Modular Physical Models in a Real-Time Interactive Application

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ABSTRACT

Through recent advances in processing power, physical modelling using finite-difference time-domain (FDTD) methods has gained an increased popularity. Though many different musical instrument models based on these methods exist, nearly all are based on the same underlying systems and interactions between them. This paper presents an application where individual resonator modules, such as strings, bars, membranes and plates, can be connected and interacted with in real time. Various excitations, including the bow, hammer and pluck, are implemented as well, allowing for expressive control and a wide sonic palette. Existing and non-existing model configurations can easily be implemented, modified and experimented with, as well as the parameters describing them.

1. INTRODUCTION

Modularity in physical modelling sound synthesis is not a new concept. The earliest example of a modular system for sound synthesis was due to Cadoz *et al.* [1], where their CORDIS system allowed complex instruments to be created using simple mass-spring systems. Later, Morrison and Adrien created Mosaic [2], a modular environment using modal synthesis [3]. Rabenstein *et al.* presented presented modular physical models with a block-based approach using wave digital filters [4] and digital waveguides [5].

Finite-difference time-domain (FDTD) methods, first used in a musical context by Ruiz [6], Hiller and Ruiz [7, 8] and later by Chaigne [9], also lend themselves to a modularity. In [10], Bilbao presents a modular environment where bars and plates are connected by nonlinear springs, and in [11] Bilbao *et al.* propose a modular environment including higher dimensional systems. Neither of these environments run in real time.

Due to a recent increase in computational power, FDTD methods have gained popularity in real-time applications [12]. A real-time modular environment using strings and lumped objects is presented in [13], and a real-time implementation of connected strings and bars using the FAUST programming language due to Südholt *et al.* in [14].

The body of stringed instruments can be simplified to a 2D system, as done in e.g. [15, 16]. The current work presents a real-time interactive modular environment where FDTD implementations strings, bars, membranes and plates can be connected and played by the user.

This work is part of a larger project, which will see this work in virtual reality (VR) in a "build your own instrument" environment, where it will be used as a sound engine.

As many instruments consist of the same components, one can avoid re-implementation of all models

an attempt to collect various implementations of physical models based on FDTD methods.

Consumer hardware

Resonator exciter [17]

linear resonator, nonlinear exciter

2. MODELS

This section describes the continuous-time equations of the systems used in the application. The transverse displacement of a system can be described by state variable $q(\boldsymbol{x}, t)$ with time $t \geq 0$ and spatial coordinate $\boldsymbol{x} \in \mathcal{D}$. Here, the dimensions and definition of domain $\mathcal{D}$ depend on the system at hand. The dynamics of a system can then be written using the following general form:

$$\mathcal{L}q = 0, \tag{1}$$

where linear partial differential operator $\mathcal{L}$ describes the dynamics of a model in isolation.

2.1 Stiff String and Bar

With reference to Eq. (1), consider a (damped) stiff string of length L (in m), its transverse displacement described by state variable $q = u(\chi, t)$ (in m). Here spatial coordinate χ is defined over domain $\mathcal{D} = \mathcal{D}_\text{s} = [0, L]$. Furthermore $\mathcal{L} = \mathcal{L}_\text{s}$ and is defined as [18]

$$\mathcal{L}_\text{s} = \rho_\text{s} A\partial_t^2 - T_\text{s}\partial_\chi^2 + E_\text{s} I\partial_\chi^4 + 2\sigma_{0,\text{s}}\rho_\text{s} A\partial_t - 2\sigma_{1,\text{s}}\rho_\text{s} A\partial_t\partial_\chi^2, \tag{2}$$

where ∂_t and ∂_χ denote partial differentiation with respect to time and space respectively. The model is parameterised by material density ρ_s (in kg/m^3), cross-sectional area $A = \pi r^2$ (in m^2), radius r (in m), tension T (in N), Young's modulus E_s (in Pa), area moment of inertia $I = \pi r^4/4$ and loss coefficients $\sigma_{0,\text{s}}$ (in s^{-1}) and $\sigma_{1,\text{s}}$ (in m^2/s). If $T = 0$, the model reduces to a bar. Note that a circular cross-section is assumed here.

