Quantum Systems

(Lecture 3: Quantum states and computation)

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The principles

Quantum computation explores the laws of quantum theory as computational resources.

Thus, the principles of the former are directly derived from the postulates of the latter.

- The state **space** postulate
- The state **evolution** postulate
- The state **composition** postulate
- The state **measurement** postulate

The underlying maths is that of Hilbert spaces.

The underlying maths: Hilbert spaces

Complex, inner-product vector space

A complex vector space with inner product which measures how much two vectors overlap:

$$\langle - | - \rangle : H \times H \longrightarrow \mathbb{C}$$

such that

$$\begin{array}{ll} (1) & \langle v | \sum_i \lambda_i \cdot |w_i\rangle\rangle \; = \; \sum_i \lambda_i \langle v | w_i \rangle \\ (2) & \langle v | w \rangle = \overline{\langle w | v \rangle} \\ (3) & \langle v | v \rangle \geq 0 \;\; (\text{with equality iff } |v\rangle = 0) \end{array}$$

Note: $\langle - | - \rangle$ is conjugate linear in the first argument:

$$\langle \sum_i \lambda_i \cdot |w_i\rangle | v \rangle \; = \; \sum_i \overline{\lambda_i} \langle w_i | v \rangle$$

Notation: $\langle v | w \rangle \; \equiv \; (|v\rangle, |w\rangle)$

Dirac's notation

Dirac's bra/ket notation is a handy way to represent elements and constructions on an Hilbert space

$|u\rangle$ A ket stands for a vector in an Hilbert space H. In $\mathcal{C}^n$, it is a column vector of complex entries. Note that the identity for $+$ (the zero vector) is just written 0.

$\langle u|$ A bra is a vector in the dual space H^*, i.e. scalar-valued linear maps in H. In $(\mathcal{C}^n)^*$ it is the adjoint, i.e. the conjugate transpose, of the corresponding ket, therefore a row vector.

There is a bijective correspondence between $|u\rangle$ and $\langle u|$

$$|u\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} \overline{u}_1 \cdots \overline{u}_n \end{bmatrix} = \langle u|$$

Inner product: examples

In $\mathcal{C}$

$$\langle a+bi|c+di\rangle = (a-bi)(c+di) = ac+adi-bci+bd$$

In $\mathcal{C}^n$: The dot product

Amost useful example of a inner product is the dot product

$$\langle u|v\rangle = \underbrace{\begin{bmatrix}\overline{u_1} & \overline{u_2} & \cdots & \overline{u_n}\end{bmatrix}}_{\langle u|} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} \overline{u_i} v_i$$

where $\overline{c} = a - ib$ is the complex conjugate of $c = a + ib$

.

Old friends: The dual space

H^*

If H is a Hilbert space, H^* is the space of linear maps from H to $\mathcal{C}$.

Elements of H^* are denoted by

$$\langle u| : H \longrightarrow \mathcal{C} \text{ and defined as } \langle u|(|v\rangle) = \langle u|v\rangle$$

In a matricial representation $\langle u|$ is obtained as the Hermitian conjugate (i.e. the transpose of the vector composed by the complex conjugate of each element) of $|u\rangle$, therefore the dot product of $|u\rangle$ and $|v\rangle$.

Old friends: Norms and orthogonality

- The inner product measures the *degree of overlapping*: $|v\rangle$ and $|w\rangle$ are orthogonal if $\langle v|w\rangle = 0$
- The "length" of a vector uses the measure of its overlap with itself to yield the (Euclidean) norm:

$$\| \, |v\rangle \, \| = \sqrt{\langle v|v\rangle}$$

  (generalizing the distance between two points)
- $|v\rangle$ is a unit vector if $\| \, |v\rangle \, \| = 1$
- normalization: $\frac{|v\rangle}{\||v\rangle\|}$
- A set of vectors $\{|i\rangle, |j\rangle, \cdots\}$ is orthonormal if each $|i\rangle$ is a unit vector and

$$\langle i|j\rangle \;=\; \delta_{i,j} \;=\; \begin{cases} i = j & \Rightarrow 1 \\ \text{otherwise} & \Rightarrow 0 \end{cases}$$

Old friends: Bases

Orthonormal basis

A orthonormal basis for a Hilbert space H of dimension n is a set $B = \{|i\rangle \mid i \in n-1\}$ of n linearly independent elements of H st

- $\langle i|j\rangle = \delta_{i,j}$ for all $|i\rangle, |j\rangle \in B$
- and B spans H, i.e. every $|v\rangle$ in H can be written as

$$|v\rangle \;=\; \sum_i \alpha_i |i\rangle \quad \text{for some } \alpha_i \in \mathcal{C}$$

Note that the amplitude or coefficient of $|v\rangle$ wrt $|i\rangle$ satisfies

$$\alpha_i \;=\; \langle i|v\rangle$$

Why?

