Ecuații diferențiale omogene

Ideea de lucru: Rezolvăm prin substituție a.î să aducem ecuația la una ce știm să rezolvăm.

Cum realizăm acest lucru?

Dacă putem scrie ec dif ca $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ atunci putem face o substituție.

Fie $v = \frac{y}{x} \Rightarrow y = vx$ Dacă derivăm obținem:

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \cdot 1$$

Pașii de rezolvare:

1. Scriem ec ca $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

2. Substituim: $v = \frac{y}{x}$

3. Rezolvăm pt "y": $y = v \cdot x$

4. Găsim $\frac{dy}{dx} = x\frac{dv}{dx} + v$

5. Substituim în ec dif: x + v DOAR (fără y)

6. Folosim o tehnică pe care o cunoaștem deja.

7. Resubstituim pt a obține înapoi "y"
$v = \frac{y}{x}$

Obs: 1) Trebuie să înlocuim toți y

Exerciții:

① $2xy\frac{dy}{dx} = x^2 + 2y^2 \quad \Big| \cdot \frac{1}{2xy} \Rightarrow$

$\Rightarrow 2\cancel{x}y\frac{dy}{dx} \cdot \frac{1}{2\cancel{x}y} = \frac{x^{\cancel{2}}}{2\cancel{x}y} + \frac{\cancel{2}y^{\cancel{2}}}{\cancel{2}x\cancel{y}} \Rightarrow \frac{dy}{dx} = \frac{x}{2y} + \frac{y}{x} \Rightarrow$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x}\right)^{-1} + \left(\frac{y}{x}\right) \Rightarrow x\frac{dv}{dx} + v = \frac{1}{2}v^{-1} + v$

subst: $v = \frac{y}{x} \Rightarrow y = v \cdot x$ $\qquad \Rightarrow x\frac{dv}{dx} = \frac{1}{2}v^{-1} + \cancel{v} - \cancel{v} \Rightarrow$

$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$ $\qquad \Rightarrow x\frac{dv}{dx} = \frac{1}{2}\cdot\frac{1}{v} \quad \Big| \cdot \frac{v\,dx}{x}$

$\Rightarrow \cancel{x}\cdot\frac{dv}{\cancel{dx}}\cdot\frac{v\,\cancel{dx}}{\cancel{x}} = \frac{1}{2}\cdot\frac{1}{\cancel{v}}\cdot\frac{\cancel{v}\,dx}{x} \Rightarrow v\,dv = \frac{1}{2x}\,dx \Rightarrow$

$= \int v\,dv = \int\frac{1}{2x}\,dx \Rightarrow \frac{v^2}{2} = \frac{1}{2}\ln|x| + C_1 \quad \Big| \cdot 2 \Rightarrow$

$\Rightarrow \cancel{2}\cdot\frac{v^2}{\cancel{2}} = \cancel{2}\cdot\frac{1}{\cancel{2}}\ln|x| + 2C_1 \Rightarrow v^2 = \ln|x| + \underbrace{2C_1}_{=C} \Rightarrow$

$\Rightarrow v^2 = \ln|x| + C \xrightarrow{\text{înlocuim}} \left(\frac{y}{x}\right)^2 = \ln|x| + C \Rightarrow \frac{y^2}{x^2} = \ln|x| + C \quad \Big| \cdot x^2$

$\Rightarrow \frac{y^2}{\cancel{x^2}}\cdot\cancel{x^2} = x^2\ln|x| + C \Rightarrow y^2 = x^2(\ln|x| + C)$

② $x\frac{dy}{dx} = y + 2\sqrt{xy}$

$$x\frac{dy}{dx} = y + 2\sqrt{xy} \;\Big|\cdot\frac{1}{x} \Rightarrow \frac{1}{\cancel{x}}\cdot\cancel{x}\frac{dy}{dx} = \frac{1}{x}\cdot y + \frac{1}{x}\cdot 2\sqrt{xy}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) + \frac{2\sqrt{xy}}{x} \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) + 2\sqrt{\frac{\cancel{x}y}{x^{\cancel{2}}}} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) + 2\left(\frac{y}{x}\right)^{1/2} \Rightarrow x\frac{dv}{dx} + v = v + 2v^{1/2}$$

subst: $v = \frac{y}{x} \Rightarrow y = vx$ $\qquad \Rightarrow x\frac{dv}{dx} = \cancel{v} + 2v^{1/2} - \cancel{v}$

