## Homework 6 - Programming

Lecture: Prof. Qiang Liu

## 1. Kernel Regression

Given a training data  $\{x_i, y_i\}_{i=1}^n$ , kernel regression approximates the unknown nonlinear relation between x and y with a function of form

$$y \approx f(\boldsymbol{x}; \ \boldsymbol{w}) = \sum_{i=1}^{n} w_i k(\boldsymbol{x}, \ \boldsymbol{x}_i),$$

where  $k(\boldsymbol{x}, \boldsymbol{x}')$  is a positive definite kernel specified by the users, and  $\{\boldsymbol{w}_i\}$  is a set of weights, estimated by minimizing a regularized mean square error:

$$\min_{\boldsymbol{w}} \left\{ \sum_{i=1}^{n} (y_i - f(\boldsymbol{x}; \boldsymbol{w}))^2 + \beta \boldsymbol{w}^{\top} \boldsymbol{K} \boldsymbol{w} \right\},\,$$

where  $\boldsymbol{w}$  is the column vector formed by  $\boldsymbol{w} = [w_i]_{i=1}^n$  and  $\boldsymbol{K}$  the  $n \times n$  matrix by  $\boldsymbol{K} = [k(\boldsymbol{x}_i, \boldsymbol{x}_j)]_{ij=1}^n$ , and  $\beta$  is a positive regularization parameter. We use a simple Gaussian radius basis function (RBF) kernel,

$$k(\boldsymbol{x}, \boldsymbol{x}') = \exp\left(-\frac{||\boldsymbol{x} - \boldsymbol{x}'||^2}{2h^2}\right),$$

where h is a bandwith parameter. A common way to set h in practice is the so called "median trick", which set h to be the median of the pairwise distance on the training data, that is,

$$\hat{h}_{\text{med}} = \text{median}(\{||x_i - x_j||: i \neq j, i, j = 1, \dots, n\}).$$

(1) [10 points] Complete the code of kernel regression following the instruction of the attached Python notebook. Specifically, you need to complete all the code necessary for function

kernel\_regression\_fit\_and\_predict to run.

- (2) [10 points] Run the algorithm with  $\beta = 1$  and  $h \in \{0.1\hat{h}_{\text{med}}, \hat{h}_{\text{med}}, 10\hat{h}_{\text{med}}\}$ . Show the curve learned with different h in the notebook and comment on how h influences the smoothness of the curve.
- (3) [10 points] Use 5-fold cross validation to find the optimal combination of h and  $\beta$  within  $h \in \{0.1\hat{h}_{\text{med}}, \hat{h}_{\text{med}}, 10\hat{h}_{\text{med}}\}$  and  $\beta \in \{0.1, 1\}$ .