JOURNAL OF NUMBER THEORY 54, 190–202 (1995)

# Modular Lattices in Euclidean Spaces

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*Communicated by J. S. Hsia*

Received August 25, 1993

Even lattices similar to their duals are discussed in connection with modular forms for Fricke groups. In particular, lattices of level 2 with large Hermite number are considered, and an analogy between the seven levels $l$ such that $1 + l$ divides 24 is stressed. © 1995 Academic Press, Inc.

## INTRODUCTION

The title refers to lattices in euclidean $n$-space which are even and similar to their duals. When the similarly factor is 1, such lattices are, of course, unimodular and form a frequently discussed genus. When that factor is a prime $l$, they form a subset (of similar interest) in a specific genus of level $l$. The aim of the paper is to point out the connection with modular forms for the Fricke group of level $l$ and describe some consequences.

One will not meet here new dense sphere packings (except for the appearance of an even counterpart to $\Lambda_{28}$). Results in this direction, using nonintegral lattices similar to their duals, are given in a recent paper by Conway and Sloane [CS].

This work was mainly done during a stay at SFB 343, Universität Bielefeld. I gratefully acknowledge the hospitality of this institution and the cooperation with Rudolf Scharlau and Boris Venkov. I am also indebted to Rainer Schulze-Pillot for a helpful discussion which led to the result in Section 1.4.

## 1. THE NOTION OF MODULAR LATTICES

### 1.1. *Basic Definitions*

Let $V$ be an $n$-dimensional euclidean vector space, $l > 0$, and $\sigma\colon V \to V$ a similarity of norm $l$, i.e.,

$$\sigma v \cdot \sigma w = l v \cdot w \qquad \text{for} \quad v, w \in V.$$

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0022-314X/95 $12.00

An integral lattice $\Lambda$ on $V$ will be called *$\sigma$-modular* if it is equal to $\sigma\Lambda^*$; here the dual $\Lambda^*$ consists of all $v \in V$ such that $v \cdot \Lambda \subset \mathbb{Z}$, and "integral" means $\Lambda \subset \Lambda^*$. The equation above for $v, w \in \Lambda^*$ implies that for such a lattice $l\Lambda^*$ is contained in $\Lambda^{**} = \Lambda$, so $l$ must be an integer. Furthermore, $\Lambda$ is isometric to $\sqrt{l}\,\Lambda^*$, and so it has determinant $l^{n/2}$. In particular, $n$ must be even unless $l$ is a square of an integer.

An integral lattice which is $\sigma$-modular for some similarity $\sigma$ will simply be called *modular*. In case $n = 2$, e.g., all integral lattices are obviously modular. In the following $n$ will always be even, and we shall only deal with even lattices $\Lambda$ (i.e., $v \cdot v$ is an even integer for all $v \in \Lambda$).

1.2. *Connection with Modular Forms*

Let $l \in \mathbb{N}$ be fixed. As usual, $\Gamma_0(l)$ denotes the subgroup of matrices in $SL_2(\mathbb{Z})$ whose lower left entry is a multiple of $l$. The normalizer of $\Gamma_0(l)$ in $SL_2(\mathbb{R})$ contains an element $t_l$ which extends $\Gamma_0(l)$ to the *Fricke group*

$$\Gamma_*(l) = \Gamma_0(l) \cup \Gamma_0(l)\, t_l, \qquad t_l = \begin{pmatrix} 0 & 1/\sqrt{l} \\ -\sqrt{l} & 0 \end{pmatrix}.$$

This is a Fuchsian group of first kind, and so we have for it the usual notion of modular forms, i.e., automorphic forms which are holomorphic both on the upper half plane and at the cusps; see [Mi], § 2.1. The theta function of an even lattice $\Lambda$ on euclidean $n$-space is

$$\Theta_\Lambda(z) = \sum_{v \in \Lambda} q^{(v \cdot v)/2}, \qquad q = e^{2\pi i z}.$$

THEOREM 1. *Let $n = 2k$, and let $\Lambda$ be an even $\sigma$-modular lattice for some similarity $\sigma$ of norm $l$. Then $\Theta_\Lambda(z)$ is a modular form of weight $k$ for the group $\Gamma_*(l)$ and its character $\chi$ given by*

$$\chi(s) = \left(\frac{(-l)^k}{a}\right) \quad \textit{for} \quad s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(l), \qquad \chi(t_l) = i^k.$$

*Proof.* As $\Lambda$ and $\sqrt{l}\,\Lambda^*$ are isometric, $\Theta_\Lambda(z)$ is a modular form for $\Gamma_0(l)$ which satisfies

$$\Theta_\Lambda(z) = \Theta_{\sqrt{l}\,\Lambda^*}(z) = i^{\,k}(-\sqrt{l}\,z)^{-k}\,\Theta_\Lambda\left(-\frac{1}{lz}\right)$$

