THE D-BASE OF FINITE CLOSURE SYSTEMS

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Journées smartFCA - 21/06/24

Joint work with:

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Lhouari Nourine (LIMOS, Université Clermont Auvergne)

(arXiv link)

Who am I

2013 ISIMA : IT engineering school 2018

Université Clermont Auvergne (UCA)

10/18 PhD : lattices, enumeration 12/21

LIMOS, UCA, with Lhouari Nourine

02/22 Post-doc : databases, functional dep. 01/23

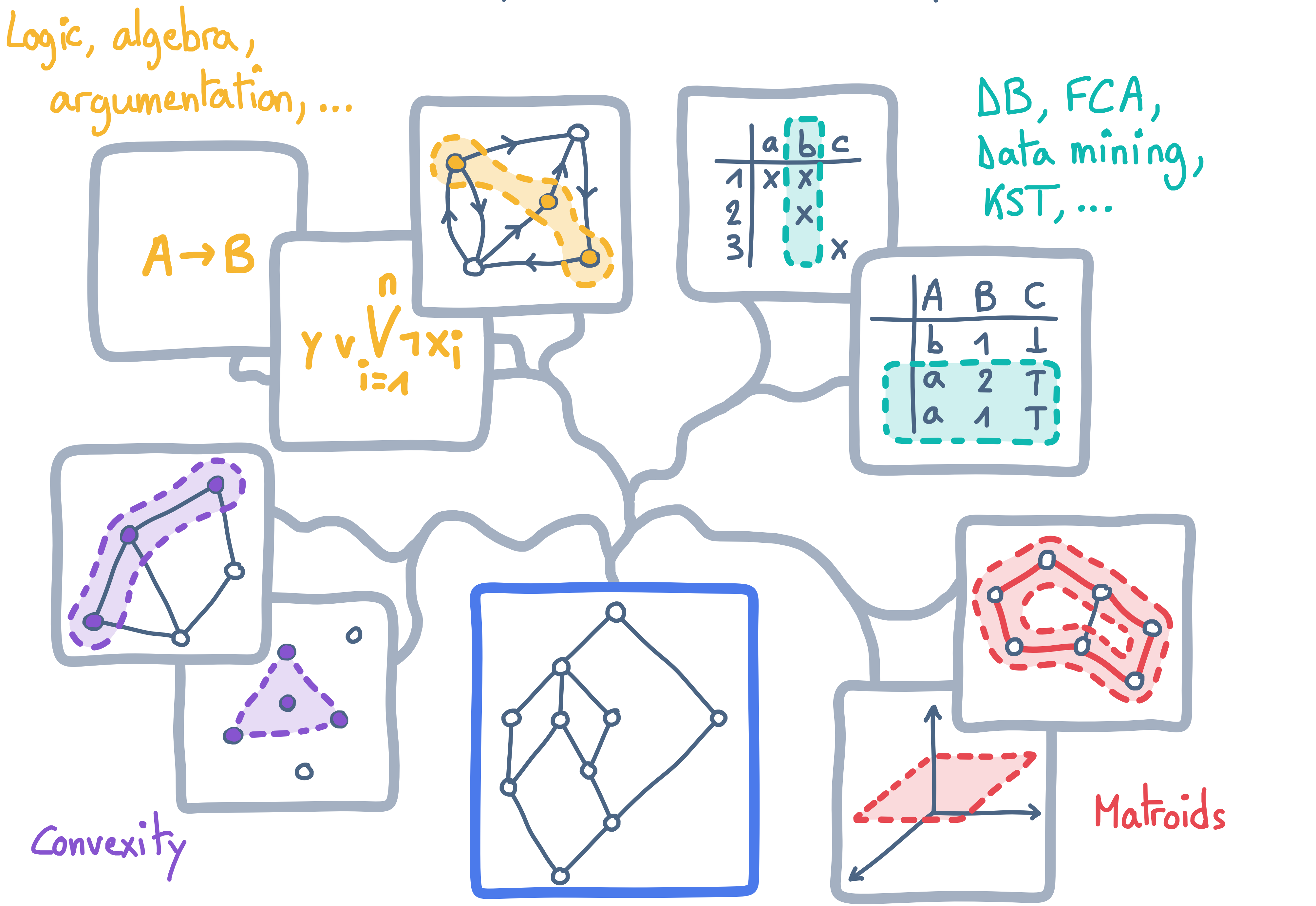
LIRIS, INSA Lyon, with Jean-Marc Petit

03/23 Post-doc : median lattices 08/23

LORIA, with Amedeo, Miguel

09/23 MCF LIS

Research : representations of closure systems and lattices



Outline

PART I: what is the D-base

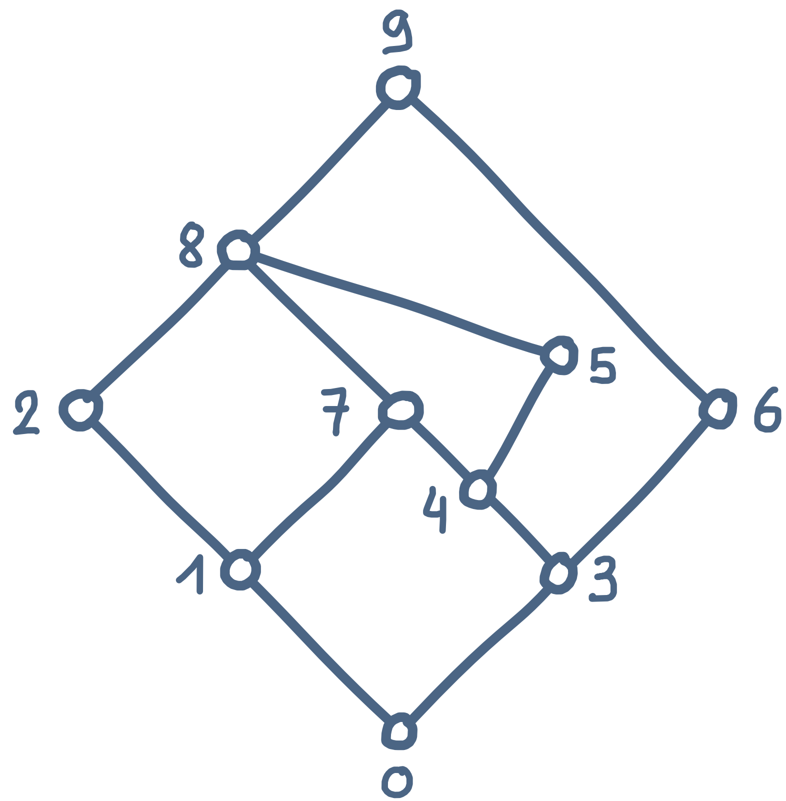
- from lattices ...
- ... to closure systems (and minimal generators)

PART II: computing the D-base

- from implications
- from meet-irreducible elements

PART I: What is the D-base

starting point: lattices and join covers



IDEA: describe the (structure of the) lattice by means of join covers of join-irreducible elements

A long-standing question

Dilworth, 40 : unique irreducible decompositions in locally distributive lattices ($\simeq$ convex geometries)

Finkbeiner, 51 : dependence relation (= implications)

Jónsson, Nation, 77 : join-refinement, minimal join covers (D-base) for free lattices

Gaskill, Rival, 78 : use of "minimal pairs" in modular lattices (= D-base)

A long-standing question

Day, 79 : study of (lower, upper) bounded lattices with relations on join-covers (D-relation, D- base)

Faigle, 86 : minimal pairs in geometric lattices

Nation, 90 : OD - graph (D-base) — in this talk

Freese et al., 95 : book on free lattices

in this talk

Bertet, Monjardet, 10 : survey on minimal generators and canonical direct base ($\supseteq$ D - base)

Leading to the Δ-base

in this talk

Adaricheva et al., 13 : introduction of the Δ-base

Rodriguez et al., 15, 17 : computing Δ-base with simplification logic (from implications)

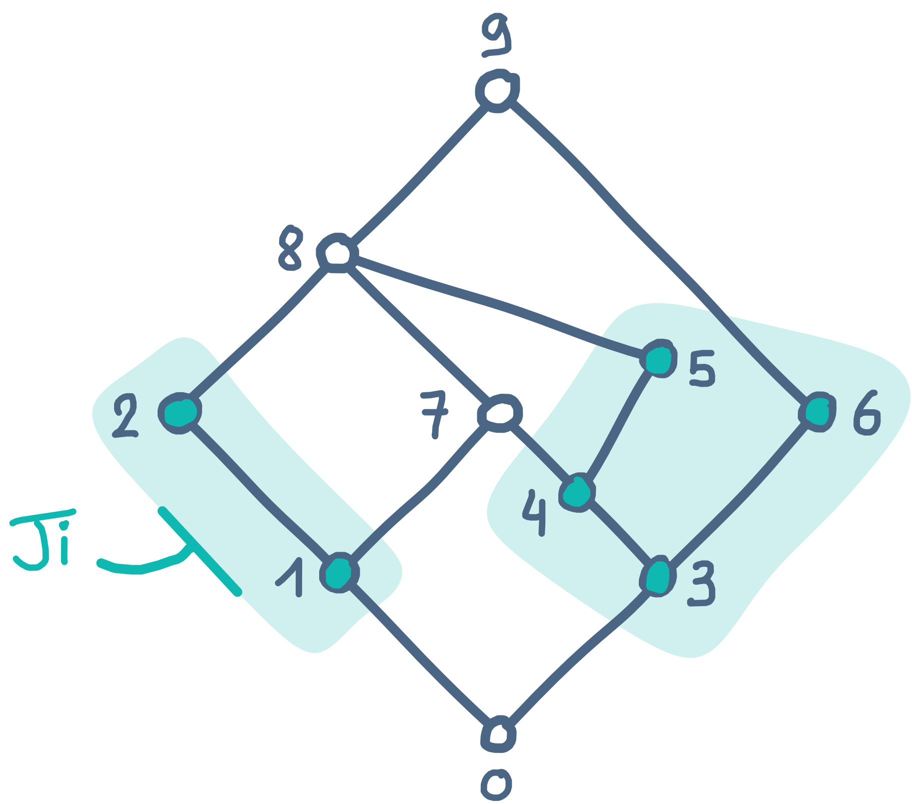
Adaricheva, Nation, 17 : computing Δ-base with hypergraph dualization (from context)

In this talk

Adaricheva, Nourine, Vilmin, 24+ : output-sensitive study of Δ-base computation, with implications or meet-irreducibles (= context)

