Exercise 1

TMA4300 Computer Intensive Statistical Models

Mads Adrian Simonsen, William Scott Grundeland Olsen

29 januar, 2021

Problem A: Stochastic simulation by the probability integral transform and bivariate techniques

1.

Let $X \sim \text{Exp}(\lambda)$, with the cdf

$$F_X(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

Then the random variable $Y := F_X(X)$ has a Uniform$(0,1)$ distribution. The probability integral transform becomes

$$Y = 1 - e^{-\lambda X} \Leftrightarrow X = -\frac{1}{\lambda}\log(1 - Y). \tag{1}$$

Thus, we sample Y from `runif()` and transform it using (1), to sample from the exponential distribution. Figure 1 shows one million samples drawn from the `generate_from_exp()` function defined in the code chunk below.

```
set.seed(123)

generate_from_exp <- function(n, rate = 1) {
  Y <- runif(n)
  X <- -(1 / rate) * log(1 - Y)
  X
}

# sample
n <- 1000000  # One million samples
lambda <- 4.32
x <- generate_from_exp(n, rate = lambda)

# plot
hist(x,
  breaks      = 80,
  probability = TRUE,
  xlim        = c(0, 2)
)
curve(dexp(x, rate = lambda),
  add = TRUE,
  lwd = 2,
  col = "red"
)
```

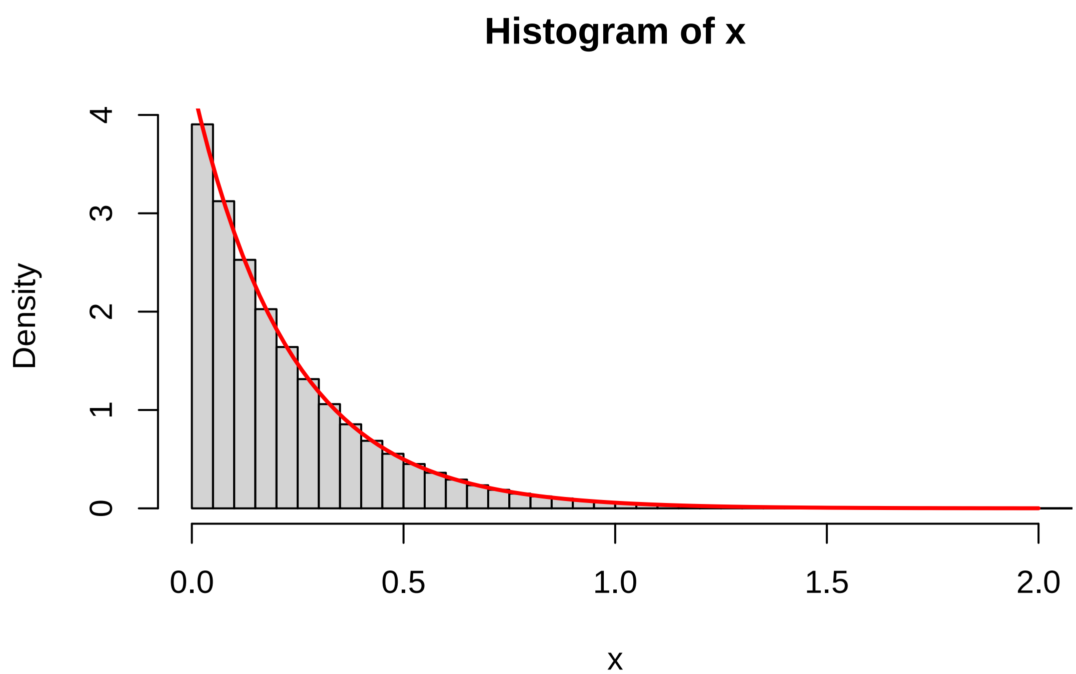

Figure 1: Normalized histogram of one million samples drawn from the exponential distribution, together with the theoretical pdf, with $\lambda = 4.32$.

2.

(a)

(b)

3.

(a)

(b)

(c)

4.

5

Problem B: The gamma distribution

1.

