Half-space separation in monophonic convexity

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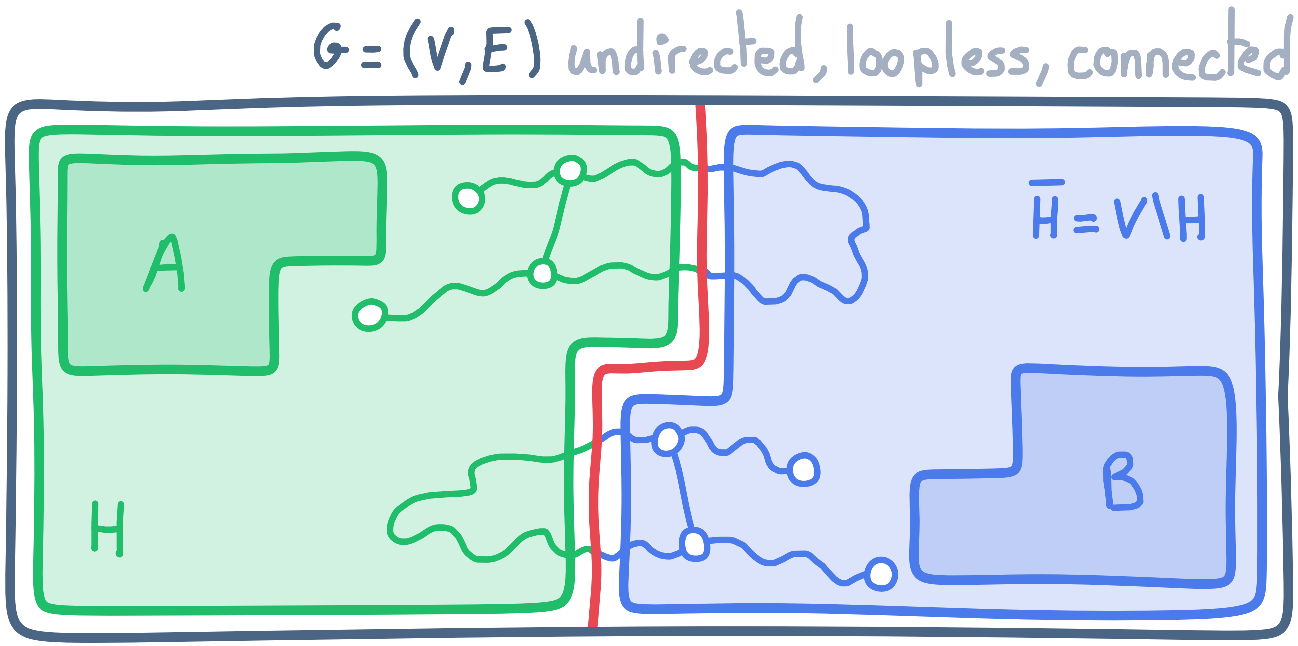
Lhouari Nourine, Université Clermont Auvergne, CNRS, LIMOS, France

Mohammed Elaroussi, Université de Bejaia, LAMOS, Algeria

Simon Vilmin, Aix-Marseille Université, CNRS, LIS, France

Introduction : the problem in a nutshell



PROB : given $G = (V, E)$ and $A, B \subseteq V$, is there a partition of V into monophonic half-spaces $H, \bar{H}$ separating A, B: $A \subseteq H$, $B \subseteq \bar{H}$ and any path with endpoints in H ($\bar{H}$) and vertices in $\bar{H}$ (H) has a chord in H ($\bar{H}$)?

Introduction : take-away message

(1) What is the problem ?

PROB : given $G=(V,E)$ and $A, B \subseteq V$, is there a partition of V into monophonic half-spaces $H, \overline{H}$ separating A, B ?

(2) Where does it come from ?

Separating points by (convex) half-spaces in $\mathbb{R}^d$, recently extended to graph convexities Seiffarth et al., 20

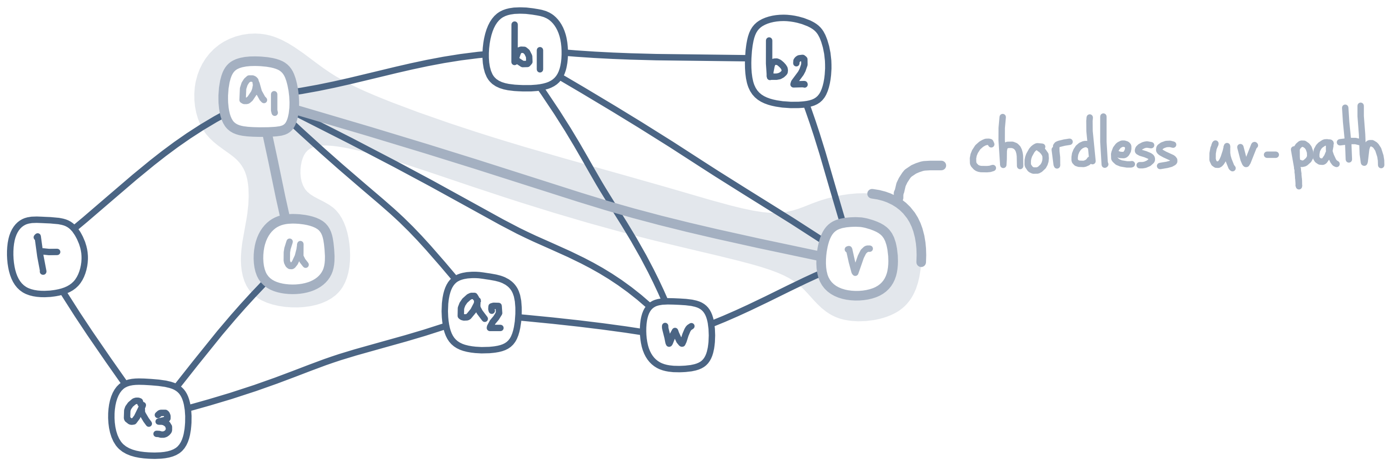
(3) What do we show ?

THM : the problem can be solved in polynomial time

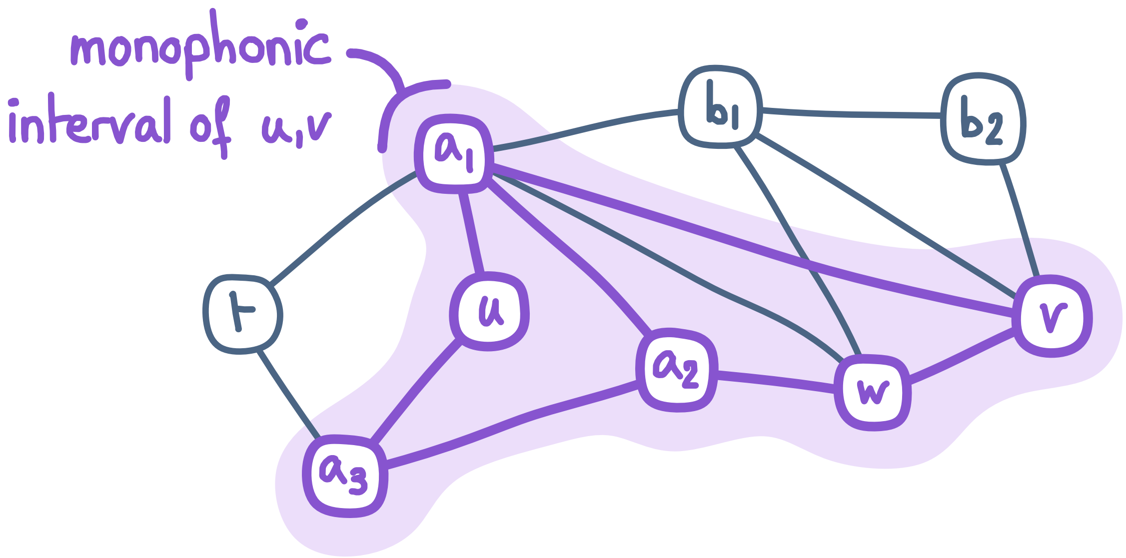
ENV. 24+

(1) What is the problem ?