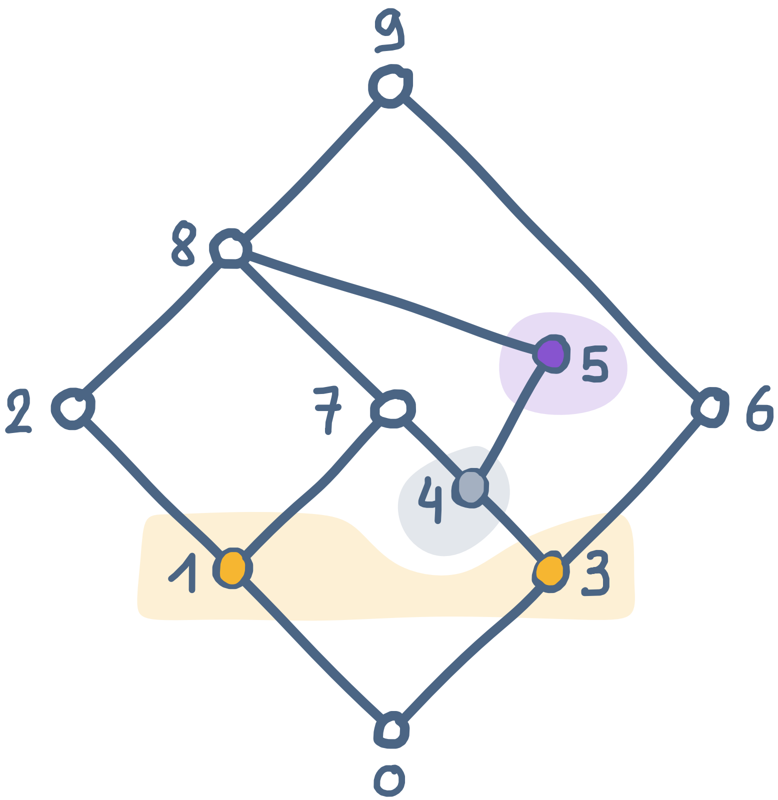
DEF (join-irr): in a lattice $\mathcal{L}$, $j \in \mathcal{L}$ is join-irreducible if $j \neq$ bot and $j = a \vee b$ entails $j = a$ or $j = b$, for $a, b \in \mathcal{L}$.

$$\mathcal{J}i = \{ j \in \mathcal{L} : j \text{ join-irreducible} \}$$



DEF (join-cover): given $j \in J_i$, a join-cover of j is a subset A of J_i such that $\boxed{j \leqslant \bigvee A}$

implication $A \to j$



Two "types" of join covers

- trivial, based on $(J_i, \leqslant)$

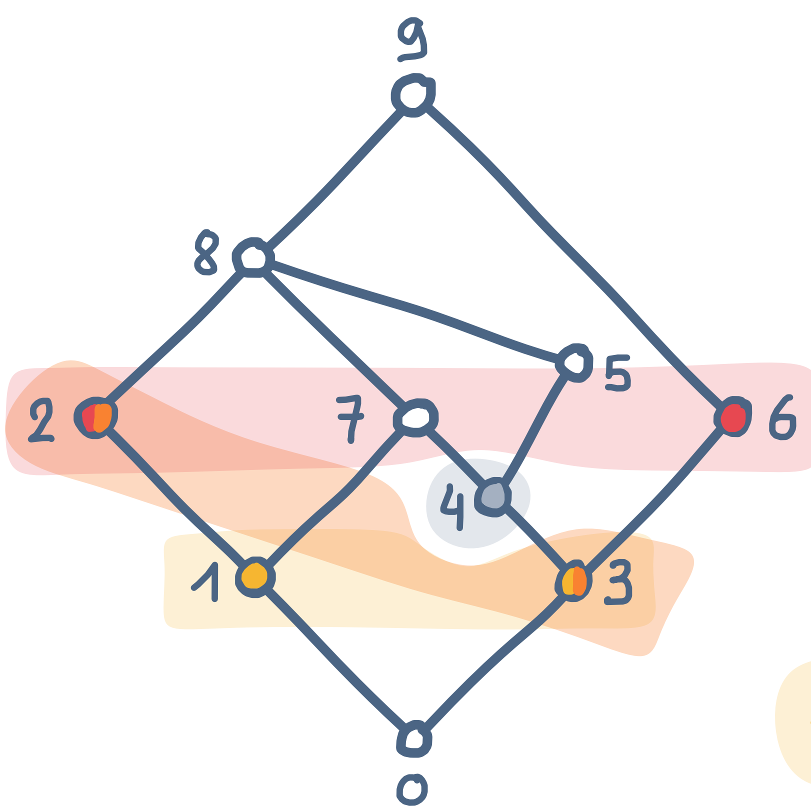
  $4 \leqslant 5$

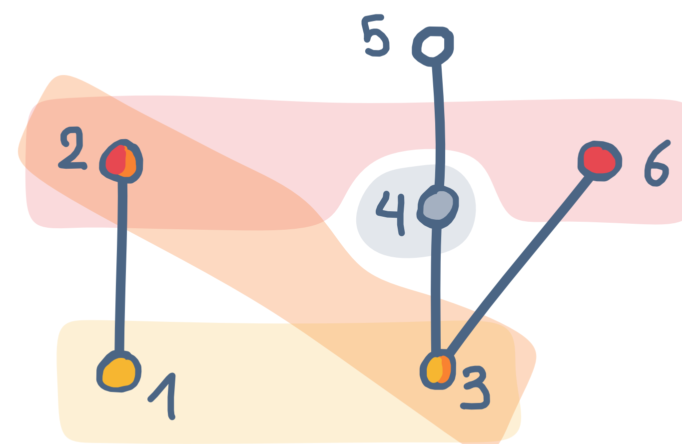
- non-trivial, $j \not\leqslant a$ for $a \in A$

  $4 \leqslant 1 \vee 3$

Minimal non-trivial join-cover : intuition

IDEA : among non-trivial join-covers of j, keep the lowest ones w.r.t. $\ll$ (or $(J_i, \leq)$ equivalently)





$(J_i, \leq)$

13 lower than 23 lower than 26 among covers of 4

Minimal non-trivial join-cover: formalization

DEF (refinement): given $A, B \subseteq J_i$, A **refines** B if for each $a \in A$, there is some $b \in B$ s.t. $a \leq b$

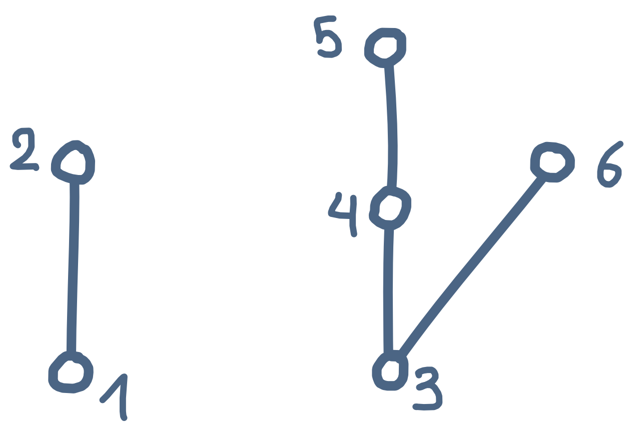
DEF (D-generator): a **minimal non-trivial join-cover** of j, or **D-generator of j**, is a ntjc of j that **cannot be refined** to another ntjc of j.

RMK:

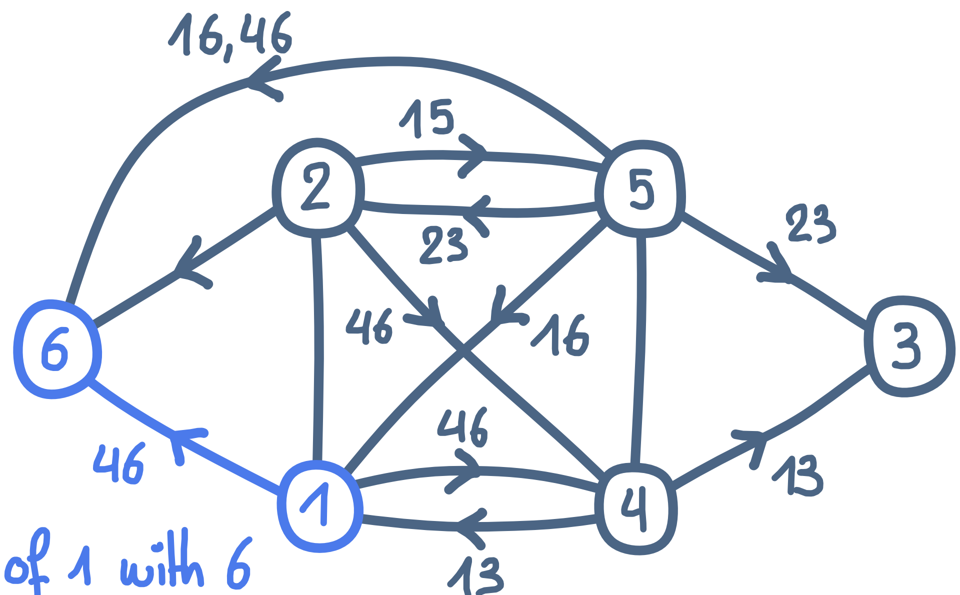
- D-generators: minimal join-cov w.r.t. refinement
- $A_1 \subseteq A_2$ implies that A_1 refines A_2 !

