Curs 3, EDDP, ID, 29.11.2020

Sisteme liniare cu coeficienți constanți (continuare)

(1) $y' = Ay$, $A \in \mathcal{M}_n(\mathbb{R})$

$$\frac{dy}{dx} = Ay \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

S_A = mulț. soluțiilor

S_A este spațiu vectorial real

o bază în S_A este numită sistem fundamental de soluții

Exemple:

1. $$\begin{cases} y_1' = 2y_1 + 3y_2 \\ y_2' = 3y_1 + 2y_2 \end{cases}$$

Se cere un sistem fundam. de soluții.

- se scrie forma matriceală a sistemului:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad ; \quad y' = Ay$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} ; \ n = 2$$

- se determină valorile proprii pt A:

$$\det(A - \lambda I_2) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow (2-\lambda)^2 - 9 = 0 \Rightarrow (2-\lambda)^2 - 3^2 = 0 \Rightarrow$$

$$\Rightarrow (2-\lambda-3)(2-\lambda+3) = 0 \Rightarrow (-\lambda-1)(-\lambda+5) = 0$$

$$\Rightarrow \underbrace{(\lambda+1)(\lambda-5)}_{} = 0 \Rightarrow$$

$$P_A(\lambda) = (\lambda+1)^1(\lambda-5)^1$$

$$\Rightarrow \lambda + 1 = 0 \Rightarrow \boxed{\lambda_1 = -1, \ m_1 = 1}$$

$$\lambda - 5 = 0 \Rightarrow \boxed{\lambda_2 = 5, \ m_2 = 1} \quad ; \ m_1 + m_2 = 2 = n.$$

- pt $\boxed{\lambda_1 = -1, m_1 = 1}$ $\Rightarrow$ determinăm $u \in \mathbb{R}^2$, $u \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a.î.

$$Au = \lambda_1 \cdot u \Rightarrow \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (-1)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 2u_1 + 3u_2 = -u_1 \\ 3u_1 + 2u_2 = -u_2 \end{cases} \Rightarrow \begin{matrix} 3u_2 = -3u_1 \Rightarrow \underline{u_2 = -u_1} \Rightarrow \\ 3u_2 = -3u_1 \end{matrix}$$

$$\Rightarrow u = \begin{pmatrix} u_1 \\ -u_1 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, u_1 \in \mathbb{R}. \Rightarrow$$

$\Rightarrow$ o soluție pt sist fundam de soluții:

$$\varphi_1(x) = e^{\lambda_1 t} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \boxed{\varphi_1(x) = \begin{pmatrix} e^{-x} \\ -e^{-x} \end{pmatrix}}$$

pt $\boxed{\lambda_2 = 5, m_1 = 1} \Rightarrow u \in \mathbb{R}^2, u \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ aî $Au = \lambda_2 u \Rightarrow$

$$\Rightarrow \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 5 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \begin{cases} 2u_1 + 3u_2 = 5u_1 \\ 3u_1 + 2u_2 = 5u_2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 3u_2 = 3u_1 \\ -3u_2 = -3u_1 \end{cases} \Rightarrow u_2 = u_1 \Rightarrow u = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \varphi_2(x) = e^{\lambda_2 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \boxed{\varphi_2(x) = \begin{pmatrix} e^{5x} \\ e^{5x} \end{pmatrix}}$$

Rezultă $\{\varphi_1, \varphi_2\}$ sistem fundam. de soluții.

$$\phi(x) = \text{coloane}\,(\varphi_1, \varphi_2)(x) = \begin{pmatrix} e^{-x} & e^{5x} \\ -e^{-x} & e^{5x} \end{pmatrix}$$

$$S_A = \{ c_1\varphi_1 + c_2\varphi_2 \mid c_1, c_2 \in \mathbb{R} \} =$$

$$= \left\{ \phi(x) C \mid C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

<u>OBS</u>: Dacă $\tilde{\phi}(x)$ este matrice de soluții, adică:

$\tilde{\phi}(x) = \text{coloane}\,(\varphi_1, \dots, \varphi_k)(x)$, $\varphi_1, \dots, \varphi_k$ soluții pt $y' = Ay$, $A \in \mathcal{M}_n(\mathbb{R})$, $\tilde{\phi}(x) \in \mathcal{M}_{n,k}(\mathbb{R})$, atunci

$$\boxed{\tilde{\phi}'(x) = A\phi(x)} \quad (2)$$

② $\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = 4y_2 - y_1 \end{cases}$; $S_A = ?$, $\phi = ?$

$$y' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} y \quad ; \quad A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}; n=2$$

$$\det(A-\lambda I_2)=0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix}=0$$

$$(2-\lambda)(4-\lambda)+1=0.$$

$$8-2\lambda-4\lambda+\lambda^2+1=0$$

$$\lambda^2-6\lambda+9=0 \Rightarrow \underbrace{(\lambda-3)^2}_{\parallel}=0$$

$$P_A(\lambda)=(\lambda-3)^2 \Rightarrow$$

$\Rightarrow \underline{\lambda_1=3 \quad , m_1=2>1}$ $(k=1)$ $\Rightarrow$ determinăm $p_0, p_1 \in \mathbb{R}^2$ nu amândoi nuli a.î.

$$\varphi(x)=(p_0+p_1x)e^{3x}$$

să fie soluție pt $y'=Ay \Rightarrow$

$$\Rightarrow \left((p_0+p_1x)\cdot e^{3x}\right)'=A(p_0+p_1x)e^{3x} \Rightarrow$$

$$(p_0+p_1x)'e^{3x}+(p_0+p_1x)(e^{3x})'=(Ap_0+Ap_1\cdot x)e^{x}$$

$$p_1e^{3x}+(p_0+p_1x)e^{3x}\cdot 3=(Ap_0+Ap_1\cdot x)e^{x} \;\Big|:e^{3x} \Rightarrow$$

$\Rightarrow \underline{p_1+3p_0}+3p_1x=\underline{Ap_0}+Ap_1x$ Identificând coef lui x $\Big\} \Rightarrow \begin{cases} p_1+3p_0=Ap_0 \\ 3p_1=Ap_1 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} p_1=Ap_0-3p_0 \\ 0_{\mathbb{R}^2}=Ap_1-3p_1 \end{cases} \Rightarrow \begin{cases} p_1=(A-3I_2)p_0 \\ 0_{\mathbb{R}^2}=(A-3I_2)p_1 \end{cases} \Big|\cdot (A-3I_2) \text{ în stg}$$

$$\Rightarrow \underbrace{(A-3I_2)p_1}_{0_{\mathbb{R}^2}}=(A-3I_2)^2p_0 \Rightarrow (A-3I_2)^2p_0=0_{\mathbb{R}^2} \Rightarrow$$

$\Rightarrow p_0 \in \underbrace{\ker\left((A-3I_2)^2\right)}$ nucleul aplicației liniare $\underbrace{(A-3I_2)^2}_{g}:\mathbb{R}^2\to\mathbb{R}^2$

$$\left.\begin{aligned} &\ker g=\{v\in\mathbb{R}^2 \mid g(v)=0_{\mathbb{R}^2}\} \\ &(A-3I_2)^2=\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \ker g=\mathbb{R}^2 \Rightarrow$ pentru p_0 considerăm elementele unei baze din $\ker g$. Cum $\ker g=\mathbb{R}^2$, luăm pt p_0, vectorii bazei canonice: $\begin{pmatrix}1\\0\end{pmatrix}; \begin{pmatrix}0\\1\end{pmatrix}$.

• pt $p_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow p_1 = (A - 3I_2)p_0 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow$

$\Rightarrow \varphi_1(x) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} t\right) e^{3t} = \begin{pmatrix} 1-t \\ -t \end{pmatrix} e^{3t}$

$$\boxed{\varphi_1(x) = \begin{pmatrix} (1-x)e^{3x} \\ -xe^{3x} \end{pmatrix}}$$

• pt $p_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow p_1 = (A - 3I_2)p_0 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$

$\Rightarrow \varphi_2(x) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x\right) e^{3x} \Rightarrow \boxed{\varphi_2(x) = \begin{pmatrix} xe^{3x} \\ (1+x)e^{3x} \end{pmatrix}}$

Se obține matricea fundam. de soluții:

$$\phi(x) = \begin{pmatrix} (1-x)e^{3x} & xe^{3x} \\ -xe^{3x} & (1+x)e^{3x} \end{pmatrix}$$

și $S_A = \left\{ \phi(x)c \;\middle|\; c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2 \right\}$.

