper into the damping terms in the PDE of the stiff string presented in Chapter 4, especially the frequency-dependent damping term  $2\sigma_1 \partial_t \partial_x^2 u$ . Recalling the compact version of the PDE of the stiff string in Eq. (4.5):

$$\partial_t^2 u = c^2 \partial_x^2 u - \kappa^2 \partial_x^4 u - 2\sigma_0 \partial_t u + 2\sigma_1 \partial_t \partial_x^2 u \quad (\text{D.1})$$

Consider first the frequency-independent damping term  $-2\sigma_0 \partial_t u$ . The more positive the velocity  $\partial_t u$  is, i.e., the string is moving upwards the more this term applies a negative, or downwards force (/effect) on the string. Vice versa, a more negative velocity will make this term apply a more positive force on the string.

As for the frequency-dependent damping term, apart from the obvious  $\sigma_1$ , the effect of the term increases with an increase of  $\partial_t \partial_x^2 u$  which literally describes the ‘rate of change of the curvature’ of the string.

Let’s first talk about positive and negative curvature, i.e., when  $\partial_x^2 u > 0$  or  $\partial_x^2 u < 0$ . Counterintuitively, in the positive case, the curve points downwards. Think about the function  $f(x) = x^2$ . It has a positive curvature (at any point), but has a minimum. This can be proven by taking  $x = 0$  and setting grid spacing  $h = 1$ .

$$\begin{aligned} \delta_{xx} f(x) &= \frac{1}{h^2} (f(-1) - 2f(0) + f(1)), \\ &= \frac{1}{1^2} ((-1)^2 - 2 \cdot 0^2 + 1^2), \\ &= (1 - 0 + 1) = 2. \end{aligned} \quad (\text{D.2})$$

In other words, the second derivative of the function  $f(x) = x^2$  around  $x = 0$  is positive.

As the term does not only include a second-order spatial derivative but also a first-order time derivative, this is referred to as a positive or negative *rate of change* of the curvature, i.e., when  $\partial_t \partial_x^2 u > 0$  or  $\partial_t \partial_x^2 u < 0$ . A positive rate of change of curvature means that the string either has a positive curvature and is getting more positive, i.e., the string gets more curved over time, or that the string has a negative curvature and is getting less negative, i.e., the string gets less curved or ‘loosens up’ over time. In the same way, a negative rate of change of curvature means that the string either has a negative curvature and is getting more negative, or that the string has a positive curvature and is getting less positive.

Let’s see some examples. Take the same function described before, but now  $f$  changes over time,  $f(x, t) = tx^2$ . When  $t$  increases over time, the curvature gets bigger. Repeating the above with  $x = 0$  and grid spacing  $h = 1$ , but now with  $t = 2$  and step size  $k = 1$ , but now with a backwards time derivative yields:

$$\begin{aligned}\delta_{t-} \delta_{xx} f(x, t) &= \frac{1}{kh^2} \left( f(-1, 2) - 2f(0, 2) + f(1, 2) \right. \\ &\quad \left. - (f(-1, 1) - 2f(0, 1) + f(1, 1)) \right), \\ &= \frac{1}{1 \cdot 1^2} \left( 2 \cdot (-1)^2 - 2 \cdot 2 \cdot (0)^2 + 2 \cdot 1^2 \right. \\ &\quad \left. - (1 \cdot (-1)^2 - 2 \cdot 1 \cdot (0) + 1 \cdot (1^2)) \right), \\ &= 2 + 2 - (1 + 1) = 2.\end{aligned}$$

So the rate of change of the curvature is positive, i.e., the already positively curved function  $x^2$  gets more curved over time.

If the curvature around a point along a string gets more positive (or less negative) over time, the force applied to that point will be positive, effectively ‘trying’ to reduce the curvature. Vice versa, if the curvature around a point along a string gets more negative (or less positive) over time, the force applied will be negative, again ‘trying’ to reduce the curvature.

From an auditory point of view, higher curvature generally means higher frequency. As the frequency-dependent damping term reduces curvature along the string it effectively damps higher frequencies.

# Appendix E

## Derivations

### E.1 Webster's update equation (5.9)

This section derives the update equation for Webster's equation in Eq. (5.9):