OΔ-graph Nation, 90

IDEA: O order of join irreducible $(J_i, \leq)$ + Δ directed labeled (multi-)graph based on Δ-generators



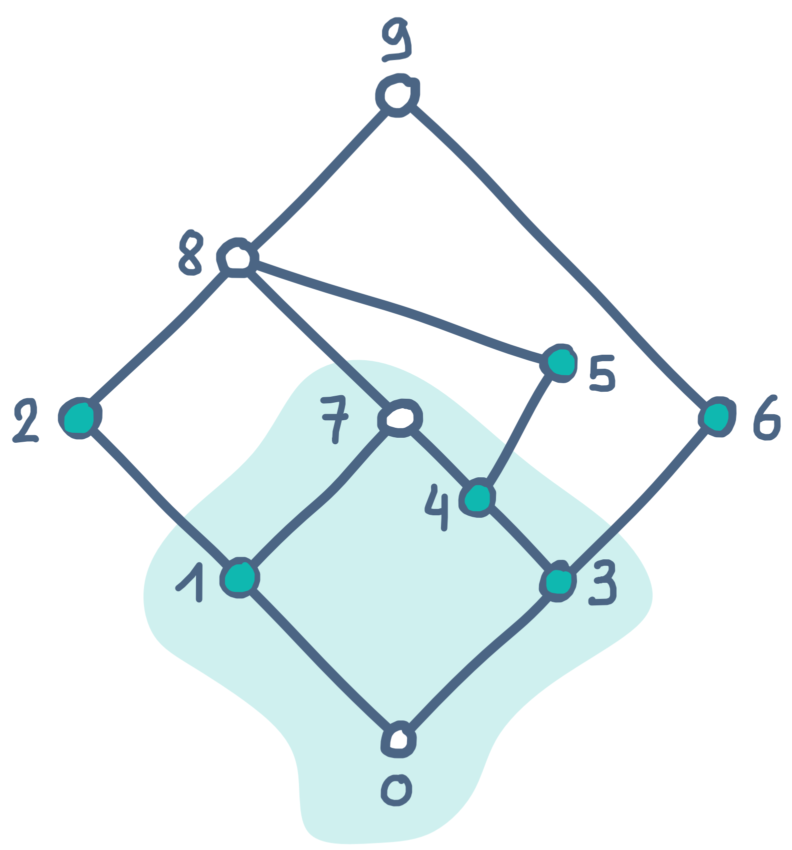
$(J_i, \leq)$

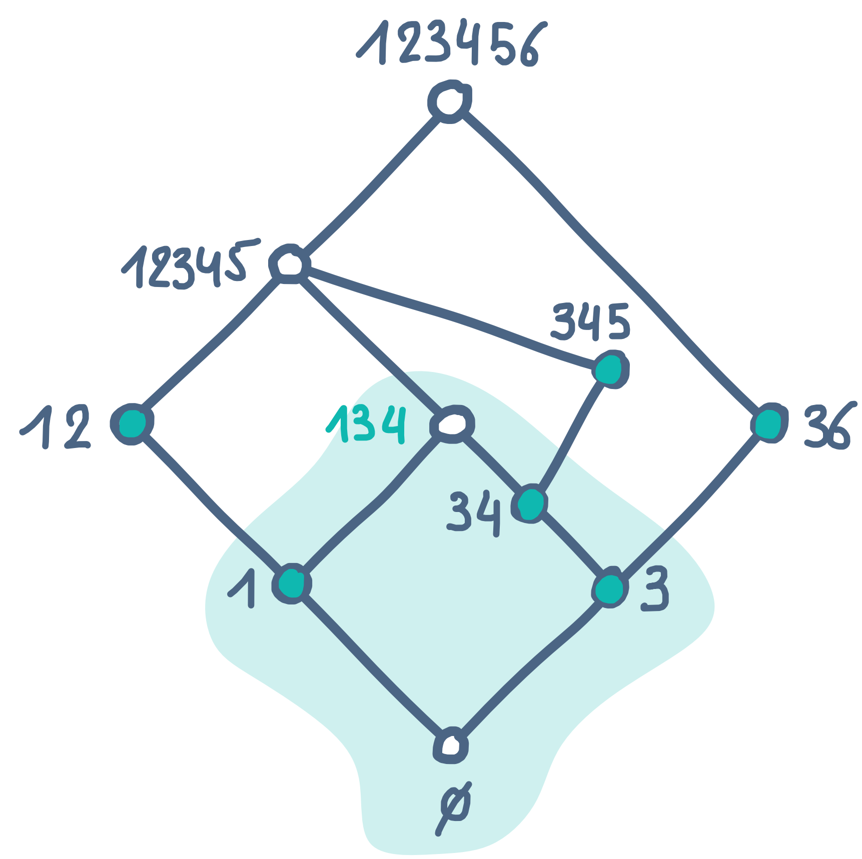


46 Δ-gen of 1 with 6

allows to rebuild the lattice as a closure system

From lattices to closure systems





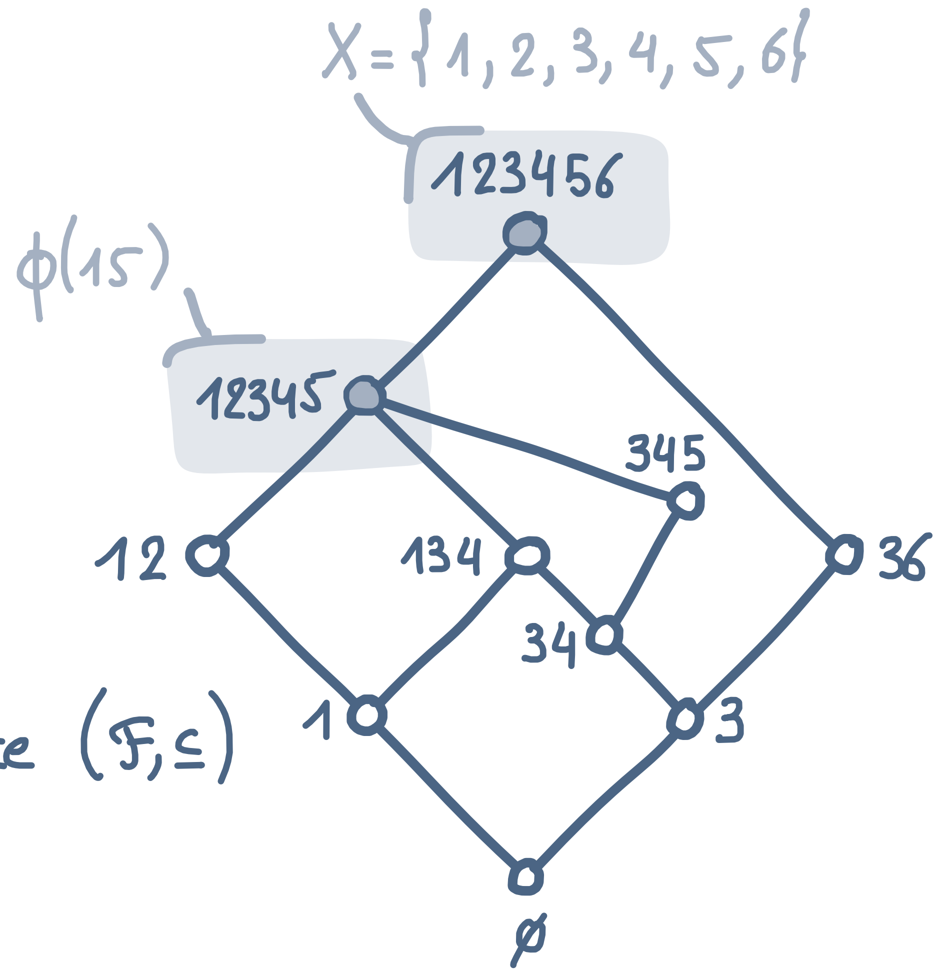
$4 \leq 5,\ 4 \leq 1 \vee 3 \longrightarrow 5 \to 4,\ 13 \to 4$

DEF (closure system): set system $\mathcal{F}$ over groundset X s.t. $X \in \mathcal{F}$ and $F_1 \cap F_2 \in \mathcal{F}$ for $F_1, F_2 \in \mathcal{F}$.

Sets in $\mathcal{F}$: closed sets

$\mathcal{F}$ induces a closure operator:

$$\phi(A) = \bigcap \{F \in \mathcal{F} : A \subseteq F\}$$

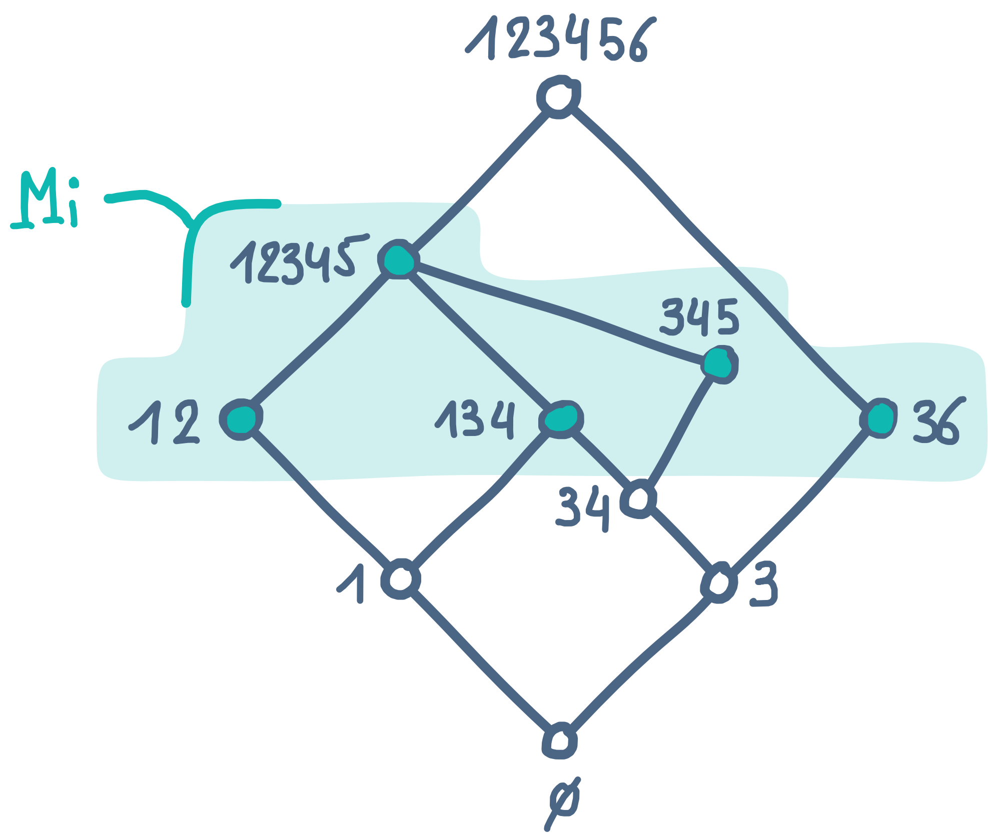


closure lattice $(\mathcal{F}, \subseteq)$

Meet-irreducible closed sets

DEF (meet-irreducible): a closed set $M \in \mathcal{F}$ distinct from X is (meet-)irreducible if for all $F_1, F_2 \in \mathcal{F}$ $M = F_1 \cap F_2$ entails $M = F_1$ or $M = F_2$

$$\underline{Mi} = \{ M \in \mathcal{F} : M \text{ is irreducible} \}$$



Implications

(or $A \to B$, $B \subseteq X$) "If I have A, I have b"

DEF (implications):

- **implication**: statement $\boxed{A \to b}$ with $A \subseteq X$, $b \in X$
- **implicational base (IB)**: set Σ of implications on X