Monophonic interval



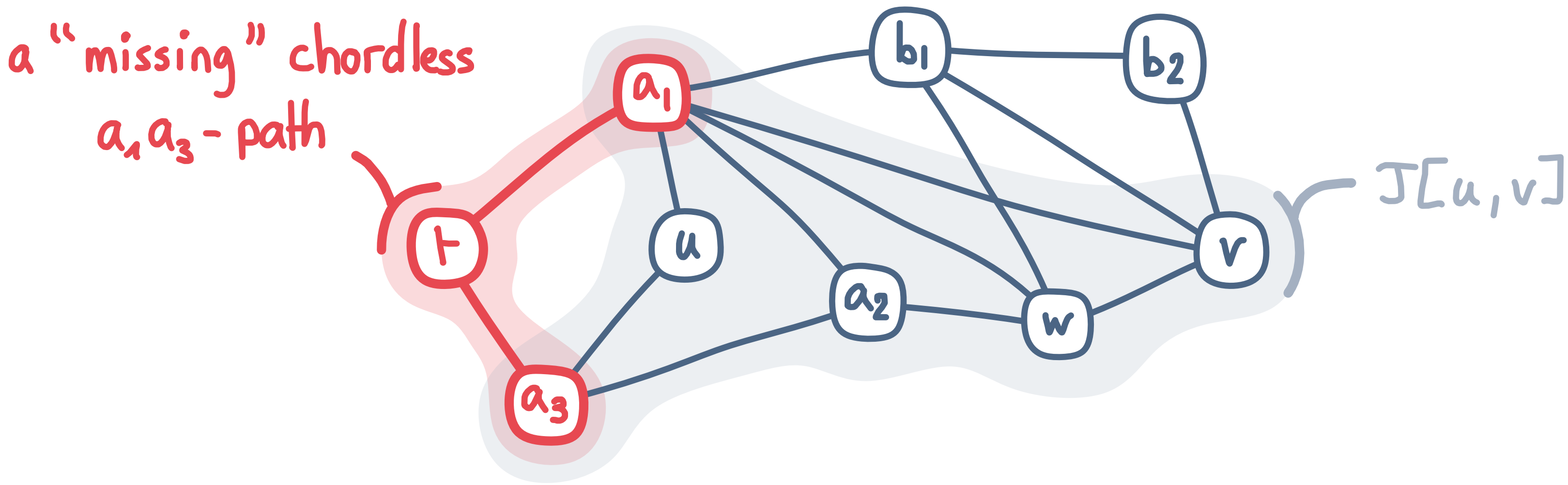
Monophonic interval



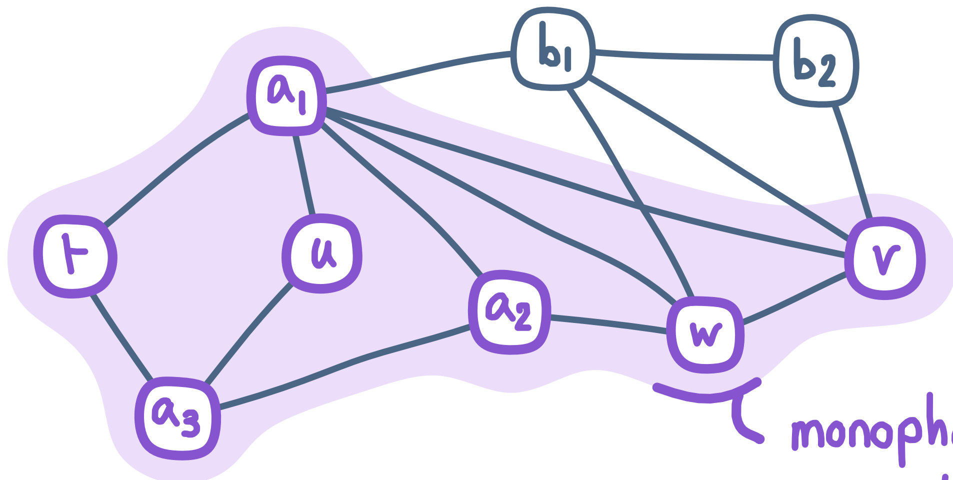
DEF: the monophonic interval of $u, v \in V$ is the set $J[u,v]$ of vertices on chordless uv-paths, i.e.,

$$J[u,v] = \{w : w \text{ on a chordless } uv\text{-path}\}$$

Monophonic convexity



Monophonic convexity



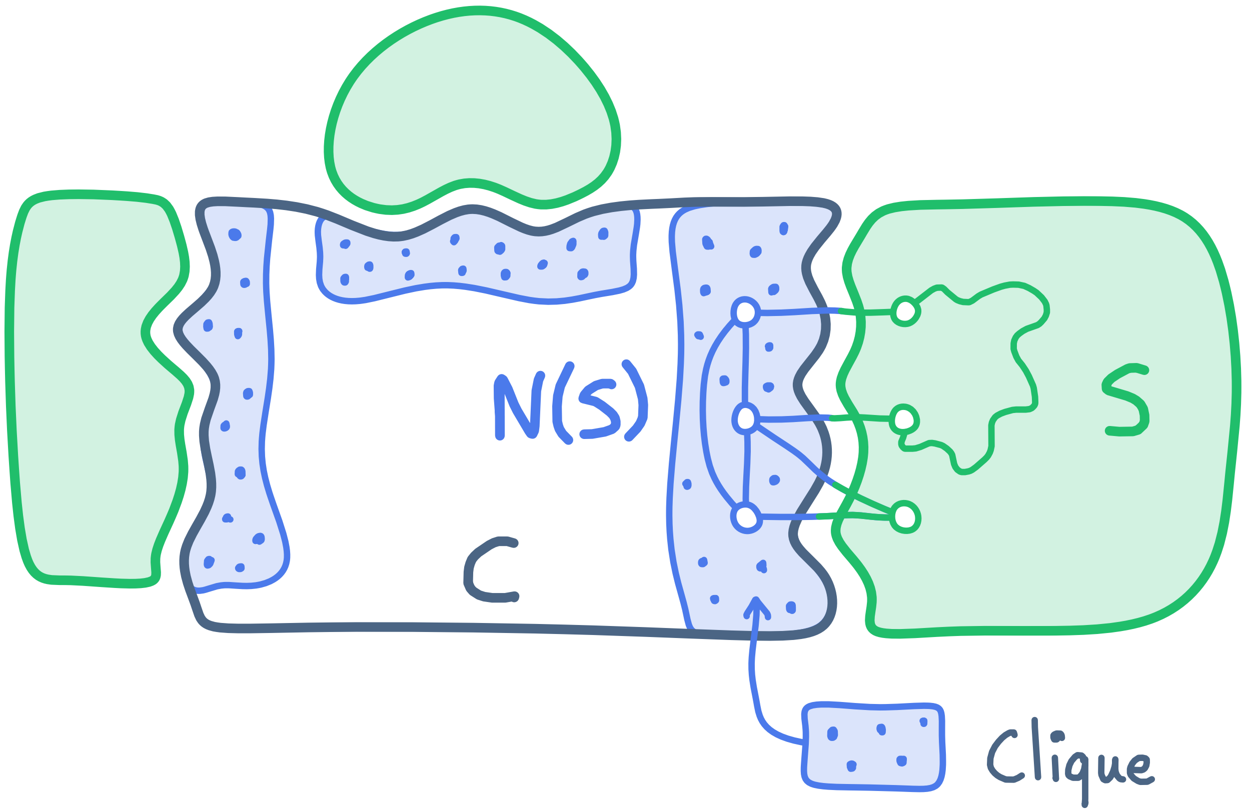
monophonically convex set
i.e. closed w.r.t. $J[\cdot]$

DEF: $C \subseteq V$ is (monophonically) convex if $J[u,v] \subseteq C$ for $u, v \in C$. The family

$$\mathscr{C} = \{C : C \subseteq V,\ C \text{ is convex}\}$$

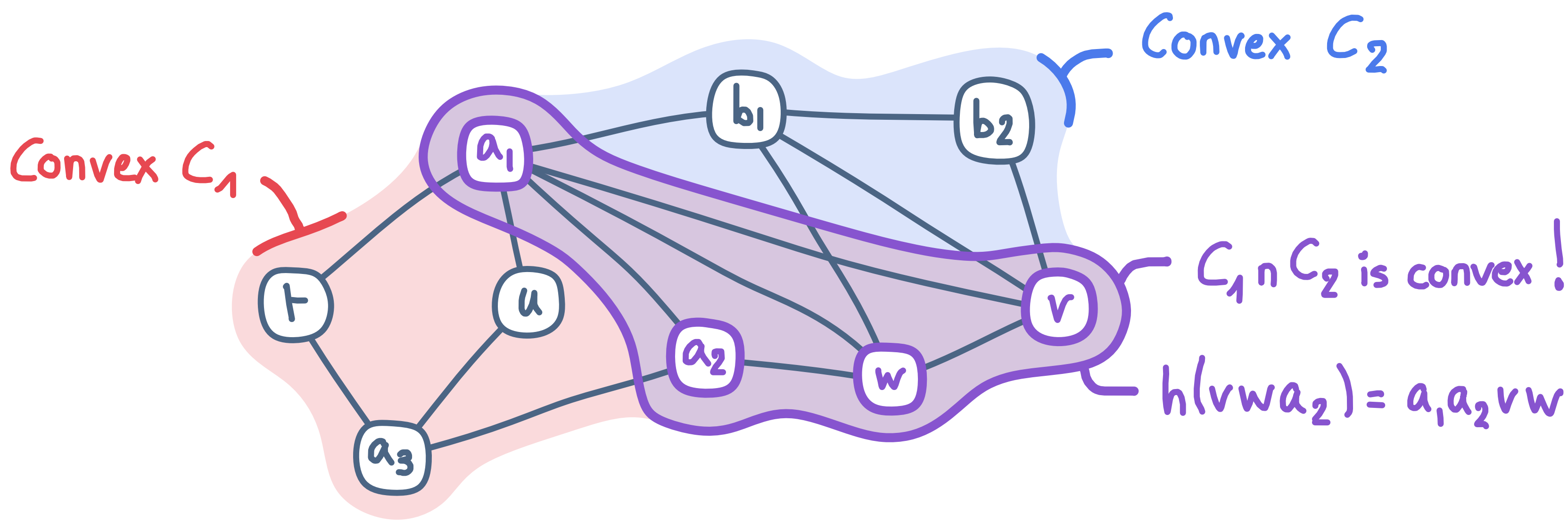
is the monophonic convexity associated to G

What are convex sets ?



