Building up a heap in linear time

Building up a heap

Building up a heap

- n standard insert-operations for a heap result in $O(n \log(n))$ time.

Building up a heap

- n standard insert-operations for a heap result in $O(n \log(n))$ time.

- Can we build up a heap for n given elements faster? Is $O(n)$ possible?

150 70 30 10 20 60 40 50 | 140 100 80 90 130 110 120

150 70 30 10 20 60 40 50 140 100 80 90 130 110 120

150 70 30 10 20 | 60 40 50 | 140 100 80 90 130 110 120

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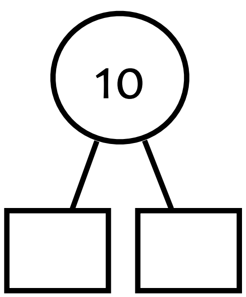

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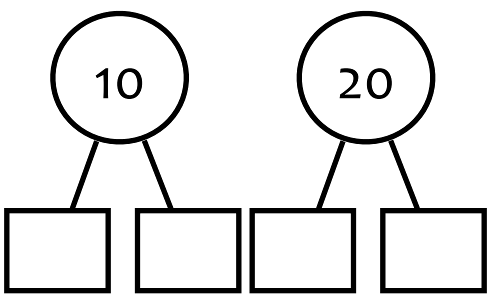

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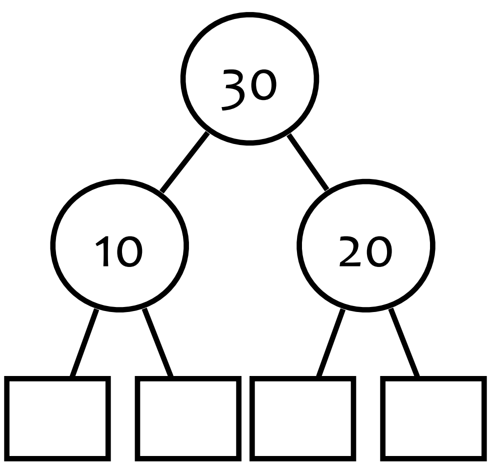

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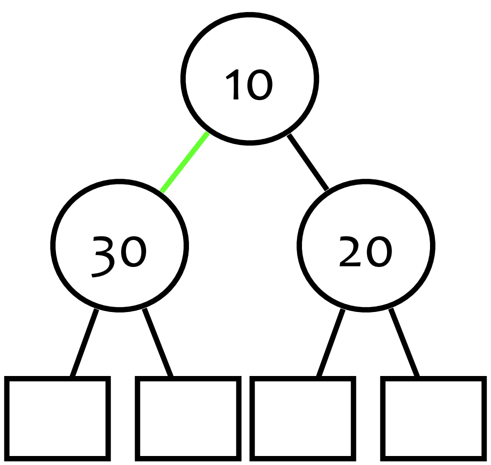

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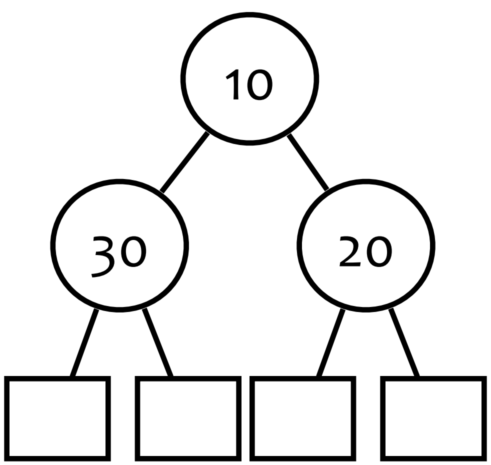

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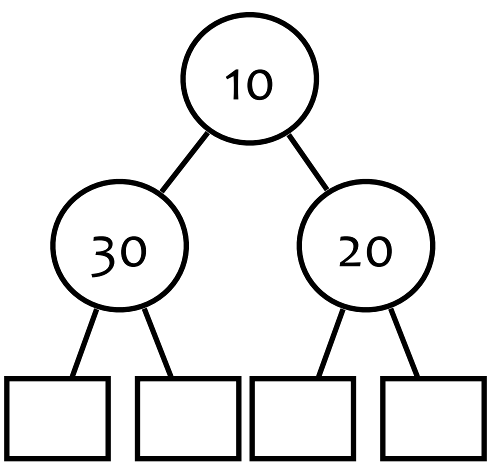

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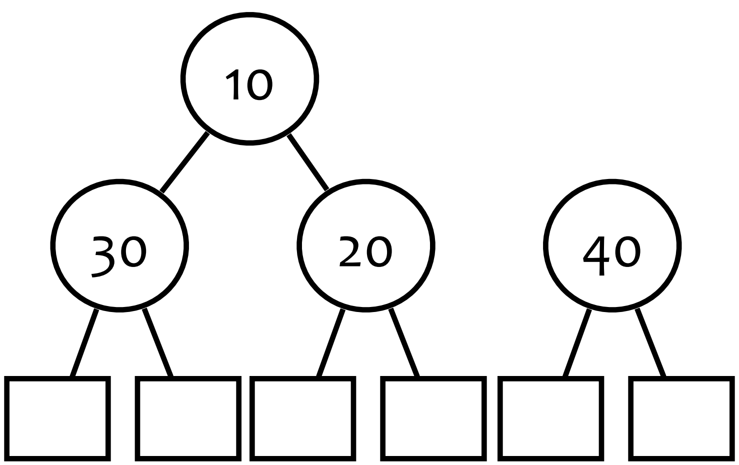

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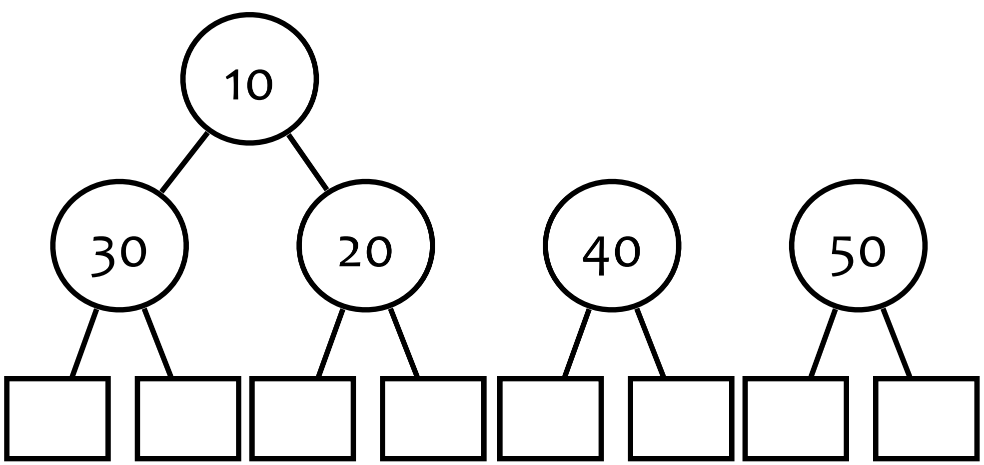

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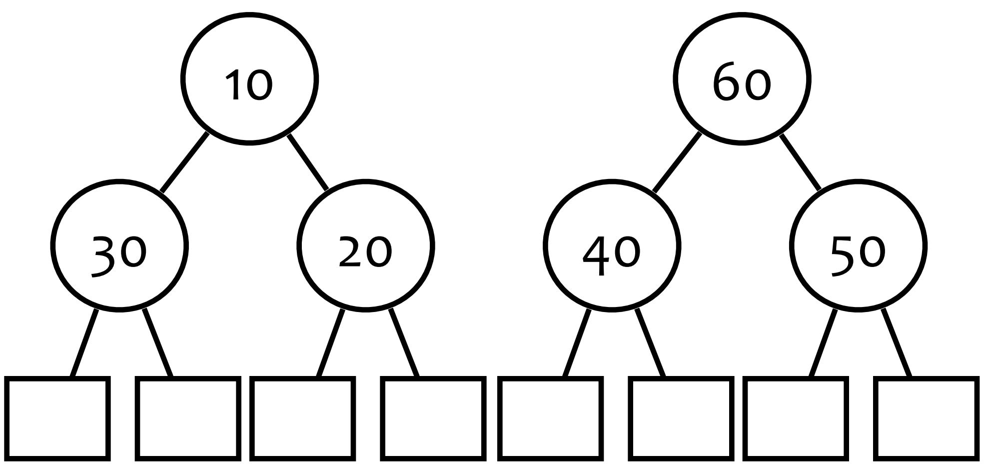

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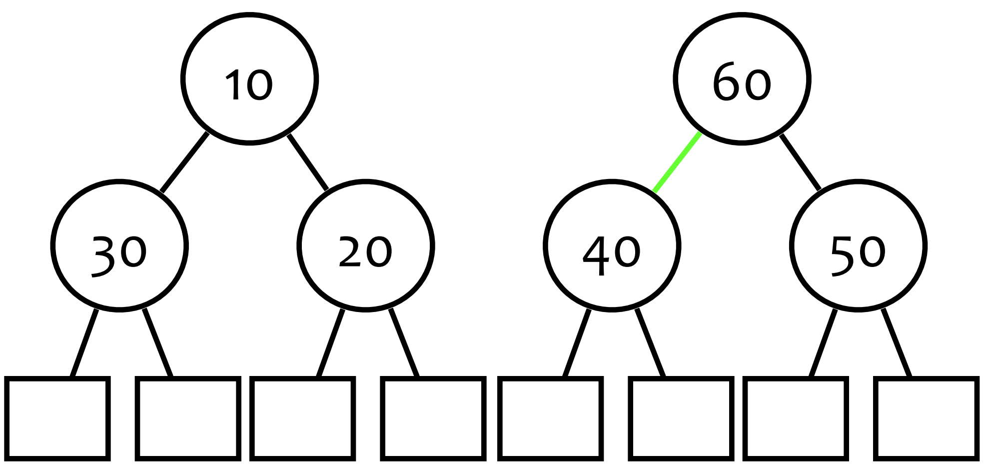

