Horn Minimization

An overview of some existing algorithms

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Introduction

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- correlation relations (*implications*)

  e.g: movies and genres, *cyber-punk* $\longrightarrow$ *sci-fi*

- minimization without loss of knowledge

  e.g: *sci-fi* $\longrightarrow$ *sci-fi* is useless

- *study* of existing algorithms

Outline

Closure operator and systems

1.1 - Closure and implications

Set Σ of attributes. A map $\varphi : 2^\Sigma \longrightarrow 2^\Sigma$ is a *closure operator* if, $\forall X, Y \subseteq \Sigma$:

- $X \subseteq \varphi(X)$ *(increasing)*
- $X \subseteq Y \longrightarrow \varphi(X) \subseteq \varphi(Y)$ *(isotone)*
- $\varphi(\varphi(X)) = \varphi(X)$ *(idempotent)*

Associated terminology:

- X is *closed* if $X = \varphi(X)$,
- Σ^φ set of closed sets: *closure system*.
- Σ^φ is closed under *intersection*, contains Σ.

$A, B \subseteq \Sigma$. An *implication* is:

- $A \longrightarrow B$, A *premise*, B *conclusion*.
- $M \subseteq \Sigma$ *model* of $A \longrightarrow B$
  - $B \subseteq M \vee A \not\subseteq M \quad (\simeq B \vee \neg A)$
  - $M \models A \longrightarrow B$, $A \longrightarrow B$ follows from M.

Set of implications $\mathcal{L}$: *implication system*.

- $\mathcal{L} \models A \longrightarrow B$: all models of $\mathcal{L}$ are models of $A \longrightarrow B$,
- $\mathcal{L} \models A \longrightarrow B$ iff $B \subseteq \mathcal{L}(A)$.

Implications and closure

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$\mathcal{L}$ an implication system:

- models of $\mathcal{L}$ form a *closure system*:

$$\Sigma^{\mathcal{L}} = \{M \subseteq \Sigma \mid M = \mathcal{L}(M)\}$$

- *closure operator* $\mathcal{L}(X)$: *smallest model* (inclusion wise) of $\mathcal{L}$ containing X, $X \subseteq \Sigma$.

$$\mathcal{L}(X) = \bigcap \{M \in \Sigma^{\mathcal{L}} \mid X \subseteq M\}$$

Small Example

1.1 - Closure and implications

Let $\mathcal{L}$ be an implication system:

- $\Sigma = \{a,\ b,\ c,\ d,\ e\}$,
- $\mathcal{L} = \{ab \longrightarrow c,\ bd \longrightarrow a, ce \longrightarrow abd\}$

We have:

- $\mathcal{L}(b) = b$, b is *closed*, hence a *model* of $\mathcal{L}$,
- $\mathcal{L}(bd) = abcd$, bd is *not* a model $(abcd\ is)$.
- $\emptyset$ is also a model.

Closure Example

1.1 - Closure and implications

- Directed graph $G = (V, E)$.
- *Closure* $\varphi(X)$ of $X \subseteq V$: every vertices reachable from X.
- $\varphi(\{A, B\}) = \{A, B, C, D\}$.
- $\varphi(\{F\}) = \{F\}$, $\{F\}$ is *closed*.

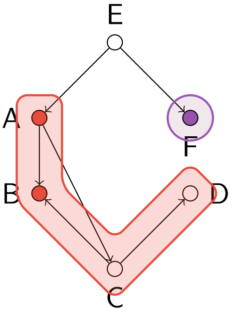

Figure: Closure of a vertex in a directed graph

Redundancy, equivalence

1.2 - Minimization task

$\mathcal{L}$ and $\mathcal{L}'$ implication systems:

- $A \longrightarrow B \in \mathcal{L}$ *redundant* if $\mathcal{L} - \{A \longrightarrow B\} \models A \longrightarrow B$,
- $\mathcal{L}' \models \mathcal{L}$: all implications of $\mathcal{L}$ follow from $\mathcal{L}'$,
- $\mathcal{L}' \models \mathcal{L}$ and $\mathcal{L} \models \mathcal{L}'$: *equivalent* bases.

Remind property: $(\mathcal{L}^- := \mathcal{L} - \{A \longrightarrow B\})$

- $\mathcal{L}^- \models A \longrightarrow B$ iff $B \subseteq \mathcal{L}^-(A)$

Minimum basis
1.2 - Minimization task

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$\mathcal{L}$ *minimum* if no possible $\mathcal{L}'$ such that:

- $\mathcal{L}'$ equivalent to $\mathcal{L}$,
- $\mathcal{L}'$ has fewer implications than $\mathcal{L}$.

Particular minimum basis:

- $P \subseteq \Sigma$ *pseudo-closed* in $\mathcal{L}$ if:
  - $P \neq \mathcal{L}(P)$,
  - if $Q \subset P$ and Q pseudo-closed, then $\mathcal{L}(Q) \subseteq P$.
- *Duquenne-Guigues* (*canonical*) base:

$$\{P \longrightarrow \mathcal{L}(P) \mid P \text{ pseudo-closed } \}$$

Pouet

2.0 -

Pouf

2.0 -

UCI Datasets

3.1 - Real datasets