THM : (Dourado et al., 10) in a connected graph G, a set $C \subseteq V$ is convex iff for each connected component S of $G - C$, $N(S)$ is a clique

Monophonic convex hull



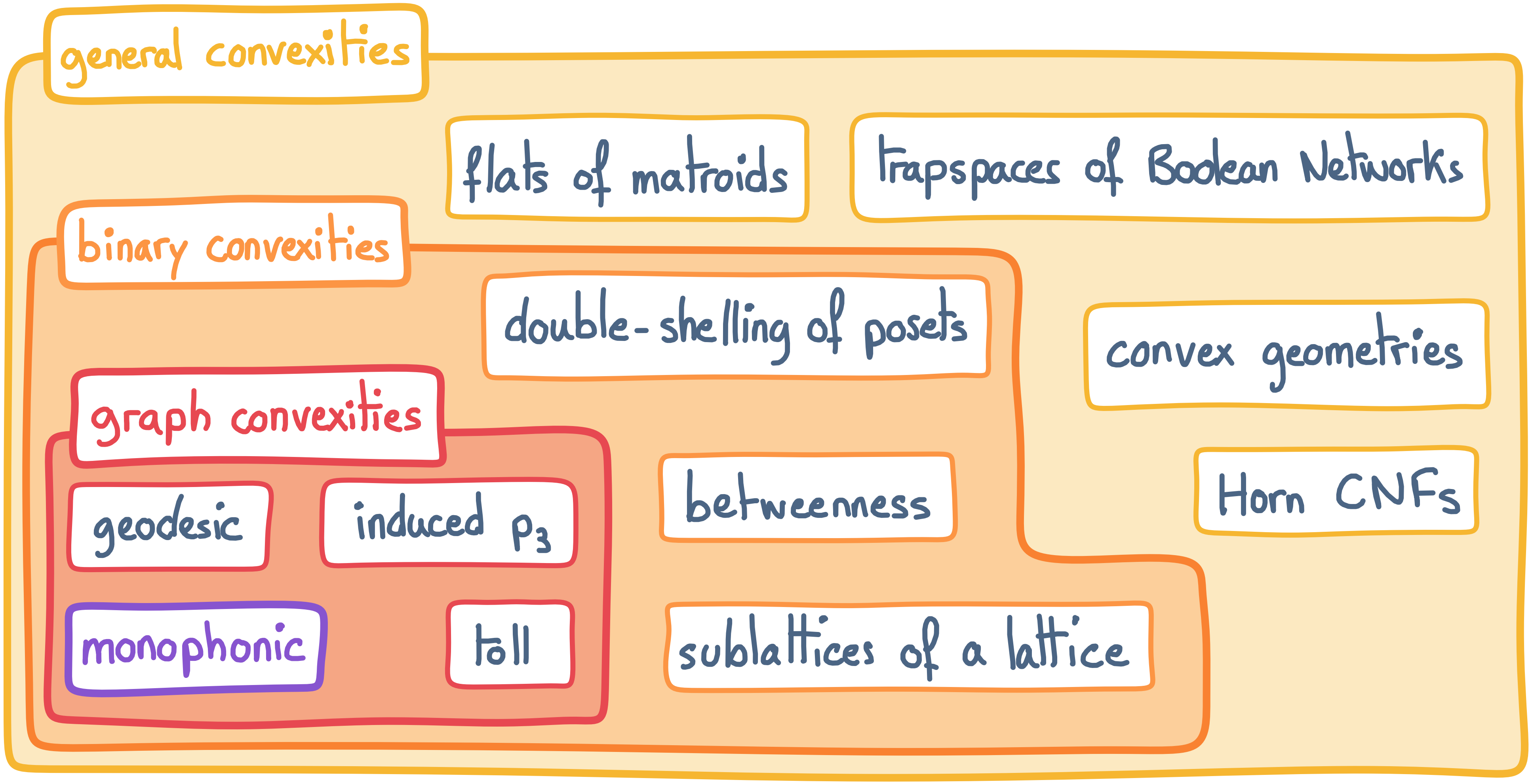
RMK: $\mathscr{C}$ is closed under taking set-intersection

DEF: the (convex) hull of $X \subseteq V$ is the $\min_{\subseteq}$ convex set including X:

$$h(X) = \min_{\subseteq} \{C : C \in \mathscr{C}, X \subseteq C\} = \bigcap \{C : C \in \mathscr{C}, X \subseteq C\}$$

RMK: finding $h(X)$ takes polynomial time Dourado et al., 10

Convexities (= closure systems)

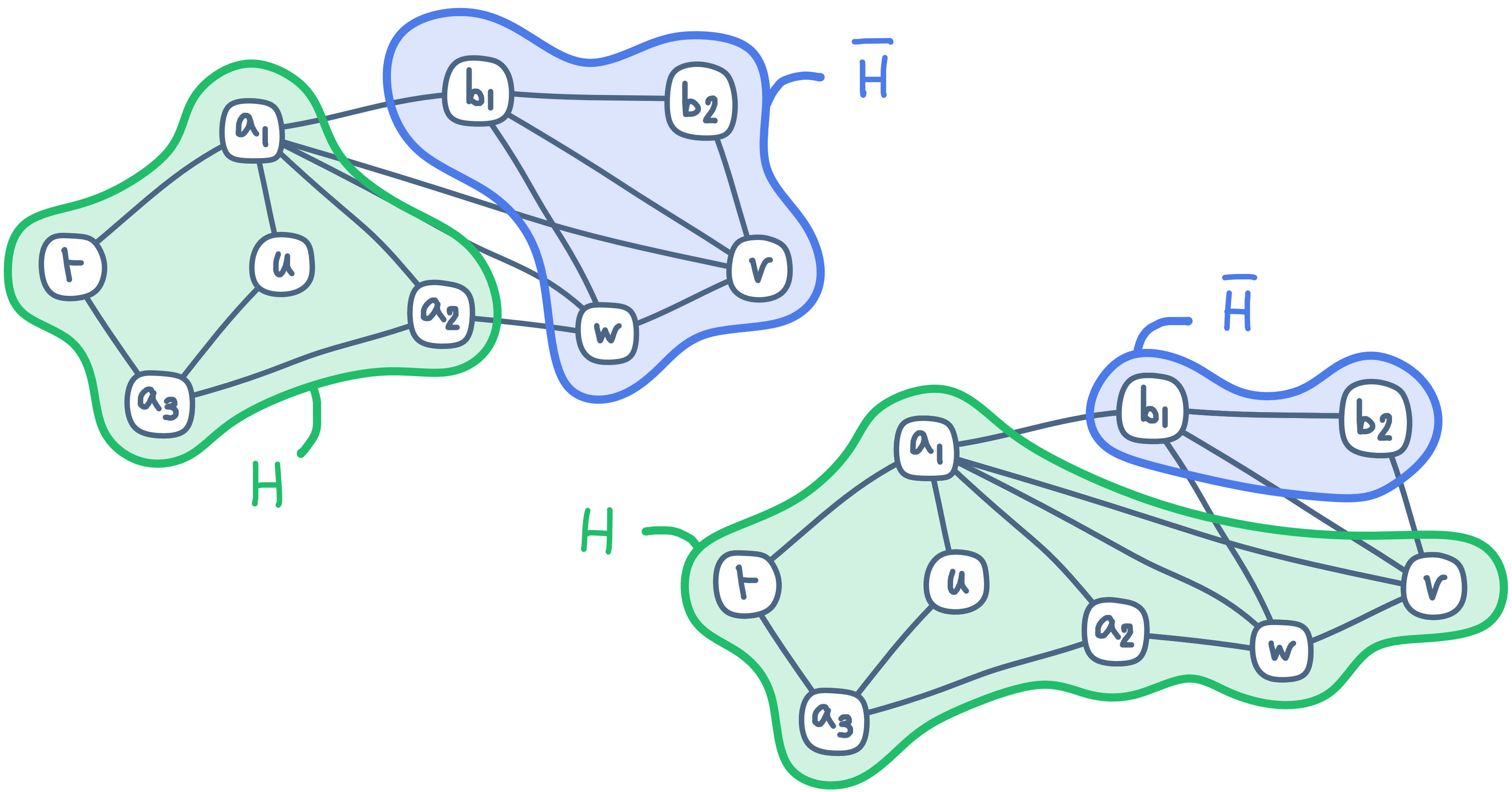


hull (= closure) operators are defined for general convexities

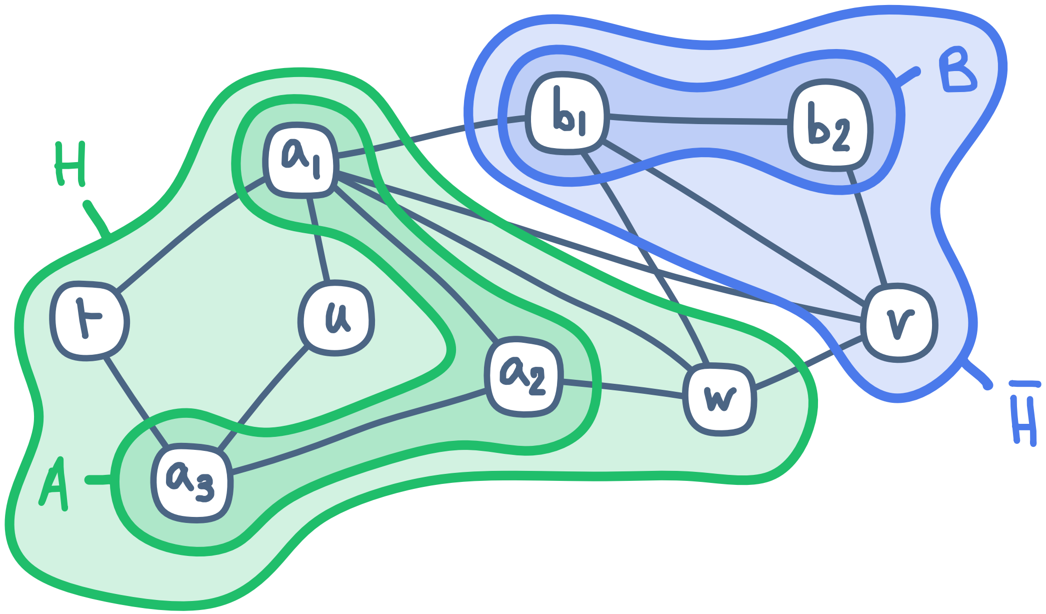
Van de Vel, 93 , Grätzer, 11 , Korte et al., 12 , ...