The boundary conditions are chosen to be simply supported as

$$u = \partial_\chi^2 u = 0, \quad \text{for}, \quad \chi = 0, L. \tag{3}$$

2.2 Thin Plate and (Stiff) Membrane

Consider a rectangular stiff membrane with side lengths L_x and L_y (both in m). With reference to Eq. (1) its transverse displacement can be described by $q = w(x, y, t)$ (in m), which is defined for domain $\mathcal{D} = \mathcal{D}_\text{p} = [0, L_x] \times [0, L_y]$, and $\mathcal{L} = \mathcal{L}_\text{p}$ is defined as [19]

$$\begin{aligned}\mathcal{L}_\text{p} = \rho_\text{p} H \partial_t^2 - T_\text{p}\Delta + D\Delta\Delta + 2\sigma_{0,\text{p}}\rho_\text{p} H \partial_t \\ - 2\sigma_{1,\text{p}}\rho_\text{p} H \partial_t \Delta.\end{aligned} \tag{4}$$

Parameters are: material density ρ_p (in kg/m^3), thickness H (in m), tension per unit length T_p (in N/m), stiffness coefficent $D = E_\text{p} H^3/12(1-\nu^2)$ (in kg $\cdot$ m$^2 \cdot$s^{-2}), Young's modulus E_p (in Pa), dimensionless Poisson's ratio ν, and loss coefficients $\sigma_{0,\text{p}}$ (in s^{-1}) and $\sigma_{1,\text{p}}$ (in m^2/s). If $T_\text{p} = 0$, the model reduces to a thin plate, and if $D = 0$ it reduces to the 2D wave equation, which can be used to model a membrane (without stiffness).

Boundary conditions are chosen to be clamped, such that

$$w = \mathbf{n} \cdot \nabla w = 0, \tag{5}$$

where ∇ denotes 'the gradient of', and $\mathbf{n}$ is a normal to the plate area at the boundary.

2.3 Connections

One can add connections to the models presented above by extending the general form in Eq. (1). Consider M models q_m indexed by $m \in \mathcal{M}$, where $\mathcal{M} = \{1, \dots, M\}$ and C connections between them. A connection is indexed by $c \in \mathcal{C}$ with $\mathcal{C} = \{1, \dots, C\}$, and is characterised by the indices of the models it connects – $r_c \in \mathcal{M}$ and $s_c \in \mathcal{M}$ – and the locations where these models are connected – $\boldsymbol{x}_{r,c} \in \mathcal{D}_{r_c}$ and $\boldsymbol{x}_{s,c} \in \mathcal{D}_{s_c}$. Here, domains $\mathcal{D}_{r_c}$ and $\mathcal{D}_{s_c}$ are the (spatial) domains that models q_{r_c} and q_{s_c} are defined for.

Model q_{r_c} will be placed 'below' q_{s_c} such that the connection force acts positively on the former and negatively on the latter. The general form in Eq. (1) can then be extended to include connections according to

$$\mathcal{L}_m q_m = \sum_{\substack{c\in\mathcal{C}\\ r_c = m}} \delta(\boldsymbol{x}_m - \boldsymbol{x}_{r,c}) f_c - \sum_{\substack{c\in\mathcal{C}\\ s_c = m}} \delta(\boldsymbol{x}_m - \boldsymbol{x}_{s,c}) f_c, \tag{6}$$

where $f_c = f_c(t)$ is the force (in N) of the c^th connection.

To illustrate, consider two models ($M = 2$), a string and a membrane, such that $q_1 = u(\chi, t)$ and $q_2 = w(x, y, t)$, and a single connection between them ($C = 1$). Placing the string below the membrane, we get that $r_1 = 1$ and $s_1 = 2$, i.e., the 'below' model of connection 1 has index 1, the 'above' model of connection 1 has index 2. Filling in the partial differential operators for the string and membrane found in Eqs. (2) and (4) respectively, Eq. (20) becomes for the string and the membrane respectively

$$\mathcal{L}_\text{s} u = \delta(\chi - \chi_\text{c}) f_1, \tag{7a}$$
$$\mathcal{L}_\text{s} w = -\delta(x - x_\text{c}, y - y_\text{c}) f_1. \tag{7b}$$

In this work, three possible connections are considered: rigid, the linear spring and the nonlinear spring. First, the rigid connection assumes that the connected systems have an identical displacement at their respective connection locations. Applied to Eq. (7a), the following indicates a rigid connection [20]:

$$u(\chi_\text{c}, t) = w(x_\text{c}, y_\text{c}, t). \tag{8}$$

Alternatively, as done in [10, 20], one can use a spring to connect two systems. A connection force due to a nonlinear damped spring is

$$f_\text{c} = K_1\eta + K_3\eta^3 + R\partial_t\eta, \tag{9}$$

where, K_1 is a linear spring coefficient (in N/m), K_3 is a nonlinear spring coefficient (in N/m^3), and R is a damping coefficient (in s^{-1}). If $K_3 = 0$, Eq. (9) reduces to a linear spring. Furthermore, $\eta = \eta(t) = w(x_\text{c}, y_\text{c}, t) - u(\chi_\text{c} t)$ is the relative displacement of the two systems at their respective connection locations (in m). [1] Section 3.2 will elaborate on how to calculate the connection forces in all cases. check