<u>OBS</u>: În $\mathbb{R}^n$, baza canonică este $\{e_1, \dots, e_n\}$

cu $e_j = {}^t(0, \dots, 0, \underset{\substack{\uparrow \\ \text{pe poz } j}}{1}, 0, \dots, 0)$, $j = \overline{1,n}$

(3) $\begin{cases} y_1' = 2y_1 - y_2 - y_3 \\ y_2' = 3y_1 - 2y_2 - 3y_3 \\ y_3' = -y_1 + y_2 + 2y_3 \end{cases}$, $\underline{n=3}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$y' = \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix} y$$

• $\det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 & -1 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$

$C1 \leftarrow C1 + C3$
$C2 \leftarrow C2 - C3$

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & -3 \\ 1-\lambda & -1+\lambda & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1-\lambda)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow$

$\Rightarrow (1-\lambda)^2 (\cancel{2}-\lambda + 0 + 0 \cancel{+1} \cancel{-3} - 0) = 0 \Rightarrow (1-\lambda)^2 \cdot (-\lambda) = 0 \Rightarrow$

$\Rightarrow p_A(\lambda)=(-1)^3\cdot\lambda\cdot(\lambda-1)^2=(-1)^3(\lambda-0)^1(\lambda-1)^2$

$$\begin{cases}\lambda_1=0, & m_1=1\\ \lambda_2=1, & m_2=2\end{cases}$$

$\boxed{\lambda_1=0,\ m_1=1} \Rightarrow u\in\mathbb{R}^3,\ u\neq\begin{pmatrix}0\\0\\0\end{pmatrix}$ a.î. $Au=\lambda_1 u \Rightarrow$

$$\Rightarrow \begin{pmatrix}2&-1&-1\\3&-2&-3\\-1&1&2\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=0\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}\Rightarrow$$

$$\Rightarrow\begin{cases}2u_1-u_2-u_3=0\\3u_1-2u_2-3u_3=0\\-u_1+u_2+2u_3=0\end{cases}$$

$$\begin{vmatrix}2&-1&-1\\3&-2&-3\\-1&1&2\end{vmatrix}=-8-3-3+2+6+6=-14+14=0$$

$$\Delta_p=\begin{vmatrix}2&-1\\3&-2\end{vmatrix}=-4+3=-1\neq0\Rightarrow\begin{cases}2u_1-u_2=u_3\\3u_1-2u_2=3u_3\end{cases}\Rightarrow$$

$\Rightarrow u_2=2u_1-u_3$

$3u_1-4u_1+2u_3=3u_3 \Rightarrow -u_1=u_3\Rightarrow \boxed{u_1=-u_3}$

$\Rightarrow \boxed{u_2=2u_1-u_3=-2u_3-u_3=-3u_3}$

$$\Rightarrow u=\begin{pmatrix}-u_3\\-3u_3\\u_3\end{pmatrix}=u_3\begin{pmatrix}-1\\-3\\1\end{pmatrix}\Rightarrow \varphi_1(x)=e^{\lambda_1 x}\begin{pmatrix}-1\\-3\\1\end{pmatrix}\Rightarrow$$

$$\Rightarrow\boxed{\varphi_1(x)=\begin{pmatrix}-1\\-3\\1\end{pmatrix}}$$

$\boxed{\lambda_2=1,\ m_2=2} \Rightarrow p_0,p_1\in\mathbb{R}^3$, nu amândoi nuli, a.î.

$\varphi(x)=(p_0+p_1x)e^x$ sol. pt $y'=Ay$

$\Rightarrow \varphi'(x)=A\varphi(x) \Rightarrow p_1e^x+(p_0+p_1x)e^x=(Ap_0+Ap_1x)e^x$

împărțim prin $e^x \Rightarrow p_1+p_0+p_1x=Ap_0+Ap_1x\Rightarrow$

$$\Rightarrow\begin{cases}p_1+p_0=Ap_0\\p_1=Ap_1\end{cases}\Rightarrow\begin{cases}p_1=(A-I_3)p_0 & \mid\cdot(A-I_3) \text{ la stg}\\ 0_{\mathbb{R}^3}=(A-I_3)p_1\end{cases}\Rightarrow$$

$\Rightarrow (A-I_3)p_1=(A-I_3)^2p_0\Rightarrow (A-I_3)^2p_0=0_{\mathbb{R}^3}\Rightarrow$

$\Rightarrow p_0 \in \underbrace{\ker\left((A-I_3)^2\right)}_{\| } \subset \mathbb{R}^3$

$$\| \left\{ \underbrace{(A-I_3)^2 v = 0_{\mathbb{R}^3}} \mid v \in \mathbb{R}^3 \right\}$$

$$(A-I_3)^2 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 3 \\ 1 & -1 & -1 \end{pmatrix}$$

Scriem ec.: $(A-I_3)^2 v = 0_{\mathbb{R}^3} \Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ -3 & 3 & 3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{cases} -v_1 + v_2 + v_3 = 0 \\ -3v_1 + 3v_2 + 3v_3 = 0 \quad | :3 \\ v_1 - v_2 - v_3 = 0 \quad | \cdot(-1) \end{cases} \Rightarrow v_1 = v_2 + v_3 \Rightarrow \quad v_2, v_3 \in \mathbb{R}.$$

$\Rightarrow \ker\left((A-I_3)^2\right) = \left\{ v = {}^t(v_2+v_3, v_2, v_3) \mid v_2, v_3 \in \mathbb{R} \right\} \Rightarrow$

$\Rightarrow v$ din $\ker\left((A-I_3)^2\right)$ sunt: $v = \begin{pmatrix} v_2+v_3 \\ v_2 \\ v_3 \end{pmatrix} \Rightarrow$

$\Rightarrow v = v_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ generează (este bază) $\ker\left((A-I_3)^2\right) \Rightarrow$

$\Rightarrow$ pt p_0 se iau pe rând valorile: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$:

- pt $p_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow p_1 = (A-I_3)p_0 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\Rightarrow \varphi_1(x) = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} x\right) e^x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^x \Rightarrow \boxed{\varphi_2(x) = \begin{pmatrix} e^x \\ e^x \\ 0 \end{pmatrix}}$$

- pt $p_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow p_1 = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\Rightarrow \varphi_2(x) = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} x\right) e^x \Rightarrow \boxed{\varphi_3(x) = \begin{pmatrix} e^x \\ 0 \\ e^x \end{pmatrix}}$$

Avem:
$$\phi(x) = \begin{pmatrix} -1 & e^x & e^x \\ -3 & e^x & 0 \\ 1 & 0 & e^x \end{pmatrix}$$

$$S_A = \left\{ \phi(x)\, C \mid C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in \mathbb{R}^3 \right\}.$$

(4) $\begin{cases} y_1' = y_1 - 3y_2 \\ y_2' = 3y_1 + y_2 \end{cases}$; $S_A = ?$; $\phi = ?$

$n = 2$; $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$; $y' = \underbrace{\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}}_{A} y$

$\det(A - \lambda I_2) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 + 9 = 0 \Rightarrow$

$\Rightarrow (1-\lambda)^2 - (3i)^2 = 0 \Rightarrow (1-\lambda-3i)(1-\lambda+3i) = 0 \Rightarrow$

$(i^2 = -1)$ $\Rightarrow p_A(\lambda) = (\lambda - (1-3i))^1 (\lambda - (1+3i))^1 \Rightarrow$

$\Rightarrow \begin{cases} \lambda_1 = 1+3i, & m_1 = 1 \\ \lambda_2 = 1-3i, & m_2 = 1 \end{cases}$, $\lambda_2 = \overline{\lambda_1}$ | se det. 2 sol în sist fundam. coresp pt λ_1 și $\overline{\lambda_1}$