$\Rightarrow y = x\frac{dv}{dx} + v$ $\qquad \Rightarrow x\frac{dv}{dx} = 2v^{1/2} \;\Big|\cdot\frac{dx}{xv^{1/2}}$

$$\Rightarrow \cancel{x}\frac{dv}{\cancel{dx}}\cdot\frac{\cancel{dx}}{\cancel{x}\cdot v^{1/2}} = 2\cancel{v^{1/2}}\cdot\frac{dx}{x\cancel{v^{1/2}}} \Rightarrow \frac{1}{v^{1/2}}dv = \frac{2}{x}dx \Rightarrow$$

$$\Rightarrow \int\frac{1}{v^{1/2}}dv = \int\frac{2}{x}dx \Rightarrow \int v^{-1/2}dv = 2\int\frac{1}{x}dx \Rightarrow$$

$$\Rightarrow \frac{v^{-\frac{1}{2}+1^{\frac{2}{2}}}}{-\frac{1}{2}+1^{\frac{2}{2}}} = 2\ln|x| + C_1 \;\Big|\cdot\frac{1}{2} \Rightarrow \frac{1}{\cancel{2}}\cdot\cancel{2}\cdot v^{1/2} = \frac{1}{\cancel{2}}\cdot\cancel{2}\ln|x| + \frac{C_1}{\cancel{2}} \;\Big|^2$$

$$\Rightarrow \left(v^{\frac{1}{\cancel{2}}}\right)^{\cancel{2}} = (\ln|x| + C)^2 \Rightarrow v = (\ln|x| + C)^2$$

blavimo C $\Rightarrow$

$$\Rightarrow \frac{y}{x} = (\ln|x| + c)^2 \Rightarrow y = x(\ln|x| + c)^2$$

③ $(x-y)\frac{dy}{dx} = x + y \;\Big|:(x-y)$

$$\frac{1}{\cancel{x-y}} \cdot \cancel{(x-y)} \frac{dy}{dx} = \frac{x+y}{x-y} \quad \Big| \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{\cancel{x}}{\cancel{x}} + \frac{y}{x}}{\frac{\cancel{x}}{\cancel{x}} - \frac{y}{x}} \Rightarrow \frac{dy}{dx} = \frac{1+\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)} \Rightarrow x\frac{dv}{dx} + v = \frac{1+v}{1-v}$$

subst: $v = \frac{y}{x} \Rightarrow y = vx$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} \frac{1-v}{-v} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 + \cancel{v} - \cancel{v} + v^2}{1-v} \Rightarrow$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1-v} \quad \Big| \frac{dx}{x\left(\frac{1+v^2}{1-v}\right)} \Rightarrow \frac{\cancel{dx}}{\cancel{x}\left(\frac{1+v^2}{1-v}\right)} \cdot \cancel{x}\frac{dv}{\cancel{dx}} = \cancel{\frac{1+v^2}{1-v}} \cdot \frac{dx}{x\cancel{\left(\frac{1+v^2}{1-v}\right)}}$$

$$\Rightarrow \frac{1-v}{1+v^2} dv = \frac{1}{x} dx \Rightarrow \int \frac{1-v}{1+v^2} dv = \int \frac{1}{x} dx \Rightarrow$$

$$\Rightarrow \int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv = \int \frac{1}{x} dx \Rightarrow \tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) =$$

$$= \ln|x| + C \xrightarrow{\text{hlacuim}} \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left[\left(1+\left(\frac{y}{x}\right)^2\right)\right] = \ln|x| + C$$

$$\int \frac{v}{1+v^2} dv \qquad u = 1+v^2 \qquad \Rightarrow \int \frac{1}{u} \cdot \frac{du}{2} = \frac{1}{2}\int \frac{1}{u} du$$