(cf. [Mi], p. 192). This formula just means automorphy with respect to the Fricke involution $t_l$. ∎

It is a remarkable property of the group $\Gamma_*(l)$ for primes $l$ that its function field has genus zero if and only if $l$ divides the order of the monster (cf. [CN]). For $l=2$ and 3, conjugation by $\left(\begin{smallmatrix}1 & 0\\ 0 & \sqrt{l}\end{smallmatrix}\right)$ takes $\Gamma_*(l)$ onto the Hecke group $G(\sqrt{l})$ generated by

$$\begin{pmatrix}1 & \sqrt{l}\\ 0 & 1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

([He], p. 53), and in these cases one can use Hecke's results to describe the algebra of modular forms as explicitly as for $l=1$. Namely, for $l=1, 2, 3$ put $k_0=4, 2, 1$ respectively. Then the character $\chi$ of Theorem 1 is defined when $k$ is a multiple of $k_0$, and in this case the corresponding space of modular forms has dimension $1+[k/k_1]$, where $k_1=24/(1+l)$; see [He], p. 42. Let $\Theta_{2k_0}(z)$ denote the theta function of the $E_8, D_4$ and $A_2$ root lattice, respectively; these lattices are modular. Let $\eta(z)$ be the Dedekind eta function, and put

$$\Delta_{2k_1}(z)=(\eta(z)\,\eta(lz))^{k_1},$$

which is a nontrivial cusp form. Then it follows that a basis of our space is given by the functions

$$\Theta_{2k_0}^{\lambda}\Delta_{2k_1}^{\mu} \qquad \text{with integers} \quad \lambda, \mu \geqslant 0, k_0\lambda+k_1\mu=k.$$

As the $q$-expansions of these functions begin with $q^{\mu}$, the space contains a unique function $F_{2k,l}$ whose $q$-expansion has $1, 0, ..., 0$ as its first $1+[k/k_1]$ coefficients. An even modular lattice $\Lambda$ of dimension $2k$ and similarly norm $l$ will be called *extremal* if $\Theta_\Lambda=F_{2k,l}$ holds; this means that

$$\min \Lambda=\min\{v\cdot v \mid v\in\Lambda, v\neq 0\}$$

is as large as possible, namely (at least) equal to $2+2[k/k_1]$. Up to now, there are five dimensions $n=2k$ for which $k$ is multiple of $k_1$ and an extremal modular lattice $\Lambda$ is known: $n=12, 16, 24$ (with $l=3, 2$ and 1, respectively), where $\min\Lambda=4$; furthermore, $n=32$ $(l=2)$ and $n=48$ $(l=1)$, where $\min\Lambda=6$. These are just the cases where for $n>8$ a modular lattice gives rise to the densest sphere packing known. See [Bi], Ch. 1; the notion of extremal lattices stems from Ch. 7.

1.3. *Genera of Modular Lattices*

More generally, for fixed $l\in\mathbb{N}$ we now consider even lattices $\Lambda$ in euclidean $\mathbb{R}^n$ such that

$$\sqrt{l}\,\Lambda^* \text{ is even, } \det\Lambda=l^{n/2}. \tag{0}$$

Equivalently, the discriminant quadratic form

$$\varphi\colon \Lambda^*/\Lambda\to\mathbb{Q}/\mathbb{Z}, \qquad v+\Lambda\mapsto\tfrac{1}{2}v\cdot v+\mathbb{Z}$$

takes its values in $\frac{1}{l}\mathbb{Z}/\mathbb{Z}$, and $\Lambda^*/\Lambda$ has order $l^{n/2}$. The genus of an even

$n$-dimensional lattice is given by the isometry class of this form (cf. [Ni]). We collect some rather well-known facts.

THEOREM 2. *Let $l$ be a prime and $n$ an even natural number. If in euclidean $n$-space there is an even lattice $\Lambda$ with property* (0) *above, then $l \equiv 3 \,(mod\, 4)$ or $n \equiv 0 \,(mod\, 4)$ must hold. Conversely, let the pair $(l, n)$ satisfy this condition. Then there is a unique genus of such lattices. On the set of classes in this genus one has the involution*

$$class(\Lambda) \mapsto class(\sqrt{l}\, \Lambda^*)$$

*whose fixed point set consists of the classes of modular lattices and is non-empty.*

*Proof.* The first two statements can be quickly verified by using the formula of Braun-Milgram

$$\sum_{v \in \Lambda^*/\Lambda} e^{2\pi i \varphi(v)} = l^{n/4} e^{2\pi i n/8}$$