Half-spaces

DEF: a half-space is a convex set H whose complement $\overline{H} = V \setminus H$ is convex (i.e. $H \cap \overline{H} = \emptyset$, $H \cup \overline{H} = V$ and $H, \overline{H} \in \mathscr{C}$)



Our problem: half-space separation
(in monophonic convexity)



PROB: given (connected) $G = (V, E)$ and $A, B \subseteq V$, is there a pair of half-spaces $(H, \overline{H})$ separating A and B $(A \subseteq H, B \subseteq \overline{H})$?

(2) Where does it come from?

Separation

Initially, separation studied as structural aspect of convexity spaces with separation axioms (e.g., S_3 or S_4/Kakutani)

- convex sets in $\mathbb{R}^d$ Kakutani, 37
- other convexity spaces Ellis, 52 Chepoi, 94, 24+

Computational problem of separation more recent

- generalizes binary linear classification in $\mathbb{R}^d$
- NP-complete for geodesic convexity Seiffarth et al., 23
- learning of geodesic and monophonic half-spaces Thiessen, Gartner, 21 Bressan et al., 24+

QUES: what about other (graph) convexities?

(3) What do we show ?

Main theorem

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THM: half-space separation in monophonic convexity can be solved in polynomial time

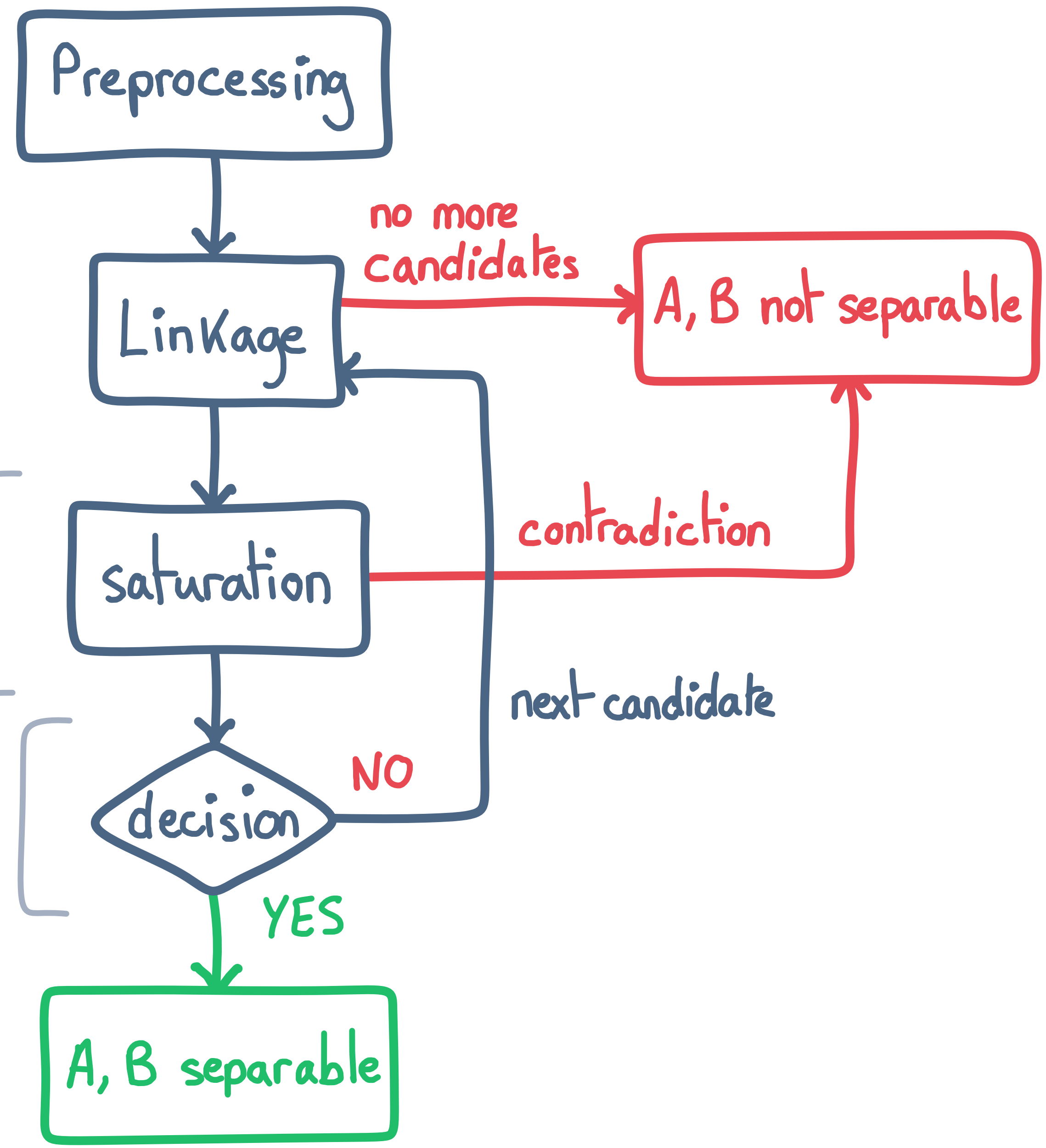
RMK: theorem obtained independently in two recent contributions Chepoi, 24+, Bressan et al., 24+