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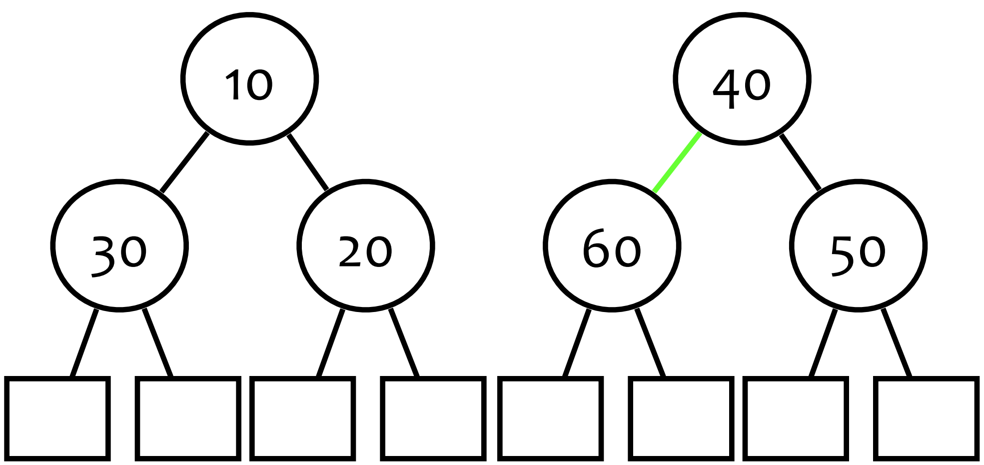

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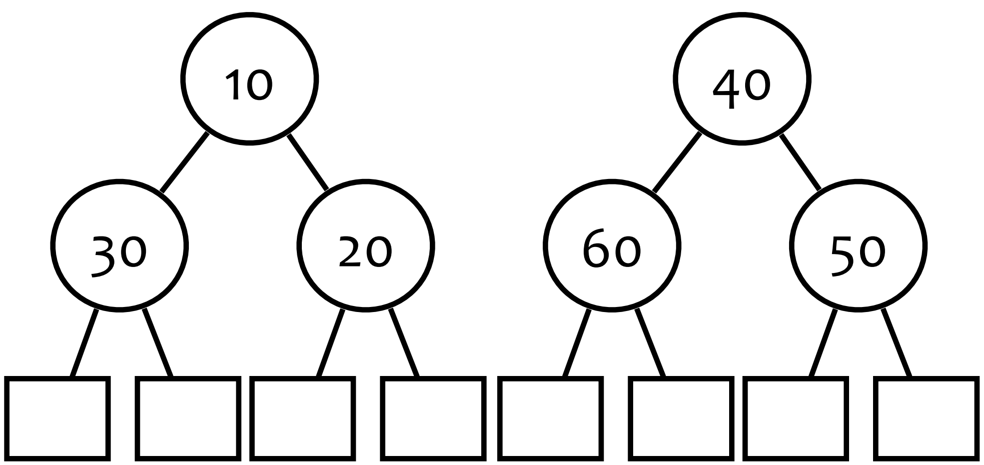

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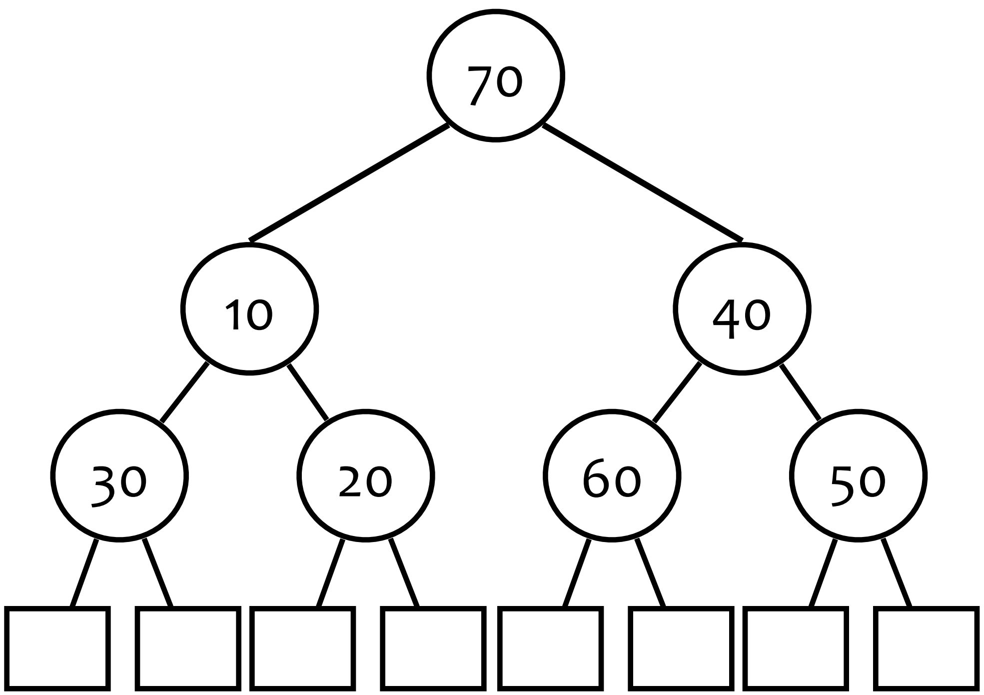

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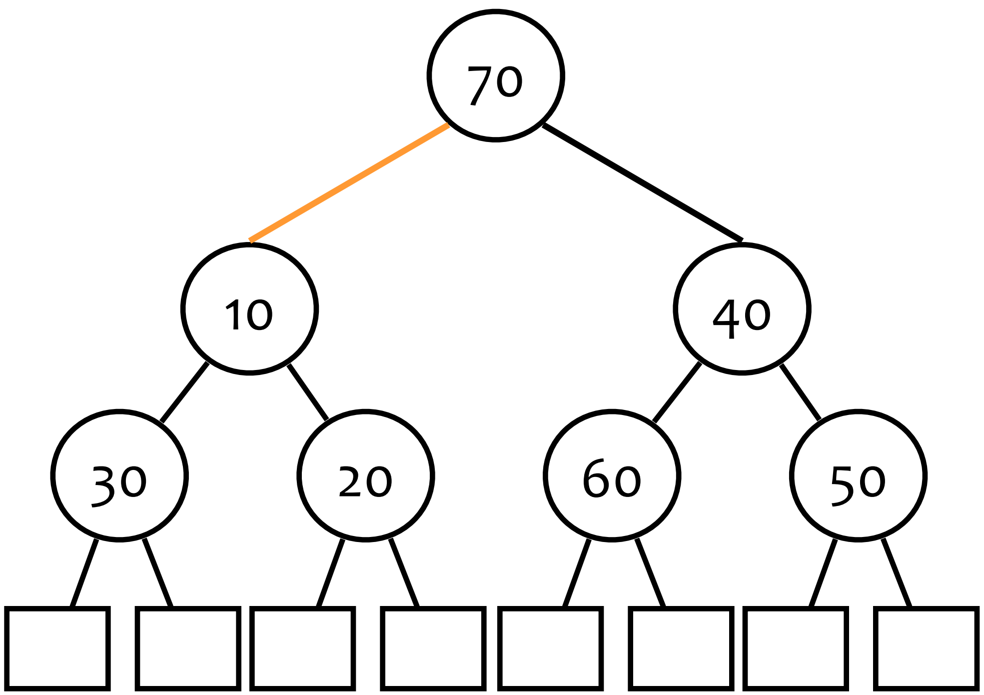

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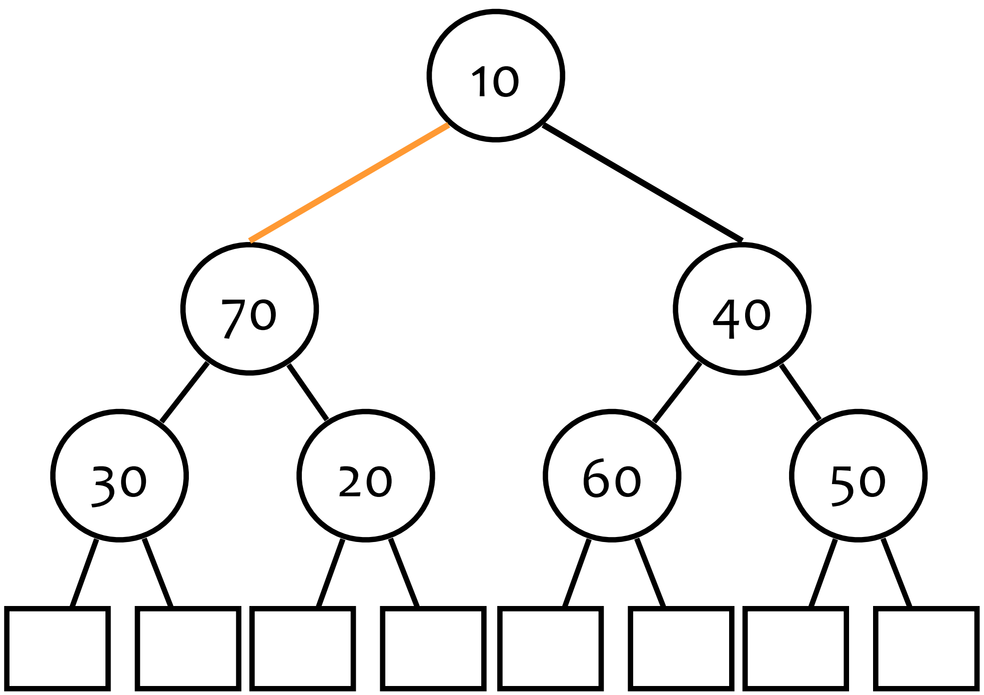