(cf. [MH], Appendix 4). For $l = 2$ decompose $\Lambda^*/\Lambda$ into orthogonal summands of dimension 2 over $\mathbb{F}_2$. A hyperbolic (resp. nonhyperbolic) summand contributes a factor 2 (resp. $-2$) to the left-hand side of the formula. So $n$ must be divisible by 4, and $\Lambda^*/\Lambda$ is hyperbolic if and only if $n \equiv 0 \,(mod 8)$. Now let $l$ be odd, and take an orthogonal basis of $\Lambda^*/\Lambda$ over $\mathbb{F}_l$. A basis vector of square (resp. nonsquare) norm contributes

$$\sum_{m=0}^{l-1} e^{2\pi i m^2/l} = \begin{cases} \sqrt{l} & \text{if } \ l \equiv 1 \,(mod\, 4) \\ i\sqrt{l} & \text{if } \ l \equiv 3 \,(mod\, 4) \end{cases}$$

(resp. the negative of this). For $l \equiv 1 \,(mod 4)$ again $n$ must be divisible by 4, and $\varphi$ is the unit form (i.e., the sum of $n/2$ squares) if and only if $n \equiv 0 \,(mod\, 8)$. For $l \equiv 3 \,(mod\, 4)$ one always gets the unit form.

As regards to the involution, it remains to construct some modular lattices for $n = 2$ or 4. In case $l \equiv 3 (mod\, 4)$ take $V = \mathbb{C}$ with inner product $v \cdot w = v\bar{w} + w\bar{v}$, similarity $\sigma v = i\sqrt{l}\, v$ and $\Lambda = \mathbb{Z} + \mathbb{Z}(1 + i\sqrt{l})/2$. In case $l \equiv 1 \,(mod\, 4)$ take $V = \mathbb{H}$ (Hamilton's quaternions), $v \cdot w$ and $\sigma$ as before, and $\Lambda$ a suitable maximal order in the quaternion skew field of reduced discriminant $l$ over $\mathbb{Q}$ (see below). With $\sigma v = (1 + i)\, v$ the latter choice is possible also for $l = 2$. ∎

1.4. *On Dimension* 4

It would perhaps be worthwhile to make a thorough study of the involution in Theorem 2 for $n = 4$. However, here we shall only deal with the question when this involution is trivial. The answer follows from results by Fricke, Deuring and Eichler.

THEOREM 3. *Let $l$ be a prime and $n=4$. Then the following conditions are equivalent*:

(i) *All lattices in the genus* (0) *are modular.*

(ii) *All cusp forms of weight* 2 *and trivial character for* $\Gamma_0(l)$ *are modular forms for* $\Gamma_*(l)$ *and the character* $\chi$ *in Theorem* 1.

(iii) *Either* $l<37$ *or* $l=41, 47, 59$ *or* 71 *holds.*

*Proof.* (i)$\Rightarrow$(ii) This follows from Theorem 1 and Eichler's basis theorem ([Ei]) which implies that the cusp forms in (ii) are linear combinations of theta functions from the genus in (i).

(ii)$\Rightarrow$(iii) Since $\chi$ takes the value $-1$ on the Fricke element, condition (ii) is equivalent to saying that nonzero cusp forms of weight 2 and trivial character for $\Gamma_*(l)$ do not exist, or that the function field of $\Gamma_*(l)$ has genus zero. As remarked already after Theorem 1, this happens precisely for the fifteen primes in (iii); cf. [Og].

(iii)$\Rightarrow$(i) These primes are also just the ones for which the two-sided ideals in all maximal orders of the quaternion skew field of reduced discriminant $l$ over $\mathbb{Q}$ are one-sided principal (or equivalently, type number and class number coincide; cf. [Pi]). One knows by the Clifford algebra construction that any lattice in our genus arises from some left ideal $I$ in such a maximal order $A$, the inner product being $(v\bar{w}+w\bar{v})/N(I)$ where $N(I)$ denotes the reduced norm. The dual lattice is given by the product with the inverse different, i.e., it is $J^{-1}I$ where $J$ denotes the maximal two-sided ideal such that $J^2=Al$. In case $J=A\lambda$ left multiplication by $\lambda$ gives the desired similarity. ∎

At the end of the proof in Section 1.3 we needed the fact that there always exists at least one maximal order $A$ such that, as above, the different $J$ is principal. To get this for $l>2$, observe first that our quaternion skew field contains $\lambda=\sqrt{-l}$. Choosing $A$ such that $\lambda\in A$, and comparing norms, we obtain $A\lambda=J$.

For weights $k>2$ the analogue of condition (ii) in Theorem 3 need not imply (i). E.g., it does not for $k=8$, $l=2$, as we shall see in the next section.

## 2. LATTICES OF LEVEL TWO

### 2.1. *The Genus of* $D_4'$

For $l=2, n=4$ the genus discussed before consists of the $D_4$ lattice, corresponding to the Hurwitz quaternion order of class number one. Now let $n = 4r$, and consider the following hierarchy: even lattices in euclidean $\mathbb{R}^n$ which are

(0) in the genus of $D_4^r$, i.e., $\sqrt{2}\,\Lambda^*$ is even and $\det \Lambda = 4^r$

(1) in the genus of $D_4^r$ and satisfying $\Theta_\Lambda = \Theta_{\sqrt{2}\,\Lambda^*}$

(2) modular with respect to some similarity of norm 2

(3) modular with respect to $id + \iota$ for some $\iota \in \mathrm{Aut}(\Lambda)$, $\iota^2 = -id$

(4) admitting a unimodular hermitian module structure over the Hurwitz order.