General (constructive) algorithm

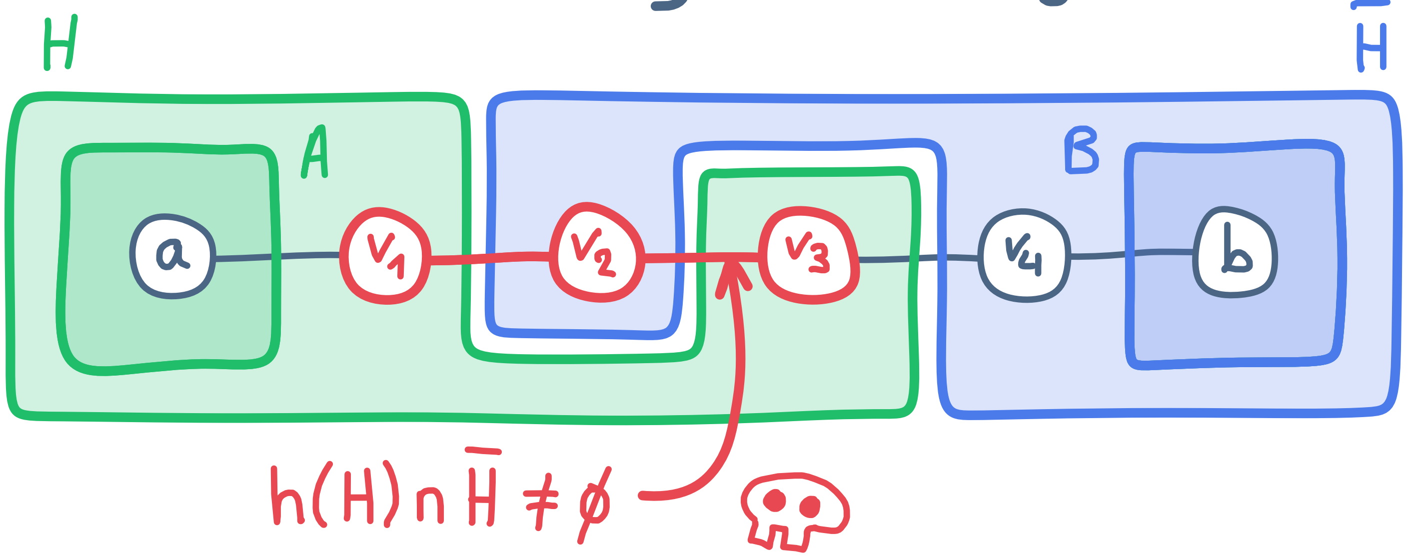
Find a linear number of candidate sets

Use the hull operator to assign vertices

Use the remaining vertices and graph structure to decide



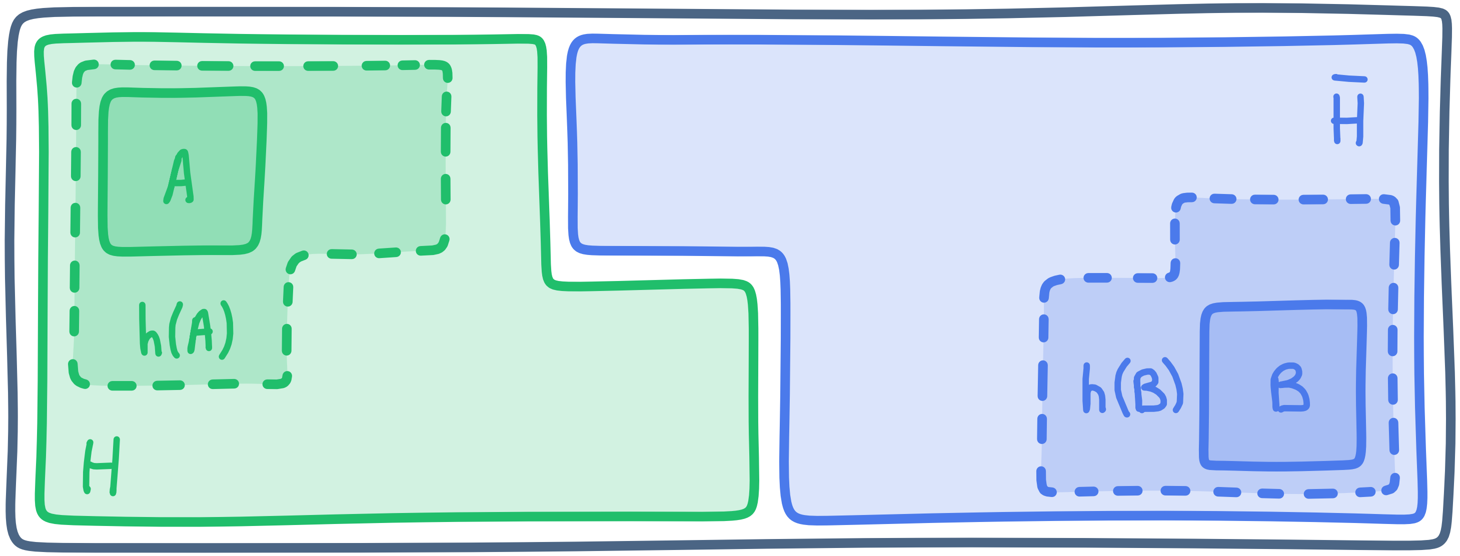
Preprocessing and linkage



LEM: let $a \in A$, $b \in B$ and $a = v_0, \dots, v_k = b$ a shortest ab-path. A and B are separable iff $A \cup \{v_0, \dots, v_i\}$ and $B \cup \{v_{i+1}, \dots, v_k\}$ are separable for some $0 \leq i \leq k$.

We can assume A and B are linked: there is $a \in A$, $b \in B$ s.t. a and b are adjacent

Saturation : first remark



RMK : (1) A, B separable $\Leftrightarrow$ $h(A), h(B)$ separable

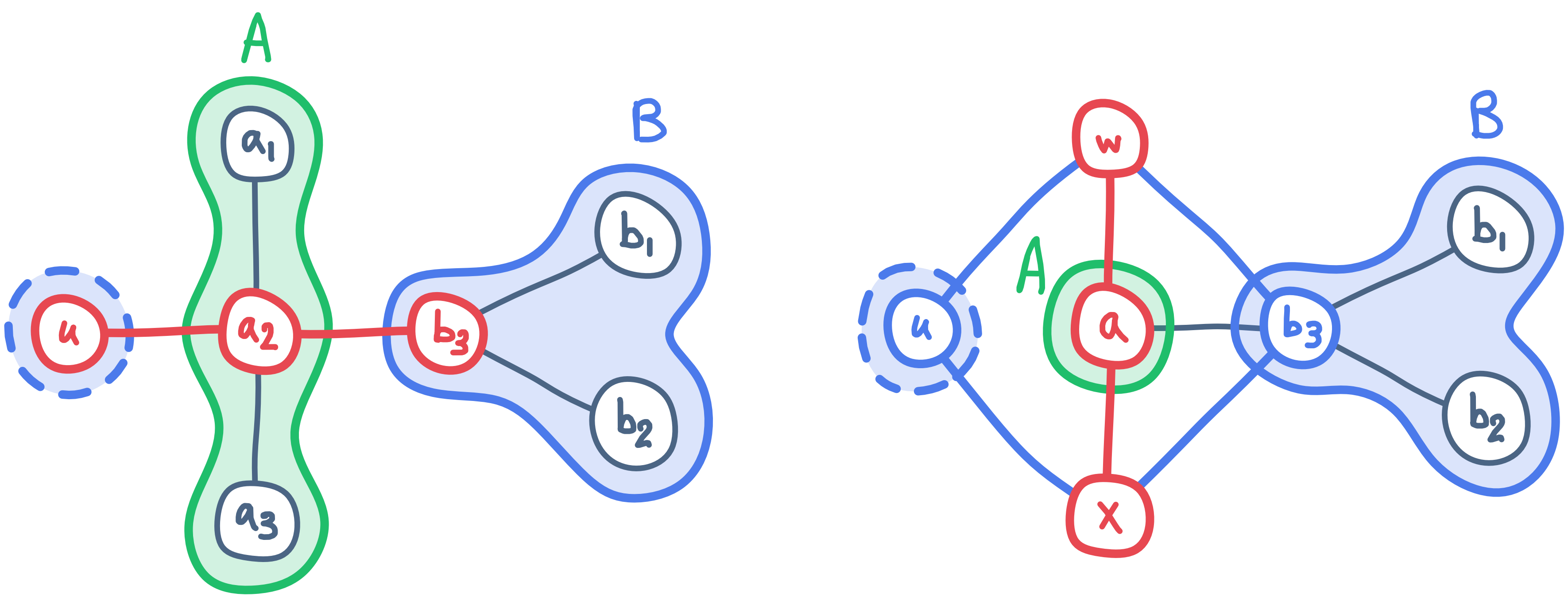
(2) $h(x)$ in poly-time Dourado et al., 10

We can assume A and B are convex

Saturation : shadows

DEF: the shadow of A w.r.t. B is

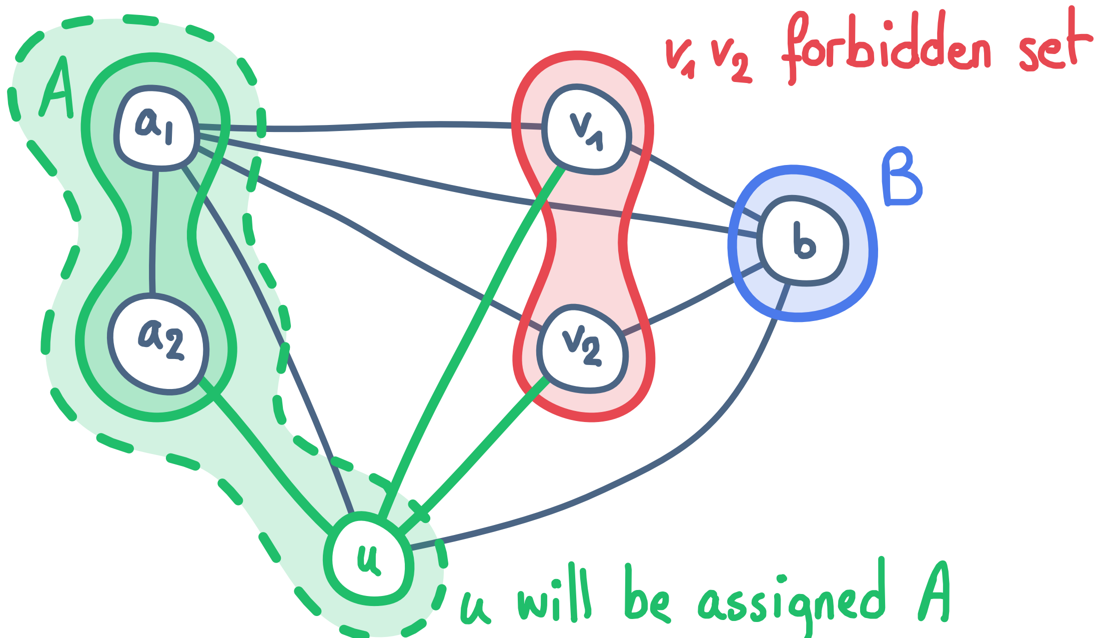
$$A/B = \{ v : v \in V, \; h(B \cup v) \cap A \neq \emptyset \}$$



RMK: any vertex of A/B (B/A) must be assigned A (B).

Saturation: forbidden sets

DEF: a forbidden set is a set $X \subseteq V \setminus (A \cup B)$ s.t. $h(X) \cap A \neq \emptyset$ and $h(X) \cap B \neq \emptyset$



If X is forbidden, its vertices must be split. Hence $\bigcap_{x \in X} h(A \cup x)$ must be assigned A (and similarly for B).

The cost of finding forbidden sets

: is there an exponential amount of forbidden sets ?
Are they easy to find ?

ANS: we need only $\subseteq$-minimal ones, which are of size at most 4

THM: Duchet, 88 let $G=(V,E)$, $v \in V$ and $X \subseteq V$.
If $v \in h(X)$, then there exists $\underbrace{u,w \in X \text{ s.t. } v \in h(uw)}_{\text{2-element subset } X' \text{ of } X \text{ s.t. } v \in h(X')}$

RMK: Duchet, 88 uses the Carathéodory number: monophonic convexities have Carathéodory number at most 2.