Se det $u \in \mathbb{C}^2$, $u \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a.î. $Au = \lambda_1 u \Rightarrow$

$\Rightarrow \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (1+3i) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} u_1 - 3u_2 = (1+3i)u_1 \\ 3u_1 + u_2 = (1+3i)u_2 \end{cases} \Rightarrow \begin{cases} -3u_2 = (\cancel{1}+3i-\cancel{1})u_1 \\ 3u_1 = (\cancel{1}+3i-\cancel{1})u_2 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} -3u_2 = 3i u_1 & \mid :(-3) \\ 3u_1 = 3i u_2 & \mid : 3i \end{cases} \Rightarrow \begin{cases} u_2 = -i u_1 \\ u_2 = \frac{1}{i} u_1 = \frac{i}{-1} u_1 \end{cases}$

$\Rightarrow u_2 = -i u_1 \Rightarrow u = \begin{pmatrix} -i u_1 \\ u_1 \end{pmatrix} = u_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} \Rightarrow$

$\Rightarrow \varphi_1(x) = \mathrm{Re}\left(e^{\lambda_1 x} \begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$; $\varphi_2(x) = \mathrm{Im}\left(e^{\lambda_1 x} \begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$

coef părții imaginare

Calculăm $e^{\lambda_1 x} \begin{pmatrix} -i \\ 1 \end{pmatrix} = e^{x + 3x \cdot i} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) =$

$= e^x \cdot e^{3x \cdot i} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) = e^x \underbrace{(\cos 3x + i \sin 3x) \cdot \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right)} \Rightarrow$

! $\boxed{e^{\alpha i} = \cos\alpha + i \sin\alpha}$

$$\Rightarrow \varphi_1(x) = e^{x}\left(\cos 3x \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin 3x \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) \Rightarrow \boxed{\varphi_1(x) = e^{x}\begin{pmatrix} \sin 3x \\ \cos 3x \end{pmatrix}}$$

$$\varphi_2(x) = e^{x}\left(\cos 3x \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin 3x \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \Rightarrow \boxed{\varphi_2(x) = e^{x}\begin{pmatrix} -\cos 3x \\ \sin 3x \end{pmatrix}}$$

$$\Rightarrow \phi(x) = \begin{pmatrix} e^{x}\sin 3x & -e^{x}\cos 3x \\ e^{x}\cos 3x & e^{x}\sin 3x \end{pmatrix}$$

$$S_A = \left\{ \phi(x)C \;\middle|\; C \in \mathbb{R}^2,\ C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right\}.$$

Tema 1: Determinați mulț. sol. sistemelor:

1) $\begin{cases} y_1' = 5y_1 + 3y_2 \\ y_2' = -3y_1 - y_2 \end{cases}$; 2) $\begin{cases} y_1' = y_1 + y_2 \\ y_2' = 3y_2 - 2y_1 \end{cases}$;

3) $\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$; 4) $\begin{cases} y_1' = 2y_1 - y_2 + 2y_3 \\ y_2' = y_1 + 2y_3 \\ y_3' = -2y_1 + y_2 - y_3 \end{cases}$

OBS: Dacă matricea A din sist. de ecuații diferențiale liniare depinde de x, atunci nu se poate rezolva cu algoritmul cu valori proprii. Sunt cazuri în care printr-o schimbare de variabilă se poate ajunge la sistem cu coef. constanți.

De exemplu:

① $\boxed{y' = \frac{1}{x} B \cdot y}$; $B \in \mathcal{M}_n(\mathbb{R})$ (3)

Prin schimbarea de variabilă: $|x| = e^{s}$

avem $(x, y) \longrightarrow (s, z)$

$\boxed{y(x) = z(s(x))}$; $s(x) = \ln|x|$

Din (3) $\Rightarrow$ $xy' = By$

din s.v. (schimbarea de variabilă) $\Rightarrow$ $y'(x) = z'(s) \cdot s'(x) = z' \cdot \frac{1}{x} \Rightarrow \boxed{xy' = z'}$ $\Bigg\} \Rightarrow$

$\Rightarrow \boxed{z' = Bz}$

OBS: $s(x) = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases} \Rightarrow s'(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-x)', & x < 0 \end{cases} =$

$= \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-1), & x < 0 \end{cases} = \frac{1}{x}, \quad \forall x \neq 0.$

② $\boxed{y' = (2k+1)x^{2k} B y}$, $B \in \mathcal{M}_n(\mathbb{R})$ (4)

cu $k \in \mathbb{N}^*$.

Prin schimbarea de variab: $x^{2k+1} = s$

$$(x, y) \longrightarrow (s, z)$$

$\boxed{y(x) = z(s(x))}$; $s(x) = x^{2k+1}$

Avem (4) $\Rightarrow$ $\frac{1}{(2k+1)x^{2k}} y' = By$

Dar

$y(x) = z(s(x)) \Rightarrow y'(x) = z'(s(x)) \cdot s'(x)$

dar $s'(x) = (x^{2k+1})' =$
$= (2k+1)x^{2k}$ $\Bigg] \Rightarrow$

$\Rightarrow \frac{1}{(2k+1)x^{2k}} y' = z' \Rightarrow$ sistemul în variab (s, z) este

$z' = Bz$.

Tema 2: Folosind schimbări de variabilă convenabile să se determine mult. sol. sistemelor:

1) $\begin{cases} y_1' = \frac{1}{x} y_2 \\ y_2' = \frac{1}{x} y_1 \end{cases}$; $A = \begin{pmatrix} 0 & \frac{1}{x} \\ \frac{1}{x} & 0 \end{pmatrix} = \frac{1}{x}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2) $\begin{cases} y_1' = 3x^2 y_2 \\ y_2' = -3x^2 y_1 \end{cases}$; $A = \begin{pmatrix} 0 & 3x^2 \\ -3x^2 & 0 \end{pmatrix} = 3x^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3) $\begin{cases} y_1' = -\frac{1}{x} y_1 + \frac{2}{x} y_2 \\ y_2' = \frac{3}{x} y_1 + \frac{4}{x} y_2 \end{cases}$

Reducerea dimensiunii unui sistem liniar $y'=A(x)y$

Se dă sistemul liniar:

$$y' = A(x)\,y \qquad (5)$$

cu $A : I \subset \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$

Se presupune că sunt cunoscute $k<n$ soluții independente al sistemului: $\varphi_j = \begin{pmatrix} \varphi_{1j} \\ \vdots \\ \varphi_{nj} \end{pmatrix}$, $j=\overline{1,k}$

astfel încât:

$$\Delta(x) = \det \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1k} \\ \vdots & & \vdots \\ \varphi_{k1} & & \varphi_{kk} \end{pmatrix}(x) \neq 0, \quad \forall x \in I. \quad (6)$$

Prin schimbarea de variabilă:

$$y = Z(x)\, w \qquad (7)$$

$$(x,y) \longrightarrow (x,w)$$

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

unde $Z(x) = \text{coloane}(\varphi_1(x), \dots, \varphi_k(x), e_{k+1}, \dots, e_n) \in \mathcal{M}_n(\mathbb{R})$

cu $e_{k+1}, \dots, e_n$ vectori din baza canonică din $\mathbb{R}^n$

Avem: $\det Z(x) = \Delta(x)$ pt că:

$$Z(x) = \begin{pmatrix} \varphi_{11}(x) & \cdots & \varphi_{1k}(x) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \\ \varphi_{k1}(x) & \cdots & \varphi_{kk}(x) & 0 & \cdots & 0 \\ \varphi_{k+1,1}(x) & \cdots & \varphi_{k+1,k}(x) & 1 & \cdots & 0 \\ \vdots & & \vdots & 0 & \ddots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_{n1}(x) & \cdots & \varphi_{n,k}(x) & 0 & \cdots & 1 \end{pmatrix}$$

Prop. 1: Prin schimbarea de variabilă (7) sistemul (5) devine

$$w' = B(x)\, w \quad (8)$$

unde $B(x)$ are primele k coloane zero.