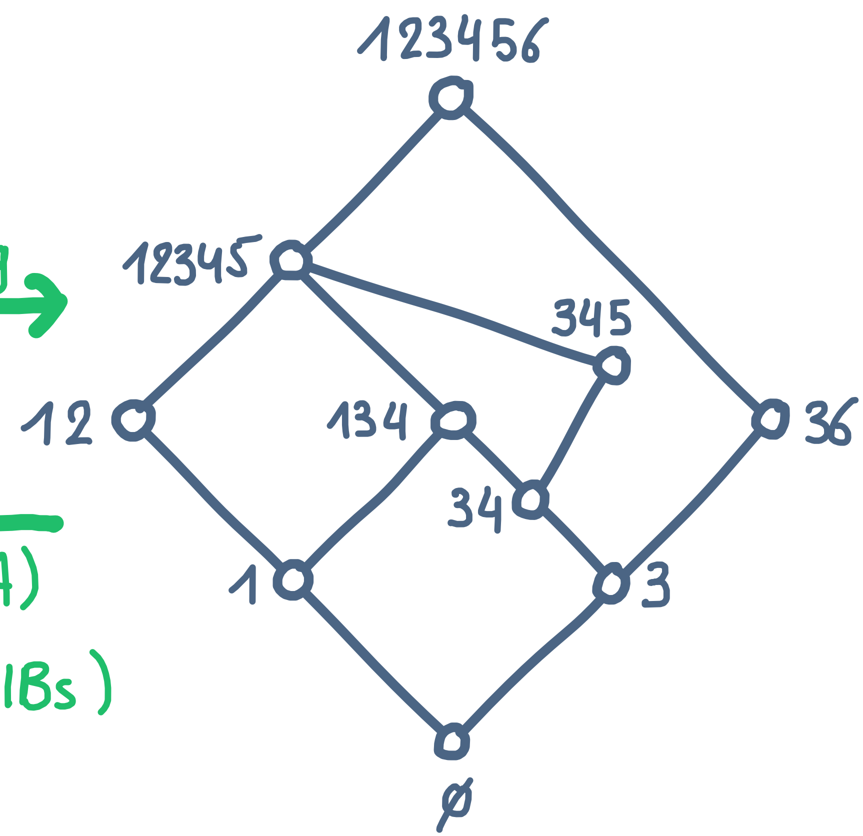
Binary implications

$$\Sigma = \begin{bmatrix} 2 \to 1, & 4 \to 3, \\ 5 \to 4, & 6 \to 3, \\ 13 \to 4, & 23 \to 5, \\ 15 \to 2, & 46 \to 2, \\ 16 \to 5 & \end{bmatrix}$$

forward chaining $\longrightarrow$

$\longleftarrow$ $b \in \phi(A)$

(many equiv. IBs)



Minimal generators, canonical direct base Bertet, Monjardet, 10

DEF (minimal generator): $A \subseteq X$ is a minimal generator of x if it is an $\subseteq$-min subset of X satisfying $x \in \phi(A)$

The canonical direct base of a closure system is

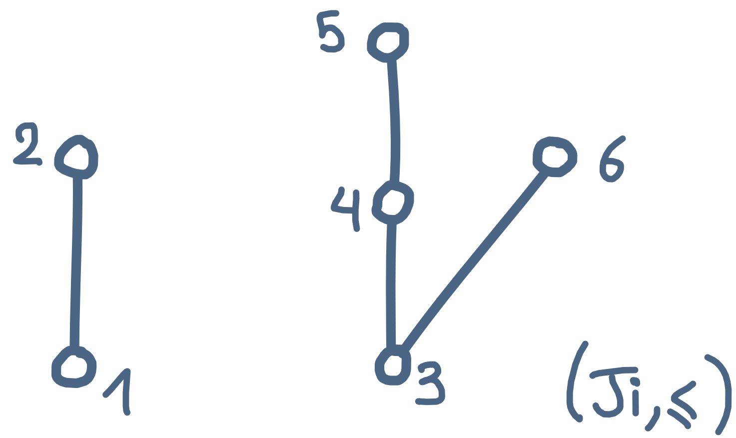
$$\Sigma_{cd} = \{ A \to x : A \text{ minimal generator of } x,\ x \notin A \}$$

Our running example:

$$\Sigma_{cd} = \begin{bmatrix} 2 \to 1,\ 46 \to 1,\ 56 \to 1, \\ 46 \to 2,\ 15 \to 2,\ 16 \to 2, \\ 5 \to 3,\ 4 \to 3,\ 6 \to 3, \\ 13 \to 4,\ 23 \to 4,\ 16 \to 4,\ 26 \to 4 \\ 14 \to 5,\ 16 \to 5,\ 24 \to 5,\ 26 \to 5 \end{bmatrix}$$

What about the D-base then ?

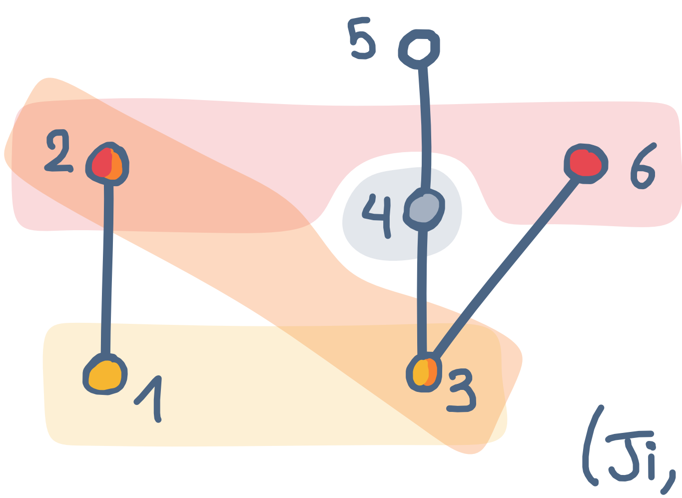
1st ingredient : order to binary implications



$$\Sigma_b = \left[2 \to 1,\ 5 \to 4,\ 5 \to 3,\ 4 \to 3,\ 6 \to 3 \right]$$

with closure operator ϕ_b

2nd ingredient : translate D-generators



non-singleton minimal generators of x minimal w.r.t. ϕ_b !

$13 \subseteq 123 \subseteq 1236$

$\phi_b(13) \subseteq \phi_b(23) \subseteq \phi_b(26)$

At last, the D-grail Adaricheva et al., 13

DEF (D-generator): $A \subseteq X$ is a D-generator of x if $x \notin \phi_b(A)$ and A is a minimal generator of x being ϕ_b-minimal among min. gen. of x

DEF (D-base): the D-base of a closure system is:

$$\Sigma_D = \Sigma_b \cup \{A \to x : A \text{ } D\text{-gen of } x\}$$

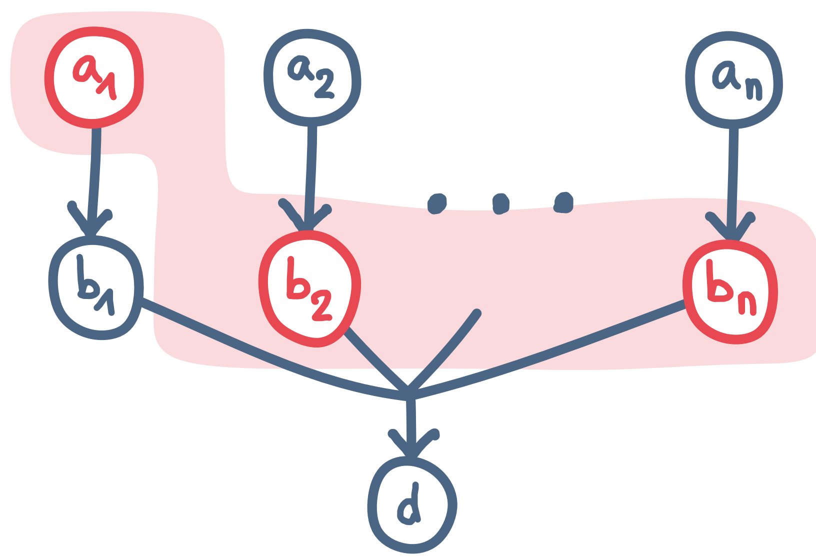
$$\Sigma_D = \begin{bmatrix} 2 \to 1, \ 5 \to 4, \ 5 \to 3, \\ 4 \to 3, \ 6 \to 3 \end{bmatrix} \cup \begin{bmatrix} 46 \to 1, \\ 46 \to 2, \ 15 \to 2, \ 16 \to 2, \\ 13 \to 4, \\ 14 \to 5, \ 16 \to 5 \end{bmatrix}$$

Bonus: gap between Σ_{cd} and Σ_Δ

RMK: we have $\Sigma_\Delta \subseteq \Sigma_{cd}$, but what is the gap?

$X = \{a_1, \dots, a_n, b_1, \dots, b_n, d\}$

$\Sigma = \{a_i \to b_i : 1 \leq i \leq n\} \cup \{b_1 \dots b_n \to d\}$

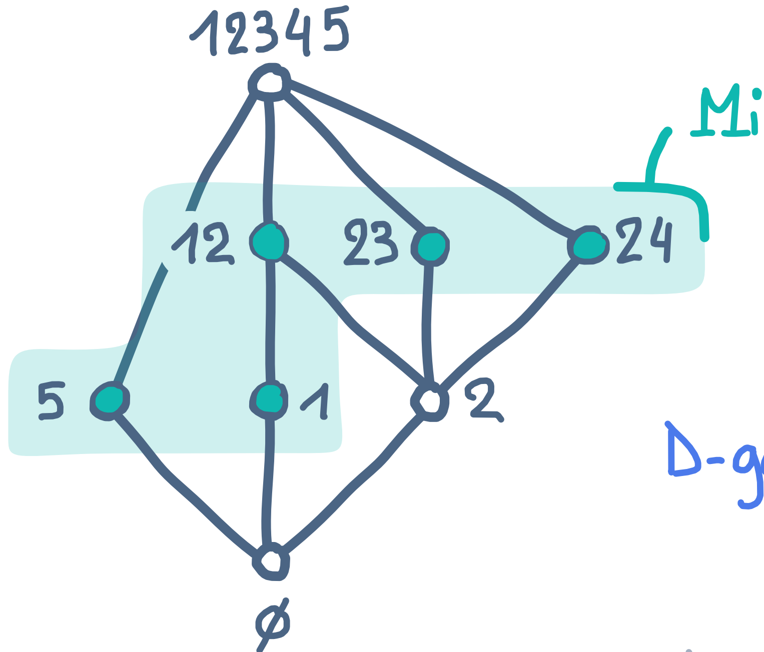


$\Sigma_\Delta = \Sigma \qquad \Sigma_{cd} = \Sigma_\Delta \cup \{A \to d : A \in \prod_{i=1}^{n} \{a_i, b_i\}\}$

exponential gap w.r.t. Σ_Δ!