Property $(p+1)$ always implies $(p)$, and except for small $n$ is actually stronger. We shall see the latter by examples, but first prove the following

LEMMA. *For $r \leqslant 5$ property* (0) *implies property* (1).

*Proof.* It is a well-known consequence of the transformation formulae that the behaviour of theta functions at the cusps depends only on the genus. Let $\Lambda$ be in the genus of $D_4^r$. Then $\Theta_\Lambda - \Theta_4^r$ is a cusp form for $\Gamma_0(2)$. Now up to weight 10 there are no cusp forms for $\Gamma_0(2)$ other than those for $\Gamma_*(2)$ (cf. [Mi], p. 296). Consequently, $\Theta_\Lambda$ for $r \leqslant 5$ is a modular form for $\Gamma_*(2)$, i.e., (1) holds. ∎

A counterexample for $r = 6$ is given by $\Lambda = D_8^2 \oplus \sqrt{2}\,E_8$. This lattice clearly satisfies (0), and it has 224 norm 2 vectors, while $\sqrt{2}\,\Lambda^*$ has 272 such vectors, so (1) does not hold. The other counterexamples will be for $r = 4$ already. We begin with $E_8 \oplus \sqrt{2}\,E_8$ which clearly satisfies (2), but not (3). Next we consider its neighbour $M = L + \mathbb{Z}\,\frac{1}{2}w$, where

$$L = D_8 \oplus \sqrt{2}\,E_8, \qquad w = 2e_1 + \frac{\sqrt{2}}{2}(e_9 + \cdots + e_{16}).$$

Since $w \cdot w = 8$, $M$ is even, and since $v \cdot v \in \mathbb{Z}$ for all $v \in L^*$, also $\sqrt{2}\,M^*$ is even. So this lattice $M$ satisfies (0), and then also (1) by the lemma. But $M$ does not satisfy (2) because its norm 2 vectors belong to $D_8 \cup (D_8 \pm \frac{1}{2}w)$ and span a 9-dimensional subspace, whereas those of $\sqrt{2}\,M^*$ are easily seen to span the whole space.

Besides $D_4^4$ two other 16-dimensional lattices of type (3) were described in [Qu], namely one of minimum 4 (see below), and one with precisely 16 pairs $\pm v$ of norm 2 vectors. In case of type (4) the number of such vectors must be a multiple of 24, the number of units in the Hurwitz order.

## 2.2. *The* 16-*Dimensional Barnes–Wall Lattice*

A lot of information about this lattice can be found in [Bi], but yet the following characterization appears to be not in the literature.

THEOREM 4. *The genus of $D_4^4$ contains an up to isometry unique lattice of minimum* 4; *this is the Barnes–Wall lattice $BW_{16}$.*

*Proof.* Let $\Lambda$ be in the genus of $D_4^4$ and have minimum 4. By the lemma in 2.1, the theta function is a modular form for $\Gamma_*(2)$, and so (cf. 1.2)

$$\Theta_{\sqrt{2}\Lambda^*}(z) = \Theta_\Lambda(z) = F_{16,2}(z) = 1 + 4320q^2 + \cdots .$$

As pointed out to me by B. B. Venkov, one could continue by making use of what is known about involutions in the Conway group. The following method, however, is more elementary; the arguments will be quite similar to Conway's in his characterization of the Leech lattice (Ch. 12 of [Bi]). We consider pairs $\pm v \in \Lambda^*$ of norm $v \cdot v = 2$. Note that $v + \Lambda = -v + \Lambda$. If $\pm v$ and $\pm v'$ are different pairs in a fixed class modulo $\Lambda$, say with $v \cdot v' \geqslant 0$, then

$$4 = v \cdot v + v' \cdot v' \geqslant (v - v') \cdot (v - v') \geqslant 4.$$

Such pairs are therefore orthogonal to each other and at most 16 in number. The number of elements $\bar{v}$ in $\Lambda^*/\Lambda$ represented by norm 2 vectors must be then at least $4320/32 = 135$. On the other hand, each such $\bar{v}$ is isotropic with respect to the discriminant form $\varphi$ discussed in 1.3. In coordinates $\lambda_i$ with respect to a suitable $\mathbb{F}_2$-basis of $\Lambda^*/\Lambda$, $\varphi$ is given by $\lambda_1\lambda_2 + \cdots + \lambda_7\lambda_8$, and there are exactly $(2^4 - 1)(2^3 + 1) = 135$ nonzero isotropic vectors. It follows that the previous inequalities must have been exact. So we can choose pairwise orthogonal norm 2 vectors $v_1, ..., v_{16}$ from $\Lambda^*$ satisfying $v_i - v_j \in \Lambda$. Put