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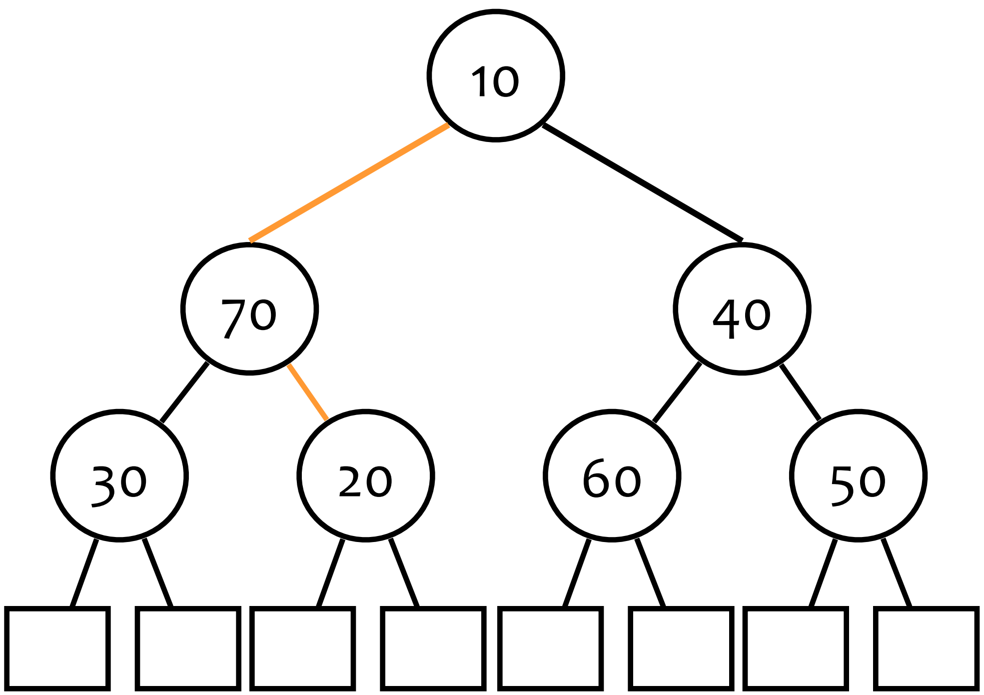

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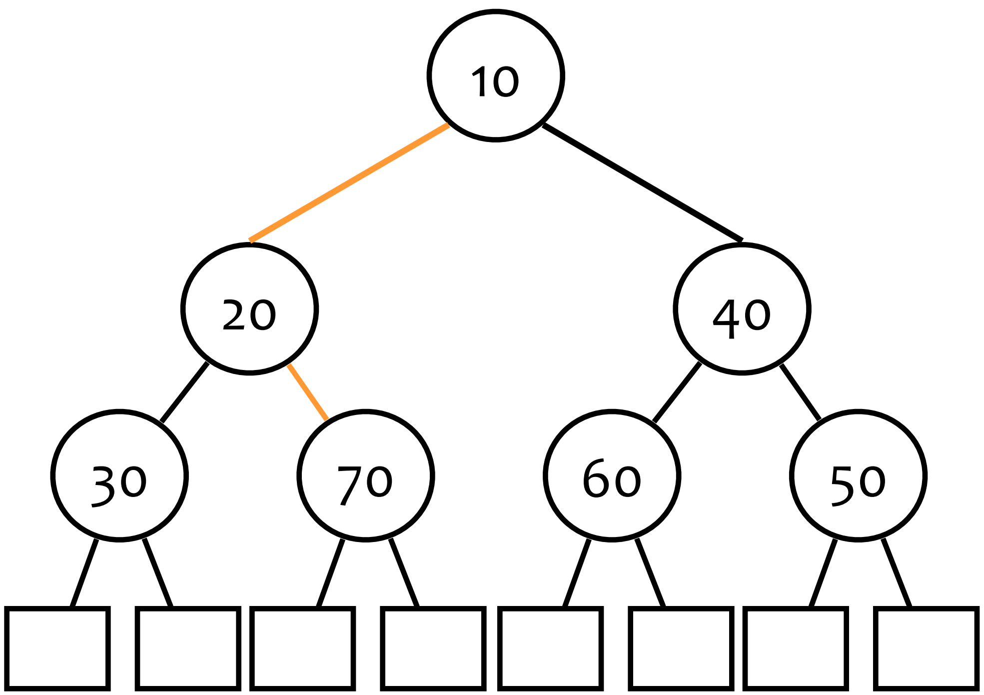

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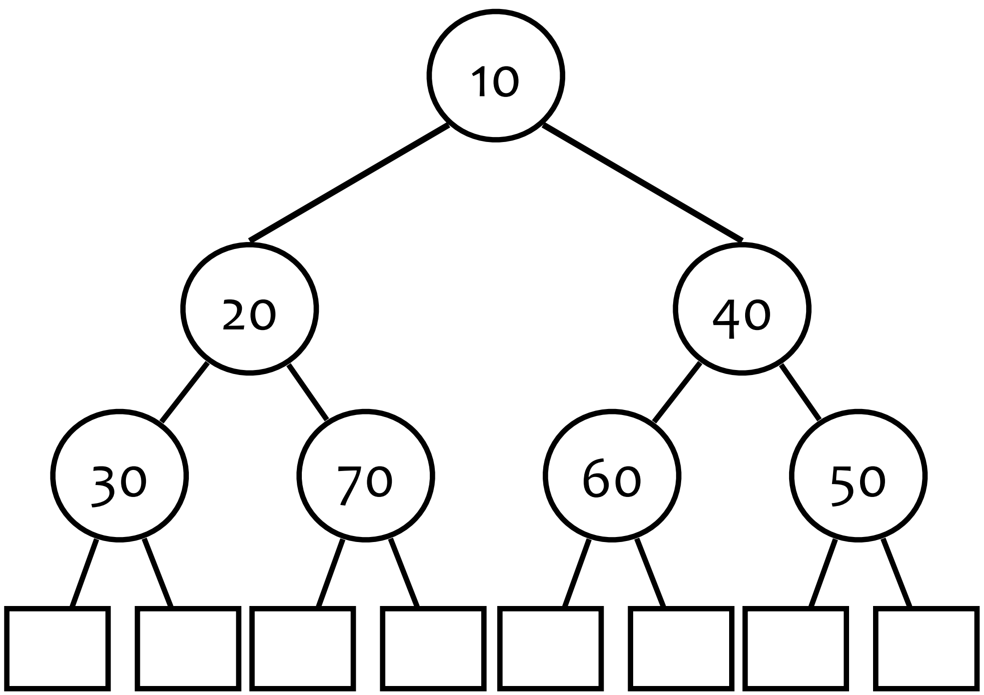

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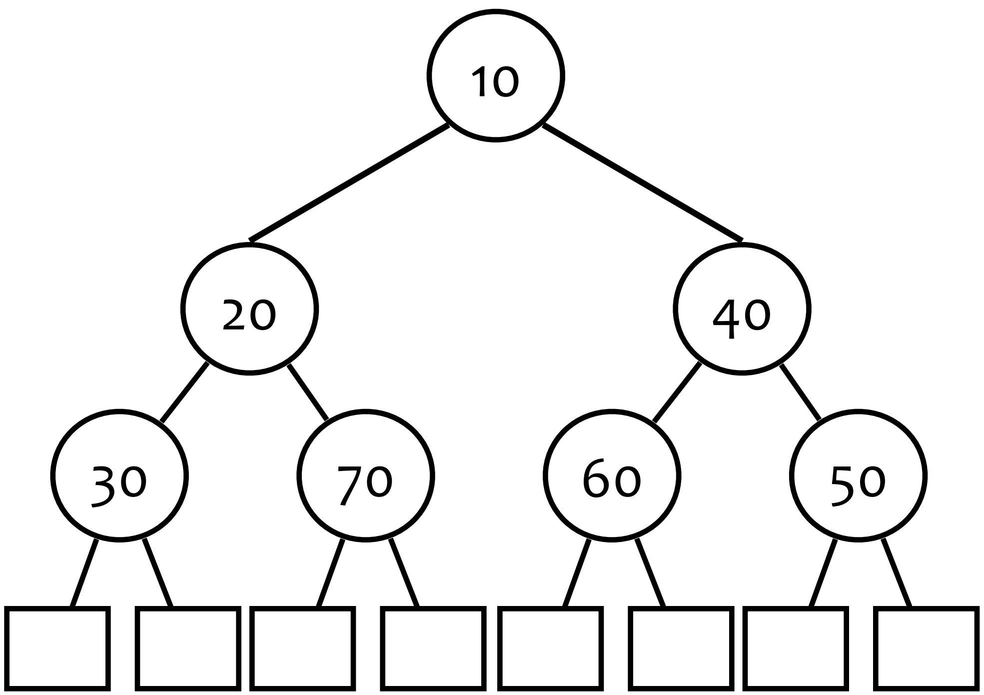

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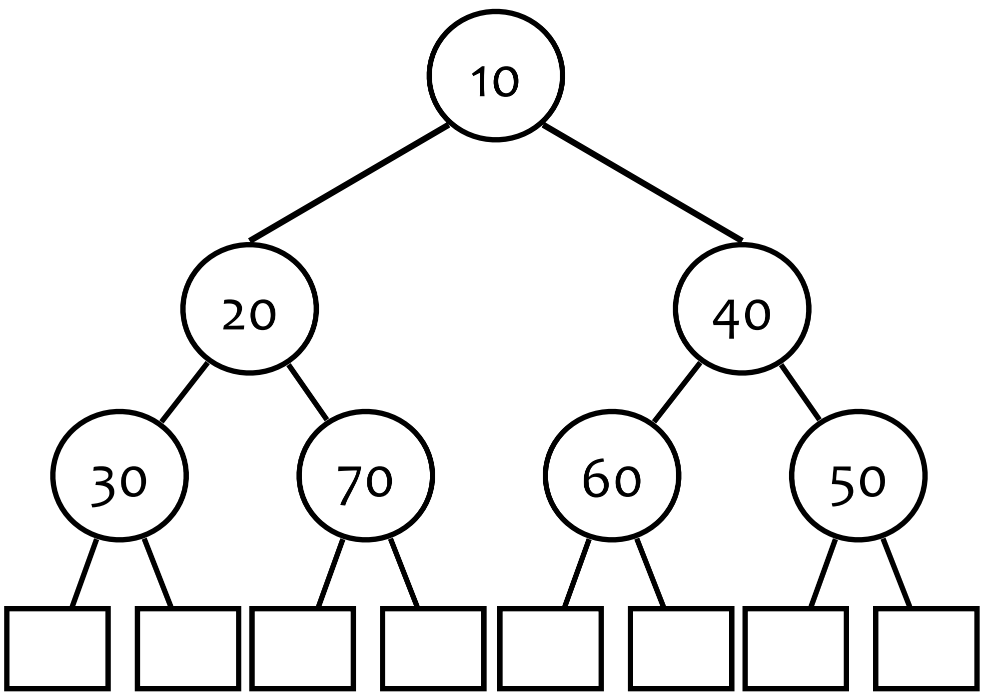

