Horn Minimization

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Introduction

- Correlation relations (*implications*).

  e.g: movies and genres, *cyber-punk* $\longrightarrow$ *sci-fi*

- Minimization without loss of knowledge.

  e.g: *sci-fi* $\longrightarrow$ *sci-fi* is useless

- *Study* of existing algorithms.

Outline

Closure operator and systems

Set Σ of attributes. A map $\varphi : 2^{\Sigma} \longrightarrow 2^{\Sigma}$ is a *closure operator* if, $\forall X, Y \subseteq \Sigma$:

- $X \subseteq \varphi(X)$ *(extensive)*
- $X \subseteq Y \longrightarrow \varphi(X) \subseteq \varphi(Y)$ *(monotone)*
- $\varphi(\varphi(X)) = \varphi(X)$ *(idempotent)*

Associated terminology:

- X is *closed* if $X = \varphi(X)$,
- Σ^{φ} set of closed sets: *closure system*,
- Σ^{φ} is closed under *intersection*, contains Σ.

Implications

1.1 - Closure and implications

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$A, B \subseteq \Sigma$. An *implication* is:

- $A \longrightarrow B$, A *premise*, B *conclusion*,
- $M \subseteq \Sigma$ *model* of $A \longrightarrow B$:
  - $B \subseteq M \vee A \not\subseteq M$, $\quad (\simeq B \vee \neg A)$
  - $M \models A \longrightarrow B$.

Set of implications $\mathcal{L}$: *implication system*.

- $\mathcal{L} \models A \longrightarrow B$: all models of $\mathcal{L}$ are models of $A \longrightarrow B$,

Implications and closure

1.1 - Closure and implications

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$\mathcal{L}$ an implication system:

- models of $\mathcal{L}$ form a *closure system*:

$$\Sigma^{\mathcal{L}} = \{M \subseteq \Sigma \mid M = \mathcal{L}(M)\}$$

- *closure operator* $\mathcal{L}(X)$: *smallest model* (inclusion wise) of $\mathcal{L}$ containing X, $X \subseteq \Sigma$:

$$\mathcal{L}(X) = \bigcap \{M \in \Sigma^{\mathcal{L}} \mid X \subseteq M\}$$

- $\mathcal{L} \models A \longrightarrow B$ iff $B \subseteq \mathcal{L}(A)$.

Redundancy, minimality

1.2 - Minimization task

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$\mathcal{L}$ and $\mathcal{L}'$ implication systems:

- $A \longrightarrow B \in \mathcal{L}$ *redundant* if $\mathcal{L} - \{A \longrightarrow B\} \models A \longrightarrow B$,
- $\mathcal{L}' \models \mathcal{L}$: all implications of $\mathcal{L}$ follow from $\mathcal{L}'$,
- $\mathcal{L}' \models \mathcal{L}$ and $\mathcal{L} \models \mathcal{L}'$: *equivalent* systems.
- $\mathcal{L}$ *minimum* if no possible $\mathcal{L}'$ such that:
  - $\mathcal{L}'$ equivalent to $\mathcal{L}$,
  - $\mathcal{L}'$ has fewer implications than $\mathcal{L}$.

Recall property: $(\mathcal{L}^- := \mathcal{L} - \{A \longrightarrow B\})$

- $\mathcal{L}^- \models A \longrightarrow B$ iff $B \subseteq \mathcal{L}^-(A)$.

Canonical basis

1.2 - Minimization task

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Particular sets:

- $P \subseteq \Sigma$ *pseudo-closed* in $\mathcal{L}$ if:
  - $P \neq \mathcal{L}(P)$,
  - if $Q \subset P$ and Q pseudo-closed, then $\mathcal{L}(Q) \subseteq P$.
- $Q \subseteq \Sigma$ *quasi-closed* in $\mathcal{L}$ if:
  - $\forall P \subseteq Q, \mathcal{L}(P) \subseteq Q$ or $\mathcal{L}(P) = \mathcal{L}(Q)$,
- pseudo-closed $\longrightarrow$ quasi-closed.

Duquenne-Guigues (*canonical*) base:

$$\{P \longrightarrow \mathcal{L}(P) \mid P \textit{ pseudo-closed } \}$$

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Let $\mathcal{L}$ be an implication system:

- $\Sigma = \{a,\ b,\ c,\ d,\ e,\ f\}$,
- $\mathcal{L} = \{ab \longrightarrow cde, cd \longrightarrow f, c \longrightarrow a, d \longrightarrow b, abcd \longrightarrow ef\}$.