Bases

$\alpha_i \;=\; \langle i|v\rangle$ because

$$\begin{aligned}\langle i|v\rangle &= \langle i|\sum_j \alpha_j j\rangle \\ &= \sum_j \alpha_j \langle i|j\rangle \\ &= \sum_j \alpha_j \delta_{i,j} \\ &= \alpha_i\end{aligned}$$

Note

If $|v\rangle$ is expressed wrt an orthonormal basis $\{|i\rangle \mid i \in n\}$, i.e. $|v\rangle = \sum_i \alpha_i |i\rangle$, then

$$\| \, |v\rangle \, \| = \sum_i \| \alpha_i \|^2$$

Example: The Hadamard basis

One of the infinitely many orthonormal bases for a space of dimension 2:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Check, e. g.

$$\langle +|-\rangle \;=\; \frac{1}{2}(|0\rangle + |1\rangle, |0\rangle - |1\rangle) \;=\; \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right) \;=\; \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} \;=\; 0$$

$$\|\,|+\rangle\,\| \;=\; \sqrt{\langle +|+\rangle} \;=\; \sqrt{\frac{1}{2}(|0\rangle + |1\rangle, |0\rangle + |1\rangle)} \;=\; \sqrt{\frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)} \;=\; 1$$

Bases

A basis for H^*

If $\{|i\rangle \mid i \in n\}$ is an orthonormal basis for H, then

$$\{\langle i| \mid i \in n\}$$

is an orthonormal basis for H^*.

Hilbert spaces

The complete picture

An Hilbert space is an inner-product space H st the metric defined by its norm turns H into a complete metric space, i.e.any Cauchy sequence

$$|v_1\rangle, |v_2\rangle, \cdots$$

$$\forall_{\epsilon>0}\ \exists_N\ \forall_{m,n>N} \quad \|\,|v_m - v_n\rangle\,\| \leq \epsilon$$

converges
(i.e. there exists an element $|s\rangle$ in H st $\forall_{\epsilon>0}\ \exists_N\ \forall_{n>N} \quad \|\,|s - v_n\rangle\,\| \leq \epsilon$)

The completeness condition is trivial in finite dimensional vector spaces

The state space postulate

Postulate 1

The state space of a quantum system is described by a unit vector in a Hilbert space

- In practice, with finite resources, one cannot distinguish between a continuous state space from a discrete one with arbitrarily small minimum spacing between adjacente locations.
- One may, then, restrict to finite-dimensional (complex) Hilbert spaces.

The state space postulate

A quantum (binary) state is represented as a superposition, i.e. a linear combination of vectors $|0\rangle$ and $|1\rangle$ with complex coeficients:

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

When state $|\phi\rangle$ is measured (i.e. observed) one of the two basic states $|0\rangle, |1\rangle$ is returned with probability

$$\| \alpha \|^2 \quad \text{and} \quad \| \beta \|^2$$

respectively.
Being probabilities, the norm squared of coefficients must satisfy

$$\| \alpha \|^2 + \| \beta \|^2 = 1$$

which enforces quantum states to be represented by unit vectors.

The state space of a qubit

Global phase

Unit vectors equivalent up to multiplication by a complex number of modulus one, i.e. a phase factor $e^{i\theta}$, represent the same state.

Let

$$|v\rangle = \alpha|u\rangle + \beta|u'\rangle$$

$$\| e^{i\theta}\alpha \|^2 = (\overline{e^{i\theta}\alpha})(e^{i\theta}\alpha) = (e^{-i\theta}\overline{\alpha})(e^{i\theta}\alpha) = \overline{\alpha}\alpha = \| \alpha \|^2$$

and similarly for β.

As the probabilities $\| \alpha \|^2$ and $\| \beta \|^2$ are the only measurable quantities, global phase has no physical meaning.

Representation redundancy

qubit state space $\neq$ complex vector space used for representation

The state space of a qubit

Relative phase

It is a measure of the angle between the two complex numbers.
Thus, it cannot be discarded!

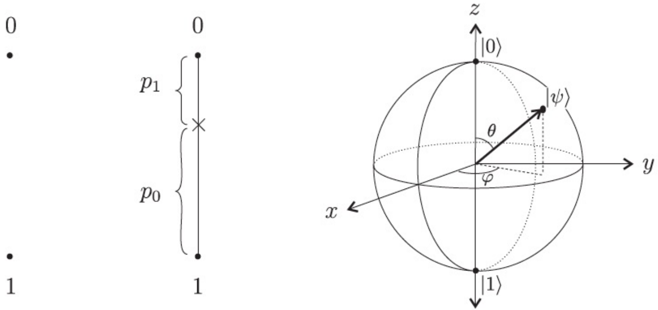
Those are different states

$$\frac{1}{\sqrt{2}}(|u\rangle + |u'\rangle) \quad \frac{1}{\sqrt{2}}(|u\rangle - |u'\rangle) \quad \frac{1}{\sqrt{2}}(e^{i\theta}|u\rangle + |u'\rangle)$$

...