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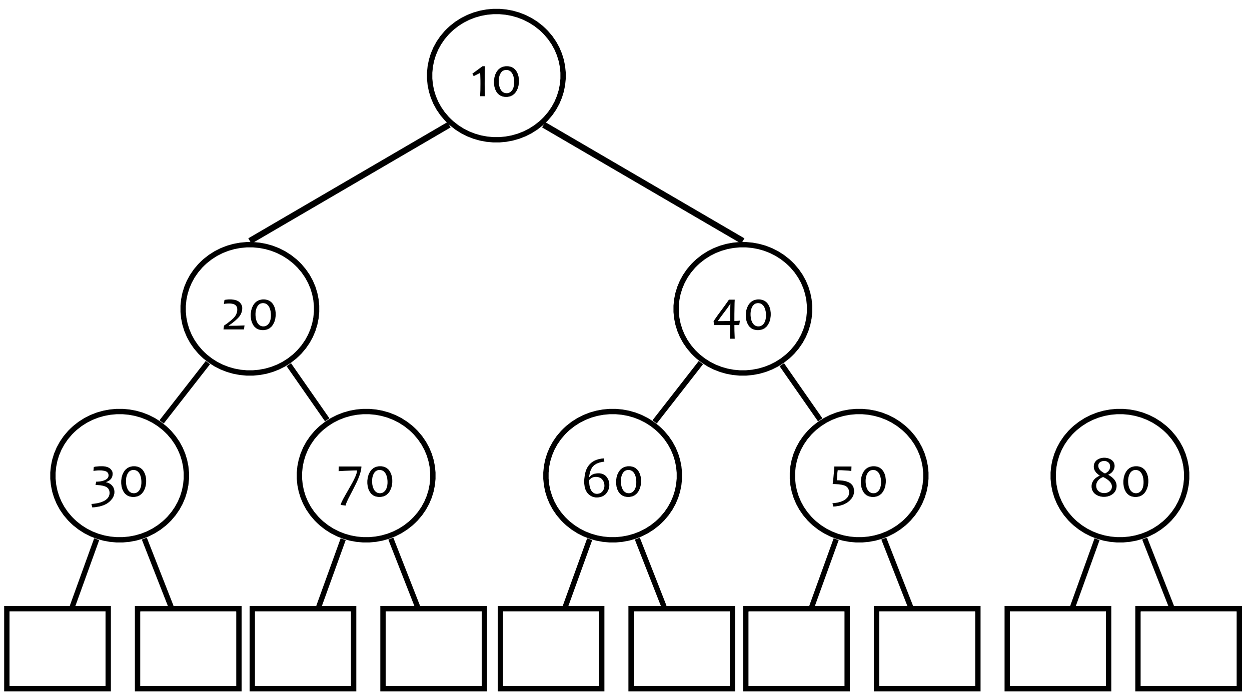

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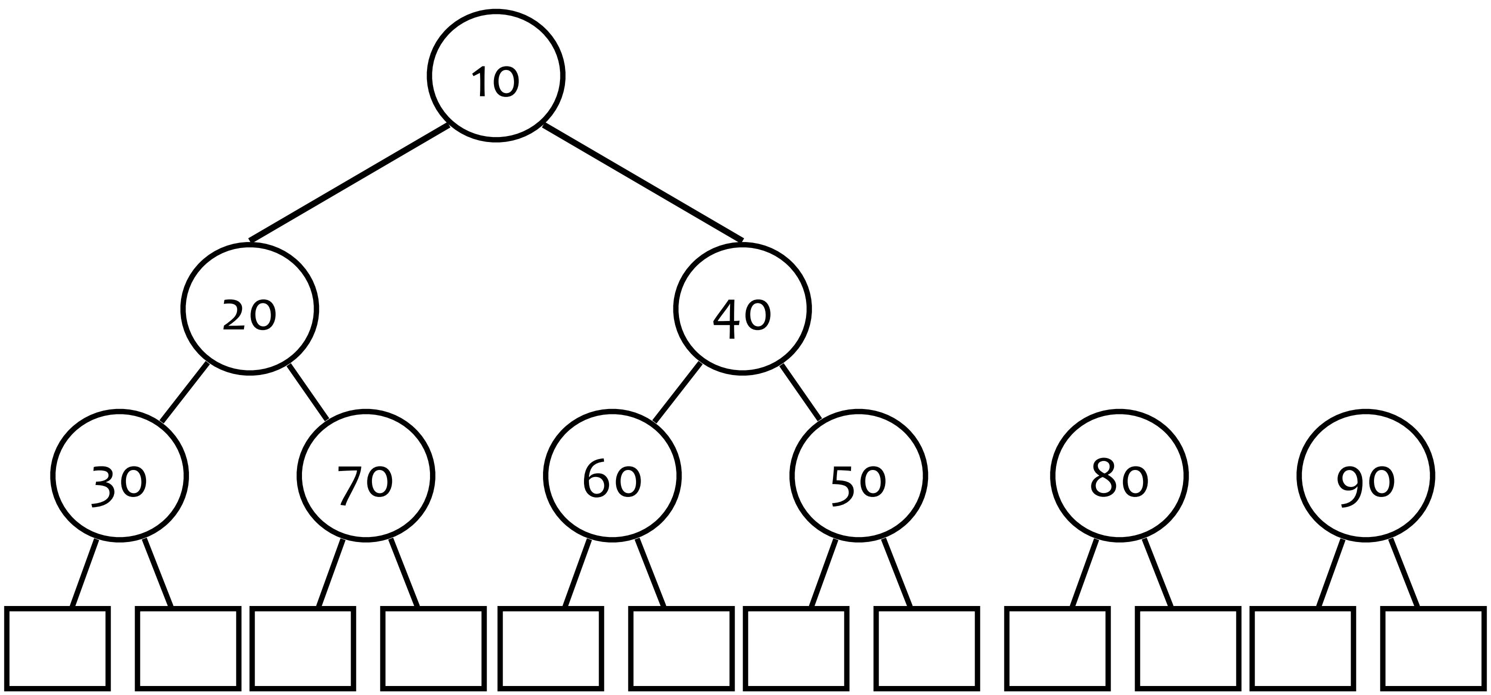

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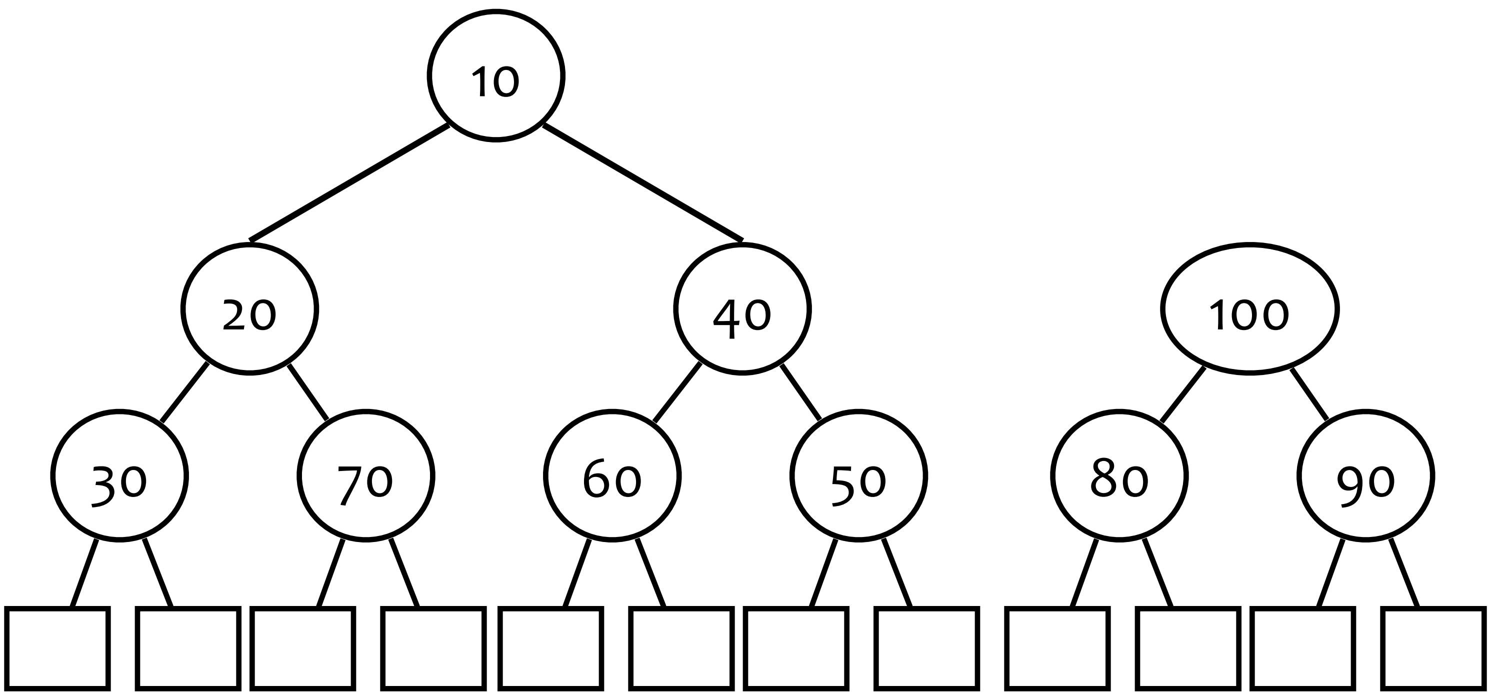