PART II: Computing the Δ-base

Brand new toy example



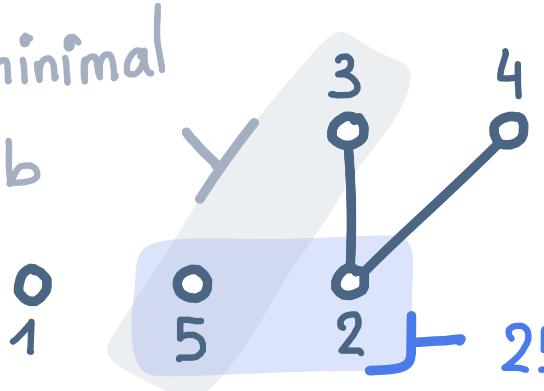
Binary implications Σ_b

$$\Sigma = \begin{bmatrix} 4 \to 2, & 25 \to 3, \\ 3 \to 2, & 34 \to 1, \\ 25 \to 4, & 14 \to 3, \\ 15 \to 4, & 13 \to 5, \\ 35 \to 4 & \end{bmatrix}$$

D-generator of 4

minimal generators of 4
(no subset of 35 gives 4)

35 not minimal w.r.t. Σ_b



Σ_b as a poset

25 minimal w.r.t. $\Sigma_b \Rightarrow$ D-generator of 4

PROB : given irreducibles M_i over X, find the D-base Σ_D

PROB : given an implicational base Σ over X, find the D-base Σ_D

RMK : enumeration tasks, listing implications without repetitions

Further motivations

Theoretical / algorithmic properties :

- convey structural information of closure systems
- ordered direct (fast forward chaining)
- much smaller than the set of all minimal generators

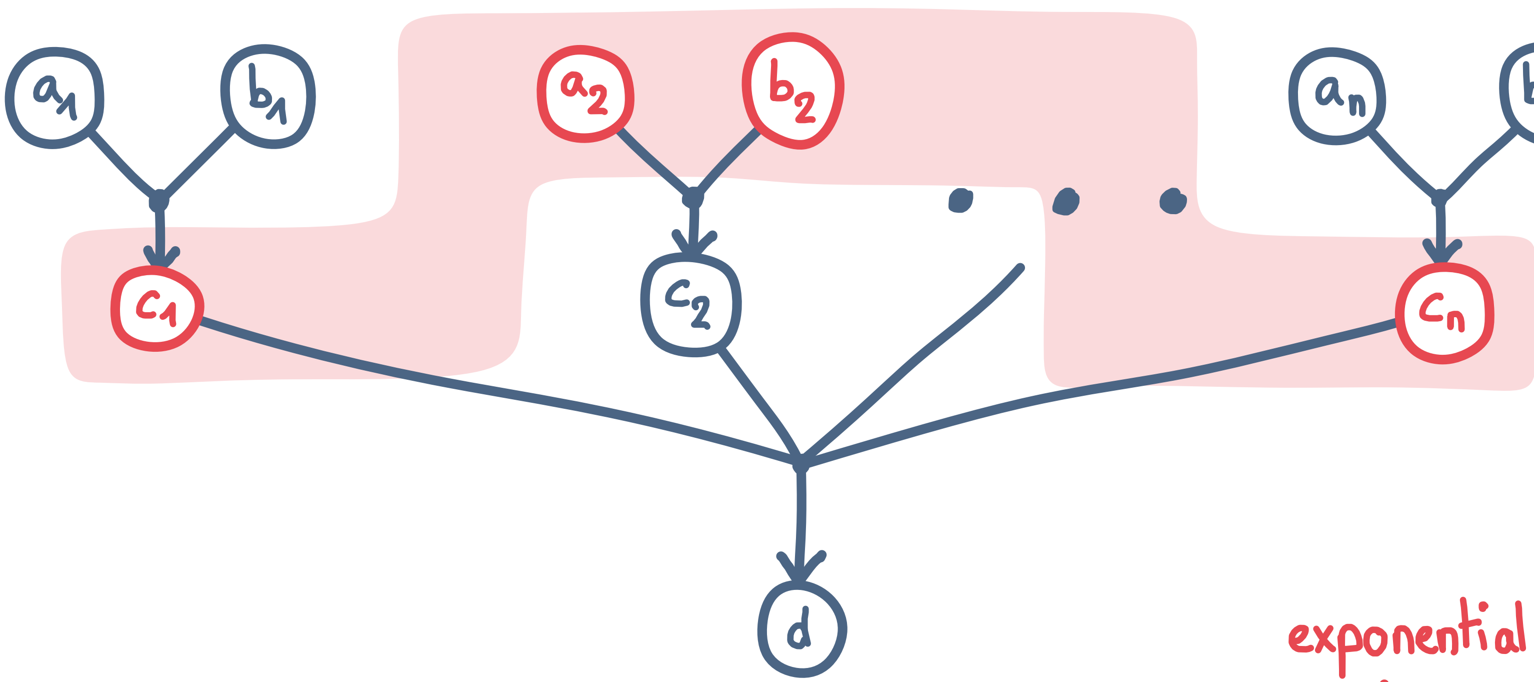
Practical uses :

- seabreeze forecast Adaricheva et al., 23
- stomach cancer risk estimation Nation et al., 21

Exponential blow up

$$X = \{a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n, d\}$$

$$\Sigma = \{a_i b_i \to c_i : 1 \leq i \leq n\} \cup \{c_1 \cdots c_n \to d\}$$



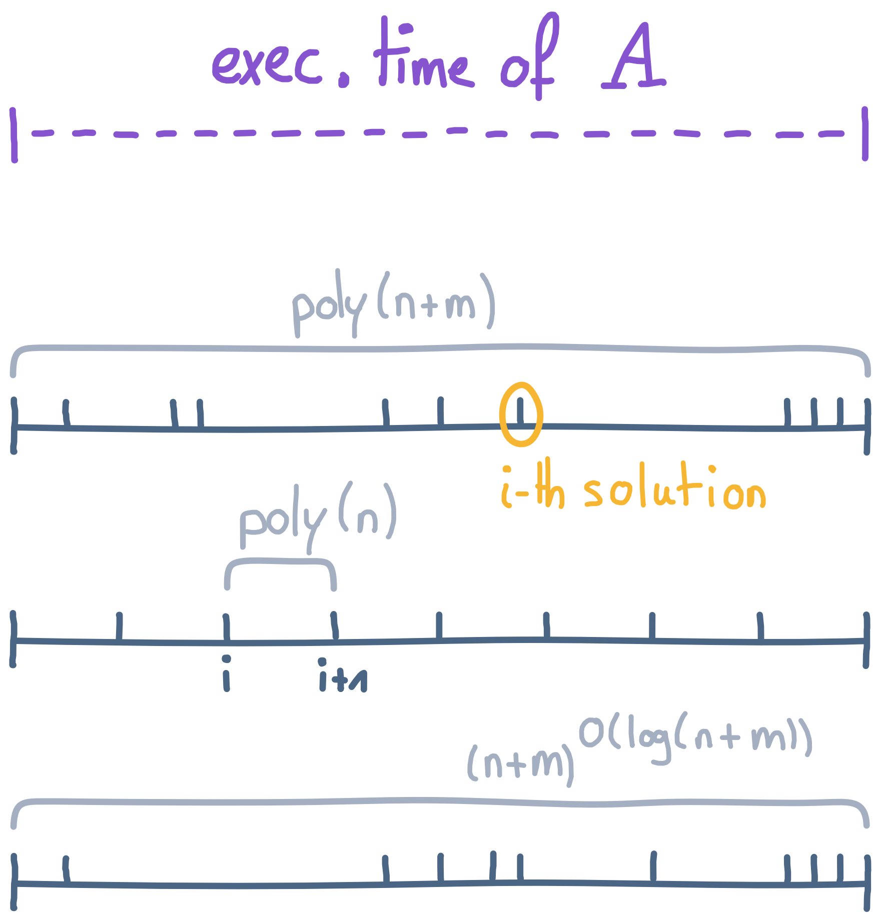
exponential gap w.r.t. Σ!

$$\Sigma_D = \Sigma \cup \boxed{\left\{A \to d : A \in \prod_{i=1}^{n} \{a_i b_i, c_i\}\right\}}$$

Enumeration: output-sensitive complexity

Each of size poly(x)

Enumeration task: with input x, list a set of solutions $R(x)$



Enumeration algorithm A

input x of size n

output $R(x)$ of size m

Output polynomial time

polynomial delay

Output quasi-polynomial time

Related works : part I

PROB : given irreducibles M_i over X,
find the Δ-base Σ_Δ

- algorithm based on Hypergraph dualization Adaricheva, Nation, 17
produces (possibly large) superset of Δ-base

PROB : given an implicational base Σ over X, find the D-base Σ_D

- algorithm using simplification logic <u>Rodriguez et al., 15, 17</u>
  no (output-sensitive) complexity analysis
- poly-delay algorithm listing D-minimal keys <u>Ennaoui, Nourine, 16</u>
  based on solution-graph traversal
  ($\simeq$ D-gen of some x)