Consecința prop. 1: Sistemul se descompune într-un sistem de dimensiune $n-k$ în necunoscute $y_{k+1}, \dots, y_n$, și k ecuații de tip primitivă (după rezolvarea sist. în n-k ec.)

$$\begin{pmatrix} w_1' \\ \vdots \\ w_k' \\ w_{k+1}' \\ \vdots \\ w_n' \end{pmatrix} = \begin{array}{c} \qquad\qquad\qquad\quad \overset{k}{\downarrow} \qquad\qquad\qquad\qquad B_{1,k}(x) \\ \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & b_{1,k+1}(x) & \cdots & b_{1,n}(x) \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & 0 & b_{k,k+1}(x) & \cdots & b_{k,n}(x) \\ \hline 0 & \cdots & 0 & b_{k+1,k+1}(x) & \cdots & b_{k+1,n}(x) \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & 0 & b_{n,k+1}(x) & \cdots & b_{n,n}(x) \end{array}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \overset{\parallel}{B_{k+1,n}(x)} \end{array} \begin{pmatrix} w_1 \\ \vdots \\ w_k \\ w_{k+1} \\ \vdots \\ w_n \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} w_1' \\ \vdots \\ w_k' \end{pmatrix} = B_{1,k}(x) \begin{pmatrix} w_{k+1} \\ \vdots \\ w_n \end{pmatrix}$$

$$\left.\begin{pmatrix} w_{k+1}' \\ \vdots \\ w_n' \end{pmatrix} = B_{k+1,n}(x) \begin{pmatrix} w_{k+1} \\ \vdots \\ w_n \end{pmatrix} \right\} \text{sist de } n-k \text{ ecuații}$$

<u>Dem. prop. 1</u> S.v. (*) în (5) $\Rightarrow$

$$\left(Z(x)\cdot w\right)' = A(x)\cdot Z(x)\, w \Rightarrow$$

$$\Rightarrow \underbrace{Z'(x)\cdot w} + Z(x)\, w' = A(x)\, Z(x)\, w \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow Z(x)\, w' = \left(A(x)Z(x) - Z'(x)\right) w \\ \text{dar } \det Z(x) \neq 0 \Rightarrow \exists \left(Z(x)\right)^{-1} \end{array}\right\} \Rightarrow$$

$$\Rightarrow w' = \underbrace{\left(Z(x)\right)^{-1}\left(A(x)Z(x) - Z'(x)\right)}_{B(x)} w.$$

Calculăm $A(x)Z(x) - Z'(x) =$

$= A(x)\cdot \text{coloane}\left(\varphi_1(x), \varphi_2(x), \dots, \varphi_k(x), e_{k+1}, \dots, e_n\right) -$

$- \text{coloane}\left(\varphi_1'(x), \varphi_2'(x), \dots, \varphi_k'(x), 0_{\mathbb{R}^n}, \dots 0_{\mathbb{R}^n}\right) =$

$= \text{coloane}\left(A(x)\varphi_1(x), A(x)\varphi_2(x), \dots, A(x)\varphi_k(x), A(x)e_{k+1}, \dots, A(x)e_n\right) -$

$- \text{coloane}\left(\varphi_1'(x), \varphi_2'(x), \dots, \varphi_k'(x), 0_{\mathbb{R}^n}, \dots, 0_{\mathbb{R}^n}\right) =$

$= \text{coloane}\left(A(x)\varphi_1(x) - \varphi_1'(x), \dots, A(x)\varphi_k(x) - \varphi_k'(x), A(x)e_{k+1}, \dots, A(x)e_n\right)$

Cum $\varphi_1, \dots, \varphi_k \in S_A \Rightarrow A(x)\varphi_j(x) - \varphi_j'(x) = 0_{\mathbb{R}^n}, \ \forall x \in I, \ \forall j = \overline{1,k} \Rightarrow$

$$\Rightarrow A(x)Z(x) - Z'(x) = \text{coloane}\Big(0_{\mathbb{R}^n}, \dots, \underset{\substack{\uparrow \\ k}}{0_{\mathbb{R}^n}}, A(x)e_{k+1}, \dots, A(x)e_n\Big)$$

$$\Rightarrow B(x) = \text{coloane}\Big(0_{\mathbb{R}^n}, \dots, \underset{\substack{\uparrow \\ k}}{0_{\mathbb{R}^n}}, (Z(x))^{-1}A(x)e_{k+1}, \dots, (Z(x))^{-1}A(x)e_n\Big)$$

<u>Exemplu</u>: Fie sistemul $\begin{cases} y_1' = 3x^2 y_2 \\ y_2' = 3x^2 y_1 \end{cases}$ (9)

a) Arătați că $\varphi_1(x) = \begin{pmatrix} e^{x^3} \\ e^{x^3} \end{pmatrix}$ este soluție a sistemului (9).

b). Determinați S_A și $\varphi_2 \in S_A$ a.î. $\{\varphi_1, \varphi_2\}$ să fie sistem fundam. de soluții pt (9).

a) $\varphi_{11}(x) = e^{x^3}$; $\varphi_{21}(x) = e^{x^3}$

$\varphi_{11}'(x) = 3x^2\varphi_{21}(x) \iff e^{x^3}\cdot(x^3)' = 3x^2\cdot e^{x^3}$ (A).

$\varphi_{21}'(x) = 3x^2\varphi_{11}(x) \iff e^{x^3}\cdot(x^3)' = 3x^2\cdot e^{x^3}$ (A).

b) $k=1$, $n=2$: $\varphi_1(x) = \begin{pmatrix} e^{x^3} \\ e^{x^3} \end{pmatrix}$; $\Delta(x) = \det(\varphi_{11}(x)) = e^{x^3} \neq 0, \ \forall x \in \mathbb{R}$

$$Z(x) = \begin{pmatrix} e^{x^3} & 0 \\ e^{x^3} & 1 \end{pmatrix} \ ; \ \det Z(x) = e^{x^3} \neq 0.$$

$$\begin{array}{cc} & \uparrow \\ & e_2 \end{array}$$

$$(x,y) \xrightarrow[y = Z(x)w]{} (x,w) \ ; \ w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\Rightarrow w' = \underbrace{(Z(x))^{-1}\left[A(x)Z(x) - Z'(x)\right]}_{B(x)} w$$

$$A(x) = \begin{pmatrix} 0 & 3x^2 \\ 3x^2 & 0 \end{pmatrix} \ ; \ Z'(x) = \begin{pmatrix} e^{x^3}\cdot 3x^2 & 0 \\ e^{x^3}\cdot 3x^2 & 0 \end{pmatrix}$$

$$A(t)\,Z(t) - Z'(t) = \begin{pmatrix} 0 & 3t^2 \\ 3t^2 & 0 \end{pmatrix}\begin{pmatrix} e^{t^3} & 0 \\ e^{t^3} & 1 \end{pmatrix} - \begin{pmatrix} e^{t^3}\cdot 3t^2 & 0 \\ e^{t^3}\,3t^2 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 3t^2 e^{t^3} & 3t^2 \\ 3t^2 e^{t^3} & 0 \end{pmatrix} - \begin{pmatrix} e^{t^3}\,3t^2 & 0 \\ e^{t^3}\,3t^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3t^2 \\ 0 & 0 \end{pmatrix}$$

pt $(Z(t))^{-1}$:

$${}^{T}(Z(t)) = \begin{pmatrix} e^{t^3} & e^{t^3} \\ 0 & 1 \end{pmatrix}$$

$$(Z(t))^{*} = \begin{pmatrix} 1 & 0 \\ -e^{t^3} & e^{t^3} \end{pmatrix}$$

$$(Z(t))^{-1} = \frac{1}{\det(Z(t))}\cdot (Z(t))^{*} = \frac{1}{e^{t^3}}\begin{pmatrix} 1 & 0 \\ -e^{t^3} & e^{t^3} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-t^3} & 0 \\ -1 & 1 \end{pmatrix}$$