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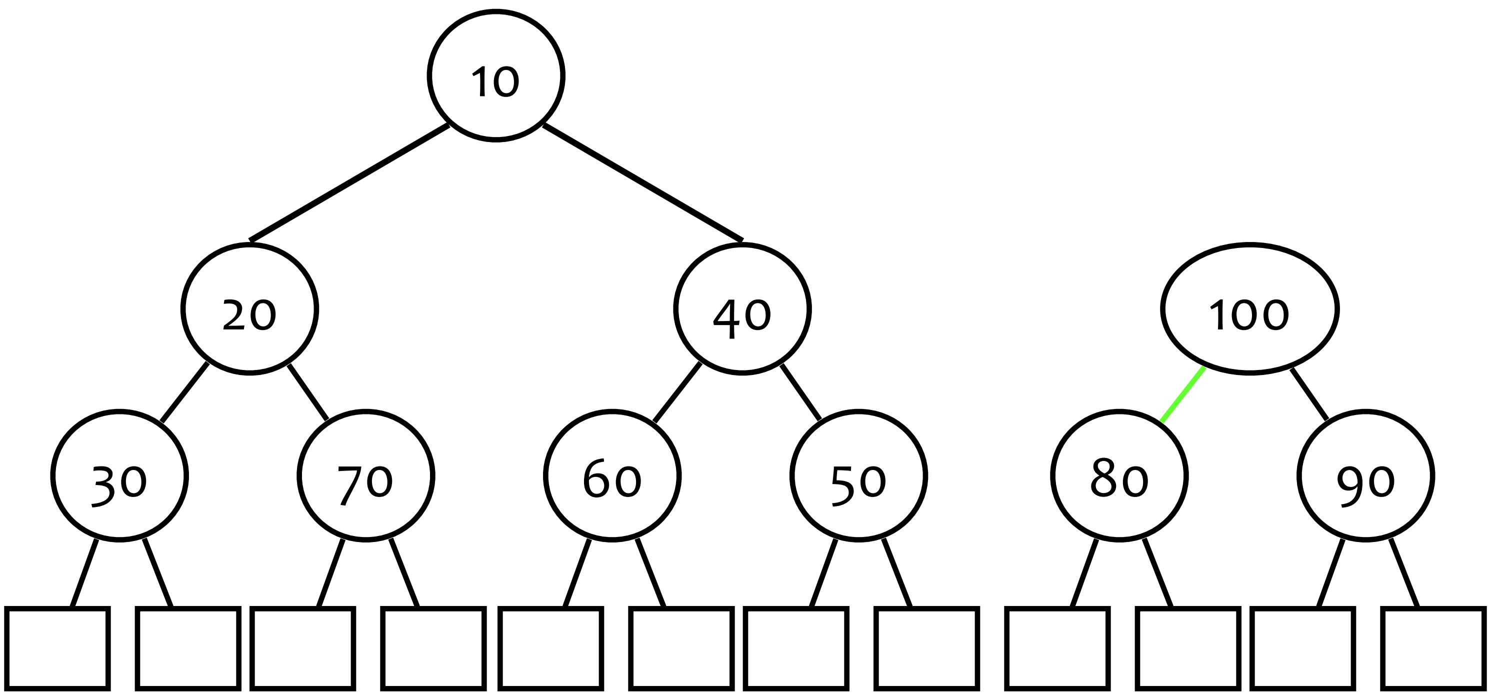

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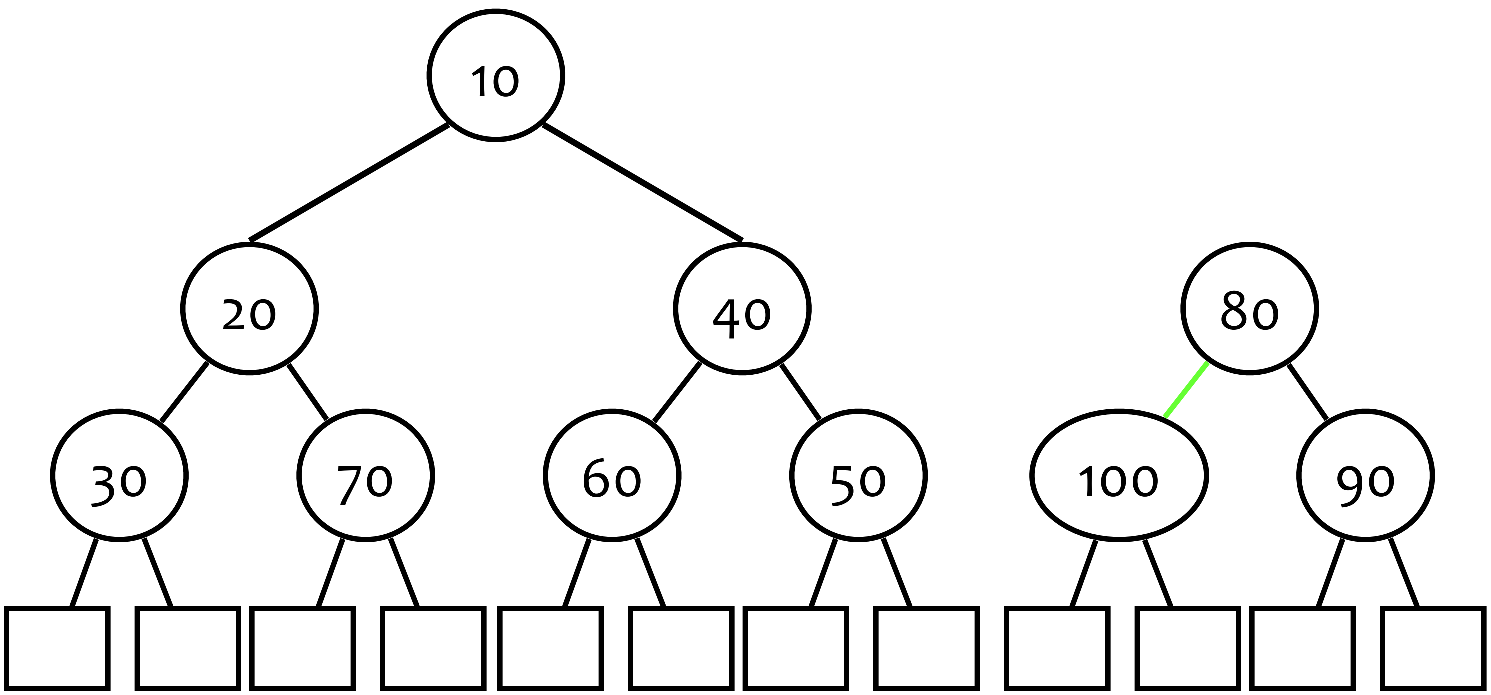

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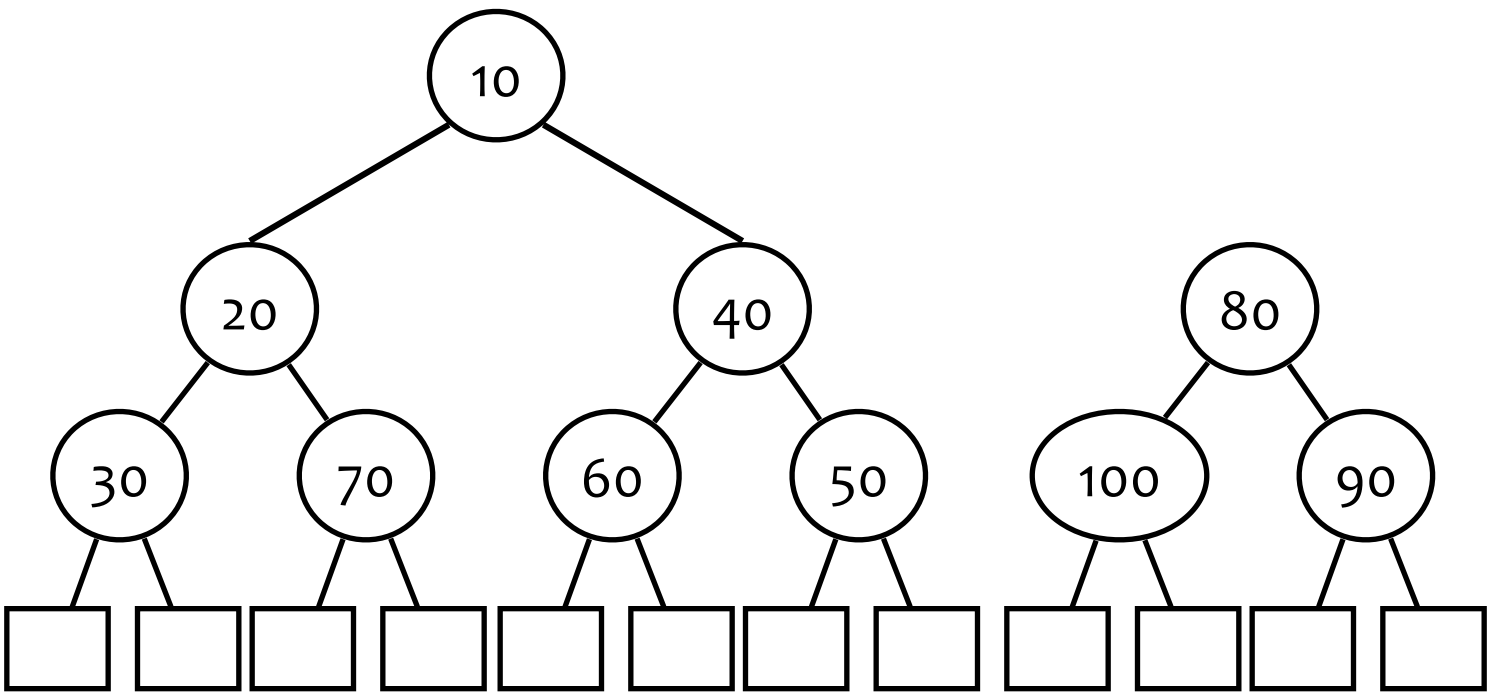