2.4 Excitations

In this work, as excitations will only be applied to 1D systems, the state variable of the stiff string, i.e., $u(\chi, t)$, will be used for the presentation of the various excitations. For the string, the general form in Eq. (1) can thus be extended to

$$\mathcal{L}_\text{s} u = e_\text{e} f_\text{e}, \tag{10}$$

where $e_\text{e} = e_\text{e}(\chi)$ is an excitation distribution and $f_\text{e} = f_\text{e}(t)$ is the externally supplied excitation force (in N).

2.4.1 *The Bow*

As done in previous work, see e.g. [15], one can include a bowing interaction by introducing a static friction model. With reference to Eq. (10), the excitation force can be defined as [20]

$$f_\text{e} = -f_\text{B}\sqrt{2a}v_\text{rel}e^{-av_\text{rel}^2 + 1/2} \tag{11}$$

with externally supplied bow force $f_\text{B} = f_\text{B}(t)$ (in N), dimensionless free parameter a and

$$v_\text{rel} = \partial_t u(\chi_\text{B}, t) - v_\text{B} \tag{12}$$

is the relative velocity (in m/s) between the string at externally supplied bowing location $\chi_\text{B} = \chi_\text{B}(t)$ (in m) and the externally supplied bow velocity $v_\text{B} = v_\text{B}(t)$ (in m/s).

Furthermore, the excitation distribution is set to be a single point along the string:

$$e_\text{e} = \delta(\chi - \chi_\text{B}). \tag{13}$$

[1] If the string would be placed above the membrane, $\eta(t) = u(\chi_\text{c} t) - w(x_\text{c}, y_\text{c}, t)$.

2.4.2 *Hammer*

Another way of exciting a system is to use a hammer, which can be modelled as a simple mass-spring system with state variable $z(t)$. The interaction between the hammer and the string is then modelled as a collision, using the following collision potential [21]:

$$\phi(\eta_\mathrm{e}) = \frac{K_\mathrm{e}}{\alpha_\mathrm{e}+1}[\eta_\mathrm{e}]_+^{\alpha_\mathrm{e}+1}, \tag{14}$$

with collision stiffness $K_\mathrm{e} \geq 0$ (in N/m$^{\alpha_\mathrm{e}}$) and nonlinear collision coefficient $\alpha_\mathrm{e} \geq 1$. Furthermore, $\eta_\mathrm{e} = \eta_\mathrm{e}(t) = \theta\left(u(\chi_\mathrm{e}, t) - z(t)\right)$ is the relative displacement between the string at collision location χ_e (in m) and the hammer (in m), and $[\cdot] = 0.5(\cdot + |\cdot|)$ describes the 'positive part of', making the collision one-sided. Furthermore, $\theta = \tau$ if the string should be excited from above, and $\theta = -\tau$ if it should be excited from below. Here, $\tau = 1$ if the hammer interaction is triggered, and $\tau = 0$ if not.

Using a change of variables based on energy quadratisation presented in [22], one can rewrite the collision potential as

$$f_\mathrm{e} = -\theta\psi\psi' \tag{15}$$

where $\psi = \psi(\eta) = \sqrt{2\phi}$, and using dots to denote a temporal derivative, $\psi' = \dot{\psi}/\dot{\eta}$. This change of variables ultimately allows for an explicit implementation of the nonlinear collision.

Using the above, the dynamics of the hammer, including the collision with the string, can be described by the following PDE

$$M\partial_t^2 z = -Kz + \theta\psi\psi' + f_\mathrm{off} \tag{16}$$

with mass M (in kg), spring constant K (in N/m) and $f_\mathrm{off} = f_\mathrm{off}(t) = Kz_\mathrm{off}$ is an external force (in N) used to counteract the spring force, where $z_\mathrm{off} = z_\mathrm{off}(t)$ is an externally supplied offset (in m). This force should keep the mass away from the equilibrium (and thus the system it excites) when controlling the application without wanting to excite the system. If the hammer excitation is triggered, z_off will be set to 0 and the spring force will pull the hammer towards the system, colliding with it in the process. After the collision, z_off will be set to be user-controlled again to avoid continuous collision between the hammer and the string.