Deci: $$B(t) = \begin{pmatrix} e^{-t^3} & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3t^2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3t^2 e^{-t^3} \\ 0 & -3t^2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & 3t^2 e^{-t^3} \\ 0 & -3t^2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Rightarrow \begin{cases} w_1' = 3t^2 e^{-t^3} w_2 \\ \underline{w_2' = -3t^2 w_2} \end{cases}$$

Din $w_2' = -3t^2 w_2$

ec. dif. liniară omogenă $\Big| \Rightarrow$

$$\Rightarrow \boxed{w_2(t) = C_1 \cdot e^{-t^3}, \ C_1 \in \mathbb{R}} \Rightarrow$$

$$\int (-3t^2)\,dt = -t^3 + K$$

$$\Rightarrow w_1' = 3t^2 e^{-t^3}\cdot C_1 e^{-t^3} \Rightarrow w_1' = 3C_1 t^2 e^{-2t^3}$$

ec. de tip primitivă

$$\Rightarrow w_1(t) = \int C_1\, 3t^2 e^{-2t^3}\,dt = -C_1\tfrac{1}{2}\cdot\int (-2t^3)' e^{-2t^3}\,dt =$$

$$= -\frac{C_1}{2}\int \left(e^{-2t^3}\right)' dt = -\frac{C_1}{2}e^{-2t^3} + C_2 \Rightarrow \boxed{w_1(t) = -\frac{C_1}{2}e^{-2t^3} + C_2 \quad C_1, C_2 \in \mathbb{R}}$$

Rezultă $$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{t^3} & 0 \\ e^{t^3} & 1 \end{pmatrix}\begin{pmatrix} -\frac{C_1}{2}e^{-2t^3} + C_2 \\ C_1 e^{-t^3} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} -\frac{c_1}{2}e^{-x^3} + c_2 e^{x^3} \\ \underbrace{-\frac{c_1}{2}e^{-x^3}} + c_2 e^{x^3} + \underbrace{\frac{2}{c_1}e^{-x^3}} \end{pmatrix} = \begin{pmatrix} -\frac{c_1}{2}e^{-x^3} + c_2 e^{x^3} \\ \frac{c_1}{2}e^{-x^3} + c_2 e^{x^3} \end{pmatrix} =$$

$$= c_1 \underbrace{\begin{pmatrix} -\frac{1}{2}e^{-x^3} \\ \frac{1}{2}e^{-x^3} \end{pmatrix}}_{\varphi_2(x)} + c_2 \underbrace{\begin{pmatrix} e^{x^3} \\ e^{x^3} \end{pmatrix}}_{\varphi_1(x)}, \quad c_1, c_2 \in \mathbb{R}.$$

Tema 3: Fie sistemul:

$$(10) \begin{cases} y_1' = \frac{1}{2}\left(1 + \frac{1}{t} - \frac{1}{t^2}\right) y_1 + \frac{1}{2}\left(1 + \frac{1}{t} + \frac{1}{t^2}\right) y_2 \\ y_2' = -\frac{1}{2}\left(1 - \frac{1}{t} + \frac{1}{t^2}\right) y_1 - \frac{1}{2}\left(1 - \frac{1}{t} - \frac{1}{t^2}\right) y_2 \end{cases}, t > 0.$$

a) Verificați că $\varphi_1(t) = \begin{pmatrix} t+1 \\ 1-t \end{pmatrix}$ este soluție a sistemului (10)

b) Determinați S_A și φ_2 ai $\{\varphi_1, \varphi_2\}$ să fie sist. fundam. de soluții pt (10).

Sisteme afine de ecuații diferențiale (liniare neomogene)

$$y' = A(x)y + b(x) \quad (11)$$

unde $A : I \subset \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$

$b : I \subset \mathbb{R} \to \mathbb{R}^n$

I) Dacă cunoaștem o soluție φ_0 a sistemului (11), atunci

$$S_{A,b} = S_A + \varphi_0 = \{\varphi + \varphi_0 \mid \varphi \in S_A\} \quad (12)$$

unde $S_{A,b}$ = mulț. sol sist. (11)

S_A = mulț. sol. sistemului liniar atașat lui 11:

$$\bar{y}' = A(x)\bar{y}. \quad (13)$$

OBS: Reducerea la sistemul (13) se face din (11) prin schimbarea de variabilă: $y = \bar{y} + \varphi_0$.

II) În general, nu se cunoaște o soluție pentru (11). În acest caz, se aplică metoda variației constantelor:

- rezolvăm sistemul liniar omogen atașat lui (11):

$$\overline{y}' = A(x)\overline{y} \qquad (13)$$

și determinăm $\{\varphi_1, \dots, \varphi_n\}$ sistem fundamental de soluții pt (13) $\Rightarrow$ $\phi(x) =$ coloane $(\varphi_1(x), \dots, \varphi_n(x))$ este matrice fundam. de soluții $\Rightarrow$

$$\Rightarrow S_A = \{\phi(x)C \mid C \in \mathbb{R}^n\}$$

- aplicăm met variației constantelor:

determinăm $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} : I \to \mathbb{R}^n$ a.î. $y(x) = \phi(x)C(x)$ să fie soluție pt (11) $\Rightarrow$

$$\Rightarrow (\phi(x)C(x))' = A(x)\phi(x)C(x) + b(x) \Rightarrow$$

$$\Rightarrow \underbrace{\phi'(x)}_{\overset{(2)}{=} A(x)\phi(x)} C(x) + \phi(x)C'(x) = A(x)\phi(x)C(x) + b(x)$$

$$\Rightarrow \cancel{A(x)\phi(x)C(x)} + \phi(x)C'(x) = \cancel{A(x)\phi(x)C(x)} + b(x)$$

$$\Rightarrow \phi(x)C'(x) = b(x)$$

sistem alg. liniar în $C_1', \dots, C_n'$

dar $\det \phi(x) \neq 0 \Rightarrow \exists\, (\phi(x))^{-1}$ $\Big\} \Rightarrow \underline{C'(x) = (\phi(x))^{-1} b(x)}$

$$\Rightarrow C_j'(x) = \left((\phi(x))^{-1} b(x)\right)_j, \quad j = \overline{1,n}$$

n ec. de tip primitivă pt a determina $C_1, \dots, C_n$ ca funcții de x.

<u>Exemplu</u>: ① Se dă sistemul: $\begin{cases} y_1' = 2y_1 + 3y_2 + x \\ y_2' = 3y_1 + 2y_2 + e^x \end{cases}$

Se cere mulț soluțiilor sistemului.

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \underbrace{\begin{pmatrix} x \\ e^x \end{pmatrix}}_{b(x)} \quad ; A(x) = A = \text{const.}$$

$$y' = Ay + b(x)$$

$b: \mathbb{R} \to \mathbb{R}^n$

- rezolvăm sistemul liniar omogen asociat : $\bar{y}' = A\bar{y}$
din exemplu ① (pag. 1) am rezolvat deja $\Rightarrow$

$$\Rightarrow \bar{y}(x) = \phi(x)\, c, \quad c \in \mathbb{R}^2$$

$$\phi(x) = \begin{pmatrix} e^{-x} & e^{5x} \\ -e^{-x} & e^{5x} \end{pmatrix}$$

- aplicăm metoda variației constantelor:
determinăm $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$ a.î. $y(x) = \phi(x) C(x)$
sol. a sist. afin: $y' = Ay + b(x) \Rightarrow$

$$\Rightarrow (\phi(x)C(x))' = A\,\phi(x)C(x) + b(x) \Rightarrow \underbrace{\phi(x)\, C'(x) = b(x)}_{\text{sistem liniar în } c_1', c_2'}$$

$$\Rightarrow \begin{pmatrix} e^{-x} & e^{5x} \\ -e^{-x} & e^{5x} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} x \\ e^{x} \end{pmatrix} \Rightarrow \begin{cases} e^{-x} c_1' + e^{5x} c_2' = x \\ -e^{-x} c_1' + e^{5x} c_2' = e^{x} \end{cases} \quad (+)$$

$$2e^{5x} c_2' = x + e^{x} \quad \Big/ : 2e^{5x}$$

$$\Rightarrow \boxed{c_2' = \frac{1}{2}\left(x e^{-5x} + e^{-4x}\right)}$$

Scăzând ec $\Rightarrow$ $2e^{-x} c_1' = x - e^{x} \quad \Big| : 2e^{-x} \Rightarrow$