The Bloch sphere

Deterministic, probabilistic and quantum bits



(from [Kaeys *et al*, 2007])

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Express $|\psi\rangle$ in polar form

$$|\psi\rangle = \rho_1 e^{i\varphi_1}|0\rangle + \rho_2 e^{i\varphi_2}|1\rangle$$

- Eliminate one of the four real parameters multiplying by $e^{-i\varphi_1}$

$$|\psi\rangle = \rho_1|0\rangle + \rho_2 e^{i(\varphi_2-\varphi_1)}|1\rangle = \rho_1|0\rangle + \rho_2 e^{i\varphi}|1\rangle$$

making $\varphi = \varphi_2 - \varphi_1$,

which is possible because global phase factors are physically meaningless.

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Switching back the coefficient of $|1\rangle$ to Cartesian coordinates

$$|\psi\rangle = \rho_1|0\rangle + (a + bi)|1\rangle$$

the normalization constraint

$$\|\rho_1\|^2 + \|a+ib\|^2 = \|\rho_1\|^2 + (a-ib)(a+ib) = \boxed{\|\rho_1\|^2 + a^2 + b^2 = 1}$$

yields the equation of a unit sphere in the real tridimensional space with Cartesian coordinates: (a, b, ρ_1).

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- The polar coordinates (ρ, θ, φ) of a point in the surface of a sphere relate to Cartesian ones through the correspondence

$$
\begin{aligned}
x =& \rho \sin\theta \cos\varphi \\
y =& \rho \sin\theta \sin\varphi \\
z =& \rho \cos\theta
\end{aligned}
$$

- Recalling $\rho = 1$ (cf unit vector),

$$
\begin{aligned}
|\psi\rangle &= \rho_1|0\rangle + (a + ib)|1\rangle \\
&= \cos\theta|0\rangle + \sin\theta(\cos\varphi + i\sin\varphi)|1\rangle \\
&= \cos\theta|0\rangle + e^{i\varphi}\sin\theta|1\rangle
\end{aligned}
$$

which, with two parameters, defines a point in the sphere's surface.

The Bloch sphere

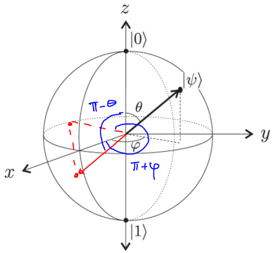
Actually, one may just focus on the upper hemisphere ($0 \leq \theta' \leq \frac{\pi}{2}$) as opposite points in the lower one differ only by a phase factor of -1, as suggested by

$$\begin{aligned}
\theta' = 0 \quad &\Rightarrow \quad |\psi\rangle \; = \; \cos 0|0\rangle + e^{i\varphi} \sin 0|1\rangle \; = \; |0\rangle \\
\theta' = \frac{\pi}{2} \quad &\Rightarrow \quad |\psi\rangle \; = \; \cos \frac{\pi}{2}|0\rangle + e^{i\varphi} \sin \frac{\pi}{2}|1\rangle \; = \; e^{i\varphi}|1\rangle \; = \; |1\rangle
\end{aligned}$$

Note that longitude (φ) is irrelevant in a pole!

The Bloch sphere

Indeed, let $|\psi'\rangle$ be the opposite point on the sphere with polar coordinates $(1, \pi - \theta, \varphi + \pi)$:



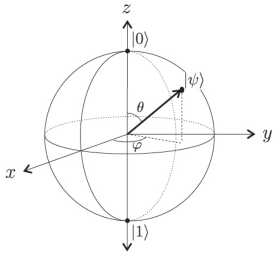
$$
\begin{aligned}
|\psi'\rangle &= \cos(\pi - \theta)|0\rangle + e^{i(\varphi+\pi)} \sin(\pi - \theta)|1\rangle \\
&= -\cos\theta|0\rangle + e^{i\varphi} e^{i\pi} \sin\theta|1\rangle \\
&= -\cos\theta|0\rangle + e^{i\varphi} \sin\theta|1\rangle \\
&= -|\psi\rangle
\end{aligned}
$$

The Bloch sphere

which leads to

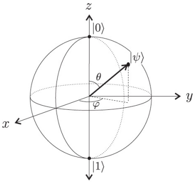
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$



The map $\frac{\theta}{2} \mapsto \theta$ is one-to-one at any point but at $\frac{\theta}{2}$:
all points on the equator are mapped into a single point: the south pole.

The Bloch sphere



- The poles represent the classical bits. In general, orthogonal states correspond to antipodal points and every diameter to a basis for the single-qubit state space.
- Once measured a qubit collapses to one of the two poles. Which pole depends exactly on the arrow direction: The angle θ measures that probability: If the arrow points at the equator, there is 50-50 chance to collapse to any of the two poles.
- Rotating a vector wrt the z-axis results into a phase change (φ), and does not affect which state the arrow will collapse to, when measured.

The state evolution postulate

If a quantum state is a ray (i.e. a unit vector in a Hilbert space H up to a global phase), its evolution is specified as a certain kind of linear maps $U : H \longrightarrow H$.