$$M = \Lambda + \mathbb{Z}v_1, \qquad N = \sum_{i=1}^{16} \mathbb{Z}v_i.$$

Then $(M : \Lambda) = 2$, and $N \subset M \subset \frac{1}{2}N$. Furthermore, $M^*$ contains a vector $w = \frac{1}{2}\sum a_i v_i$ such that $\Lambda$ consists of all $v \in M$ for which $v \cdot w$ is even. Since no $v_i$ lies in $\Lambda$, all $a_i$ must be odd and can be chosen as $\pm 1$. After replacing, if necessary, some $v_i$ by $-v_i$, we may even assume $a_i = 1$ for all $i$.

It remains to identify $M$. Use the basis $(v_i)_i$ to identify $\frac{1}{2}N/N$ with $\mathbb{F}_2^{16}$. Then $M$ corresponds to a linear binary code $C$ of length 16. As $M$ is even, $\det M = 2^6$, and $\min \Lambda = 4$, this code must be doubly-even, of dimension 5, and minimal distance 8. Now it is obvious (in contrast to the Leech lattice case) that such a linear Hadamard code is unique up to equivalence. When $C$ is taken as the first order Reed–Muller code, one gets $\Lambda = BW_{16}$ in its usual presentation. One can check that this lattice really belongs to the genus of $D_4^4$. ∎

A discussion of $BW_{16}$ with respect to property (4) will be found in [Ma], Ch. VIII, and an enumeration of all classes in its genus in [SV].

2.3. *Extremal Lattices of Higher Minimum*

Modular lattices of level 2 and minimum 6 are first possible for $n=32$, but much less than for $n=16$ can be said. We must have

$$\Theta_{\sqrt{2}\,\Lambda^*} = \Theta_\Lambda = F_{32,2} = 1 + 261120q^3 + \cdots.$$

Continuing as in the proof of Theorem 4, we see that now pairs of norm 3 vectors $\pm v \in \Lambda^*$ in the same class modulo $\Lambda$ are orthogonal to each other, and the number of all anisotropic classes is $(2^8-1)\,2^7 = 261120/8$. This merely implies existence of four orthogonal pairs and a sublattice of $\Lambda$ isometric to $\sqrt{3}\,D_4$. One may postulate that even $\sqrt{3}\,D_4^8 \subset \Lambda$. Such a lattice $\Lambda$, named $Q_{32}$ in [Bi], was specified in [Qu]; by construction it satisfies condition (3) of Section 2.1. More recently, Elkies in his study of Mordell–Weil lattices has also found a lattice with theta function $F_{32,2}$ (cf. [Oe]).

The most wanted extremal even modular lattice is a unimodular one of minimum $m=8$ in dimension $n=72$. In the case of level 2 one may similarly ask for $m=8$, $n=48$: answer also unknown. However, if one looks at all levels $l$ for which $1+l$ divides 24 (so that, as in 1.2, the cusp for $\Delta$ of weight $k_1 = 24/(1+l)$ exists), then there is a negative answer at least for $l=23$. Namely, the Hermite number of a 6-dimensional lattice of determinant $23^3$ and minimum 8 would be just a bit too large. Although the analogy between $l=1, 2$ and 23 works for minimum $m=4$ and $m=6$ (always $n=(m-2)k_1$), one perhaps should not take this as a too serious hint.

2.4. *Largest Hermite Numbers of Level 2 Lattices, $n \leqslant 36$*

Here we consider arbitrary even lattices $\Lambda$ of level at most 2, i.e., such that $\sqrt{2}\,\Lambda^*$ is also even. By the proof of Theorem 2, we must have $n \equiv 0 \ (mod\ 4)$, $\det \Lambda = 2^d$ with $d$ even, $0 \leqslant d \leqslant n$, and the genus of $\Lambda$ is uniquely determined by $d$.

THEOREM 5. *The maximum of $\gamma(\Lambda) = (\min \Lambda)/\sqrt[n]{\det \Lambda}$, taken over all $n$-dimensional even lattices of level at most 2, is given for*

| | |
|---|---|
| $n=4, 8, 12$ | *by* $\Lambda = D_4, E_8, D_{12}$ |
| $n=16$ *and* 20 | *by* $\Lambda$ *in the genus of* $D_4^{n/4}$, $\min \Lambda = 4$ |
| $n=24$ *and* 28 | *by* $\Lambda_{24}$ *(the Leech lattice) and* |
| | $\Lambda$ *in the genus of* $D_{28}$, $\min \Lambda = 4$ |
| $n=32$ *and* 36 | *by* $\Lambda$ *in the genus of* $D_4^{n/4}$, $\min \Lambda = 6$. |

*Proof.* Let us begin with the existence of the required lattices. Those in the genus of $D_4^{n/4}$ have already been discussed or can be found in [Qu].