Saturation : packing up

DEF : A and B are saturated if they are convex and no more vertices can be assigned using shadows and forbidden sets.

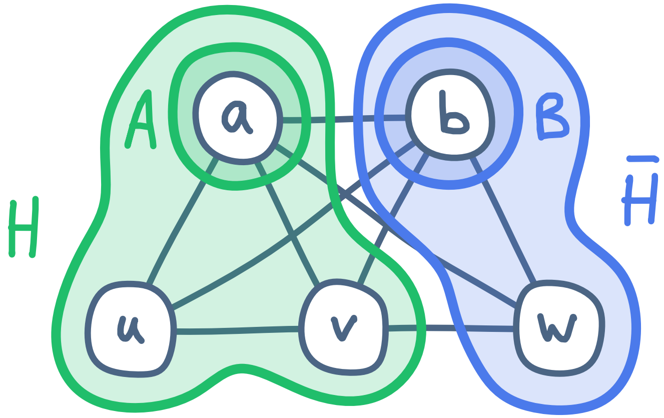
RMK : saturation is deterministic and can be computed in polynomial time.

LEM : A and B are separable iff their respective saturations are separable.

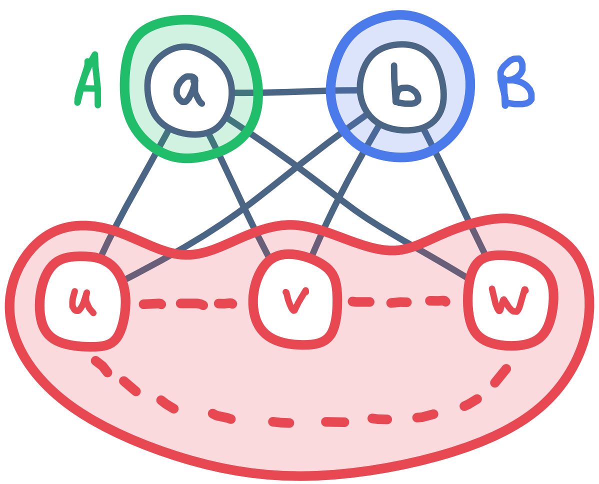
Decision : what now ?

QUES: suppose linkage and saturation did not produce any contradiction. Is there anything left to decide ? YES

A, B separable



A, B not separable



triangle of forbidden pairs

Decision : graph shape (1)

DEF: for (disjoint) $X, Y \subseteq V$, $F(X,Y) = X \cap N(Y)$ ("Frontier")

LEM: if A, B are linked and saturated then for all $v \in N(A \cup B)$, $F(A,B) \cup F(B,A) \subseteq N(v)$. Hence

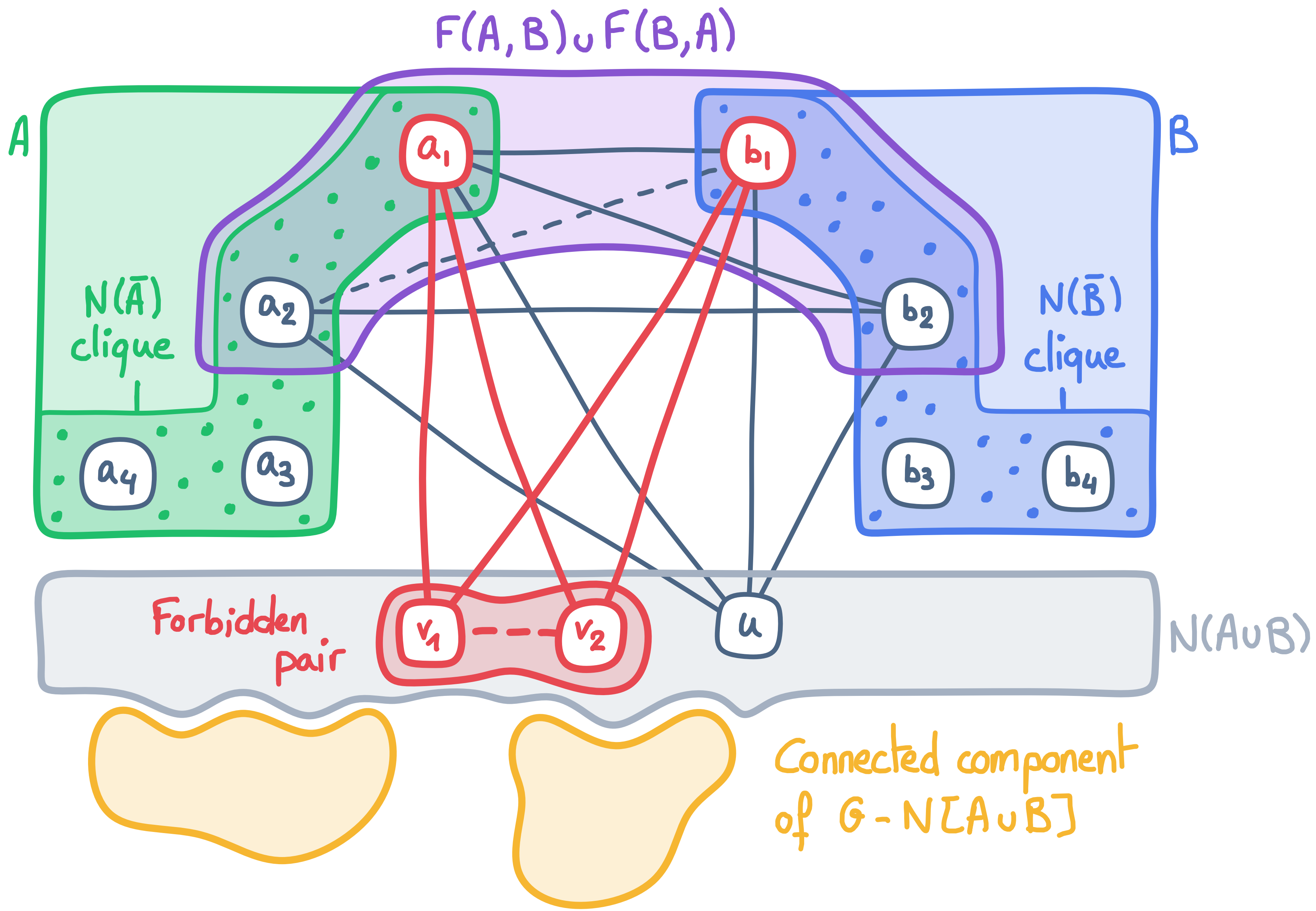
(1) $N(A) \setminus B = N(B) \setminus A = N(A \cup B)$

(2) $N(\bar{A})$ and $N(\bar{B})$ are cliques

LEM: if A, B are linked and saturated then $\subseteq$-min forbidden sets are exactly pairs $u, v \in \overline{A \cup B}$ s.t.

$h(uv) \cap N(A \cup B)$ is not a clique

illustration

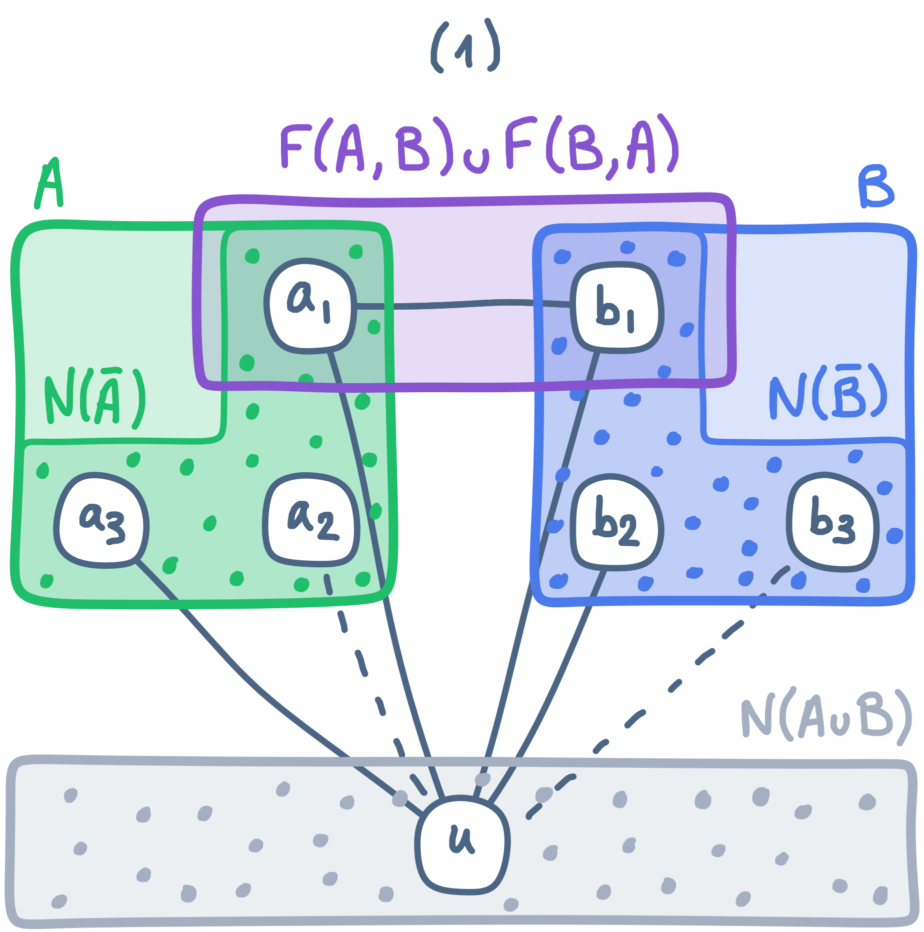


Decision: graph shape (2)