Linearity

$$U\left(\sum_j \alpha_j \,|v_j\rangle\right) = \sum_j \alpha_j \, U(|v_j\rangle)$$

just by itself has an important consequence:

quantum states cannot be cloned

The no-cloning theorem

Linearity implies that quantum states cannot be cloned

Let $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$ be a 2-qubit operator and $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ for $|a\rangle$, $|b\rangle$ orthogonal. Then,

$$
\begin{aligned}
U(|c\rangle|0\rangle) &= \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle)) \\
&= \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle) \\
&\neq \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle) \\
&= |c\rangle|c\rangle \\
&= U(|c\rangle|0\rangle)
\end{aligned}
$$

(Recall: $|x\rangle|y\rangle = |xy\rangle = |x\rangle \otimes |y\rangle$)

But, linearity is not enough ...

... we need to enforce that the norm squared of the new amplitudes still represent a probability distribution

$$\text{If } \sum_j \alpha_j \, U(|v_j\rangle) = \sum_j \alpha'_j \, |v_j\rangle \text{ then } \sum_j \| \alpha'_j \|^2 = 1$$

This is achieved by making U unitary, i.e. such that

$$U^{-1} = U^{\dagger}$$

What is $U^\dagger$? The adjoint map

Given a linear map $U : H \longrightarrow H'$, its adjoint $U^\dagger : H' \longrightarrow H$ is the unique linear map such that

$$\boxed{\langle U^\dagger a|b\rangle \;=\; \langle a|Ub\rangle}$$

Note that $(UV)^\dagger = V^\dagger U^\dagger$ and $U^{\dagger^\dagger} = U$ because

$$\langle V^\dagger U^\dagger a|b\rangle \;=\; \langle U^\dagger a|Vb\rangle \;=\; \langle a|UVb\rangle$$

and

$$\langle U^{\dagger^\dagger} a|b\rangle \;=\; \langle a|U^\dagger b\rangle$$

The state evolution postulate

Postulate 2

The evolution over time of the state of a closed quantum system is described by a unitary map.

The evolution is linear

$$U\left(\sum_j \alpha_j |v_j\rangle\right) = \sum_j \alpha_j U(|v_j\rangle)$$

and preserves the normalization constraint

$$\text{If } \sum_j \alpha_j U(|v_j\rangle) = \sum_j \alpha'_j |v_j\rangle \text{ then } \sum_j \| \alpha'_j \|^2 = 1$$

The state evolution postulate

Preservation of the normalization constraint means that unit length vectors (and thus orthogonal subspaces) are mapped by U to unit length vectors (and thus to orthogonal subspaces).

This entails a condition on valid quantum operators: they must preserve the inner product, i.e.

$$\langle Ua|Ub\rangle \;=\; \langle a|U^\dagger Ub\rangle \;=\; \langle v|w\rangle$$

which is only the case iff U is unitary, i.e. $U^\dagger$ is the inverse of U:

$$U^\dagger U = \; UU^\dagger \;=\; I$$

Unitary maps

- Preserving the inner product means that a unitary operator maps orthonormal bases to orthonormal bases.
- Conversely, any operator with this property is unitary.
- If given in matrix form, being unitary means that the set of columns of its matrix representation are orthonormal (because the jth column is the image of $U|j\rangle$). Equivalently, rows are orthonormal (why?)

Unitary maps

Unitarity is the only constraint on quantum operators: Any unitary matrix specifies a valid quantum operator.

This means that there are many non-trivial operators on a single qubit (in contrast with the classical case where the only non-trivial operation on a bit is complement).

Finally, because the inverse of a unitary matrix is also a unitary matrix, a quantum operator can always be inverted by another quantum operator

$\boxed{\text{Unitary transformations are reversible}}$

Representing linear maps

A linear map $U : H \longrightarrow H'$ is fully characterized by specifying how it acts on a basis of H. If H is finite this leads to a natural representation of U as matrix.

Let $\{|j\rangle \mid j \in n-1\}$ be a basis for a n-dimensional Hilbert space H, and similarly $\{|i\rangle \mid i \in m-1\}$ for a m-dimensional H' . Then the $m \times n$ matrix corresponding to U is defined as

$$\begin{bmatrix} U|0\rangle & U|1\rangle & \cdots & U|n-1\rangle \end{bmatrix}$$

i.e. its j^{th}-column corresponds to m-dimensional vector $U|j\rangle$.

The Dirac notations provides a handy, alternative description of matrices via outer products.