Since it will be needed for $n=28$, we recall here our construction of such a lattice of minimum 4 for $n \geqslant 16$. Let $D_4/3D_4 = X \oplus Y$ be a hyperbolic decomposition (i.e., $X = X^{\perp}$, $Y = Y^{\perp}$), and $H \subset D_4^r$ the inverse image of

$$\{x+y \mid x \in X^r, y \in Y^r, x_1 = \cdots = x_r, y_1 + \cdots + y_r = 0\}.$$

Then $(1/\sqrt{3})\, H$ belongs to the genus of $D_4^r$, has minimum 4 for $r \geqslant 4$, and its norm 4 vectors for $r \geqslant 7$ are of the form $(1/\sqrt{3})(v_1, ..., v_r)$ where $v_i = 0$ for $r-2$ indices $i$, and $v_i \cdot v_i = 6$ otherwise. Now for $r=7$ let $K \subset D_4^7$ be the inverse image of a selforthogonal binary $[7, 3, 4]$-code extended over $D_4/2D_4^*$. Then $\Lambda = (1/\sqrt{6})(H \cap K)$ is easily seen to have the required properties. (Note that the Conway–Sloane lattices $\Lambda_{28}$ also have determinant 4 and minimum 4, but are not integral.)

To prove optimality, let $\Lambda$ be of level at most 2, $\det \Lambda = 2^d$; we may assume $d < n$ (otherwise $\sqrt{2}\, \Lambda^*$ would be unimodular). The following principles will be used. At first, on $\Lambda/2\Lambda^*$ a regular quadratic form over $\mathbb{F}_2$ is induced by $\frac{1}{2} v \cdot v$, and for $d \leqslant n-4$ this form is isotropic, so we can choose some $w \in \Lambda$, $w$ not in $2\Lambda^*$, with $w \cdot w$ divisible by 4. Then the lattice $\Lambda^w = \{v \in \Lambda \mid v \cdot w \in 2\mathbb{Z}\}$ again has level 2, $\det \Lambda^w = 2^{d+2}$, $\min \Lambda^w \geqslant \min \Lambda$. After using this to reduce the number of cases to be considered, we choose some "standard" lattice $L$ in the genus of $\Lambda$ and write $\Theta_\Lambda$ as $\Theta_L$ plus a cusp form (of trivial character) for $\Gamma_0(2)$. The dimension of the space of such cusp forms is known to be $[n/8] - 1$ for $n \geqslant 8$ ([Mi], p. 60).

(a) $n \leqslant 12$. There are no nonzero cusp forms, so $D_4$, $E_8$ and $D_{12}$ (or $D_4 \oplus E_8$) are optimal.

(b) $n = 16, 20$. By what has been said, it suffices to show

$$\text{(i)} \quad \det \Lambda = 2^{n/2-2} \Rightarrow \min \Lambda = 2$$

$$\text{(ii)} \quad \det \Lambda = 2^{n-2} \Rightarrow \min \Lambda \leqslant 4.$$

Case (i) is done for $n=16$ by the proof of Theorem 4, so we let $n=20$, $\det \Lambda = 2^8$, and put $L = D_4^3 \oplus D_8$. Then

$$\begin{aligned} \Theta_\Lambda &= \Theta_L + a\Theta_4 \Delta_{16} \\ &= 1 + (184 + a)\, q + (11560 + 16a)\, q^2 + \cdots, \end{aligned}$$

$$\begin{aligned} \Theta_{\sqrt{2}\Lambda^*} &= -\frac{1}{2}\, \Theta_\Lambda \Big|_{10} t_2 = \Theta_4^3 \Theta_{\sqrt{2} D_8^*} + \frac{a}{2}\, \Theta_4 \Delta_{16} \\ &= 1 + \left(88 + \frac{a}{2}\right) q + \cdots. \end{aligned}$$

The last line gives $a \geqslant -176$, and so $\Lambda$ has at least eight norm 2 vectors. For (ii) it suffices to consider $\Theta_\Lambda$ alone.

(c) $n=24, 28$. For $n=24$ we must check that $\min \Lambda \leqslant 4$ if $\det \Lambda = 2^{14}$, and $\min \Lambda \leqslant 6$ if $\det \Lambda = 2^{22}$. Using now $L = D_4^4 \oplus \sqrt{2}\, D_8^*$ (resp. $L = \sqrt{2}\, D_{24}^*$), and $\Theta_4^2 \Delta_{16}$ and $\Delta_{24}$ as a basis for the cusp forms, this is done as above. For $n=28$ we do the same for determinants $2^{18}$ and $2^{26}$.