Finally, with reference to Eq. (10), the excitation distribution is set to be a raised cosine with centre location χ_e and excitation width e_w (in m):

$$e_\mathrm{e}(\chi) = \begin{cases} \frac{1-\cos\left(\frac{2\pi(\chi-\chi_\mathrm{e})}{e_\mathrm{w}}+\pi\right)}{2} & \text{if } \chi_\mathrm{e} - \frac{e_\mathrm{w}}{2} \leq \chi \leq \chi_\mathrm{e} + \frac{e_\mathrm{w}}{2} \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

2.4.3 *Pluck*

The pluck is modelled nearly the same way as the hammer. The main difference is that $\tau = 1$ until the collision force is larger than a certain value. This will be elaborated on in Section 3.3.

3. DISCRETE TIME

In order to implement the models described in Section 2 using FDTD methods, a spatio-temporal grid needs to be defined.

For all models, time is discretised to $t = nk$ with time index $n = 0, 1, 2\ldots$ and time step $k = 1/f_\mathrm{s}$ (in s) where f_s is the sample rate (in Hz). For the stiff string, space is subdivided into N equal intervals of length h (in m) according to $\chi = ph$ with spatial index $p \in \{0, \ldots, N\}$.

In the case of the stiff membrane, the spatial coordinate is discretised onto as $(x, y) = (lh, mh)$ where spatial indices $l \in \{0, \ldots, N_x\}$ and $m \in \{0, \ldots, N_y\}$. Here, N_x and N_y are the number of intervals in the x and y direction respectively. Notice that the same value for grid spacing h is used for both the x and y directions.

Using these definitions, the general state variable $q(\boldsymbol{x}, t)$ can be approximated to grid function $q_{\boldsymbol{l}}^n$, where for the stiff string $\boldsymbol{l} = p$ yielding grid function u_p^n and for the stiff membrane $\boldsymbol{l} = (l, m)$ yielding grid function $z_{(l,m)}^n$.

3.1 FDTD schemes

The general form in Eq. (1) can be discretised to the following FDTD scheme

$$\ell q_{\boldsymbol{l}}^n = 0 \tag{18}$$

where ℓ is the discretised version of linear partial differential operator $\mathcal{L}$. The discrete-time definitions of Eqs. (2) and (4) will not be given here for brevity, but can be found in the literature (e.g. [20], [12]). Expanding Eq. (18) for definition of ℓ yields an update equation of the form

$$aq_{\boldsymbol{l}}^{n+1} = bq_{\boldsymbol{l}}^n + cq_{\boldsymbol{l}}^{n-1} \tag{19}$$

where a, b and c depend on the system at hand.

3.2 Connections

The general form in Eq. (20) can be discretised to

$$\ell_m q_{m,\boldsymbol{l}}^n = \sum_{\substack{c\in\mathcal{C}\\ r_c=m}} J_{m,\boldsymbol{l},0}(\boldsymbol{x}_{r,c}) f_c^n - \sum_{\substack{c\in\mathcal{C}\\ s_c=m}} J_{m,\boldsymbol{l},0}(\boldsymbol{x}_{s,c}) f_c^n, \tag{20}$$

where

$$J_{m,\boldsymbol{l},0}(\boldsymbol{x}) = \begin{cases} \frac{1}{h_m^d} & \text{if } \boldsymbol{x} = \lfloor \boldsymbol{l}/h_m \rfloor \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

is a spreading operator that localises the connection force of the c^th connection f_c^n to location $\boldsymbol{x}$. Here, d is the number of spatial dimensions the system is defined over ($d = 1$ for the string, $d = 2$ for the membrane).

For a rigid connection, the force can be solved for by

$$f_c^n = \frac{I_{s_c,\boldsymbol{l},c}(\boldsymbol{x})q_{s_c}^\star - I_{r_c,\boldsymbol{l},c}(\boldsymbol{x})q_{r_c}^\star}{\frac{\|J_{r_c,\boldsymbol{l},c}(\boldsymbol{x}_{r,c})\|_{d_{r_c}}^2}{a_{r_c}} + \frac{\|J_{r_c,\boldsymbol{l},c}(\boldsymbol{x}_{s,c})\|_{d_{s_c}}^2}{a_{s_c}}} \tag{22}$$

Using the following FDTD operators:

$$\mu_{t\cdot} q_{\boldsymbol{l}}^n = \frac{1}{2}\left(q_{\boldsymbol{l}}^{n+1} + q_{\boldsymbol{l}}^{n-1}\right) \tag{23}$$

$$\delta_{t\cdot} q_{\boldsymbol{l}}^n = \frac{1}{2k}\left(q_{\boldsymbol{l}}^{n+1} - q_{\boldsymbol{l}}^{n-1}\right) \tag{24}$$

the connection force in Eq. (9) can be discretised according to

$$f_{\mathrm{c}}^{n} = K_1 \mu_{t\cdot}\eta^n + K_3(\eta^n)^2\mu_{t\cdot}\eta^n + R\delta_{t\cdot}\eta^n. \tag{25}$$

It can be shown that this discretisation yields a stable system [20].