Our results: part I

PROB: given irreducibles M_i over X, find the D-base Σ_D

Dualization of distributive lattices + Elbassioni 22

ANV, 24+

THM: given M_i over X, Σ_D can be computed in output quasi-polynomial time

Our results : part II

PROB : given an implicational base Σ over X, find the D-base Σ_D

↓

Solution-graph traversal
\+ Ennaoui, Nourine 16

↓

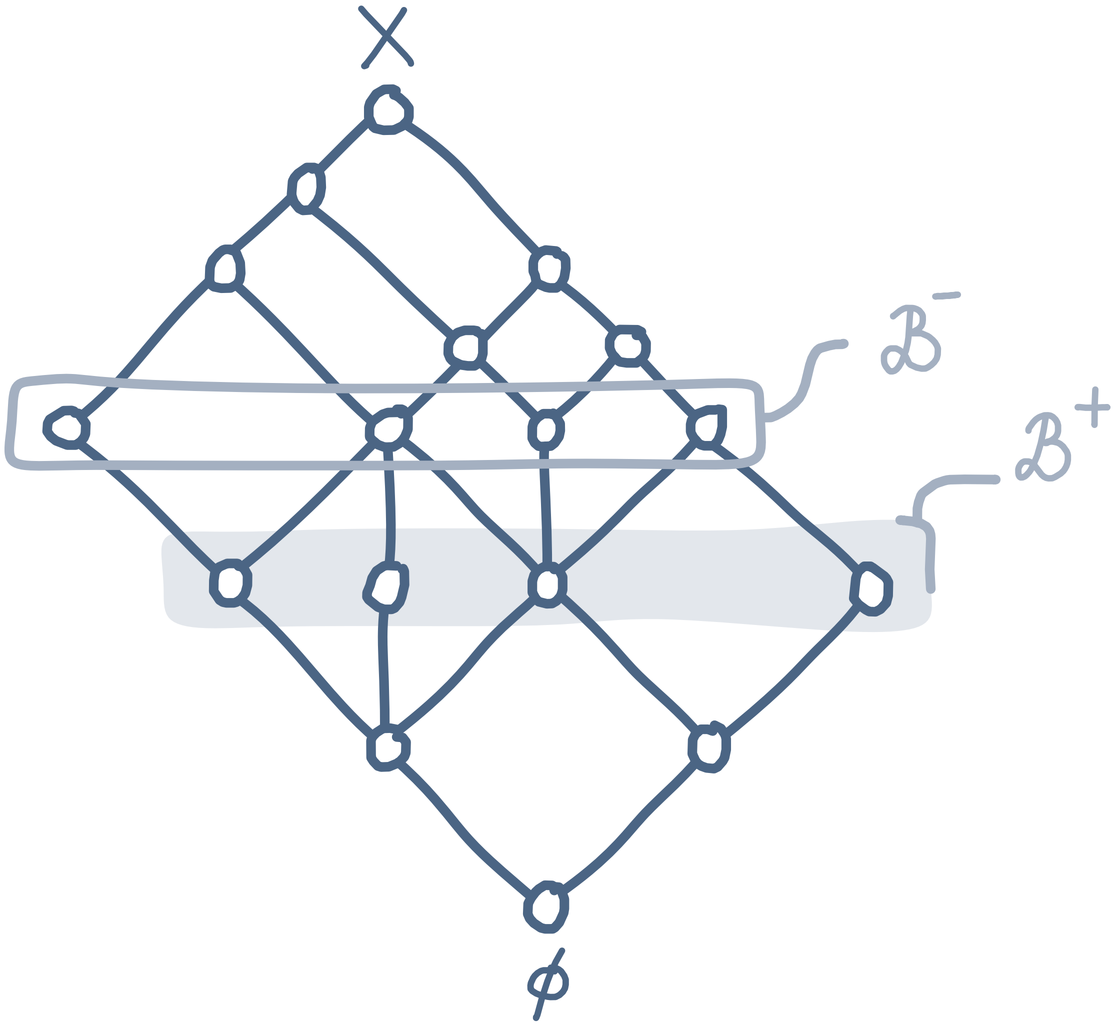
ANV, 24+

THM : given Σ over X, Σ_D can be computed with polynomial delay

PROB: given irreducibles M_i over X,

find the Δ-base Σ_Δ

Dualization

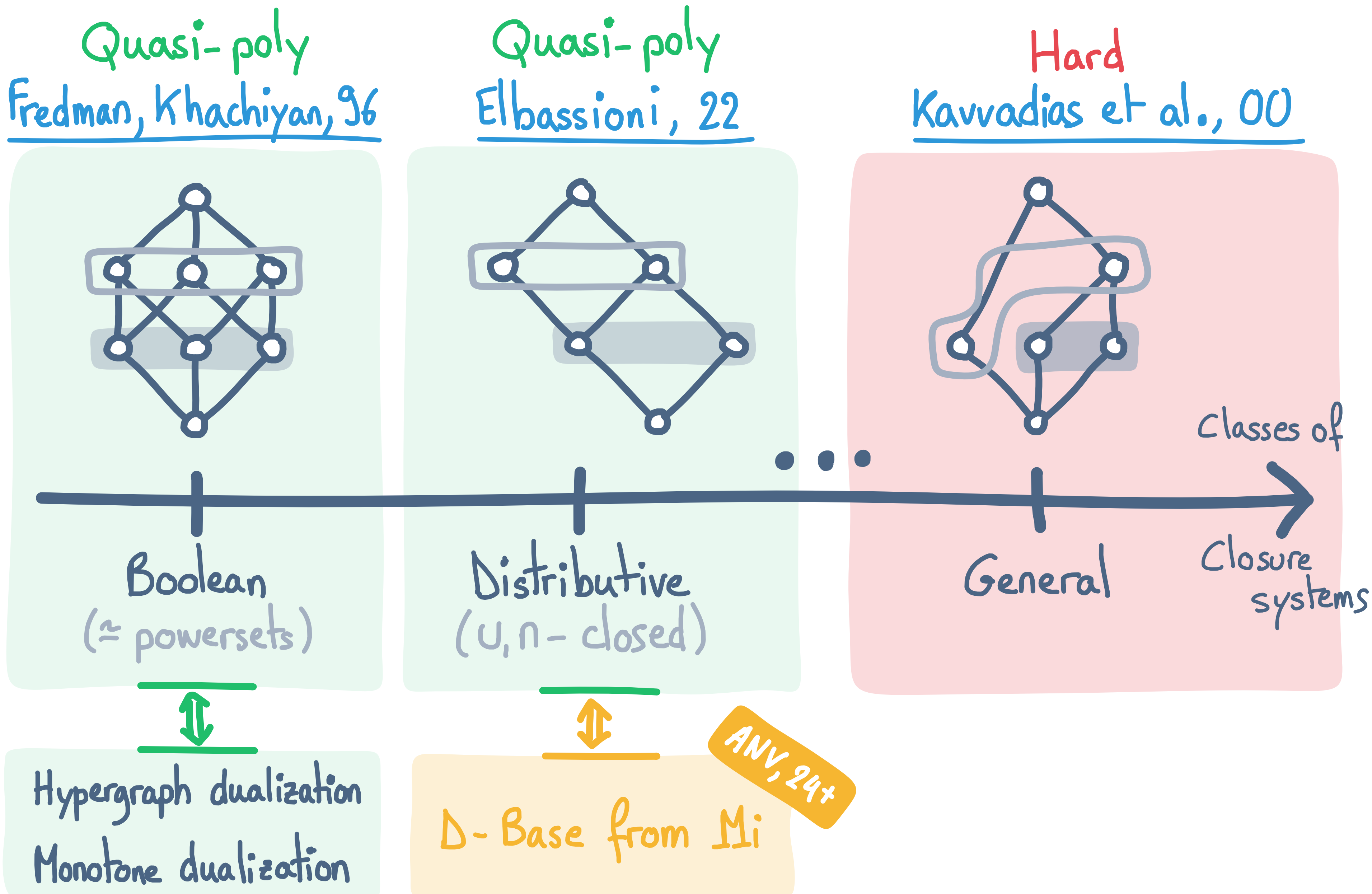


$\mathcal{B}^-$ and $\mathcal{B}^+$ are dual:

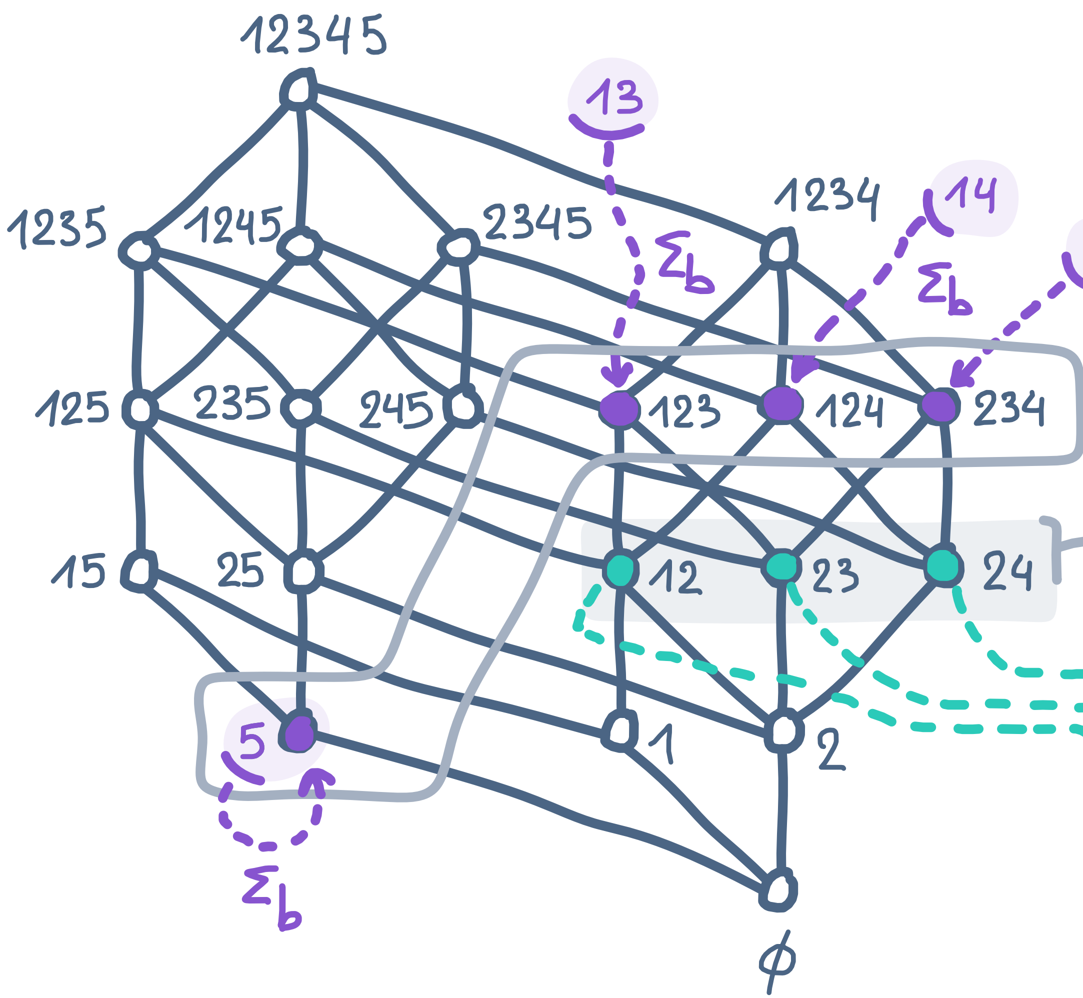
- $\downarrow\mathcal{B}^+ \cup \uparrow\mathcal{B}^- = \mathcal{F}$
- $\downarrow\mathcal{B}^+ \cap \uparrow\mathcal{B}^- = \emptyset$

PROB: with Σ over X and antichain $\mathcal{B}^+$, find antichain $\mathcal{B}^-$

Dualization complexity and D-base

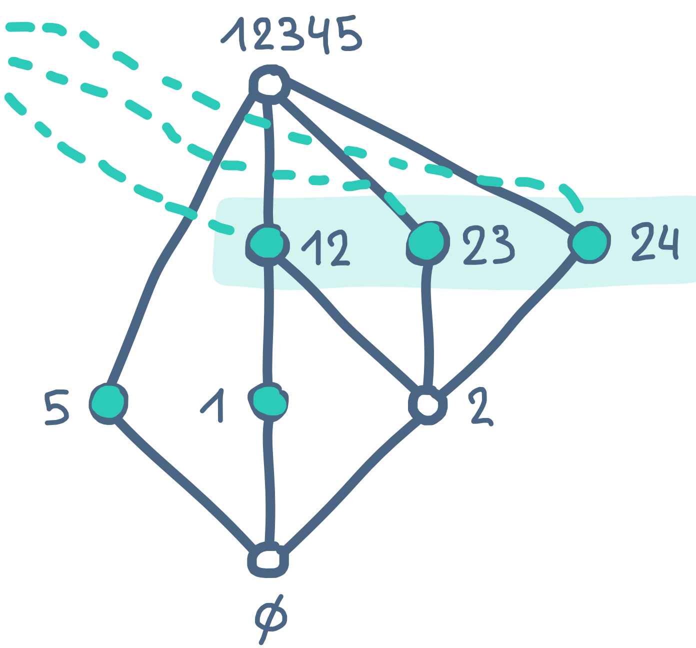


Intuition: Δ-base from Mi & Dualization Distr.