Representing linear maps

Outer product

... is computed straightforwardly by matrix multiplication, e.g.

$$|0\rangle\langle 0| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$
$$|1\rangle\langle 0| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

In general, for vectors $|i\rangle, |j\rangle$ in an orthonormal basis, $|i\rangle\langle j|$ is a square matrix with 1 in position (i, j) and 0 elsewhere. As an operator, $|i\rangle\langle j|$ maps $|j\rangle$ into $|i\rangle$ because

$$|i\rangle\langle j|\,|j\rangle = |i\rangle\langle j|j\rangle = |i\rangle$$

A linear map $U : H \longrightarrow H'$ can be represented as a matrix

$$\sum_{i\in m-1, j\in n-1} U_{i,j}\,|i\rangle\langle j|$$

Representing linear maps

Decomposition of the identity (for an orthonormal basis)

$$I_H = \sum_{i \in n-1} |i\rangle\langle i|$$

Thus,

$$\begin{aligned} U &= I_{H'} U I_H = \sum_{i \in m-1} |i\rangle\langle i| \; U \sum_{j \in n-1} |j\rangle\langle j| \\ &= \sum_{i \in m-1, j \in n-1} |i\rangle\langle i| \, U \, |j\rangle\langle j| \\ &= \sum_{i \in m-1, j \in n-1} \langle i|U|j\rangle \; |i\rangle\langle j| \end{aligned}$$

Clearly,

$$U_{i,j} = \langle i|U|j\rangle$$

Representing linear maps

because

$$\begin{aligned}
\langle i|U|j\rangle &= \langle i| \left(\sum_{i'\in m-1, j'\in n-1} U_{i',j'} \, |i'\rangle\langle j'| \right) |j\rangle \\
&= \sum_{i'\in m-1, j'\in n-1} U_{i',j'} \, \langle i|i'\rangle\langle j|j'\rangle \\
&= \sum_{i'\in m-1, j'\in n-1} U_{i',j'} \, \delta_{ii'}\delta_{jj'} \; = \; U_{i,j}
\end{aligned}$$

Representing linear maps

Any orthonormal provides a decomposition of the identity.

Is there a standard way to provide a decomposition for an arbitrary operator U over a Hilbert H?

Yes, if U is normal operator, i.e. $UU^\dagger = U^\dagger U$, because of the

Spectral theorem

Any normal operator on a finite, n-dimensional Hilbert space H provides a basis for H consisting of its eigenvectors. Thus,

$$U = \sum_{i \in n-1} \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

where each $(\lambda_i, |\lambda_i\rangle)$ is a eigenvalue / eigenvector pair.

Typical quantum gates on 1 qubit

The $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$ gate

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

As $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$, its spectral decomposition yields

$$X = |+\rangle\langle +| - |-\rangle\langle -|$$

Typical quantum gates on 1 qubit

Acts as

$$Z|0\rangle = |0\rangle \text{ and } Z|1\rangle = -|1\rangle$$

i.e. leaves $|0\rangle$ invariant, but injects a phase $e^{i\pi} = -1$ to $|0\rangle$, corresponding to a rotation of π radians around the Z axis.

Clearly, its spectral decomposition yields:

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Typical quantum gates on 1 qubit

The phase shift gate

$$P_\phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

i.e. $P_\phi |0\rangle = |0\rangle$ and $P_\phi |1\rangle = e^{i\phi}|1\rangle$.

The probability of measuring a $|0\rangle$ or $|1\rangle$ remains unchanged, but it modifies the phase of the quantum state.

This corresponds to a rotation of ϕ radians around the Z axis (i.e. along a line of latitude on the Bloch sphere) by ϕ radians.

Typical quantum gates on 1 qubit

Examples

- $Z = P_\pi$
- $S = P_{\frac{\pi}{2}} = \sqrt{Z} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
- $T = P_{\frac{\pi}{4}} = \sqrt{S}$ (also called the $\frac{\pi}{8}$ gate)

$$T \; = \; P_{\frac{\pi}{4}} \; = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

which, up to a global phase factor $e^{i\frac{\pi}{8}}$, is equivalent to

$$\begin{bmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$$

Typical quantum gates on 1 qubit

Pauli gates

X, Y, Z specify a rotation by π radians around the corresponding axes on the Bloch sphere.

$$
\begin{aligned}
I &= |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
X &= |1\rangle\langle 0| + |0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
Z &= |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
Y &= i(-|1\rangle\langle 0| + |0\rangle\langle 1|) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}
\end{aligned}
$$

Typical quantum gates on 1 qubit

Rotation gates

Correspond to arbitrary rotations around the three axes of the Bloch sphere

$$R_e(\theta) \,\widehat{=}\, e^{\frac{-i\theta E}{2}} = \cos\left(\frac{\theta}{2}\right) I - i \sin\frac{\theta}{2} E$$

where $e \,\widehat{=}\, x, y, z$ and $E \,\widehat{=}\, X, Y, Z$.

because, for any real number θ and matrix R st $R^2 = I$, which is the case for X, Y, and Z,

$$e^{i\theta R} = cos(\theta) I + i \sin(\theta) R$$

Typical quantum gates on 1 qubit

Rotation gates as matrices in the computational basis

$$R_x(\theta) = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix}$$

Typical quantum gates on 1 qubit

Compute $R_z(\theta)|\psi\rangle$ for $|\psi\rangle = \cos\left(\frac{\sigma}{2}\right)|0\rangle + e^{i\gamma}\sin\left(\frac{\sigma}{2}\right)|1\rangle$