$$\Rightarrow \boxed{c_1' = \frac{1}{2}\left(x e^{x} - e^{2x}\right)}$$

Se obține :

$$c_1(x) = \frac{1}{2}\int \underbrace{x e^{x}}_{x(e^{x})'} dx - \frac{1}{2}\int e^{2x} dx = \frac{1}{2}\left(x e^{x} - \int x' e^{x} dx\right) - \frac{1}{2}\frac{e^{2x}}{2} \Rightarrow$$

$$\Rightarrow \underline{c_1(x) = \frac{1}{2}\left(x e^{x} - e^{x}\right) - \frac{1}{4} e^{2x} + K_1}, \quad K_1 \in \mathbb{R}.$$

$$c_2(x) = \frac{1}{2}\int x e^{-5x} dx + \frac{1}{2}\int e^{-4x} dx = \frac{1}{2}\int x \left(\frac{e^{-5x}}{-5}\right)' dx + \frac{1}{2}\frac{e^{-4x}}{-4} =$$

$$= \frac{1}{2}\left(-\frac{1}{5} x e^{-5x} + \frac{1}{5}\int x' e^{-5x} dx\right) - \frac{1}{8} e^{-4x} \Rightarrow$$

$$\Rightarrow \underline{c_2(x) = \frac{1}{2}\left(-\frac{1}{5} x e^{-5x} + \frac{1}{5}\frac{e^{-5x}}{(-5)}\right) - \frac{1}{8} e^{-4x} + K_2}, \quad K_2 \in \mathbb{R}$$

Soluția sistemului afin este:

$$y(x) = \begin{pmatrix} e^{-x} & e^{5x} \\ -e^{-x} & e^{5x} \end{pmatrix} \begin{pmatrix} \frac{1}{2}(x-1)e^{x} - \frac{e^{2x}}{4} + K_1 \\ -\frac{1}{10}\left(x+\frac{1}{5}\right)e^{-5x} - \frac{1}{8}e^{-4x} + K_2 \end{pmatrix}$$

$K_1, K_2 \in \mathbb{R}$.

Tema 4: Determinați mulțimea soluțiilor sistemelor:

1) $\begin{cases} y_1' = y_2 + 2e^{x} \\ y' = y_1 \end{cases}$

2) $\begin{cases} y_1' = y_1 - y_2 + 2\sin x \\ y_2' = 2y_1 - y_2 \end{cases}$

3) $\begin{cases} y_1' = -y_1 + 2y_2 \\ y_2' = y_2 - y_1 + \frac{1}{\cos x} \end{cases}$, $x \in (0, \frac{\pi}{2})$

Ecuații diferențiale de ordin n

$$y^{(n)} = f(x, y, y^{(1)}, \ldots, y^{(n-1)}) \quad (14)$$

cu $f : D \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

Ecuației (14) i se poate asocia un sistem (vezi cursul 2) prin notația:

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}; \quad \begin{cases} z_1 = y \\ z_2 = y^{(1)} \\ \vdots \\ z_{n-1} = y^{(n-2)} \\ z_n = y^{(n-1)} \end{cases} \Rightarrow \begin{cases} z_1' = z_2 \\ z_2' = z_3 \\ \vdots \\ z_{n-1}' = z_n \\ z_n' = f(x, z_1, \ldots, z_n) \end{cases} \quad (15)$$

În cazul ec. liniare omogene de ordin n avem:

$$f(x, y, y^{(1)}, \ldots, y^{(n-1)}) = \sum_{j=0}^{n-1} a_j(x)\, y^{(j)} \quad (16)$$

iar pentru ec. afină (liniară neomogenă):

$$f(x, y, y^{(1)}, \ldots, y^{(n-1)}) = \sum_{j=0}^{n-1} a_j(x)\, y^{(j)} + g(x) \quad (17)$$

Rezolvarea ec. liniare omogene de ordin n:

$$y^{(n)} = \sum_{j=0}^{n-1} a_j(x)\, y^{(j)} \quad (18)$$

Ec (18) i se asociază sistemul: $\begin{cases} z_1' = z_2 \\ z_2' = z_3 \\ \vdots \\ z_{n-1}' = z_n \\ z_n' = \sum_{j=0}^{n-1} a_j(x) z_{j+1} \end{cases} \Rightarrow$

$$\Rightarrow \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_{n-1}' \\ z_n' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_0(x) & a_1(x) & a_2(x) & \dots & a_{n-2}(x) & a_{n-1}(x) \end{pmatrix}}_{A(x)} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} \to z' = A(x)\, z \cdot (19)$$

Prop 2: Prin modul în care asociem sistemul (19) ec. (18), avem:

1) φ este soluție pt (18) $\Rightarrow \psi = \begin{pmatrix} \varphi \\ \varphi^{(1)} \\ \vdots \\ \varphi^{(n-1)} \end{pmatrix}$ este soluție pt (19)

2) $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$ soluție pt (19) $\Rightarrow \varphi = \psi_1$ este soluție pt (18).

Prop. 3: Dacă notăm S = mult sol. ec. (18), atunci folosind sistemul asociat avem:

1) S este spațiu vectorial real

2) $\dim S = n$.

Conform prop. 3 este suficient să determinăm o bază în S, adică un sistem fundamental de soluții pt ec. (18).

OBS: Dacă știm $\{\psi_1, \dots, \psi_n\}$ sistem fundamental de soluții pt (19), $\psi_j = \begin{pmatrix} \psi_{1j} \\ \vdots \\ \psi_{nj} \end{pmatrix}$, $j = \overline{1,n}$, atunci cf. prop. 1 $\Rightarrow (\varphi_j = \psi_{1j})_{j=\overline{1,n}}$ este sistem fundamental de soluții pt (18).

Pt. ec. de ordin n liniare, omogene, cu coef const, se poate da un algoritm de determinare a unui sistem fundamental de soluții, independent de sistemul asociat:

Fie ecuația

$$y^{(n)} = \sum_{j=0}^{n-1} a_j y^{(j)}, \quad a_0, \dots, a_{n-1} \in \mathbb{R}. \quad (20)$$

Alg de determinare a unui sistem fundam. de soluții este:

- rezolvăm ec. caracteristică:

$$r^n = \sum_{j=0}^{n-1} a_j r^j \quad (21)$$

(polinom de grad n)

Notăm $r_1, \dots, r_k$ rădăcinile distincte ale ec. caracteristice cu multiplicitățile $m_1, \dots, m_k$

Evident:
$$\begin{cases} m_1 + \dots + m_k = n \\ r^n - \sum_{j=0}^{n-1} a_j r^j = (r-r_1)^{m_1} \dots (r-r_k)^{m_k} \end{cases}$$

- pt fiecare rădăcină r_j a ec. caract. (21) se determină exact m_j soluții pentru sistemul fundamental de soluții, astfel:

1) $r_j \in \mathbb{R}$, $m_j \geq 1 \Rightarrow$
$$\boxed{\begin{aligned} \varphi_1(x) &= e^{r_j x} \\ \varphi_2(x) &= x e^{r_j x} \\ &\vdots \\ \varphi_{m_j}(x) &= x^{m_j - 1} e^{r_j x} \end{aligned}}$$

Obs: dacă $m_j = 1$, atunci avem doar soluția $\varphi_1(x) = e^{r_j x}$

2) $r_j \in \mathbb{C} \setminus \mathbb{R}$, $m_j \geq 1 \Rightarrow \bar{r}_j$ este printre soluțiile $r_1, \dots, r_k$ cu aceeași multiplicitate ca și r_j

și scriem $2m_j$ soluții:

$$\varphi_j(x) = Re(x^{j-1} e^{r_j x}) \; ; \; \tilde{\varphi}_j = Im(x^{j-1} e^{r_j x})_{\overline{j=1,m_j}}$$