② $\mathcal{B}^-$ obtained from 5 and D-generators of 5

③ $\mathcal{B}^+$ = ⊆-max irreducibles without 5

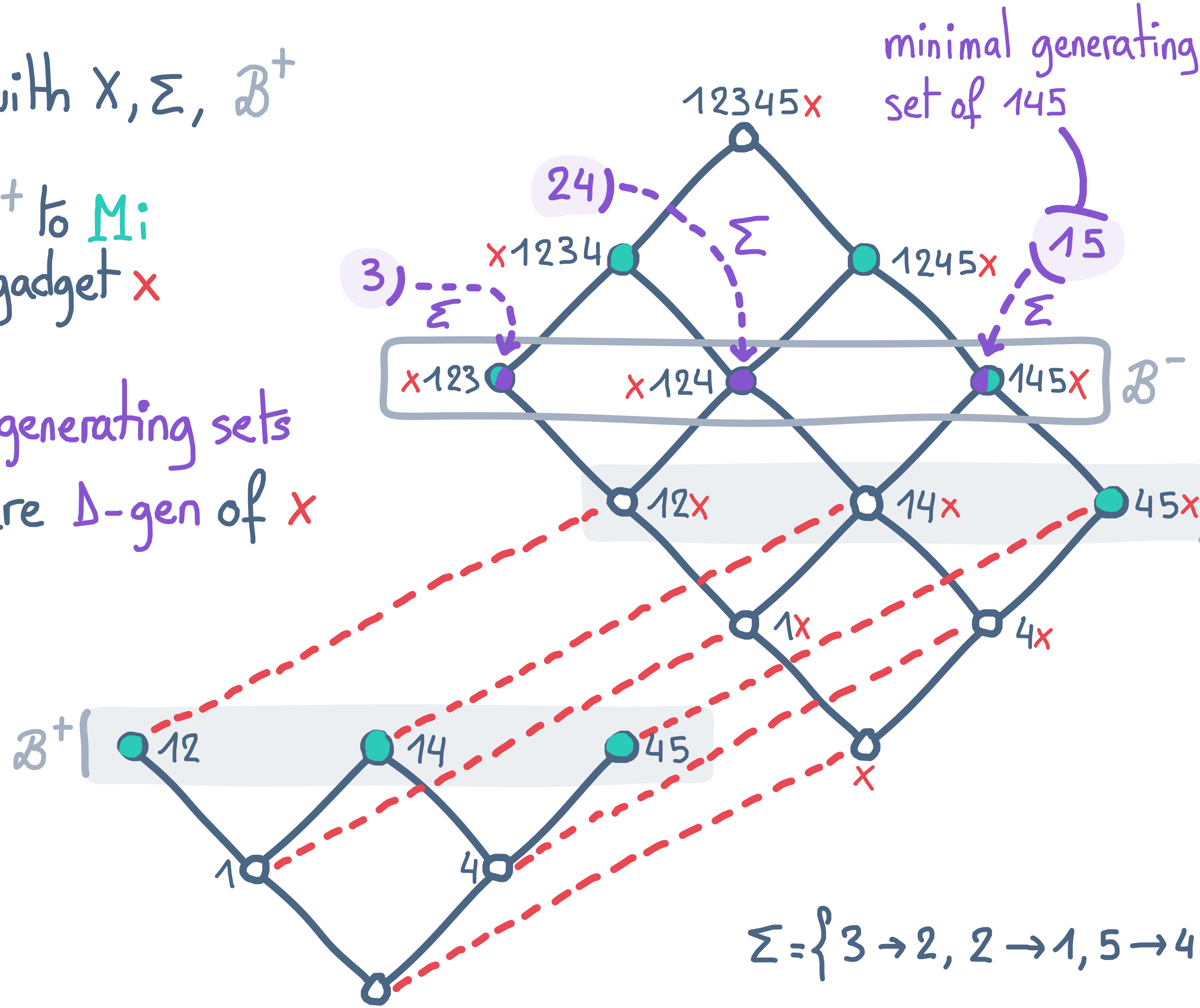


① distributive closure system of

$\Sigma_b = \{3 \to 2,\ 4 \to 2\}$

Intuition: D-base from Mi ≈ Dualization Distr.

1. Start with $X, \Sigma, \mathcal{B}^+$
2. Add $\mathcal{B}^+$ to Mi using gadget x
3. (min.) generating sets of $\mathcal{B}^-$ are D-gen of x