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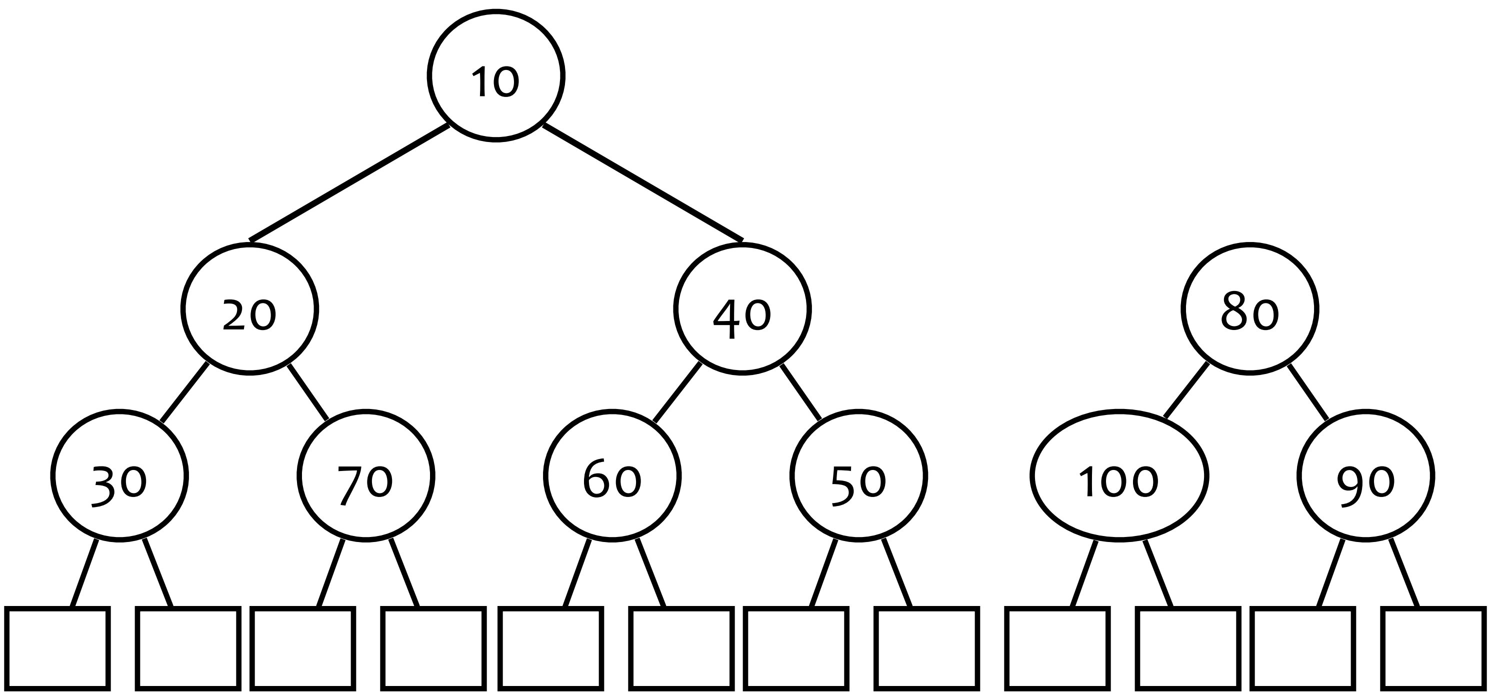

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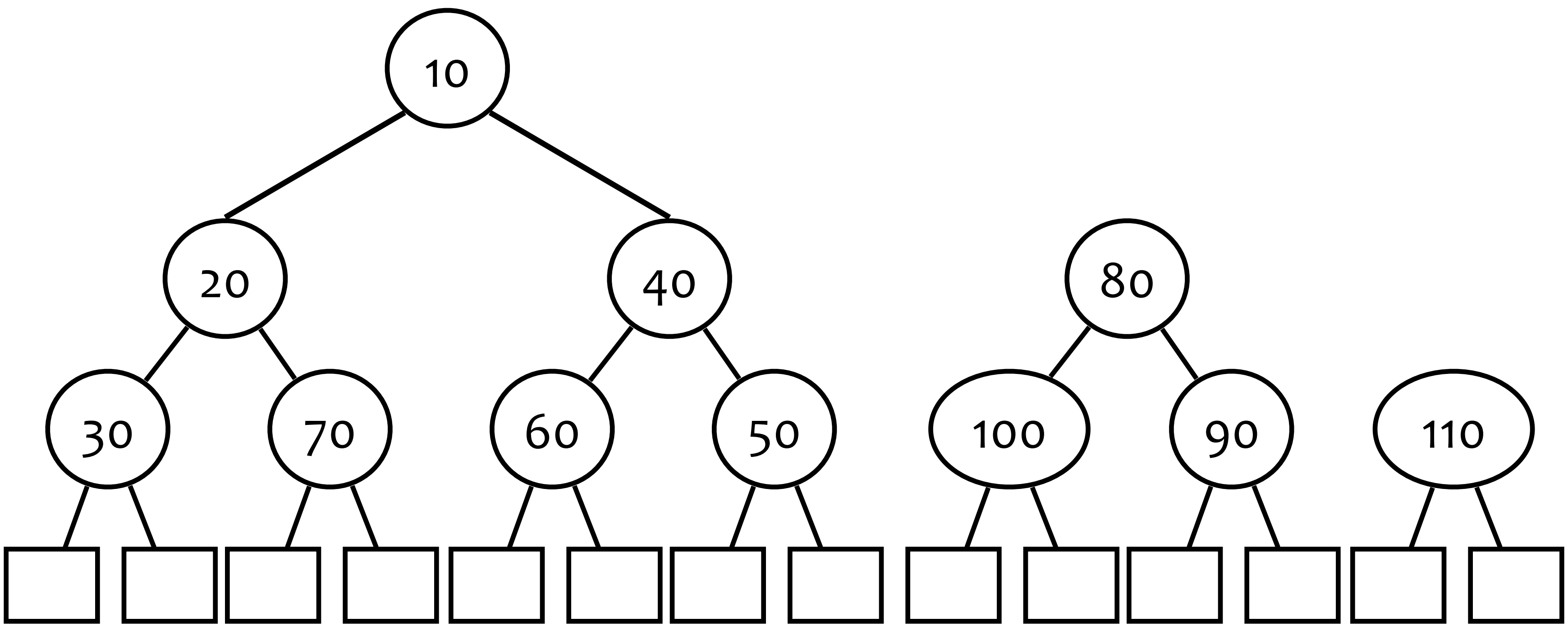

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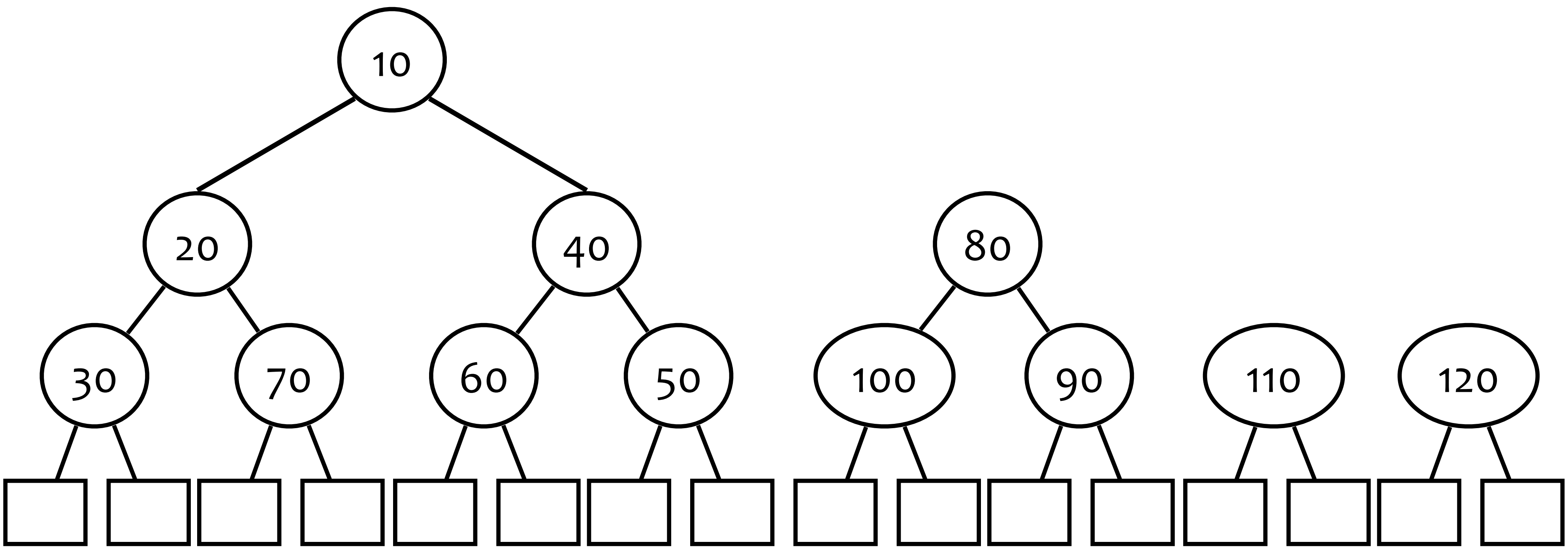

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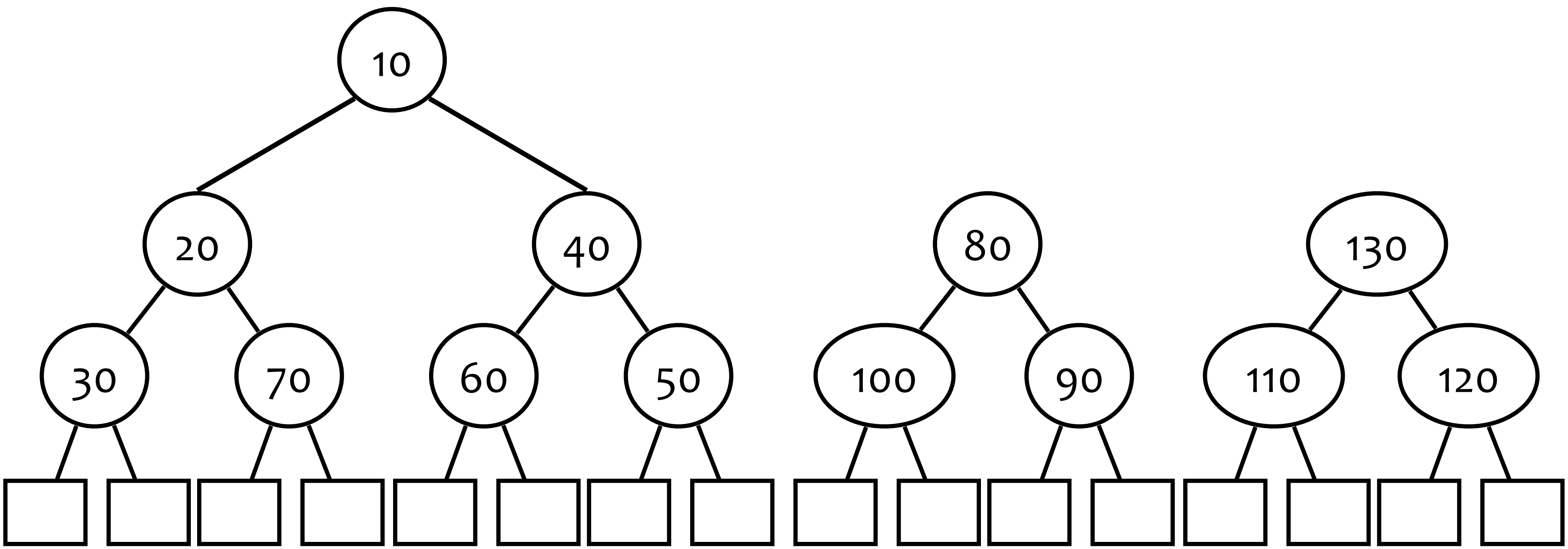