Dacă $r_j = \alpha_j + i\beta_j$, $\alpha_j, \beta_j \in \mathbb{R}$, $\beta_j \neq 0 \Rightarrow$

$$\Rightarrow e^{r_j x} = e^{\alpha_j x} \cdot e^{i\beta_j x} = e^{\alpha_j x}(\cos\beta_j x + i \sin\beta_j x)$$

$$\Rightarrow \boxed{\begin{aligned} \varphi_j(x) &= x^{j-1} e^{\alpha_j x} \cos\beta_j x \\ \tilde{\varphi}_j(x) &= x^{j-1} e^{\alpha_j x} \sin\beta_j x \end{aligned} \quad , \; j = \overline{1, m_j}}$$

Exemplu: Fie ec: $y^{(2)} = 5y^{(1)} - 6y$

a) Mulț soluțiilor

b) Soluția care verifică $\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases}$.

a) Ec. caract.

$$r^2 = 5r - 6 \qquad (y = y^{(0)})$$

$$r^2 - 5r + 6 = 0$$

$$\Delta = 25 - 24 = 1$$

$$r_{1,2} = \frac{5 \pm 1}{2} \begin{cases} r_1 = 3 \\ r_2 = 2. \end{cases}$$

$$(r-3)^1(r-2)^1 = 0 \Rightarrow \begin{matrix} m_1 = 1 \\ m_2 = 1 \end{matrix}$$

Cf alg de mai sus avem $\left\{\begin{array}{l} \text{pt } r_1 = 3, m_1 = 1 \Rightarrow \\ \qquad \Rightarrow \varphi_1(x) = e^{3x} \\ \text{pt } r_2 = 2, m_2 = 1 \Rightarrow \\ \qquad \Rightarrow \varphi_2(x) = e^{2x} \end{array}\right| \Rightarrow$

$\Rightarrow \{e^{3x}, e^{2x}\}$ este sistem fundamental de soluții $\Rightarrow y(x) = C_1 e^{3x} + C_2 e^{2x}$.

b) $y'(x) = C_1 e^{3x} \cdot 3 + C_2 e^{2x} \cdot 2$

$$\left.\begin{array}{l} y(0) = 1 \Rightarrow C_1 e^0 + C_2 e^0 = 1 \\ y'(0) = 2 \Rightarrow 3C_1 e^0 + 2C_2 e^0 = 2 \end{array}\right\} \Rightarrow \begin{cases} C_1 + C_2 = 1 \rightarrow C_2 = 1 - C_1 \\ 3C_1 + 2C_2 = 2 \end{cases} \Rightarrow$$

$$\Rightarrow 3C_1 + \cancel{2} - 2C_1 = \cancel{2} \Rightarrow \boxed{C_1 = 0} \; \boxed{C_2 = 1}$$

Cazul ec. afine:
$$\boxed{y^{(n)} = \sum_{j=0}^{n-1} a_j(x)\, y^{(j)} + g(x)} \quad (22)$$

Cazul în care cunoaștem o soluție particulară:

- fie φ_0 o soluție pt (22). Atunci mulțimea soluțiilor ec. (22) este:

$$\left\{ \overline{\varphi}(t) + \varphi_0 \;\middle|\; \overline{\varphi} \text{ sol. a ec. liniare omogene atașate} \right\}$$

unde ec. liniară omogenă atașată este:
$$\overline{y}^{(n)} = \sum_{j=0}^{n-1} a_j(x)\, \overline{y}^{(j)} \quad (23)$$

Dacă nu știm o soluție particulară, atunci se aplică metoda variației constantelor:

- se determină $\{\varphi_1, \dots, \varphi_n\}$ sistem fundamental de soluții pt ec. liniară omogenă atașată (23): $\overline{y}(x) = c_1\varphi_1(x) + \dots + c_n\varphi_n(x)$.
- aplicăm metoda variației constantelor:

se determină $c_1, \dots, c_n : I \to \mathbb{R}$ a.î.
$$y(x) = c_1(x)\varphi_1(x) + \dots + c_n(x)\varphi_n(x)$$
soluția ec. afine (22).

Asociind sistemul ec. (22) se obține că $c_1', c_2', \dots, c_n'$ trebuie să verifice sistemul:

$$\begin{cases} c_1'\varphi_1(x) + \dots + c_n'\varphi_n(x) = 0 \\ c_1'\varphi_1^{(1)}(x) + \dots + c_n'\varphi_n^{(1)}(x) = 0 \\ \vdots \\ c_1'\varphi_1^{(n-2)}(x) + \dots + c_n'\varphi_n^{(n-2)}(x) = 0 \\ c_1'\varphi_1^{(n-1)}(x) + \dots + c_n'\varphi_n^{(n-1)}(x) = g(x). \end{cases} \quad (24)$$

OBS. (24) se obține dacă se face variația constantelor în sistemul:
$$z' = A(x)\cdot z + b(x)$$
cu $A(x)$ din (19) și $b(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(x) \end{pmatrix}$

Exemplu: Fie ec.: $y^{(3)} = +3y^{(2)} - 3y^{(1)} + y + \underbrace{2e^x}_{g(x)}$

Se cere mulț. soluțiilor ec.

$a_2 = -3$; $a_1 = 3$; $a_0 = -1$; $g(x) = 2e^x$

- rezolvăm (determinăm sistem fundamental de soluții) ec. liniară omogenă atașată.

$$\bar{y}^{(3)} = +3\bar{y}^{(2)} - 3\bar{y}^{(1)} + \bar{y}$$

are coef constanți $\Rightarrow$ aplicăm alg cu ec. caract:

$$r^3 = 3r^2 - 3r + 1$$
$$r^3 - 3r^2 + 3r - 1 = 0$$
$$(r-1)^3 = 0 \rightarrow r_1 = 1 \,,\; m_1 = 3 . \Rightarrow$$

$$\Rightarrow \begin{cases} \varphi_1(x) = e^x \\ \varphi_2(x) = xe^x \\ \varphi_3(x) = x^2 e^x \end{cases} \rightarrow \bar{y}(x) = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$$

- aplicăm variația constantelor: determinăm $C_1, C_2, C_3 : \mathbb{R} \to \mathbb{R}$ a.î. C_1', C_2', C_3' verifică sistemul liniar:

$$\begin{cases} C_1' e^x + C_2'(xe^x) + C_3'(x^2 e^x) = 0 \quad /:e^x \\ C_1'(e^x)' + C_2'(xe^x)' + C_3'(x^2e^x)' = 0 \\ C_1'(e^x)'' + C_2'(xe^x)'' + C_3'(x^2e^x)'' = 2e^x \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} C_1' + C_2' x + C_3' x^2 = 0 \\ C_1' e^x + C_2'(1+x)e^x + C_3'(2x+x^2)e^x = 0 \quad /:e^x \\ C_1' e^x + C_2'(2+x)e^x + C_3'(2+4x+x^2)e^x = 2e^x \quad /:e^x \end{cases}$$

$$\Rightarrow \begin{cases} C_1' + C_2' x + C_3' x^2 = 0 \\ C_1' + C_2'(1+x) + C_3'(2x+x^2) = 0 \\ C_1' + C_2'(2+x) + C_3'(2+4x+x^2) = 2 \end{cases} \begin{matrix} \xrightarrow{(-)} C_2' + C_3' \cdot 2x = 0 . \\ \xrightarrow{(-)} C_2' + C_3'(2+2x) = 2 \end{matrix}$$

$$\overline{\quad 2C_3' = 2 \Rightarrow \quad} (-)$$

$$\Rightarrow \boxed{C_3' = 1} \;;\; \boxed{C_2' = -2x} \rightarrow C_1' = 2x^2 - x^2 \Rightarrow \boxed{C_1' = x^2} \Rightarrow$$

$$\Rightarrow \begin{cases} C_1 = \frac{x^3}{3} + K_1 \\ C_2 = -x^2 + K_2 \\ C_3 = x + K_3 \end{cases} \Rightarrow y(x) = C_1(x)\varphi_1(x) + C_2(x)\varphi_2(x) + C_3(x)\varphi_3(x) \Rightarrow$$

$$\Rightarrow y(x) = \left(\frac{x^3}{3} + K_1\right)e^x + \left(-x^2 + K_2\right)xe^x + \left(x + K_3\right)x^2e^x =$$

$$= \underbrace{\left(\frac{x^3}{3}e^x \cancel{- x^3 e^x} \cancel{+ x^3 e^x}\right)}_{\substack{\text{soluție particulară} \\ \varphi_0(x) = \frac{x^3}{3}e^x .}} + \underbrace{\left(K_1 e^x + K_2 x e^x + K_3 x^2 e^x\right)}_{\substack{\text{forma generală pt.} \\ \text{ec. liniară omogenă}}}$$

<u>Tema 5</u>: Să se determine soluțiile ec:

1) $y^{(2)} = 4y^{(1)} - 3y + \underbrace{e^x + x - 1}_{g(x)}$

2) $y^{(2)} = y^{(1)} - y + e^{\frac{x}{2}} \cos \frac{x\sqrt{3}}{2}$

3) $y^{(2)} = 2y^{(1)} + 2y + e^x$

4) $y^{(3)} = y + e^x$

5) $y^{(3)} = 6y^{(2)} - 11y^{(1)} + 6y + 3e^x$.