We have:

- $\mathcal{L}(b) = b$, b is *closed*, hence a *model* of $\mathcal{L}$,
- $\mathcal{L}(ab) = abcdef$, ab is *not* a model, *(abcdef is)*
- $abcd \longrightarrow ef$ is *redundant*,

Algorithms Core

2.1 - Minimizing the input

Two main ideas:

- *minimizing* input system
  - algorithms from FCA and databases,
- *building* a minimum system against the input
  - query learning interpretation,

Some notations, given $\mathcal{L}$:

- $|\mathcal{B}|$ number of implications, $|\Sigma|$ number of attributes,
- $|\mathcal{L}| = O(|\mathcal{B}||\Sigma|)$, size of $\mathcal{L}$.

First algorithm

2.1 - Minimizing the input

MinCover:

- from Day, Wild, (80's),
- two steps:
  1. *maximize* the conclusion
     $A \longrightarrow B$ becomes $A \longrightarrow \mathcal{L}(A)$
  2. *remove* redundant information
     if $\mathcal{L}(A) = \mathcal{L}^{-}(A)$, $A \longrightarrow \mathcal{L}(A)$ is *redundant*
- output: *canonical* base, complexity: $O(|\mathcal{B}||\mathcal{L}|)$.

> Recall: $\mathcal{L}^{-} = \mathcal{L} - \{A \longrightarrow \mathcal{L}(A)\}$

A variation

2.1 - Minimizing the input

DuquenneMin:

- variation of MinCover by Duquenne (2007),
- three steps:
  1. *quasi-closure* and *redundancy* elimination
     if $B \subseteq \mathcal{L}^-(A)$, $A \longrightarrow B$ is useless
     else $A \longrightarrow B$ becomes $\mathcal{L}^-(A) \longrightarrow (\mathcal{L}^-(A) \cup B)$
  2. sort implications in $\subseteq$-compatible order (premises)
  3. *iteratively build* Duquenne-Guigues base
     if $\mathcal{L}^-(A)$ *pseudo-closed*, add $\mathcal{L}^-(A) \longrightarrow \mathcal{L}(A)$ to the result
- output: *canonical* base, complexity: $O(|\mathcal{B}||\mathcal{L}|)$.

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MaierMin:

- functional dependency based algorithm, Maier (80's),
- steps:
  1. *redundancy* elimination
     if $B \subseteq \mathcal{L}^{-}(A)$, $A \longrightarrow B$ is useless
  3. *equivalence classes* reduction
     group implications by premises *closure*
     reduce those classes
- output: minimum basis, complexity: $O(|\mathcal{B}||\mathcal{L}|)$.

General principle

2.2 - Building the result

Query learning and Angluin algorithm (90's):

- aim: learn a theory by constructing an *hypothesis*
- *oracle* answering *queries* (questions)
- queries:
  - *equivalence*: is our hypothesis equivalent to the target?
  - *membership*: is a set a model of the target?
- improve by *counterexample*
  - *positive*: model of the target, not of the hypothesis
  - *negative*: model of the hypothesis, not of the target

AFP-Based Algorithm

2.2 - Building the result

AFP-BASED:

- derived from Angluin, *no proof yet*,
- principle:
  1. take implications one by one,
  2. refine the hypothesis:

     use *premises* to generate possible counter-examples

     add right-closed implications or refine old ones
- expected output: *canonical* base,
- *idea* of complexity: $O(|\mathcal{B}|^3|\mathcal{L}|)$.

Using minimality constraint

2.2 - Building the result

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BERCZIMIN:

- logic based, Berczi, 2017
- principle:
  1. *build* a basis $\mathcal{L}_c$ against the input $\mathcal{L}$
  2. repeat minimality selection up to *equivalence*
  3. *minimality selection*:

     select the next minimal negative counter-example A,
     add $A \longrightarrow \mathcal{L}(A)$ to $\mathcal{L}_c$
- output: *canonical* base, complexity: $O(|\mathcal{B}|^2|\mathcal{L}|)$.

Context 1

3.1 - Experiments

Practical aspect:

- context of FCA, previous study of closure operators,
- use datasets from UCI repository (*scaling*):
  - solar flare: 49 attributes,
  - SPECT: 23 attributes,
- use of CLOSURE *($\simeq$ forward chaining)*
- C++, boost, python.