$$du = 2v\,dv \quad | : 2$$

$$\frac{du}{2} = v\,dv \qquad \Rightarrow \frac{1}{2}\ln|u| = \frac{1}{2}\ln\underbrace{|1+v^2|}_{(+)}$$

$$④ \quad (x+y)\frac{dy}{dx} = x - y \quad \Big| \cdot \frac{1}{(x+y)} \Rightarrow$$

$$\Rightarrow \cancel{\frac{1}{(x+y)}} \cdot \cancel{(x+y)} \cdot \frac{dy}{dx} = \frac{x-y}{x+y} \Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y} \Big| \frac{\cdot \frac{1}{x}}{\cdot \frac{1}{x}} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cancel{\frac{x}{x}} - \frac{y}{x}}{\cancel{\frac{x}{x}} + \frac{y}{x}} \Rightarrow \frac{dy}{dx} = \frac{1 - \left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)} \Rightarrow x\frac{dv}{dx} + v = \frac{1-v}{1+v} \Rightarrow$$

subst: $v = \frac{y}{x} \Rightarrow y = vx$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v\frac{1+v}{} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v-v-v^2}{1+v} \Rightarrow$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v} \Rightarrow x\frac{dv}{dx} = \frac{-(v^2+2v-1)}{v+1} \quad \Big| \cdot \frac{dx}{\frac{-(v^2+2v-1)}{v+1} x}$$

$$\Rightarrow \frac{\cancel{dx}}{\frac{-(v^2+2v-1)}{v+1} \cdot \cancel{x}} \cdot \cancel{x}\frac{dv}{\cancel{dx}} = \cancel{\frac{-(v^2+2v-1)}{v+1}} \cdot \frac{dx}{\cancel{\frac{-(v^2+2v-1)}{v+1}} \cdot x} \quad \Big| \cdot (-1)$$

$$\Rightarrow \frac{v+1}{v^2+2v-1}\, dv = -\frac{1}{x}\, dx \Rightarrow \int \frac{v+1}{v^2+2v-1}\, dv = \underbrace{\int -\frac{1}{x}\, dx}_{-\ln|x| + C_1}$$

$$\int \frac{v+1}{v^2+2v-1}\, dv$$

$$u = v^2 + 2v - 1$$

$$du = 2v + 2\, dv \quad |:2$$

$$\frac{du}{2} = v + 1\, dv$$

$$= \int \frac{1}{u} \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int \frac{1}{u}\, du \Rightarrow \frac{1}{2}\ln|u| = \frac{1}{2}\ln|v^2+2v-1|$$

$$\Rightarrow \frac{1}{2}\ln|u| = -\ln|x| + C_1$$

$$\Rightarrow \frac{1}{2}\ln|v^2+2v-1| = -\ln|x| + C_1 \quad \Big| \cdot 2 \Rightarrow \cancel{2}\cdot\frac{1}{\cancel{2}}\ln|v^2+2v-1| =$$

$$= -2\ln|x| + 2C_1 \Rightarrow \cancel{e}^{\cancel{\ln}|v^2+2v-1|} = \cancel{e}^{-2\cancel{\ln}|x|^{2}+2C_1} \Rightarrow$$

$$\Rightarrow |v^2+2v-1| = |x|^{-2}\cdot e^{2C_1} \Rightarrow v^2+2v-1 = \underbrace{\pm e^{2C_1}}_{C}\cdot x^{-2} \Rightarrow$$

$$\Rightarrow v^2+2v-1 = \frac{C}{x^2} \xrightarrow{\text{Inlocuim}} \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1 = \frac{C}{x^2} \Rightarrow$$

$$\Rightarrow \frac{y^2}{x^2} + \frac{2y}{x} - 1 = \frac{C}{x^2} \quad \Big| \cdot x^2 \Rightarrow \cancel{x^2}\cdot\frac{y^2}{\cancel{x^2}} + x^{\cancel{2}}\cdot\frac{2y}{\cancel{x}} - x^2 = \cancel{x^2}\cdot\frac{C}{\cancel{x^2}}$$