3.3 Excitations

For the bow, a cubic interpolator J_3 is introduced

$$J_{p,3}(\chi_{\mathrm{i}}) = \frac{1}{h}\begin{cases} -\alpha_{\mathrm{i}}(\alpha_{\mathrm{i}}-1)(\alpha_{\mathrm{i}}-2)/6, & p = p_{\mathrm{i}}-1, \\ (\alpha_{\mathrm{i}}-1)(\alpha_{\mathrm{i}}+1)(\alpha_{\mathrm{i}}-2)/2, & p = p_{\mathrm{i}}, \\ -\alpha_{\mathrm{i}}(\alpha_{\mathrm{i}}+1)(\alpha_{\mathrm{i}}-2)/2, & p = p_{\mathrm{i}}+1, \\ \alpha_{\mathrm{i}}(\alpha_{\mathrm{i}}+1)(\alpha_{\mathrm{i}}-1)/6, & p = p_{\mathrm{i}}+2, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

where $l_{\mathrm{i}} = \lfloor \chi_{\mathrm{i}}/h \rfloor$ and $\alpha_{\mathrm{i}} = \chi_{\mathrm{i}}/h - l_{\mathrm{i}}$.

For the pluck, if $|J_{p,0}(\chi_{\mathrm{e}})f_{\mathrm{e}}| > \varphi$ where φ is a threshold,

Note that the force will be scaled by the grid spacing through the inclusion of $J_{p,0}$ such that the plucking interaction will be similar for strings with different values of h.

4. REAL-TIME APPLICATION

Only 1D models can be excited

The application is divided into three parts: the control panel (bottom), the excitation panel (right), and the instrument area.

An extensive demo of the application, going through all of the functionality described in this section, can be found online [2]

4.1 Model Interaction

scroll wheel is linked to $-0.2 \leq v_{\mathrm{B}} \leq 0.2$ if the bowing excitation is chosen. If instead the hammer or pluck interaction are chosen, the scrollwheel is linked to the excitation width $h \leq e_{\mathrm{w}} \leq 10h$. .

The direction of the bowing is visualised with a moving gradient

4.2 Control panel

If a button is clicked

The states of all resonator modules will be set to 0.

Instructions will be shown...

4.2.1 Modules

The various modules a user can choose are the following

- Stiff string
- Bar
- Membrane
- Thin Plate
- Stiff Membrane

[2] youtubelink here

	None	Rigid	Linear	Nonlinear
No graphics	3.8	8.3	13	13
Graphics	12.7	28.4	42.3	43.1

Table 1. CPU usage (in %) of two identical strings ($N = 118$) with all moving grid points connected in various ways.

2D models get an extra parameters: maxPoints, so as to not overload the CPU

4.2.2 Outputs

The output of any model can be obtained by listening to $q(\boldsymbol{l}_{\mathrm{o}}, t)$ for an output location $\boldsymbol{l}_{\mathrm{o}}$. In the application it is possible to retrieve

One can adjust the listening points

Left, right and stereo channels are shown in white, red and yellow respectively.

4.2.3 Connections

Rigid connections are shown in green, linear springs in orange and non-linear springs in magenta.

Although experiments with overlapping connections have been done, it has been decided to exclude these from the application. Following [10], although any number of overlapping connections can be explicitly calculated, one needs to do a matrix inverse every sample. Therefore, it was decided to exclude overlapping connections from the application.

4.2.4 Groups

Various models can be grouped together

4.2.5 Presets

4.3 Excitation

The excitation panel contains a dropdown menu including the bow, hammer and pluck excitations.

If the hammer is triggered when the mass is further away from the string, the excitation will be of higher amplitude.

5. RESULTS

6. DISCUSSION

To the best of the authors' knowledge, the way of real-time interactive excitation used for the hammer and the pluck has not been done before.

7. CONCLUSION

Applying excitations to 2D systems.

Port to Virtual Reality

Adding inputs so that the application can be used as an effect.

Acoustic tubes

Exciting resonator with exciter of another (bowed tube, lip-excited string)

Pitch change

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