$$\begin{aligned}
\begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\sigma}{2}\right) \\ e^{i\gamma}\sin\left(\frac{\sigma}{2}\right) \end{bmatrix} &= \begin{bmatrix} e^{-i\frac{\theta}{2}}\cos\left(\frac{\sigma}{2}\right) \\ e^{i\frac{\theta}{2}} e^{i\gamma}\sin\left(\frac{\sigma}{2}\right) \end{bmatrix} \\
&= e^{-i\frac{\theta}{2}} \begin{bmatrix} \cos\left(\frac{\sigma}{2}\right) \\ e^{i\theta} e^{i\gamma}\sin\left(\frac{\sigma}{2}\right) \end{bmatrix} \\
&= e^{-i\frac{\theta}{2}} \left(\cos\left(\frac{\sigma}{2}\right)|0\rangle + e^{i(\gamma+\theta)}\sin\left(\frac{\sigma}{2}\right)|1\rangle\right)
\end{aligned}$$

As global phase is insignificant, the angle mapping $\gamma \mapsto \gamma + \theta$ is a rotation of θ around the z-axis of the Bloch sphere.

Typical quantum gates on 1 qubit

Theorem

Let U be a 1-gate, and v, w any two non-parallel axes of the Bloch sphere. Then there exist real numbers $\alpha, \beta\, \gamma, \delta$ st

$$U = e^{i\alpha} R_v(\beta) R_w(\gamma) R_v(\delta)$$

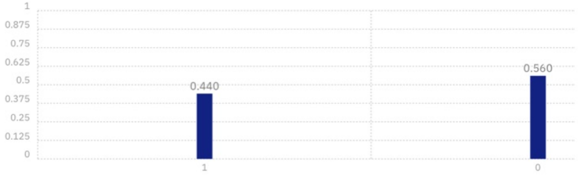
which means that any 1-gate can be expressed as a sequence of two rotations about an axis and one rotation about another non parallel axis, multiplied by a suitable phase factor.

proof hint: Recall U is unitary and unfold the definition of rotation gate.

Typical quantum gates on 1 qubit

The Hadamard gate creates superpositions

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



$$H|0\rangle = |+\rangle = \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{\text{superposition}}$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Building larger states from smaller

Operator U in the no-cloning theorem acts on a 2-dimensional state, i.e. over the composition of two qubits.

$\boxed{\text{What does composition mean?}}$

Postulate 3

The state space of a combined quantum system is the tensor product $V \otimes W$ of the state spaces V and W of its components.

Composing quantum states

State spaces in a quantum system combine through tensor: $\otimes$

$$\boxed{n \ m\text{-dimensional vectors} \quad \rightsquigarrow \quad \text{a vector in } m^n\text{-dimensional space}}$$

i.e. the state space of a quantum system grows exponentially with the number of particles: cf, Feyman's original motivation

Example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \otimes \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ b \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ c \begin{bmatrix} d \\ e \\ f \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ad \\ ae \\ af \\ bd \\ be \\ bf \\ cd \\ ce \\ cf \end{bmatrix}$$

Composing quantum states

Tensor $V \otimes W$

- $B_{V\otimes W}$ is a set of elements of the form $|v_i\rangle \otimes |w_j\rangle$, for each $|v_i\rangle \in B_V$, $|w_i\rangle \in B_W$ and $\dim(\mathrm{V} \otimes \mathrm{W}) = \dim(\mathrm{V}) \times \dim(\mathrm{W})$
- $(|u_1\rangle + |u_2\rangle) \otimes |z\rangle \;=\; |u_1\rangle \otimes |z\rangle + |u_2\rangle \otimes |z\rangle$
- $|z\rangle \otimes (|u_1\rangle + |u_2\rangle) \;=\; |z\rangle \otimes |u_1\rangle + |z\rangle \otimes |u_2\rangle$
- $(\alpha|u\rangle) \otimes |z\rangle \;=\; |u\rangle \otimes (\alpha|z\rangle) \;=\; \alpha(|u\rangle \otimes |z\rangle)$
- $\langle(|u_2\rangle \otimes |z_2\rangle)|(|u_1\rangle \otimes |z_1\rangle)\rangle \;=\; \langle u_2|u_1\rangle\langle z_2|z_1\rangle$

Composing quantum states

Clearly, every element of $V \otimes W$ can be written as

$$\alpha_1(|v_1\rangle \otimes |w_1\rangle) + \alpha_2(|v_2\rangle \otimes |w_1\rangle) + \cdots + \alpha_{nm}(|v_n\rangle \otimes |w_m\rangle)$$

Example

The basis of $V \otimes W$, for V, W qubits with the computational basis is

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

Thus, the tensor of $\alpha_1|0\rangle + \alpha_2|1\rangle$ and $\beta_1|0\rangle + \beta_2|1\rangle$ is

$$\alpha_1\beta_1|0\rangle \otimes |0\rangle \; + \; \alpha_1\beta_2|0\rangle \otimes |1\rangle \; + \; \alpha_2\beta_1|1\rangle \otimes |0\rangle \; + \; \alpha_2\beta_2|1\rangle \otimes |1\rangle$$

i.e., in a simplified notation,

$$\alpha_1\beta_1|00\rangle \; + \; \alpha_1\beta_2|01\rangle \; + \; \alpha_2\beta_1|10\rangle \; + \; \alpha_2\beta_2|11\rangle$$