$$\Rightarrow y^2 + 2xy - x^2 = C$$

5) $$x(x+y)\frac{dy}{dx} = y(x-y) \quad \Big| : x(x+y)$$

$$\Rightarrow \frac{1}{\cancel{x(x+y)}}\cdot\cancel{x(x+y)}\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)} \Rightarrow \frac{dy}{dx} = \frac{y(x-y)}{x(x+y)} \quad \Big| \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\cdot\frac{1-\frac{y}{x}}{1+\frac{y}{x}} \Rightarrow x\frac{dv}{dx} + v = v\cdot\frac{1-v}{1+v} \Rightarrow x\frac{dv}{dx} = v\cdot\frac{1-v}{1+v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\cancel{v}-v^2-\cancel{v}-v^2}{1+v} \Rightarrow$$

subst: $v = \frac{y}{x} \Rightarrow y = vx$

$$\Rightarrow x\frac{dv}{dx} = \frac{-2v^2}{1+v} \quad \Big| \cdot \frac{dx}{\left(\frac{-2v^2}{1+v}\right)\cdot x}$$

$$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\Rightarrow \frac{\cancel{dx}}{\left(\frac{-2v^2}{1+v}\right)\cdot\cancel{x}}\cdot\cancel{x}\frac{dv}{\cancel{dx}} = -\frac{2\cancel{v^2}}{\cancel{1+v}}\cdot\frac{dx}{\cancel{\left(\frac{-2v^2}{1+v}\right)}\cdot x} \Rightarrow$$

$$\frac{1+v}{-2v^2}\,dv = \frac{1}{x}\,dx \;\Big|\cdot(-2) \Rightarrow \frac{1+v}{v^2}\,dv = -\frac{2}{x}\,dx \Rightarrow$$

$$\Rightarrow \int\left(\frac{1+v}{v^2}\right)dv = \int -\frac{2}{x}\,dx = \int\frac{1}{v^2}\,dv + \int\frac{\cancel{v}}{v^{\cancel{2}}}\,dv = -2\int\frac{1}{x}\,dx$$

$$\Rightarrow \frac{v^{-2+1}}{-2+1} + \ln|v| = -2\ln|x| + C_1 \Rightarrow -\frac{1}{v} + \ln|v| = \ln|x|^{-2} + C_1$$

$$\Rightarrow \ln|v| + \ln|x|^2 = \frac{1}{v} + C_1 \Rightarrow \ln|v\cdot x^2| = \frac{1}{v} + C_1$$

Wlocuim $\Rightarrow \ln\left|\frac{y}{\cancel{x}}\cdot x^{\cancel{2}}\right| = \frac{1}{\frac{y}{x}} + C_1 \Rightarrow \ln|xy| = \frac{x}{y} + C$

⑥ $x^2\frac{dy}{dx} = xy + x^2\cdot e^{\frac{y}{x}} \;\Big|\cdot\frac{1}{x^2} \Rightarrow$

$$\Rightarrow \frac{1}{\cancel{x^2}}\cdot\cancel{x^2}\frac{dy}{dx} = \frac{1}{x^{\cancel{2}}}\cdot\cancel{x}y + \frac{1}{\cancel{x^2}}\cdot\cancel{x^2}\cdot e^{\frac{y}{x}} \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) + e^{\left(\frac{y}{x}\right)}$$

$$\Rightarrow x\frac{dv}{dx} + v = v + e^v \Rightarrow x\frac{dv}{dx} = \cancel{v} + e^v - \cancel{v} \Rightarrow$$

subst: $v = \frac{y}{x} \Rightarrow y = vx$ $\qquad\Rightarrow x\frac{dv}{dx} = e^v \;\Big|\cdot\frac{dx}{x e^v} \Rightarrow$

$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$ $\qquad\Rightarrow \frac{\cancel{dx}}{\cancel{x}e^v}\cdot\cancel{x}\frac{dv}{\cancel{dx}} = \cancel{e^v}\cdot\frac{dx}{x\cancel{e^v}} \Rightarrow$

$$\Rightarrow \frac{1}{e^v}\,dv = \frac{1}{x}\,dx \Rightarrow \int e^{-v}\,dv = \int\frac{1}{x}\,dx \Rightarrow -e^{-v} = \ln|x| + C_1$$