$$\begin{aligned}
 \frac{\bar{S}}{k^2}(\Psi_l^{n+1} - 2\Psi_l^n + \Psi_l^{n-1}) &= c^2 \left( (\delta_{x-} S_{l+1/2})(\mu_{x-} \delta_{x+} \Psi_l^n) \right. \\
 &\quad \left. + (\mu_{x-} S_{l+1/2})(\delta_{x-} \delta_{x+} \Psi_l^n) \right), \\
 \Psi_l^{n+1} - 2\Psi_l^n + \Psi_l^{n-1} &= \frac{c^2 k^2}{\bar{S}} \left( \frac{1}{h} (S_{l+1/2} - S_{l-1/2}) \frac{1}{2h} \overbrace{(\Psi_{l+1}^n - \Psi_{l-1}^n)}^{\mu_{x+} \delta_{x-} \Psi_l^n = \delta_x \cdot \Psi_l^n} \right. \\
 &\quad \left. + \frac{1}{2} (S_{l+1/2} + S_{l-1/2}) \frac{1}{h^2} (\Psi_{l+1}^n - 2\Psi_l^n + \Psi_{l-1}^n) \right), \\
 \Psi_l^{n+1} &= 2\Psi_l^n - \Psi_l^{n-1} + \underbrace{\frac{\lambda^2}{2\bar{S}}}_{\lambda = \frac{ck}{h}} \left( S_{l+1/2} \Psi_{l+1}^n - S_{l+1/2} \Psi_{l-1}^n \right. \\
 &\quad \left. - S_{l-1/2} \Psi_{l+1}^n + S_{l-1/2} \Psi_{l-1}^n + S_{l+1/2} \Psi_{l+1}^n + S_{l+1/2} \Psi_{l-1}^n \right. \\
 &\quad \left. + S_{l-1/2} \Psi_{l+1}^n + S_{l-1/2} \Psi_{l-1}^n - 2(S_{l+1/2} + S_{l-1/2}) \Psi_l^n \right), \\
 \Psi_l^{n+1} &= 2\Psi_l^n - \Psi_l^{n-1} + \frac{\lambda^2}{2\bar{S}} \left( 2S_{l+1/2} \Psi_{l+1}^n + 2S_{l-1/2} \Psi_{l-1}^n - 4\bar{S} \Psi_l^n \right), \\
 \Psi_l^{n+1} &= 2\Psi_l^n - \Psi_l^{n-1} + \frac{\lambda^2 S_{l+1/2}}{\bar{S}} \Psi_{l+1}^n + \frac{\lambda^2 S_{l-1/2}}{\bar{S}} \Psi_{l-1}^n - 2\lambda^2 \Psi_l^n, \\
 \Psi_l^{n+1} &= 2(1 - \lambda^2) \Psi_l^n - \Psi_l^{n-1} + \frac{\lambda^2 S_{l+1/2}}{\bar{S}} \Psi_{l+1}^n + \frac{\lambda^2 S_{l-1/2}}{\bar{S}} \Psi_{l-1}^n.
 \end{aligned}$$

## E.2 Boundary terms webster's equation

This section derives process of obtaining the values for  $\epsilon_l$  and  $\epsilon_r$  such that the boundary terms in Webster's equation are strictly dissipative.

Starting with the second term in the energy balance in Eq. (5.27):

$$-c^2 \langle \delta_t \cdot \Psi_l^n, \delta_{x-} (S_{l+1/2}(\delta_{x+} \Psi_l^n)) \rangle_d^{\epsilon_l, \epsilon_r}. \quad (\text{E.1})$$

Using identity (3.15a)

$$\begin{aligned} \langle f_l^n, \delta_{x-} g_l^n \rangle_d^{\epsilon_l, \epsilon_r} &= -\langle \delta_{x+} f_l^n, g_l^n \rangle_d + f_N^n g_{N-1}^n - f_0^n g_0^n \\ &\quad + \frac{\epsilon_r}{2} f_N^n (g_N^n - g_{N-1}^n) + \frac{\epsilon_l}{2} f_0^n (g_0^n - g_{-1}^n), \end{aligned}$$

this can be rewritten to (with  $f = \delta_t \cdot \Psi$  and  $g = S_{l+1/2}(\delta_{x+} \Psi)$ )

$$-c^2 \langle \delta_t \cdot \Psi_l^n, \delta_{x-} (S_{l+1/2}(\delta_{x+} \Psi_l^n)) \rangle_d^{\epsilon_l, \epsilon_r} = c^2 \langle \delta_t \cdot \delta_{x+} \Psi_l^n, (S_{l+1/2}(\delta_{x+} \Psi_l^n)) \rangle_d - b.$$

where

$$b = b_r - b_l \quad (\text{E.2})$$

with

$$\begin{aligned} b_r &= c^2 (\delta_t \cdot \Psi_N^n) \left( S_{N-1/2} \underbrace{(\delta_{x-} \Psi_{N-1}^n)}_{(\delta_{x+} \Psi_{N-1}^n)} \right) \\ &\quad + \frac{\epsilon_r}{2} (\delta_t \cdot \Psi_N^n) \left( S_{N+1/2}(\delta_{x+} \Psi_N^n) - S_{N-1/2} \underbrace{(\delta_{x+} \Psi_{N-1}^n)}_{(\delta_{x-} \Psi_N^n)} \right) \end{aligned}$$