(a)

Let $f(x)$ be the target distribution we wish to sample from, and let $g(x)$ be the proposal distribution. For the rejection sampling algorithm, we require that

$$f(x) \leq c \cdot g(x), \quad \forall x \in \mathbb{R}, \tag{2}$$

for some constant $c > 0$. Let X and U be independent samples where $X \sim g(x)$ and $U \sim \text{Uniform}(0,1)$. Then the acceptance probability is

$$
\begin{aligned}
\Pr\left(U \le \frac{f(X)}{c \cdot g(X)}\right) &= \int_{-\infty}^{\infty} \int_0^{f(x)/(c\ g(x))} f_{X,U}(x,u)\,du\,dx \\
&= \int_{-\infty}^{\infty} \int_0^{f(x)/(c\ g(x))} g(x) \cdot 1\,du\,dx \\
&= \int_{-\infty}^{\infty} \frac{f(x)}{c\ g(x)} g(x)\,dx \\
&= \frac{1}{c} \int_{-\infty}^{\infty} f(x)\,dx \\
&= \frac{1}{c}
\end{aligned}
$$

We wish to sample from $\text{Gamma}(\alpha, \beta = 1)$, using the proposal distribution $g(x)$ given in `eqref{??????}`. We want to choose c such that the acceptance probability is maximized while (2) is satisfied. We must check three cases. The trivial case when $x \le 0$, we have $f(x) = g(x) = 0$ so (2) is satisfied for all c. When $0 < x < 1$ we have

$$
\begin{aligned}
f(x) &\le c\, g(x) \\
\frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} &\le c\, \frac{1}{\alpha^{-1} + e^{-1}} x^{\alpha-1} \\
c &\ge \left(\frac{1}{\alpha} + \frac{1}{e}\right) \frac{1}{\Gamma(\alpha)} e^{-x} \\
c &\ge \left(\frac{1}{\alpha} + \frac{1}{e}\right) \frac{1}{\Gamma(\alpha)}.
\end{aligned}
$$

The last case, when $x \ge 1$, we have

$$
\begin{aligned}
f(x) &\le c\, g(x) \\
\frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} &\le c\, \frac{1}{\alpha^{-1} + e^{-1}} e^{-x} \\
c &\ge \left(\frac{1}{\alpha} + \frac{1}{e}\right) \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \\
c &\ge \left(\frac{1}{\alpha} + \frac{1}{e}\right) \frac{1}{\Gamma(\alpha)}.
\end{aligned}
$$

That is, we choose $c := (\alpha^{-1} + e^{-1})/\Gamma(\alpha)$, such that the acceptance probability becomes

$$
\Pr\left(U \le \frac{f(X)}{c \cdot g(X)}\right) = \frac{1}{c} = \frac{\Gamma(\alpha)}{\alpha^{-1} + e^{-1}}, \quad \alpha \in (0,1).
$$

(b)

```r
set.seed(137)

generate_from_gamma <- function(n, shape = 0.5) {
  c <- gamma(shape) / (1 / shape + 1 / exp(1))   # constant that minimizes the envelope
  x <- vector(mode = "numeric", length = n)
  for (i in 1:n) {
```

```
    repeat {
      x[i] <- generate_from_piecewise(1, shape)  # draw from proposal
      u <- runif(1)                              # draw from U(0, 1)
      f <- dgamma(x[i], shape = shape)           # target value
      g <- theo_PDF(x[i], alpha = shape)         # proposal value
      alpha <- (1 / c) * (f / g)
      if (u <= alpha) {
        break
      }
    }
  }
  return(x)
}

# n <- 1000000
# alpha <- 0.9
# x <- generate_from_gamma(n, shape = alpha)
# hist(x,
#   breaks      = 80,
#   probability = TRUE,
#   xlim        = c(0, 6)
# )
# curve(dgamma(x, shape = alpha),
#   add = TRUE,
#   lwd = 2,
#   col = "red"
# )
```

2.

(a)

We will now use the ratio-of-uniforms method to simulate from Gamma$(\alpha, \beta = 1)$. Additionally we have $\alpha > 1$ this time. Let us define

$$C_f = \left\{ (x_1, x_2) : 0 \leq x_1 \leq \sqrt{f^*\left(\frac{x_2}{x_1}\right)} \right\}, \quad \text{where} \quad f^*(x) = \begin{cases} x^{\alpha-1}e^{-x}, & x > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

and

$$a = \sqrt{\sup_x f^*(x)}, \quad b_+ = \sqrt{\sup_{x\geq 0}(x^2 f^*(x))} \quad \text{and} \quad b_- = -\sqrt{\sup_{x\leq 0}(x^2 f^*(x))}, \tag{4}$$

such that $C_f \subset [0, 1] \times [b_-, b_+]$.