(d) $n=32, 36$. Similar calculations as before, but with three basic cusp forms, give the result. We check, e.g., for $n=32$ that $\min \Lambda \leqslant 4$ if $\det \Lambda = 2^{14}$:

$$\begin{aligned}\Theta_\Lambda &= \Theta_4^6 \Theta_{D_8} + a\Theta_4^4 \Delta_{16} + b\Theta_4^2 \Delta_{24} + c\Delta_{16}^2 \\ &= 1 + (256 + a + b)q + (26048 + 88a + 24b + c)q^2 + \cdots, \\ \Theta_{\sqrt{2}\,\Lambda^*} &= \frac{1}{2}\,\Theta_\Lambda \Big|_{16} t_2 \\ &= 1 + \left(160 + \frac{a}{2}\right) q + \left(11456 + 44a + 32b + \frac{c}{2}\right) q^2 + \cdots.\end{aligned}$$

We may assume $\min \Lambda \geqslant 4$, so $a + b = -256$. If $\min \sqrt{2}\,\Lambda^* = 2$, then $2\Lambda^* \subset \Lambda$ implies $\min \Lambda = 4$. If $\min \sqrt{2}\,\Lambda^* > 2$, then $a = -320$, so $b = 64$, and $c \geqslant 1152$ (with equality if $\min \sqrt{2}\,\Lambda^* > 4$). Therefore $\Lambda$ has at least 576 norm 4 vectors (which is the exact number if, e.g., $\Lambda$ contains an extremal modular lattice as a sublattice of index 2). ∎

*Remark.* The lattices of minimum 4 in the genus of $D_4^{n/4}$ which were described at the beginning of the proof satisfy condition (4) of Section 2.1 (since a hyperbolic decomposition of $D_4/3D_4$ may be chosen as a left ideal decomposition of the $2 \times 2$ matrix algebra over $\mathbb{F}_3$). The extremal one for $n=20$ can also be constructed as a quaternionic neighbour of $BW_{16} \oplus D_4$; in this way Christine Bachoc has proved its uniqueness over the Hurwitz order (private communication). The 28-dimensional lattice $\Lambda$ described next inherits a hermitian structure over the Hurwitz order, and it contains its hermitian dual, a lattice isometric to $\sqrt{2}\,\Lambda^*$. Therefore also $\sqrt{2}\,\Lambda^*$ has minimum at least 4 (actually 6), and this again determines the theta function. One obtains

$$\begin{aligned}\Theta_\Lambda &= \Theta_4 \Theta_{\Lambda_{24}} - 1536\Theta_4^3 \Delta_{16} + 1512\Theta_4 \Delta_{24} \\ &= 1 + 98280q^2 + \cdots.\end{aligned}$$

## 3. Appendix: On Modular Forms for $\Gamma_*(l)$

### 3.1. *A Dimension Formula*

We have been using the fact that for $l = 1, 2, 3$ the algebra of modular forms relevant for modular lattices is a polynomial ring in two generators, but as alluded to in Section 2.3, such a structure exists also in a few other cases. For lack of an explicit reference, and since it may be of independent interest, the required dimension formula will be given in the following.

We fix a prime $l$. Let $k \in \mathbb{N}$ be even if $l \not\equiv 3\ (\mathit{mod}\ 4)$, and arbitrary otherwise. Then $\mathscr{G}_k(\Gamma_*(l), \chi)$ denotes the space of modular forms for $\Gamma_*(l)$ having weight $k$ and character $\chi$ as in Theorem 1. The case of odd $k$ is excluded in this section.

THEOREM 6. *Let $k > 0$ be even, and $l > 3$. Then*

$$\dim \mathscr{G}_k(\Gamma_*(l), \chi) = \frac{k}{2} g_0(l) - g_*(l) + 1 + \frac{1}{2}\left[\frac{k}{4}\right]\left(1 + \left(\frac{-1}{l}\right)\right) + \frac{1}{2}\left[\frac{k}{3}\right]\left(1 + \left(\frac{-3}{l}\right)\right),$$

*where $g_0(l)$ and $g_*(l)$ denote the genus of the function field of $\Gamma_0(l)$ and $\Gamma_*(l)$, respectively.*

*Proof.* Denoting by $\mathscr{G}_k(\Gamma)$ the space of modular forms of weight $k$ and trivial character, we have

$$\mathscr{G}_k(\Gamma_*(l), \chi) = \begin{cases} \mathscr{G}_k(\Gamma_*(l)) & \text{if } k \equiv 0 \pmod 4 \\ \mathscr{G}_k(\Gamma_0(l))/\mathscr{G}_k(\Gamma_*(l)) & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Therefore it suffices to apply (once or twice) the dimension formula for $\mathscr{G}_k(\Gamma)$ which in case of a Fuchsian group having cusps reads ([Mi], p. 60)

$$\dim \mathscr{G}_k(\Gamma) = (k-1)(g-1) + \sum_{\mathfrak{p}} \left[\frac{k}{2}\left(1 - \frac{1}{e_{\mathfrak{p}}}\right)\right].$$