and

$$\begin{aligned} b_l &= c^2 (\delta_t \cdot \Psi_0^n) \left( S_{1/2}(\delta_{x+} \Psi_0^n) \right) \\ &\quad - \frac{\epsilon_l}{2} (\delta_t \cdot \Psi_0^n) \left( S_{1/2}(\delta_{x+} \Psi_0^n) - S_{-1/2} \underbrace{(\delta_{x+} \Psi_{-1}^n)}_{(\delta_{x-} \Psi_0^n)} \right) \end{aligned}$$

which can be rewritten to

$$b_r = c^2 (\delta_t \cdot \Psi_N^n) \left( \frac{\epsilon_r}{2} S_{N+1/2}(\delta_{x+} \Psi_N^n) + \left( 1 - \frac{\epsilon_r}{2} \right) S_{N-1/2}(\delta_{x-} \Psi_N^n) \right), \quad (\text{E.3})$$

$$b_l = c^2 (\delta_t \cdot \Psi_0^n) \left( \frac{\epsilon_l}{2} S_{-1/2}(\delta_{x-} \Psi_0^n) + \left( 1 - \frac{\epsilon_l}{2} \right) S_{1/2}(\delta_{x+} \Psi_0^n) \right). \quad (\text{E.4})$$

Then, for the centred radiating boundary condition in Eq. (5.17) to be strictly dissipative, i.e.,  $\delta_x \cdot \Psi_l^n = 0 \Rightarrow b_r = 0$  the special choice for  $\epsilon_r =$

$S_{N-1/2}/\mu_{xx}S_N$  needs to be made:

$$\begin{aligned}
\mathfrak{b}_r &= c^2(\delta_t \cdot \Psi_N^n) \left( \frac{S_{N-1/2}}{2\mu_{xx}S_N} S_{N+1/2}(\delta_{x+} \Psi_N^n) + \left(1 - \frac{S_{N-1/2}}{2\mu_{xx}S_N}\right) S_{N-1/2}(\delta_{x-} \Psi_N^n) \right), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} \left( \frac{S_{N+1/2}}{2\mu_{xx}S_N} (\delta_{x+} \Psi_N^n) + \left(1 - \frac{S_{N-1/2}}{2\mu_{xx}S_N}\right) (\delta_{x-} \Psi_N^n) \right), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} \left( 1 - \frac{S_{N-1/2}}{2\mu_{xx}S_N} \right) \left( \frac{\frac{S_{N+1/2}(\delta_{x+} \Psi_N^n)}{2\mu_{xx}S_N}}{\left(1 - \frac{S_{N-1/2}}{2\mu_{xx}S_N}\right)} + \delta_{x-} \Psi_N^n \right), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} \left( 1 - \frac{\epsilon_r}{2} \right) \left( \frac{\frac{S_{N+1/2}(\delta_{x+} \Psi_N^n)}{2\mu_{xx}S_N}}{\left( \frac{2\mu_{xx}S_N - S_{N-1/2}}{2\mu_{xx}S_N} \right)} + \delta_{x-} \Psi_N^n \right), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} \left( 1 - \frac{\epsilon_r}{2} \right) \left( \frac{S_{N+1/2}(\delta_{x+} \Psi_N^n)}{2\mu_{xx}S_N - S_{N-1/2}} + \delta_{x-} \Psi_N^n \right), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} \left( 1 - \frac{\epsilon_r}{2} \right) \left( \frac{S_{N+1/2}(\delta_{x+} \Psi_N^n)}{S_{N+1/2} + S_{N-1/2} - S_{N-1/2}} + \delta_{x-} \Psi_N^n \right), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} \left( 1 - \frac{\epsilon_r}{2} \right) (\delta_{x+} \Psi_N^n + \delta_{x-} \Psi_N^n), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} \left( 1 - \frac{\epsilon_r}{2} \right) \left( \frac{1}{h} (\Psi_{N+1}^n - \Psi_N^n + \Psi_N^n - \Psi_{N-1}^n) \right), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} \left( 1 - \frac{\epsilon_r}{2} \right) 2(\delta_x \cdot \Psi_N^n), \\
&= c^2(\delta_t \cdot \Psi_N^n) S_{N-1/2} (2 - \epsilon_r) (\delta_x \cdot \Psi_N^n).
\end{aligned} \tag{E.5}$$

The same can be done for  $\mathfrak{b}_l$  with  $\epsilon_l = S_{1/2}/\mu_{xx}S_0$  to get

$$\mathfrak{b}_l = c^2(\delta_t \cdot \Psi_0^n) S_{1/2} (2 - \epsilon_l) (\delta_x \cdot \Psi_0^n). \tag{E.6}$$

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Digital versions of musical instruments have been created for several decades, and for good reason! They are more compact, more easy to maintain, and less difficult to play than their real-life counterparts. One way to digitise an instrument is to record it and play back the samples, but this does not capture the entire range of expression of the real instrument. Simulating an instrument based on its physics, including its geometry and material properties, is much more flexible to player control. Although it requires more computational power to generate the sound in real time, the simulation could possibly go beyond what is physically possible. A violin growing into a cello, bowing your trumpet, your imagination is the limit...