LEM: if A, B are linked and saturated, then:

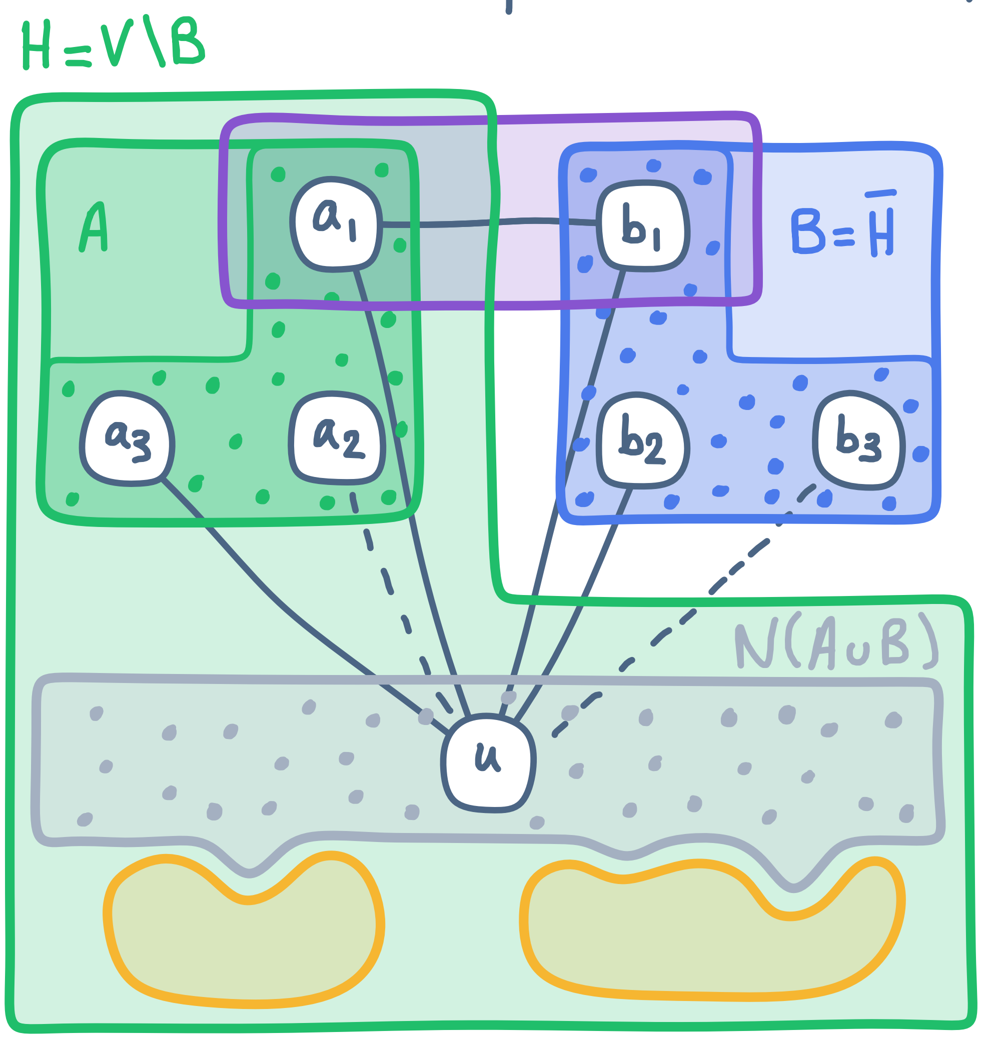
(1) either $N(A \cup B)$ is a clique

(2) or $N(\bar{A}) \cup N(\bar{B}) \subseteq N(u)$ for all $u \in N(A \cup B)$



Decision : $N(A \cup B)$ is a clique

LEM: if A, B are linked and saturated and $N(A \cup B)$ is a clique, then $H = V \setminus B$, $\bar{H} = B$ separate A and B

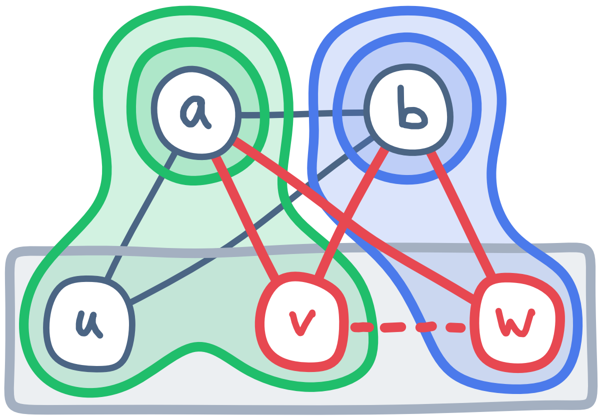


$J[u,v] \subseteq H$ for $u,v \in H$:

- $u,v \in A$: A convex
- $u \in A$, $v \notin A$: B shadow-closed
- $u,v \notin A$: no forbidden pair

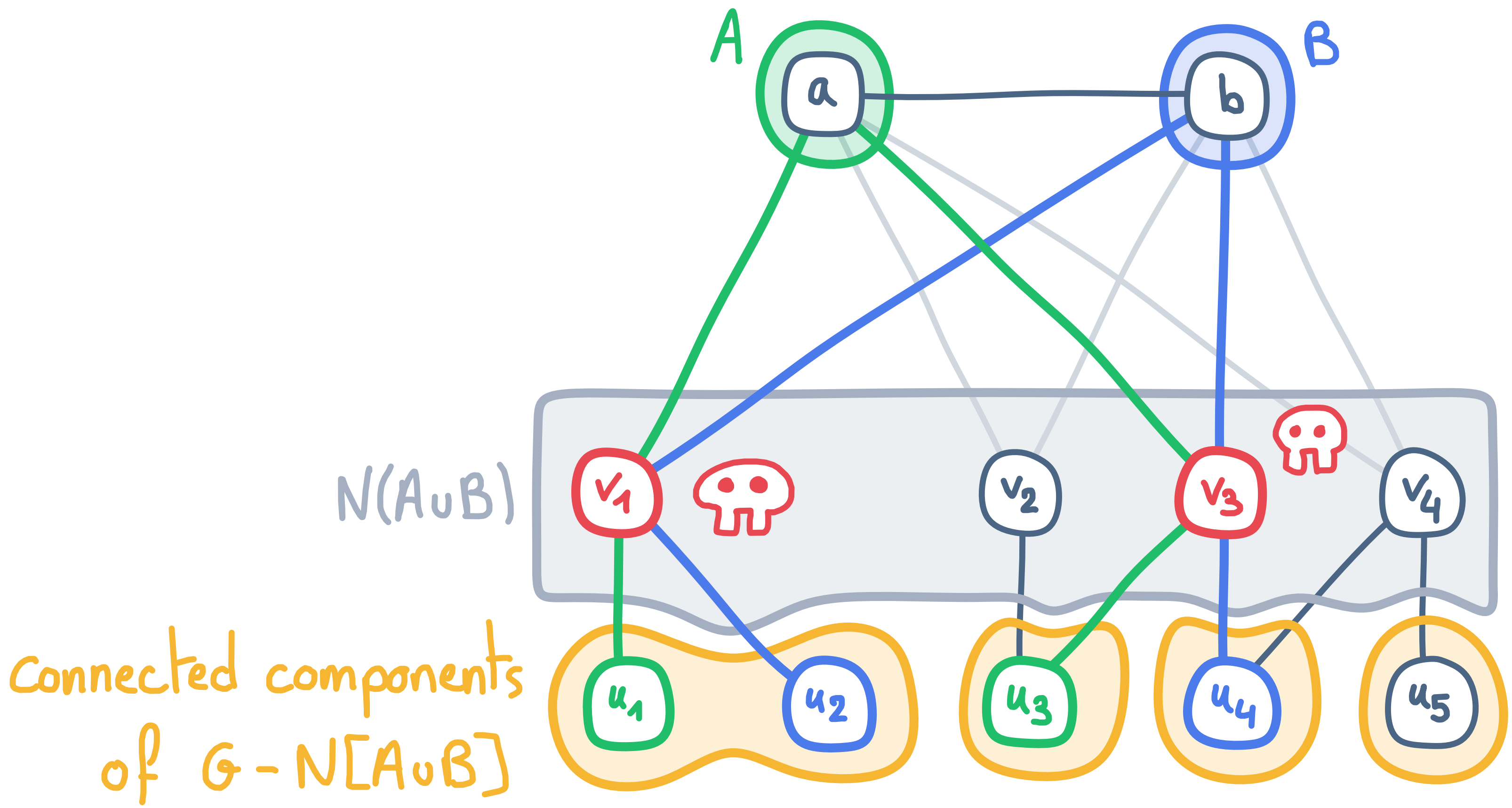
Decision: N(A∪B) not a clique (1)

RMK: in N(A∪B), u not adj. to v ⇔ uv forbidden pair



If A, B are linked and saturated, they are separable only if the subgraph induced by N(A∪B) is co-bipartite