Here $g$ denotes the genus, and $\mathfrak{p}$ runs through the elliptic points (with order $e_{\mathfrak{p}}$) and cusps ($e_{\mathfrak{p}} = \infty$). The number of cusps is 2 for $\Gamma_0(l)$ and 1 for $\Gamma_*(l)$. Elliptic points of $\Gamma_*(l)$ have order 2 or 3. This is well-known for $\Gamma_0(l)$, and is readily verified for fixed points of matrices in $\Gamma_0(l)\, t_l$ using the hypothesis $l > 3$. (The trace of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) t_l$ is $\sqrt{l}\,(c/l - b)$ which must have absolute value less than 2. So $c = bl$, and then the characteristic polynomial is $X^2 + 1$.) For a point of $\Gamma_0(l)$, the order multiplied by the ramification index over $\Gamma_*(l)$ (which is 1 or 2) gives the order of the underlying point of $\Gamma_*(l)$. In particular, only ordinary points of $\Gamma_0(l)$ can be ramified, and the number of elliptic points of order $e$ for $\Gamma_*(l)$ becomes

$$\frac{1}{2}\left(1 + \left(\frac{-1}{l}\right)\right) + 2(g_0(l) - 2g_*(l) + 1) \qquad \text{if } e = 2,$$

$$\frac{1}{2}\left(1 + \left(\frac{-3}{l}\right)\right) \qquad \text{if } e = 3,$$

using the numbers for $\Gamma_0(l)$ (cf. [Mi], p. 208) together with the Riemann–Hurwitz formula in case $e=2$. When substituted into the dimension formula above, this gives the result. (Of course, the genus is calculated from the number of ramified points, and not vice versa, but it is unnecessary to go into this calculation here.) ∎

## 3.2. *Special Levels*

For fixed $l$ we form the direct sum of the spaces $\mathscr{G}_k(\Gamma_*(l), \chi)$ introduced in the preceding section, where now $k$ runs through all (resp. all even) nonnegative integers, depending on whether $l \equiv 3 \ (mod\,4)$ or not, and $\chi$ depends on $k\ (mod\,4)$. We obtain a graded algebra which will be denoted (a bit vaguely) by $\mathscr{G}(\Gamma_*(l), \chi)$.

THEOREM 7. *Let $k_1 = 24/(1+l)$ be integral. Put $\Delta = (\eta(z)\,\eta(lz))^{k_1}$ and $\Theta = \Theta_A(z)$ where $A$ is an even modular lattice of level $l$ and lowest positive dimension. Then*

$$\mathscr{G}(\Gamma_*(l), \chi) = \mathbb{C}[\Theta, \Delta].$$

*Proof* (*for* $l > 3$). For $l = 5, 7, 11, 23$ the dimension in Theorem 6 is found to be $1 + [k/k_1]$. It follows that for even $k$ the forms $\Theta^\lambda \Delta^\mu$ of weight $k$ span $\mathscr{G}_k(\Gamma_*(l), \chi)$. Now let $f$ be in this space for odd $k$; here $l \equiv 3\ (mod\,4)$, dim $A = 2$. Then $f\Theta$ can be written as a linear combination of all $\Theta^\lambda \Delta^\mu$ of weight $k+1$. If $v = (k+1)/k_1$ is integral, $\Delta^v$ is one of these forms, but its coefficient must be zero as one sees by evaluating $f\Theta$ at a zero of $\Theta$. Therefore $\Theta$ can be cancelled. ∎

The notion of extremal modular lattices can now be applied likewise to $l = 5, 7, 11, 23$. The minimum of such a lattice in dimension $n$ must reach $2 + 2[n/2k_1]$, with $k_1$ as above. Extremal lattices of minimum 4 analogous to $A_{24}$ and $BW_{16}$ (i.e., for $n = 2k_1$) are the Coxeter–Todd lattice $K_{12}$ of level 3, the Maas lattice $Q_8(1)$ of level 5, Barnes's $P_6 = A_6^{(2)}$ of level 7, and the obvious nonprincipal ideals for $n=4$, $l=11$ and $n=2$, $l=23$. Craig's $A_{10}^{(3)}$ of level 11 is one more extremal lattice having minimum 6. It would be interesting to know whether there exists one for $n = 14$, $l = 11$, minimum 8.

*Note added in proof.* At the end of Sections 1.2 and 2.3 it was ignored that the Leech lattice gives rise to two extremal modular lattices of minimum norm 8, namely a 48-dimensional one of level 2 (cf. [Bi], p. 242) and a 24-dimensional one of level 5. New examples of extremal lattices, including one with minimum norm 8 for level 7 and dimension 20, have occurred in recent work by G. Nebe and W. Plesken (*Mem. AMS*, to appear) and C. Batut, R. Scharlau, and H.-G. Quebbemann (*Exp. Math.*, to appear).

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