Bases

The computational basis for a vector space

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$$

corresponding to the composition of n qubits (each living in V) is the set

$$\{\underbrace{|0\rangle \cdots |0\rangle|0\rangle}_{n},\ \underbrace{|0\rangle \cdots |0\rangle|1\rangle}_{n},\ \underbrace{|0\rangle \cdots |1\rangle|0\rangle}_{n},\ \cdots\ \underbrace{|1\rangle \cdots |1\rangle|1\rangle}_{n}\}$$

$$\stackrel{abv}{=}$$

$$\{\underbrace{|0\cdots 00\rangle}_{n},\ \underbrace{|0\cdots 01\rangle}_{n},\ \underbrace{|0\cdots 10\rangle}_{n},\ \cdots\ \underbrace{|1\cdots 11\rangle}_{n}\}$$

which may be written in a compressed (decimal) way as

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \cdots |2^n - 1\rangle\}$$

Bases

The computational basis for a two qubit system would be

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

with

$$|0\rangle = |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |1\rangle = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |2\rangle = |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |3\rangle = |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Bases

There are of course other bases ... besides the standard one, e.g.

The Bell basis

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

Compare with the Hadamard basis for the single qubit systems

Representing multi-qubit states

Any unit vector in a 2^n Hilbert space represents a possible n-qubit state, but for

... a certain level of redundancy

- As before, vectors that differ only in a global phase represent the same quantum state
- but also the same phase factor in different qubits of a tensor product represent the same state:

$$|u\rangle \otimes (e^{i\Phi}|z\rangle) \; = \; e^{i\Phi}(|u\rangle \otimes |z\rangle) \; = \; (e^{i\Phi}|u\rangle) \otimes |z\rangle$$

Actually, phase factors in qubits of a single term of a superposition can always be factored out into a coefficient for that term, i.e. phase factors distribute over tensors

Representing multi-qubit states

Representation

- Relative phases still matter (of course!)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{differs from} \quad \frac{1}{\sqrt{2}}(e^{i\Phi}|00\rangle + |11\rangle)$$

even if

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(e^{i\Phi}|00\rangle + e^{i\Phi}|11\rangle) = \frac{e^{i\Phi}}{\sqrt{2}}(|00\rangle + |11\rangle$$

- The complex projective space of dimension 1 (depicted in the Block sphere) generalises to higher dimensions, although in practice linearity makes Hilbert spaces easier to use.

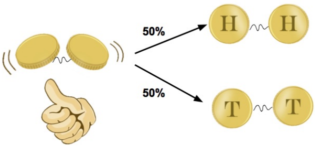
Entanglement

Most states in $V \otimes W$ cannot be written as $|u\rangle \otimes |z\rangle$

For example, the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ = \ \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

is entangled



Entanglement

Actually, to make $|\Phi^+\rangle$ equal to

$$(\alpha_1|0\rangle+\beta_1|1\rangle)\otimes(\alpha_2|0\rangle+\beta_2|1\rangle) \;=\; \alpha_1\alpha_2|00\rangle+\alpha_1\beta_2|01\rangle+\beta_1\alpha_2|10\rangle+\beta_1\beta_2|11\rangle$$

would require that $\alpha_1\beta_2 = \beta_1\alpha_2 = 0$ which implies that either

$$\alpha_1\alpha_2 = 0 \;\text{ or }\; \beta_1\beta_2 = 0$$

Note
Entanglement can also be observed in simpler structures, e.g. relations:

$$\{(a, a), (b, b)\} \subseteq A \times A$$

cannot be separated, i.e. written as a Cartesian product of subsets of A.

2-gates: *CNOT*

Acts on the standard basis for a 2-qubit system, flipping the second bit if the first bit is 1 and leaving it unchanged otherwise.

$$\begin{aligned} CNOT &= |0\rangle\langle 0| \otimes I \; + \; |1\rangle\langle 1| \otimes X \\ &= |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \; + \; |1\rangle\langle 1| \otimes (|1\rangle\langle 0| + |0\rangle\langle 1|) \\ &= |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

CNOT is unitary and is its own inverse, and cannot be decomposed into a tensor product of two 1-qubit transformations

2-gates: *CNOT*

The importance of *CNOT* is its ability to change the entanglement between two qubits, e.g.

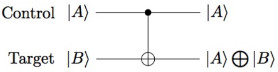
$$\begin{aligned} CNOT\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\right) &= CNOT\left(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right) \\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \end{aligned}$$

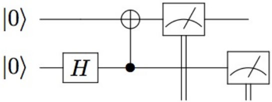
Being its own inverse, also takes an entangled state to an unentangled one.