First we find $\sup_x f^*(x)$. This must be when $x > 0$. We differentiate $f^*(x)$ and setting the expression equal to zero to find the stationary point.

$$\begin{aligned} 0 &= \frac{d}{dx} f^*(x) \\ &= \frac{d}{dx} x^{\alpha-1} e^{-x} \\ &= e^{-x} x^{\alpha-2} \left((\alpha - 1) - x \right) \\ \Rightarrow \quad x &= \alpha - 1, \quad \text{where} \quad \alpha > 1. \end{aligned}$$

Since we have only one stationary point, $f^*(x)$ is continuous, $f^*(x) > 0 \ \forall x > 0$ and $\lim_{x\to 0+} f^*(x) = \lim_{x\to\infty} f^*(x) = 0$, then $x = \alpha - 1$ must be the global maximum point. That is

$$a = \sqrt{f^*(\alpha-1)} = \sqrt{(\alpha-1)^{\alpha-1}e^{-(\alpha-1)}} = \left(\frac{\alpha-1}{e}\right)^{(\alpha-1)/2}. \tag{5}$$

We now wish to find b_+.

$$\begin{aligned} 0 &= \frac{d}{dx}x^2 f^*(x) \\ &= \frac{d}{dx}x^{\alpha+1}e^{-x} \\ &= e^{-x}x^{\alpha}\left((\alpha+1) - x\right) \\ \Rightarrow \quad x &= \alpha + 1, \quad \text{where} \quad \alpha > 1. \end{aligned}$$

Using the same reasoning as for a, we have that $x = \alpha + 1$ is a global maximum point for $x^2 f^*(x)$. Then

$$b_+ = \sqrt{(\alpha+1)^2 f^*(\alpha+1)} = \sqrt{(\alpha+1)^{\alpha+1}e^{-(\alpha+1)}} = \left(\frac{\alpha+1}{e}\right)^{(\alpha+1)/2}. \tag{6}$$

Finally, we have that

$$b_- = -\sqrt{\sup_{x\le 0}(x^2 \cdot 0)} = 0. \tag{7}$$

(b)

To avoid producing `NaN`s, we will implement the ratio-of-uniform method on a log scale. We get the following log-transformations.

$$\begin{gathered} X_1 \sim \text{Uniform}(0, a) \Rightarrow \log X_1 = \log a + \log U_1, \quad U_1 \sim \text{Uniform}(0,1); \\ X_2 \sim \text{Uniform}(b_- = 0, b_+ = b) \Rightarrow \log X_2 = \log b + \log U_2, \quad U_2 \sim \text{Uniform}(0,1); \\ y = \frac{x_2}{x_1} \Rightarrow y = \exp\{(\log x_2) - (\log x_1)\}; \\ 0 \le x_1 \le \sqrt{f^*(y)} \Rightarrow \log x_1 \le \frac{1}{2}\log f^*(y); \\ f^*(y) = \begin{cases} y^{\alpha-1}e^{-y}, & y > 0, \\ 0, & \text{otherwise}, \end{cases} \Rightarrow \log f^*(y) = \begin{cases} (\alpha-1)\log y - y, & y > 0, \\ -\infty, & \text{otherwise}. \end{cases} \end{gathered}$$

```r
set.seed(434)

lgamma_core <- function(x, alpha = 2) {
  ifelse(
    test = x <= 0,
    yes  = -Inf,
    no   = (alpha - 1)*log(x) - x
  )
}

sample_from_gamma_rou <- function(n, alpha = 2, include_trials = FALSE) {
  log_a <- ((alpha - 1) / 2) * (log(alpha - 1) - 1)
  log_b <- ((alpha + 1) / 2) * (log(alpha + 1) - 1)
  trials <- 0
```

```r
  y <- vector(mode = "numeric", length = n)
  for (i in 1:n) {
    repeat {
      log_x1 <- log_a + log(runif(1))
      log_x2 <- log_b + log(runif(1))
      y[i] <- exp(log_x2 - log_x1)
      log_f <- lgamma_core(y[i], alpha = alpha)
      if (log_x1 <= 0.5 * log_f) {
        break
      } else {
        trials <- trials + 1
      }
    }
  }
  if (include_trials) {
    return(list(x = y, trials = trials))
  }
  return(y)
}

# generate 1000 samples for each alpha and record number of trials
n <- 1000
m <- 50
alpha <- seq(2, 2000, length.out = m)
trials <- vector(mode = "integer", length = m)

for (i in 1:m) {
  trials[i] <- sample_from_gamma_rou(n, alpha[i], include_trials = TRUE)$trials
}

# plot trials wrt. alpha
ggplot(mapping = aes(x = alpha, y = trials)) +
  geom_point()
```