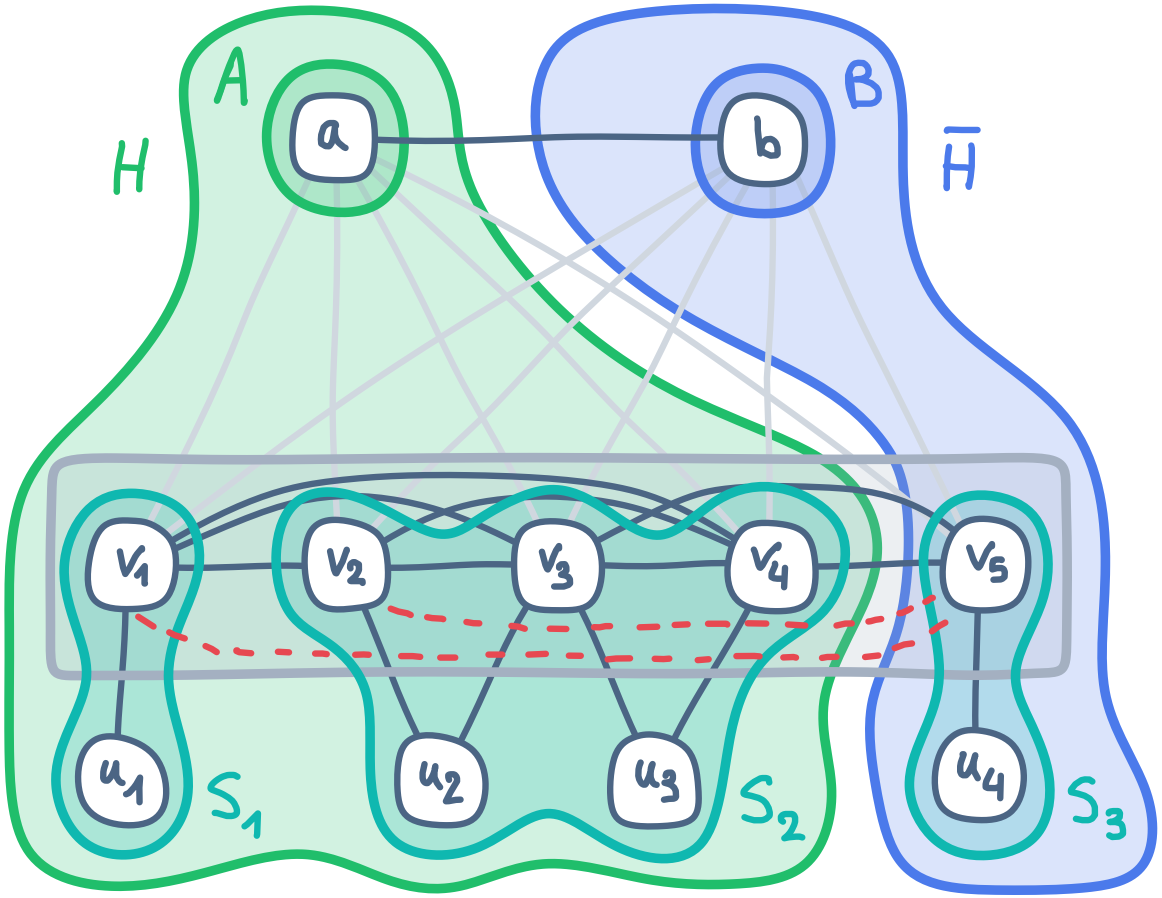
Decision: $N(A \cup B)$ not a clique (2)

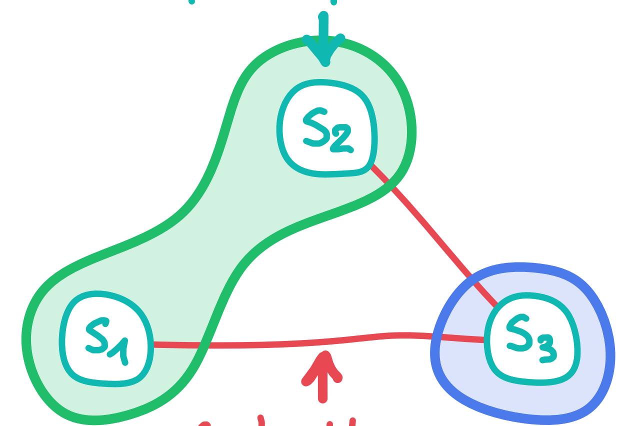


$C_1, \dots, C_m$ the connected components (CCs) of $G - N[A \cup B]$.

$\forall 1 \leq i, j \leq m$, $N(C_i) \cap N(C_j) \neq \emptyset \Rightarrow C_i \cup C_j$ cannot be split.

Decision: the graph G(A, B)





LEM: if A, B are linked and saturated, then they are separable iff G(A, B) is bipartite and loopless. Any bipartition of G(A, B) into stable sets gives a separation of A and B.

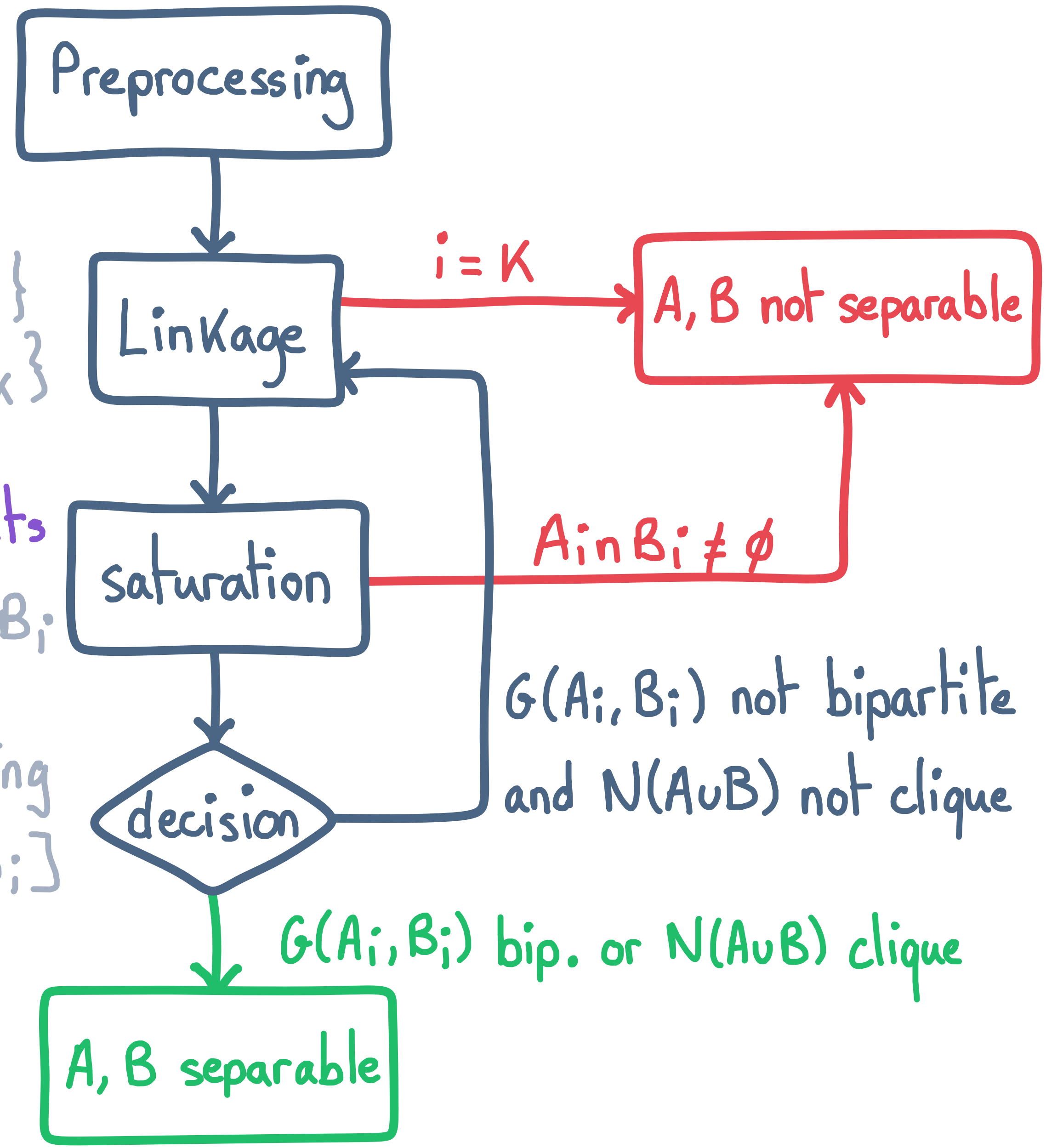
General (constructive) idea

Fix $a \in A, b \in B$ and shortest ab-path $a = v_0, \dots, v_k = b$

$A_i = A \cup \{v_0, \dots, v_i\}$
$B_i = B \cup \{v_{i+1}, \dots, v_k\}$

use shadows and forbidden sets to build A_i, B_i

build $G(A_i, B_i)$ by aggregating CCs of $G - N[A_i \cup B_i]$



Conclusion

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THM: half-space separation in monophonic convexity can be solved in polynomial time

QUES: the strategy we used is (1) use the hull operator in general and then (2) use the "structure" of the class at hand. To what extent can this strategy be applied to other convexity spaces?

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