$\int e^{-v}\,dv$ $\qquad u = -v$, $\quad du = -dv \,|\cdot(-1)$, $\quad -du = dv$ $\qquad\Rightarrow \int e^u\cdot -du = -e^{-v}$

$$\Rightarrow -e^{-v} = \ln|x| + C_1 \;\Big|\cdot(-1) \Rightarrow e^{-v} = -\ln|x| - C_1 \xRightarrow{\ln}$$

$$\ln e^{-v} = \ln\left[-\ln|x| \underbrace{- C_1}_{C}\right] \Rightarrow -v \underbrace{\ln e}_{1} = \ln\left[\ln\left|\frac{1}{x}\right| + C\right] \Rightarrow$$

$$\Rightarrow -v = \ln\left[\ln\left|\frac{1}{x}\right| + C\right]$$

$$\xRightarrow{\text{hlavní}} -\frac{y}{x} = \ln\left[\ln\left|\frac{1}{x}\right| + C\right] \Rightarrow y = -x \ln\left[\ln\left|\frac{1}{x}\right| + C\right]$$

(7) $$xy \cdot \frac{dy}{dx} = x^2 + 3y^2 \;\Big|\cdot \frac{1}{xy}$$

$$\Rightarrow \frac{1}{\cancel{xy}} \cdot \cancel{xy} \cdot \frac{dy}{dx} = \frac{x^{\cancel{2}}}{\cancel{x}y} + \frac{3y^{\cancel{2}}}{x\cancel{y}} \Rightarrow \frac{dy}{dx} = \frac{x}{y} + \frac{3y}{x} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)^{-1} + 3\left(\frac{y}{x}\right) \Rightarrow \frac{dv}{dx} \cdot x + v = v^{-1} + 3 \cdot v$$

$$\Rightarrow \frac{dv}{dx} x = v^{-1} + 3v - v$$

subst: $v = \frac{y}{x} \Rightarrow y = vx$

$$\Rightarrow \frac{dv}{dx} \cdot x = v^{-1} + 2v$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$$

$$\Rightarrow \frac{dv}{dx} \cdot x = \frac{1}{v} + 2v \cdot \frac{v}{v} \Rightarrow$$

$$\Rightarrow \frac{dv}{dx} \cdot x = \frac{1+2v^2}{v} \;\Big|\cdot \frac{dx}{x \cdot \left(\frac{1+2v^2}{v}\right)} \Rightarrow \frac{dv}{\cancel{dx}} \cdot \cancel{x} \cdot \frac{\cancel{dx}}{\cancel{x}\left(\frac{1+2v^2}{v}\right)} = \frac{\cancel{1+2v^2}}{\cancel{v}} \cdot \frac{dx}{x\left(\cancel{\frac{1+2v^2}{v}}\right)}$$

$$\Rightarrow \frac{v}{2v^2+1}\, dv = \frac{1}{x}\, dx \Rightarrow \int \frac{v}{2v^2+1}\, dv = \int \frac{1}{x}\, dx \Rightarrow$$

$$\int \frac{v\, dv}{2v^2+1} \qquad u = 2v^2+1 \qquad du = 4v\,dv \;|:4 \qquad \frac{du}{4} = v\,dv$$

$$\Rightarrow \int \frac{1}{u} \cdot \frac{du}{4} = \frac{1}{4} \ln|u| = \frac{1}{4} \underbrace{\ln|2v^2+1|}_{(+)}$$

$$\Rightarrow \frac{1}{4}\ln(2v^2+1) = \ln|x| + C_1 \quad \Big| \cdot 4 \qquad \underbrace{|x|^4}_{(+)} \Rightarrow x^4$$

$$\Rightarrow \ln(2v^2+1) = 4\ln|x| + 4C_1$$

$$\Rightarrow e^{\ln(2v^2+1)} = e^{4\ln|x|+4C_1} \Rightarrow 2v^2+1 = x^4 \cdot e^{4C_1} = C$$

$$\Rightarrow 2v^2+1 = Cx^4 \Rightarrow 2v^2 = Cx^4 - 1 \xRightarrow{\text{înlocuim}} 2\left(\frac{y}{x}\right)^2 = Cx^4 - 1 \Rightarrow$$