Ecuații Euler liniare, de ordin n.

$$\boxed{x^n y^{(n)} = \sum_{j=0}^{n-1} x^j y^{(j)} + g(x)} \quad (25)$$

<u>Prop. 5</u>: Ec. (25) se reduce la ec. afină cu coef. constanți prin s.v: $|x| = e^s$

$$\underset{\substack{\text{de ordin } n \\ \text{(Euler)} \\ \text{liniară}}}{(x, y)} \xrightarrow{\qquad\qquad} \underset{\substack{\text{de ordin } n \\ \text{afină}}}{(s, z)}$$

$$\boxed{y(x) = z(s(x))}$$

$$y'(x) = z'(s(x))\, s'(x) = z'(s)\cdot \frac{1}{x} \Rightarrow \boxed{x y' = z'}$$

$$s(x) = \ln|x| \Rightarrow s'(x) = \frac{1}{x}$$

$$y'' = (y')' = \left(\frac{1}{x} z'\right)' = -\frac{1}{x^2} z' + \frac{1}{x} z''\cdot s'(x) \Rightarrow$$

$$\Rightarrow y'' = -\frac{1}{x^2} z' + \frac{1}{x^2} z'' \Rightarrow \boxed{x^2 y'' = z'' - z'}$$

$$y''' = \left(\frac{1}{x^2}(z''-z')\right)' = \frac{-2}{x^3}(z''-z') + \frac{1}{x^2}(z'''-z'')\cdot\frac{1}{x} \Rightarrow$$

$$\Rightarrow x^3 y''' = -2z'' + 2z' + z''' - z''$$

$$\boxed{x^3 y''' = z''' - 3z'' + 2z'}$$

Exemplu: $x^3 y''' = 3xy' + 3y + x^2$, $x > 0$.

ec Euler liniară de ordin 3

$$(x, y) \xrightarrow[|x| = e^s]{} (s, z)$$

$y = z$

$xy' = z'$

$x^2 y'' = z'' - z'$

$x^3 y''' = z''' - 3z'' + 2z'$

$\Rightarrow z''' - 3z'' + 2z' = 3z' + 3z + e^{2s}$

$\Rightarrow z''' - 3z'' - 1\cdot z' - 3z = e^{2s}$.

Pb ec în (s, z) ec. liniară omogenă atașată este:

$$\bar{z}''' - 3\bar{z}'' - \bar{z}' - 3\bar{z} = 0 \quad ; g(s) = e^{2s}$$

ec. caract este: $r^3 - 3r^2 - r + 3 = 0$.

are rad $r_1 = 1 \Rightarrow$

$$\Rightarrow \underbrace{r^3 - r^2} - \underbrace{2r^2 + 2r} \underbrace{-3r + 3} = 0$$

$$r^2(r-1) - 2r(r-1) - 3(r-1) = 0.$$

$$(r-1)(r^2 - 2r - 3) = 0 \Rightarrow r_1 = 1,$$

$$r^2 - 2r - 3 = 0 \Rightarrow \Delta = 4 + 12 = 16 \quad r_2 = \frac{2+4}{2} = 3, \quad r_3 = \frac{2-4}{2} = -1$$

$$\rightarrow \quad r_1=1,\ m_1=1 \Rightarrow \varphi_1(s)=e^{s}$$
$$r_2=3,\ m_2=1 \Rightarrow \varphi_2(s)=e^{3s}$$
$$r_3=-1,\ m_3=1 \Rightarrow \varphi_3(s)=e^{-s}$$
$$\Rightarrow \bar{z}(s)=C_1e^{s}+C_2e^{3s}+C_3e^{-s}.$$

Variem const $C_1, C_2, C_3 \Rightarrow$ det C_1', C_2', C_3' din sistemul

$$\begin{cases} C_1'e^{s}+C_2'e^{3s}+C_3'e^{-s}=0 \\ C_1'(e^{s})'+C_2'(e^{3s})'+C_3'(e^{-s})'=0 \\ C_1'(e^{s})''+C_2'(e^{3s})''+C_3'(e^{-s})''=e^{2s}. \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} C_1'e^{s}+C_2'e^{3s}+C_3'e^{-s}=0 \\ C_1'e^{s}+3C_2'e^{3s}-C_3'e^{-s}=0 \\ C_1'e^{s}+9C_2'e^{3s}+C_3'e^{-s}=e^{2s} \end{cases}$$

$\Rightarrow (+)\quad 2C_1'e^{s}+4C_2'e^{3s}=0$

$\Rightarrow (+)\quad 2C_1'e^{s}+12C_2'e^{3s}=e^{2s}$

$(-)$

$$\Rightarrow 8C_2'e^{3s}=e^{2s} \Rightarrow \boxed{C_2'=\frac{1}{8}e^{-s}}$$

$$2C_1'e^{s}=-\frac{4}{8}e^{-s}\cdot e^{3s} \Rightarrow \boxed{C_1'=-\frac{1}{4}e^{s}}$$

$$C_3'e^{-s}=+\frac{1}{4}e^{s}\cdot e^{s}-\frac{1}{8}e^{-s}e^{3s}$$

$$C_3'=\frac{1}{8}e^{2s}\cdot e^{s} \Rightarrow \boxed{C_3'=\frac{1}{8}e^{3s}}$$

$$\begin{cases} C_1(s)=-\frac{1}{4}e^{s}+K_1 \\ C_2(s)=-\frac{1}{8}e^{-s}+K_2 \\ C_3(s)=\frac{1}{24}e^{3s}+K_3 \end{cases} \Rightarrow z(s)=C_1(s)e^{s}+C_2(s)e^{2s}+C_3(s)e^{-s} \Rightarrow$$

$$\Rightarrow s=\ln|x|=\ln x \quad ; \quad e^{\ln x}=x,\quad e^{2\ln x}=e^{\ln x^2}=x^2,\quad e^{-\ln x}=e^{\ln x^{-1}}=\frac{1}{x}$$

$$\Rightarrow y(x)=C_1(\ln x)\cdot x+C_2(\ln x)\,x^2+C_3(\ln x)\cdot\frac{1}{x}$$

$K_1, K_2, K_3 \in \mathbb{R}$.

Temă (6): 1) $x^3 y''' + xy' - y = x^2$

2) $x^2 y'' - 2y'x = 9x^2$

} asemănător cu ex. de mai sus.

3) Arătați că prin schimbarea de variab. $|ax+b| = e^s$,

ec.:

$$(ax+b)^n y^{(n)} = \sum_{j=0}^{n-1} (ax+b)^j y^{(j)} + g(x)$$

devine ec. liniară de ordin n în variabile (s, z); $z(s(x)) = y(x)$.

Aplicați pt:

$$(2x+3)^3 y''' + 4(2x+3)^2 y'' + 4(2x+3)y' - 8y = 8(2x+3)^2.$$

<u>Pt. examen</u>:

- sâmbătă sau duminică: 31.01.2021, 9^{00} (!)

- <u>simulare test examen</u>: 17 ianuarie 2021, 9^{00}
  test de 15 minute.