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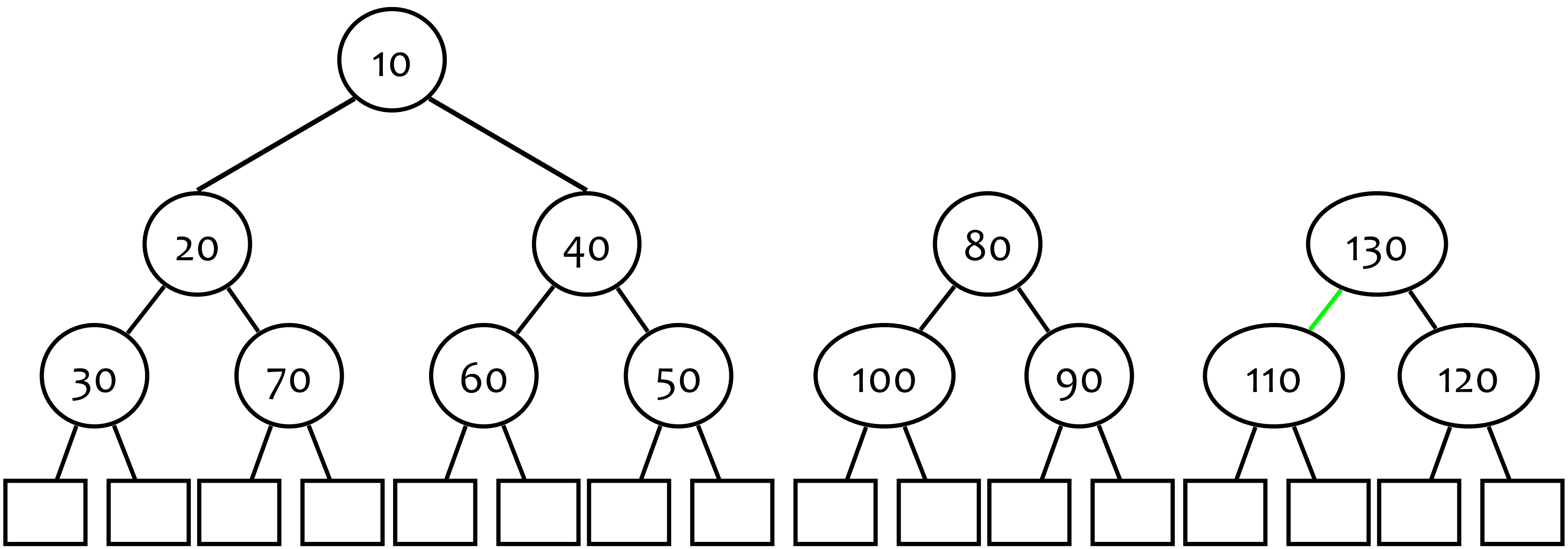

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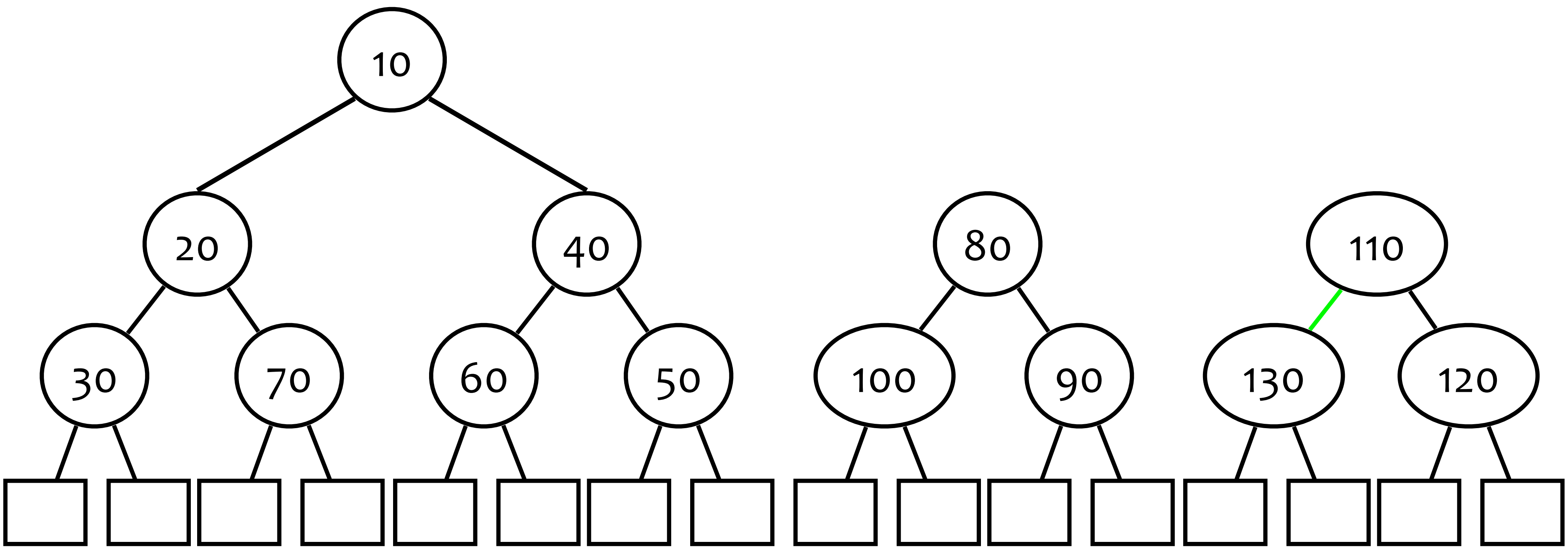