$$\Rightarrow \frac{2y^2}{x^2} = Cx^4 - 1 \Rightarrow 2y^2 = x^2(Cx^4-1)$$

$$\Rightarrow 2y^2 = Cx^6 - x^2 \Rightarrow 2y^2 + x^2 = Cx^6$$

(8) $$xy\frac{dy}{dx} = y^2 + x\sqrt{4x^2+y^2} \quad \Big| : xy$$

$$\frac{1}{\cancel{xy}} \cdot \cancel{xy}\frac{dy}{dx} = \frac{y^{\cancel{2}}}{x\cancel{y}} + \frac{\cancel{x}\sqrt{4x^2+y^2}}{\cancel{x}y} \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{4x^2+y^2}}{y} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{4x^2+y^2}{y^2}} \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{4\left(\frac{x}{y}\right)^2 + \frac{\cancel{y^2}}{\cancel{y^2}}} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{4\left(\frac{y}{x}\right)^{-2}+1} \Rightarrow \frac{dy}{dx} = v + \sqrt{4v^{-2}+1} \Rightarrow$$

$$\Rightarrow \frac{dv}{dx}x + v = v + \sqrt{4v^{-2}+1} \Rightarrow$$

subst: $v = \frac{y}{x} \Rightarrow y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v \qquad \Rightarrow \frac{dv}{dx}x = \cancel{v} + \sqrt{4 \cdot \frac{1}{v^2} + \frac{v^2}{1}} - \cancel{v} \Rightarrow$$

$$\Rightarrow \frac{dv}{dx}x = \sqrt{\frac{v^2+4}{v^2}} \Rightarrow \frac{dv}{dx}x = \frac{\sqrt{v^2+4}}{v} \quad \Big| \cdot \frac{dx}{x \cdot \frac{\sqrt{v^2+4}}{v}}$$

$$\Rightarrow \frac{\cancel{dx}}{x \cdot \frac{\sqrt{v^2+4}}{v}} \cdot \frac{dv}{\cancel{dx}} \cdot x = \frac{\cancel{\sqrt{v^2+4}}}{\cancel{v}} \cdot \frac{dx}{x \cdot \frac{\cancel{\sqrt{v^2+4}}}{\cancel{v}}} \Rightarrow$$

$$\Rightarrow \frac{v}{\sqrt{v^2+4}}\, dv = \frac{1}{x}\, dx \Rightarrow \int \frac{v}{\sqrt{v^2+4}}\, dv = \underbrace{\int \frac{1}{x}\, dx}_{\ln|x| + C_1}$$

$$\int \frac{v}{\sqrt{v^2+4}}\, dv$$

$u = v^2 + 4$

$du = 2v\, dv \mid :2$

$\frac{du}{2} = v\, dv$

$$\Rightarrow \int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2}\int u^{-\frac{1}{2}}\, du = \frac{1}{2} \cdot \frac{u^{-\frac{1}{2}+\frac{2}{1}}}{-\frac{1}{2}+\frac{2}{1}} = \frac{1}{\cancel{2}} \cdot \frac{u^{1/2}}{\frac{1}{\cancel{2}}} = (v^2+4)^{1/2}$$

$$\Rightarrow (v^2+4)^{1/2} = \ln|x| + C_1 \Big|^2 \Rightarrow (v^2+4)^{\frac{1}{\cancel{2}}\cdot\cancel{2}} = (\ln|x| + C_1)^2 \Rightarrow$$

$$\Rightarrow v^2 + 4 = (\ln|x| + \underbrace{C_1}_{C})^2 \Rightarrow v^2 = (\ln|x| + C)^2 - 4 \xRightarrow{\text{Înlocuim}}$$

$$\Rightarrow \frac{y^2}{x^2} = (\ln|x| + C)^2 - 4 \ \Big| \cdot x^2 \Rightarrow y^2 = x^2(\ln|x| + C)^2 - 4x^2$$