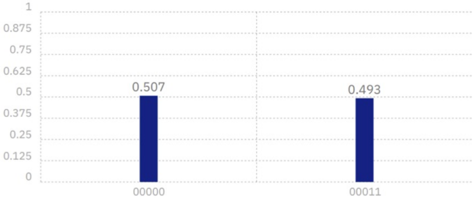
Note that entanglement is not a local property in the sense that transformations that act separately on two or more subsystems cannot affect the entanglement between those subsystems:

$(U \otimes V)\,|v\rangle$ is entangled iff $|v\rangle$ is

2-gates: *CNOT*


Control $|A\rangle$
$|A\rangle$
Target $|B\rangle$
$|A\rangle \oplus |B\rangle$


$|0\rangle$
$|0\rangle$
H

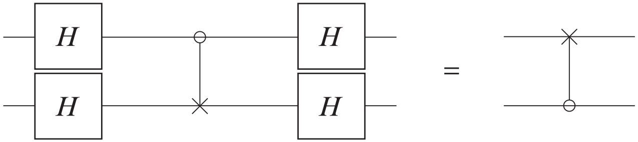

1
0.875
0.75
0.625
0.5
0.375
0.25
0.125
0
0.507
0.493
00000
00011

2-gates: *CNOT*

The notions of control/target bit in *CNOT* are arbitrary: they depend on what basis is considered. The standard behaviour is obtained in the computational basis. However, roles are interchanged in the Hadamard basis in which the effect of *CNOT* is

$$|++\rangle \mapsto |++\rangle \quad |+-\rangle \mapsto |--\rangle \quad |-+\rangle \mapsto |-+\rangle \quad |--\rangle \mapsto |+-\rangle$$

Exercise



The proof

$$\color{red}{LHS} = \frac{1}{2}\begin{bmatrix} H & H \\ H & -H \end{bmatrix} \overbrace{\color{red}{\begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}}}^{\color{red}{CNOT}} \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} H & HX \\ H & -HX \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} I + HXH & I - HXH \\ I - HXH & I + HXH \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I + Z & I - Z \\ I - Z & I + Z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

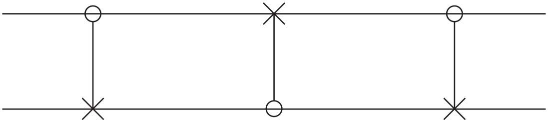
$$= I \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1| = \color{red}{RHS}$$

noting that

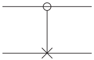
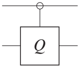
$$H \otimes H = (I \otimes H)(H \otimes I) = \frac{1}{\sqrt{2}}\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

Exercise

Discuss



Controlled Q-gates

From  to 

$$
\begin{aligned}
C_Q|0\rangle|\varphi\rangle &= |0\rangle|\varphi\rangle \\
C_Q|1\rangle|\varphi\rangle &= |1\rangle Q|\varphi\rangle
\end{aligned}
$$

$$C_Q = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Q$$

corresponding to the following matrix in the standard basis:

$$C_Q = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$$

Controlled phase shift gate

$$C_{e^{i\theta}} = |00\rangle\langle 00| + |01\rangle\langle 01| + e^{i\theta}|10\rangle\langle 10| + e^{i\theta}|11\rangle\langle 11|$$

$$C_{e^{i\theta}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix}$$

Transforming a global into a local phase

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|00\rangle + e^{i\theta}|11\rangle)$$

Actually, a unitary transformation is completely determined by its action on a basis, but not by specifying what states the states corresponding to basis states are sent to.

Example: $e^{i\theta}$ takes the four quantum states to themselves (because e.g. $|10\rangle$ and $e^{i\theta}|10\rangle$ represent the same state), but a global phase can be transformed into a local one, as above

CCNOT or Toffoli gate

A 3-bit gate corresponding to controlled *CNOT*. If the first two bits are in the state $|1\rangle$ applies X the third bit, else it does nothing:

$$|q_1 q_2 q_3\rangle \ \mapsto \ |q_1 q_2, q_3 \oplus (q_1 \wedge q_2)\rangle$$

In matrix form,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Universal set of gates?

Is there a universal set of quantum gates?

In general no: there are uncountably many quantum transformations, and a finite set of generators can only generate countably many elements. However, it is possible for finite sets of gates to generate arbitrarily close approximations to all unitary transformations.

Definitions

- The error in approximating U by V is

$$Er(U, V) \;=\; \max_{|\phi\rangle} \; \| (U - V)|\phi\rangle \|$$

- An operator U can be approximated to arbitrary accuracy if for any positive ϵ there exists another unitary transformation V st $Er(U, V) \leq \epsilon$.
- A set of gates is universal if for any integer $n \geq 1$, any n-qubit unitary operator can be approximated to arbitrary accuracy by a quantum circuit using only gates from that set.

Universal set of gates?

Some examples

- The set $\{H, T\}$ is universal for 1-gates.
- The set $\{H, T, CNOT\}$ is a universal set of gates.

How efficient is an approximation?

To approximate an unitary transformation encoding some specific computation, one would expect to use a number of gates from the universal set which is polynomial in the number of qubits and the inverse of the quality factor ϵ.

Main result: theorem of Solovay-Kitaev