Context 2
3.1 - Experiments

1 dataset give rise to 5 systems:

- Duquenne-Guigues basis (*DG*),
- minimal generators (*mingen*, right-closed),
- proper implications (*proper*)
- Maier minimum on mingen (*min 1*, right-closed),
- Maier minimum on proper (*min 2*)

$\mathcal{L}$		$\lvert\Sigma\rvert$	$\lvert\mathcal{B}\rvert$
Flare	minimum	49	3382
	mingen		39787
	proper		10692
SPECT	minimum	23	2169
	mingen		44341
	proper		8358

Table: Summary of real datasets characteristics

Overhaul results

3.1 - Experiments

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$\mathcal{L}$		MinCov	Duq	Maier	Berczi	AFP
Flare	DG	*0.097*	0.117	0.211	27.922	96.178
	min 1	*0.134*	0.194	0.288	27.750	98.145
	min 2	0.200	0.190	0.308	30.063	111.944
	proper	1.684	*0.933*	*0.917*	88.375	402.453
	mingen	16.047	*7.981*	*7.576*	*160.328*	*2514.610*
SPECT	DG	*0.045*	0.066	0.108	10.328	22.454
	min 1	*0.061*	0.080	0.134	8.156	19.438
	min 2	0.078	0.070	0.150	8.250	26.980
	proper	0.930	*0.394*	*0.451*	51.063	114.564
	mingen	24.077	*10.206*	*10.858*	*194.875*	*863.903*

Table: Comparison of the algorithms on real datasets (execution in s)

Observation

3.1 - Experiments

Observations *on these data*:

- cost of AFP and BercziMin,
- DuquenneMin, MaierMin efficient on *non-minimum cases*,
- MinCover slightly better on right-closed minimum cases.

Generating random implication (given $|\Sigma|$, $|\mathcal{B}|$):

- discrete uniform distribution on *size*, *elements*,
- premise A, conclusion B, yield $A - B \longrightarrow B$.

Minimum tests 1

3.1 - Experiments

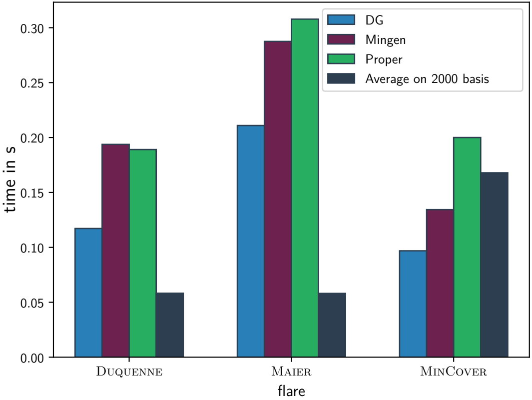

Figure: Flare - Random against and minimum basis: 49 attr, 3382 imp

Minimum tests 2

3.1 - Experiments

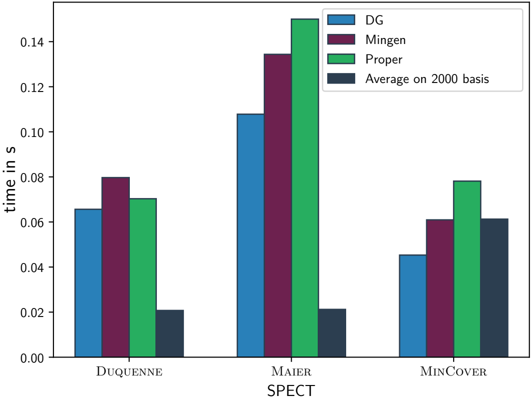

Figure: SPECT - Random against minimum basis: 23 attr, 2169 imp

Insight on random tests

3.1 - Experiments

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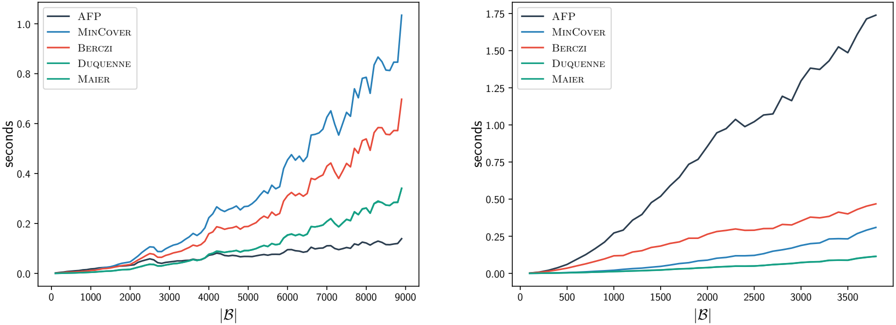

(a) Average time, $|\Sigma| = 100$ (b) Average time, $|\Sigma| = 500$

Figure: Random generated tests for fixed $|\Sigma|$ (over 500 ex)

Ideas

3.1 - Experiments

Observations:

- high speed of MaierMin, DuquenneMin on redundant cases,
- first glance at random: efficiency of AFP.

Explanations:

- *redundancy elimination* as first step,
- suggests a study of underlying *structure* (AFP).

Boundaries:

- valid in *our context*,
- random generation,
- suggests *extension* of tests.

Conclusion

Purpose:

- *study* of minimization algorithms.

Results:

- algorithms from various communities,
- in practice: *redundancy elimination*.

Perspectives:

- *theoretical* study of systems structure, AFP proof and complexity,
- *experimental* aspect, extend tests.