Mai putem scrie și:

$$y^2 + 4x^2 = x^2(\ln|x| + C)^2$$

(9) $y\frac{dy}{dx} + x = \sqrt{x^2+y^2} \quad \Big| \cdot \frac{1}{y}$

$$\Rightarrow \cancel{y}\frac{dy}{dx} \cdot \frac{1}{\cancel{y}} + x \cdot \frac{1}{y} = \frac{\sqrt{x^2+y^2}}{y} \Rightarrow \frac{dy}{dx} = -\frac{x}{y} + \sqrt{\frac{x^2+y^2}{y^2}} \Rightarrow$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{-1} + \sqrt{\frac{x^2}{y^2} + 1} \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{-1} + \sqrt{\left(\frac{y}{x}\right)^{-2} + 1}$$

Subst: $v = \frac{y}{x} \Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$

$\Rightarrow x\frac{dv}{dx} + v = -v^{-1} + \sqrt{v^{-2}+1}$

$\Rightarrow x\frac{dv}{dx} = -v - v^{-1} + \sqrt{\frac{1}{v^2} + \frac{v^2}{1}} \Rightarrow$

$\Rightarrow x\frac{dv}{dx} = -v - \frac{1}{v} + \sqrt{\frac{1+v^2}{v^2}} \Rightarrow x\frac{dv}{dx} = \frac{-v^2-1}{v} + \frac{\sqrt{v^2+1}}{v} \Rightarrow$

$\Rightarrow x\frac{dv}{dx} = \frac{-v^2-1+\sqrt{v^2+1}}{v} \Rightarrow x\frac{dv}{dx} = \frac{\sqrt{v^2+1} - v^2 - 1}{v} \Rightarrow$

$\Rightarrow x\frac{dv}{dx} = \frac{\sqrt{v^2+1} - (v^2+1)}{v} \Rightarrow \frac{v}{\sqrt{v^2+1} - (v^2+1)}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{v}{\sqrt{v^2+1} - (v^2+1)}\,dv = \underbrace{\int \frac{1}{x}\,dx}_{\ln|x| + C_1}$ $\Rightarrow$ (pagina următoare)

$\int \frac{v}{\sqrt{v^2+1} - (v^2+1)}\,dv$

$u = v^2 + 1$

$du = 2v\,dv \mid :2$

$\frac{du}{2} = v\,dv$

$w = 1 - u^{1/2}$

$dw = -\frac{1}{2}u^{-1/2}du$

$-2\,dw = \frac{1}{u^{1/2}}\,du$

$= \int \frac{1}{\sqrt{u} - u} \cdot \frac{du}{2} = \frac{1}{2}\int \frac{1}{\sqrt{u}(1-\sqrt{u})}\,du$

$= \frac{1}{\cancel{2}}\int \frac{1}{w} \cdot (-\cancel{2})\,dw = -\int \frac{1}{w}\,dw = -\ln|w| = -\ln|1-\sqrt{u}|$

$= -\ln|1 - \sqrt{v^2+1}|$

$$\Rightarrow -\ln|1-\sqrt{v^2+1}| = \ln|x| + C_1 \quad |\cdot(-1) \Rightarrow$$

$$\Rightarrow \ln|1-\sqrt{v^2+1}| = -\ln|x| - C_1 \Rightarrow \ln|1-\sqrt{v^2+1}| = \ln\left|\frac{1}{x}\right| - C_1$$

$$\Rightarrow \cancel{e}^{\cancel{\ln}|1-\sqrt{v^2+1}|} = \cancel{e}^{\cancel{\ln}\left|\frac{1}{x}\right| - C_1} \Rightarrow |1-\sqrt{v^2+1}| = \left|\frac{1}{x}\right| \cdot e^{-C_1}$$

$$\Rightarrow 1-\sqrt{v^2+1} = \underbrace{\pm e^{-C_1}}_{C} \cdot \frac{1}{x} \Rightarrow 1-\sqrt{v^2+1} = \frac{C}{x}$$

Inlocuim $\Rightarrow 1-\sqrt{\frac{y^2}{x^2}+1} = \frac{C}{x} \Rightarrow 1-\sqrt{\frac{y^2+x^2}{x^2}} = \frac{C}{x} \Rightarrow$

$$\Rightarrow 1 - \frac{\sqrt{y^2+x^2}}{x} = \frac{C}{x} \quad |\cdot x \Rightarrow x-\sqrt{y^2+x^2} = C$$