Figure 2 strongly suggests that the acceptance probability decreases with increasing α. That is, the ratio of the area of the square $a \cdot (b_+ - b_-) = ab$ and the region C_f is increasing when α increases.

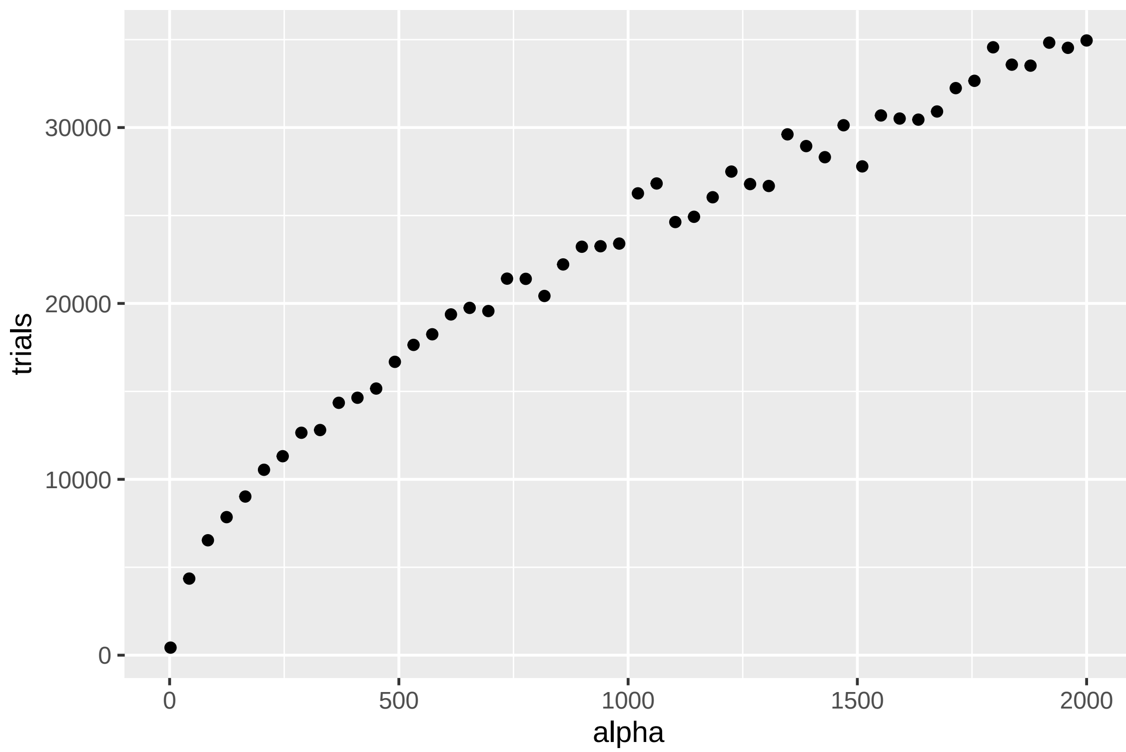

Figure 2: Number of trials before accepting $N = 1000$ simulations for various shape parameters (α) using the ratio-of-uniform method.

3.

(a)

(b)

4.

5.

(a)

(b)

Problem C: Monte Carlo integration and variance reduction

1.

2.

3.

(a)

(b)

Problem D: Rejection sampling and importance sampling

1.

2.

3.

4.