$\Sigma = \{3 \to 2,\ 2 \to 1,\ 5 \to 4\}$

PROB: given irreducibles M_i over X, find the Δ-base Σ_Δ

Dualization of distributive lattices + Elbassioni 22

THM: given M_i over X, Σ_Δ can be computed in output quasi-polynomial time

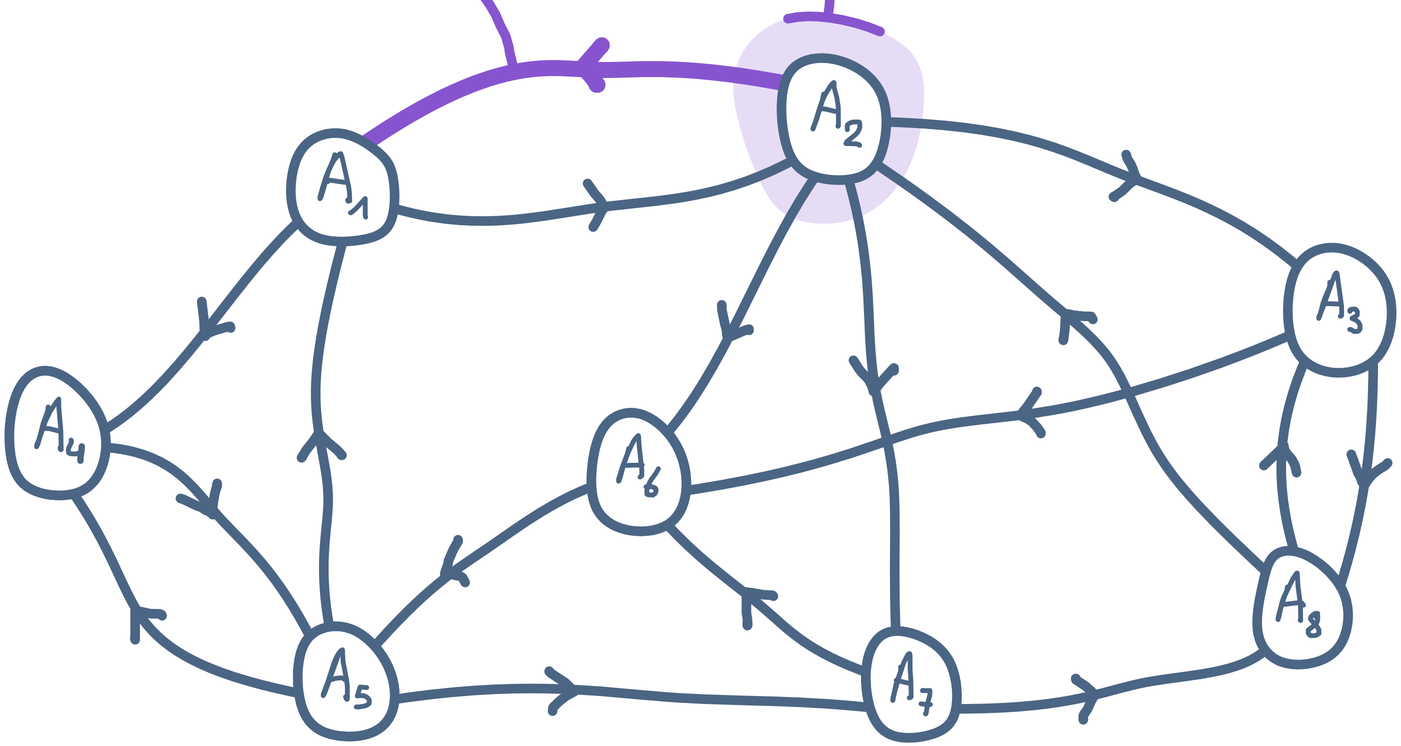
ANV, 24+

PROB: given an implicational base Σ over X, find the D-base Σ_D

Principle: Solution-graph traversal

arc = transition* which turns A_2 into A_1

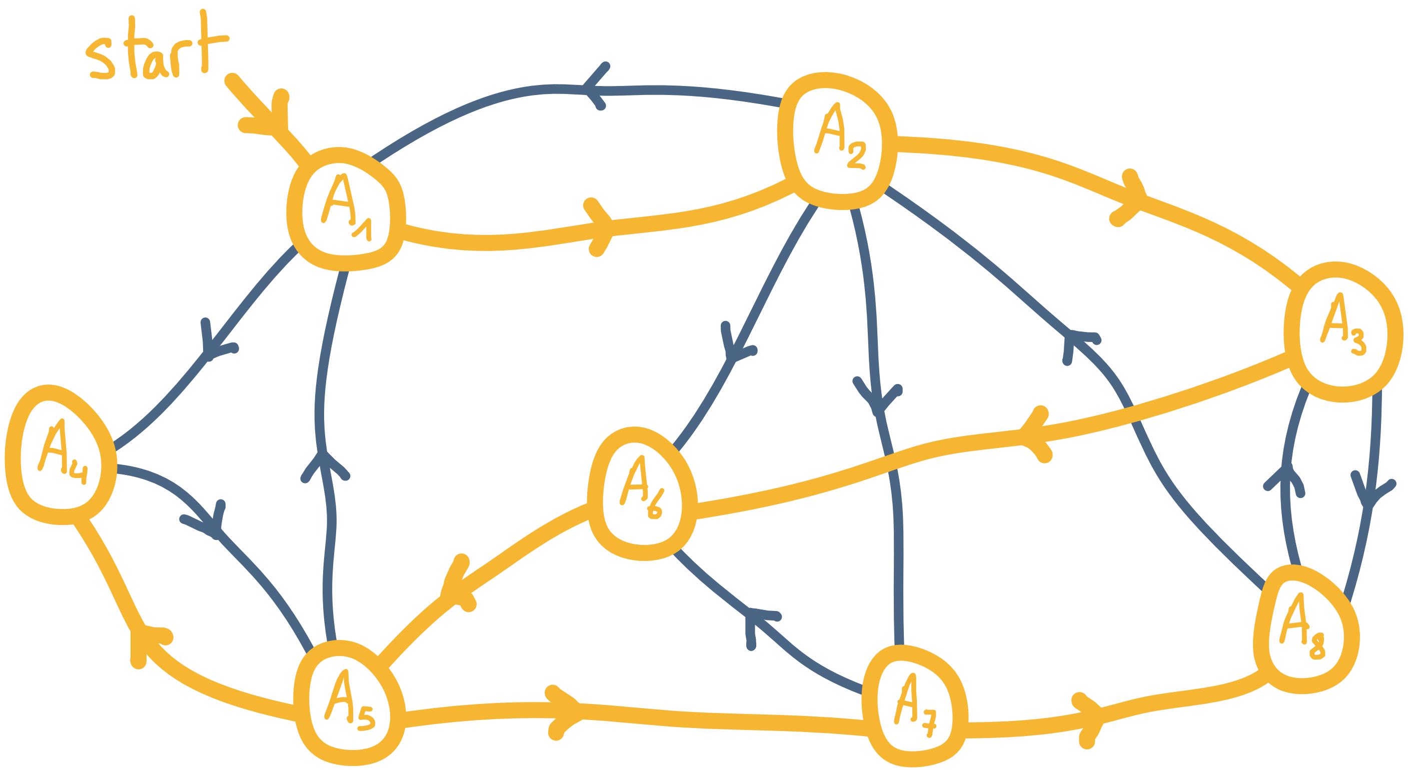
vertex = a D-generator of x



* transition key idea: substitute $a_2 \in A_2$ with B s.t. $B \to a_2 \in \Sigma$

(greedily) minimize w.r.t. Σ_b

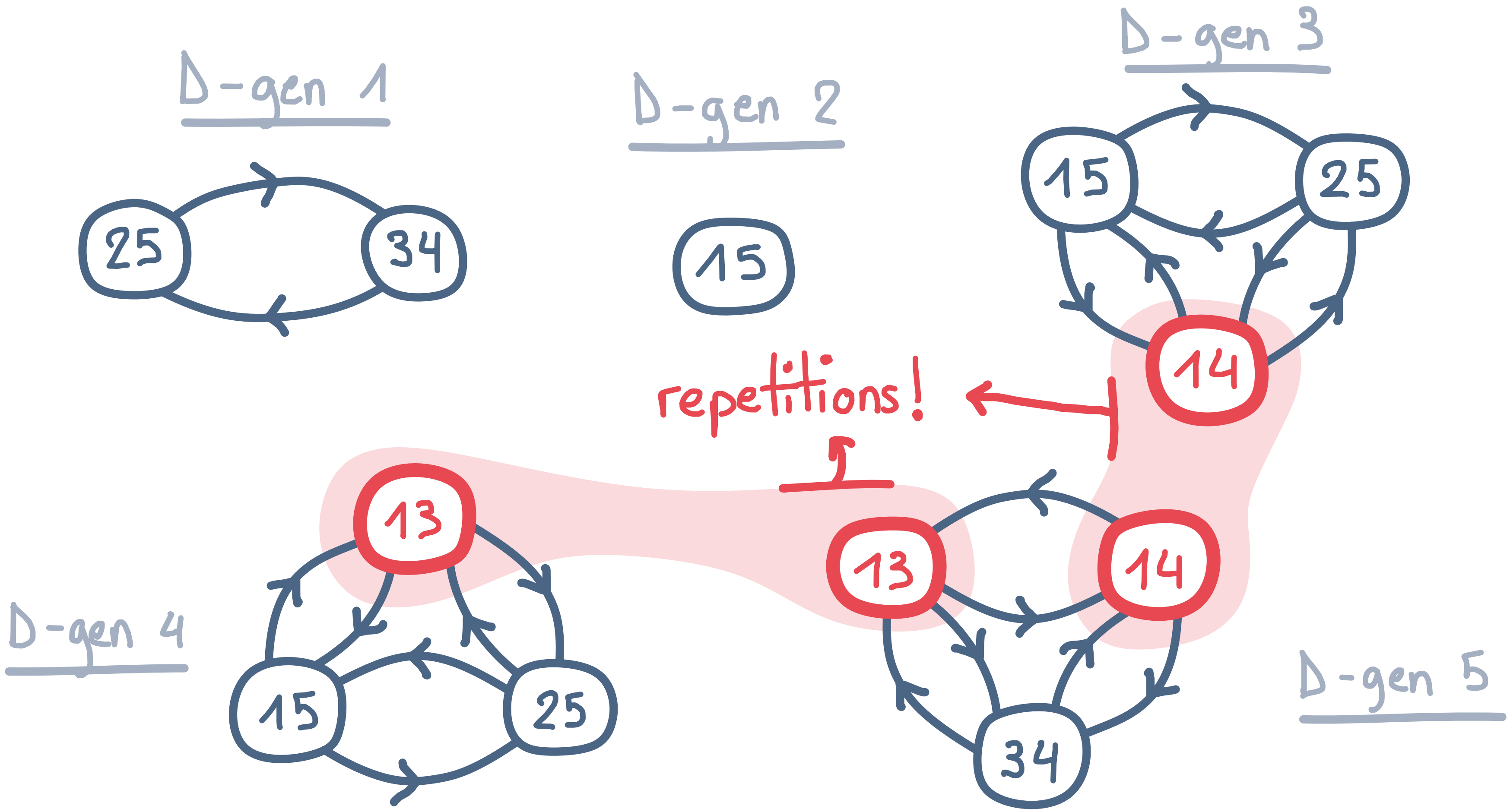
Principle: Solution-graph traversal



1st solution in poly-time + poly transitions + strongly connected

⇒ poly-delay enumeration (with DFS) of D-gen of some x

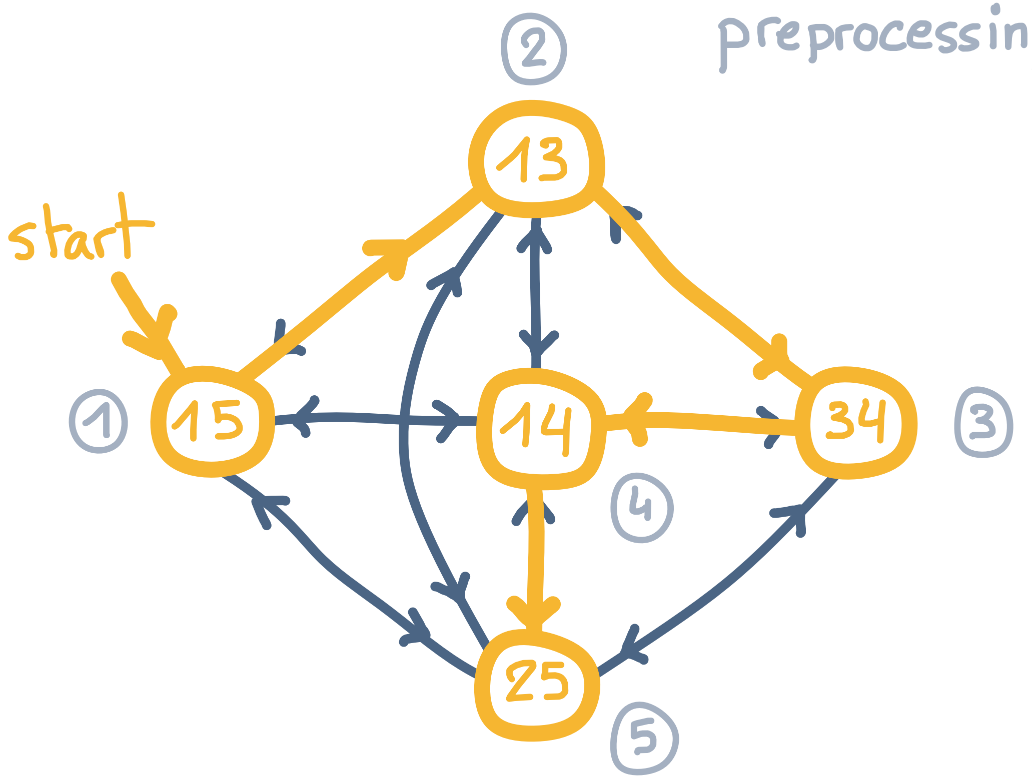
In our case (running ex)



PROB: applying algo on each $x \in X$ yields repetitions

$\Rightarrow$ no guarantee on delay

Fix: merge the graphs



preprocessing

0. 3 → 2, 4 → 2
1. 15 → 2, 15 → 3, 15 → 4
2. 13 → 2, 13 → 5
3. 34 → 5, 34 → 1
4. 14 → 3, 14 → 5
5. 25 → 3, 25 → 1, 25 → 4

FIX: take the union of supergraphs

- poly transitions
- 1st solution in poly-time $\forall x \in X$
- strongly connected components

$\Rightarrow$ poly delay enumeration of all D-gens (with DFSs)

PROB: given an implicational base Σ over X, find the D-base Σ_D

Solution graph traversal

\+ Ennaoui, Nourine 16

THM: given Σ over X, Σ_D can be computed with polynomial delay

ANV, 24+

The D-base:

- describe a lattice by minimal join covers
- ordered direct subset of canonical direct base

Finding the D-base:

- output quasi-poly from $\mathcal{M}_i$
- poly-delay from Σ

Questions regarding E-base (subset of D-base)

- Characterize systems with valid E-base
- Similar algorithms for E-base?

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2023

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