10 $(x^2-y^2)\frac{dy}{dx} = 2xy \quad |\cdot \frac{1}{(x^2-y^2)}$

$$\Rightarrow \frac{1}{\cancel{(x^2-y^2)}} \cdot \cancel{(x^2-y^2)} \cdot \frac{dy}{dx} = \frac{2xy}{(x^2-y^2)} \Rightarrow \frac{dy}{dx} = \frac{2xy}{(x^2-y^2)} \quad \Big| \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{2\cancel{x}y}{x^{\cancel{2}}}}{\frac{\cancel{x^2}}{\cancel{x^2}} - \frac{y^2}{x^2}} \Rightarrow \frac{dy}{dx} = \frac{2\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)^2} \Rightarrow x\frac{dv}{dx} = \frac{2v}{1-v^2} \; \frac{1-v^2}{-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v - v(1-v^2)}{1-v^2} \Rightarrow$$

subst: $v = \frac{y}{x} \Rightarrow y = vx$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v-v+v^3}{1-v^2}$$

$$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^3+v}{-(v^2-1)} \quad \Big| \cdot \frac{dx}{\frac{v^3+v}{-(v^2-1)}\cdot x} \quad (-1)$$

$$\Rightarrow \frac{v^2-1}{v^3+v}\,dv = -\frac{1}{x}\,dx \Rightarrow \int \frac{v^2-1}{v^3+v}\,dv = -\int \frac{1}{x}\,dx \Rightarrow$$

$$\Rightarrow \int \frac{v^2-1}{v(v^2+1)}\,dv = -\int \frac{1}{x}\,dx$$

$$* \quad \frac{v^2-1}{v(v^2+1)} = \frac{A}{v} + \frac{Bv+C}{v^2+1} \Rightarrow v^2-1 = A(v^2+1) + v(Bv+C)$$

$$\Rightarrow v^2-1 = Av^2 + A + Bv^2 + Cv \Rightarrow v^2-1 = (A+B)v^2 + Cv + A$$

$$A+B=1 \qquad C=0 \qquad A=-1 \Rightarrow -1+B=1 \Rightarrow B=2$$

$$\Rightarrow \int \frac{v^2-1}{v(v^2+1)}\,dv = \underbrace{-\int \frac{1}{v}\,dv}_{-\ln|v|} + \underbrace{\int \frac{2v}{v^2+1}\,dv}_{\ln(v^2+1)} \Rightarrow -\ln|v| + \ln(v^2+1) = -\ln|x| + C_1 \;\Big|\cdot(-1)$$

$$2\int \frac{v}{v^2+1}\,dv \qquad u = v^2+1 \qquad du = 2v\,dv \;\Big|:2 \qquad \frac{du}{2} = v\,dv$$

$$\Rightarrow \ln|v| - \ln(v^2+1) = \ln|x| - C_1 \Rightarrow$$

$$\Rightarrow \cancel{2}\int \frac{1}{u}\cdot\frac{du}{\cancel{2}} = \int \frac{1}{u}\,du = \ln|u| = \ln|\underbrace{v^2+1}_{(+)}|$$

$$\Rightarrow \ln\left|\frac{v}{v^2+1}\right| = \ln|x| - C_1 \Rightarrow e^{\cancel{\ln}\left|\frac{v}{v^2+1}\right|} = e^{\cancel{\ln}|x|-C_1}$$

$$\Rightarrow \left|\frac{v}{v^2+1}\right| = |x|\cdot e^{-C_1} \Rightarrow \frac{v}{v^2+1} = \underbrace{\pm e^{-C_1}}_{C}\cdot x \Rightarrow \frac{v}{v^2+1} = Cx$$

Vracím $\Rightarrow \dfrac{\frac{y}{x}}{\frac{y^2}{x^2} + \overset{x^2}{1}} = Cx \Rightarrow \dfrac{xy}{y^2+x^2} = Cx \;\Big|\cdot\frac{1}{x} \Rightarrow \dfrac{y}{y^2+x^2} = C \Rightarrow y = C(x^2+y^2)$