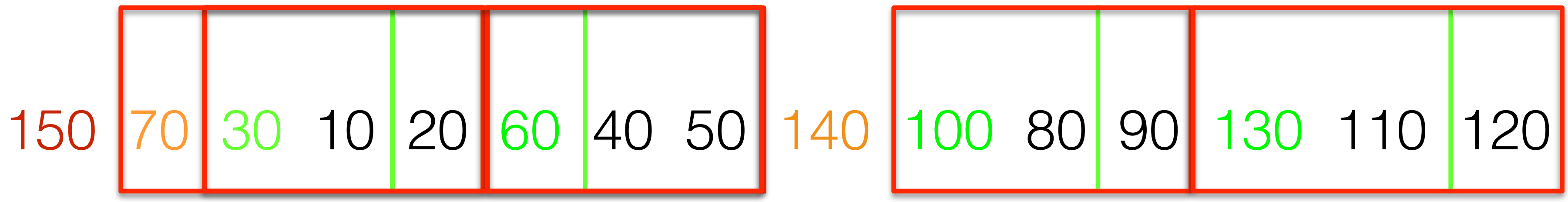
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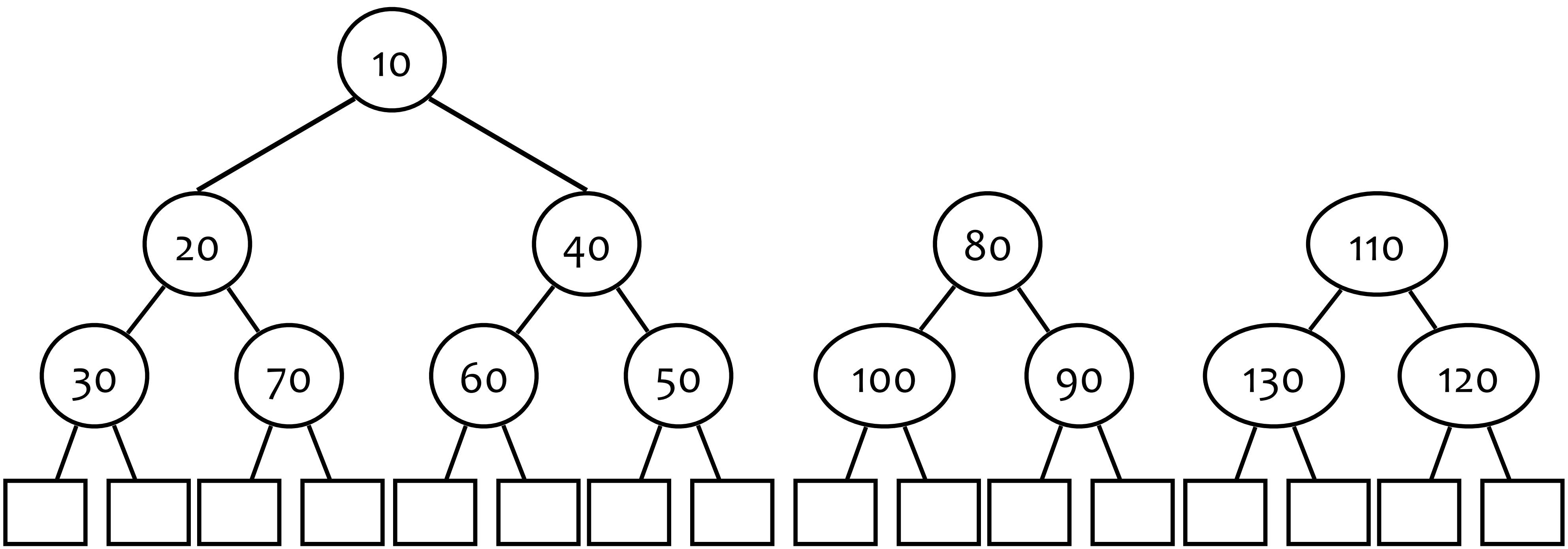
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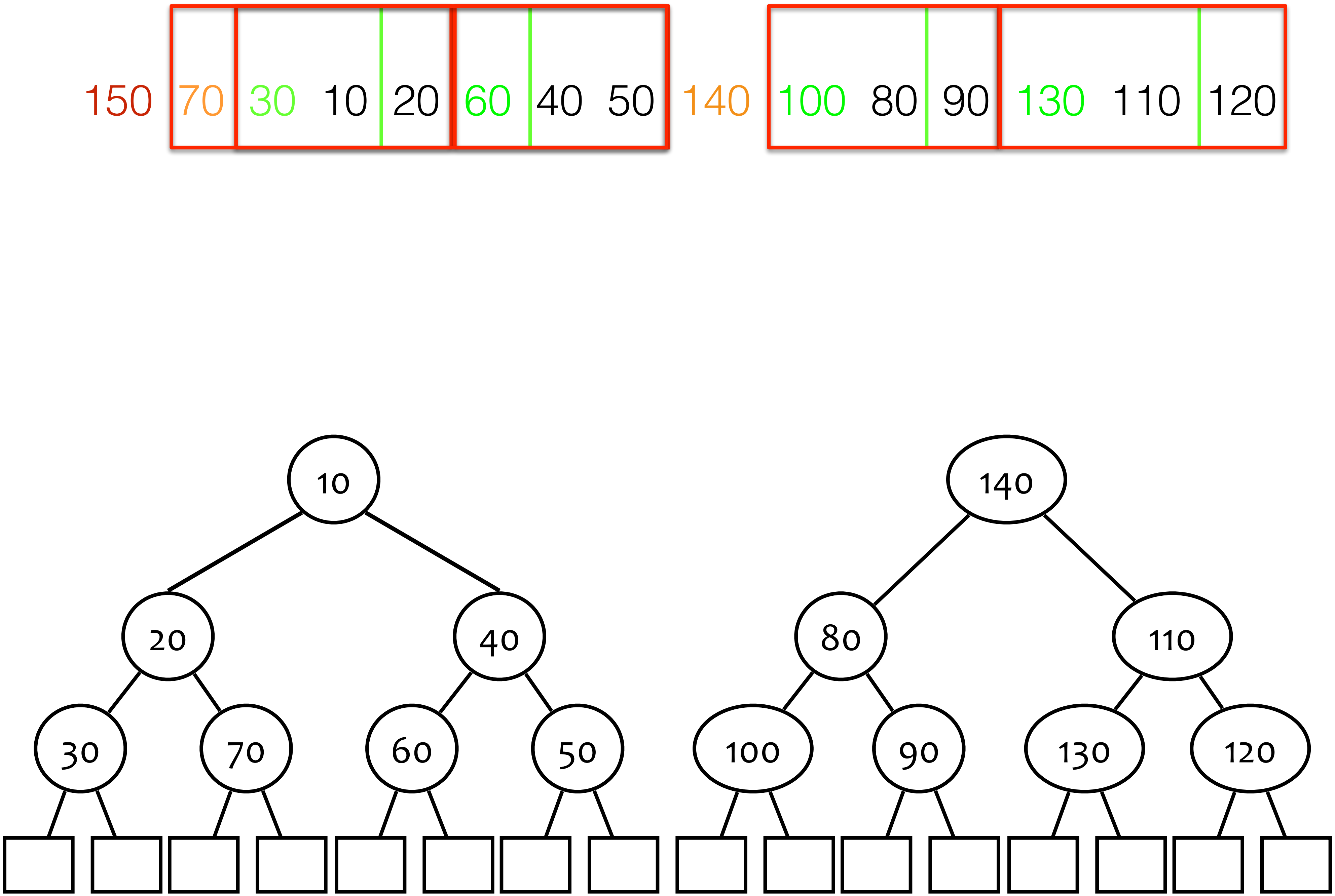
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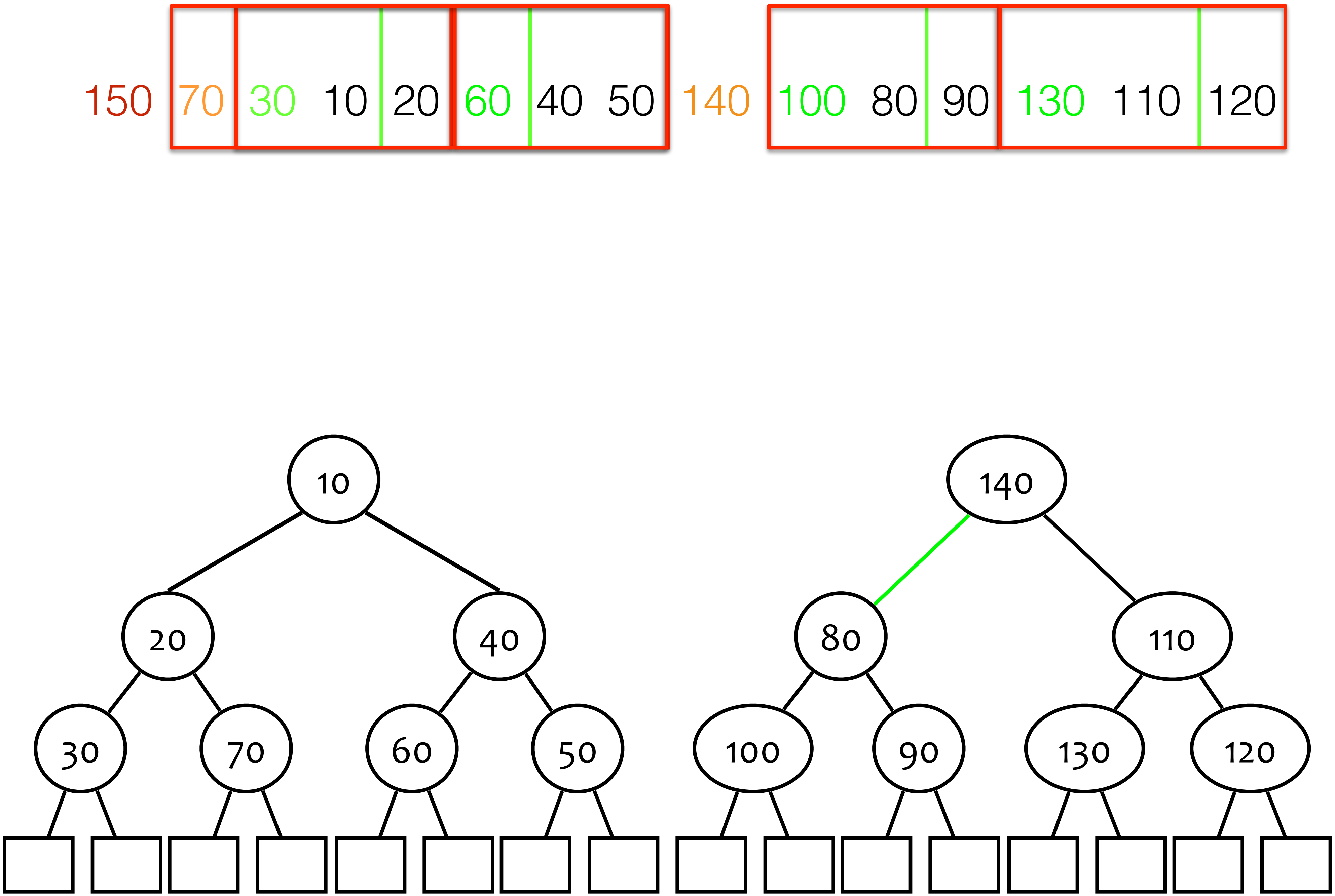
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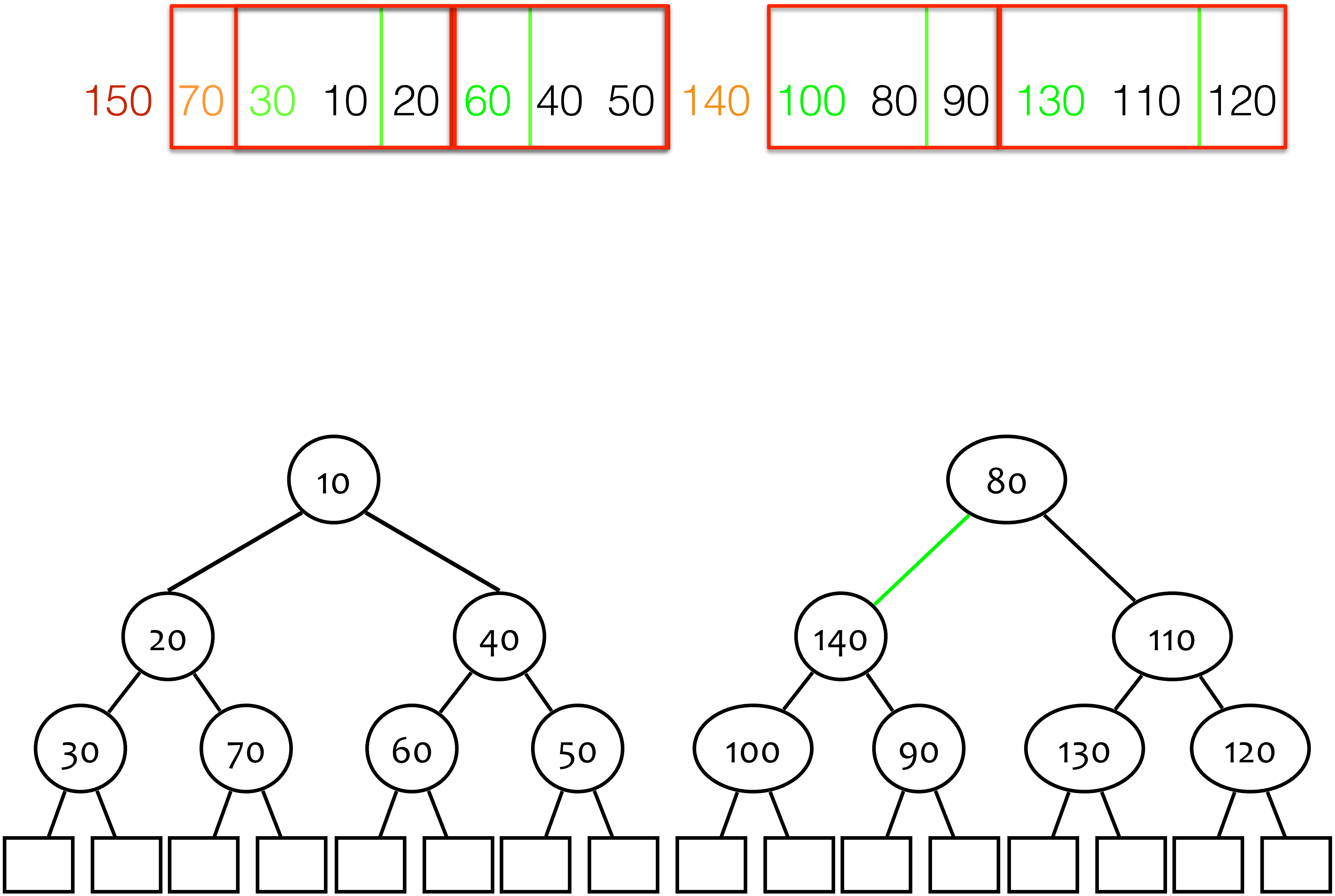
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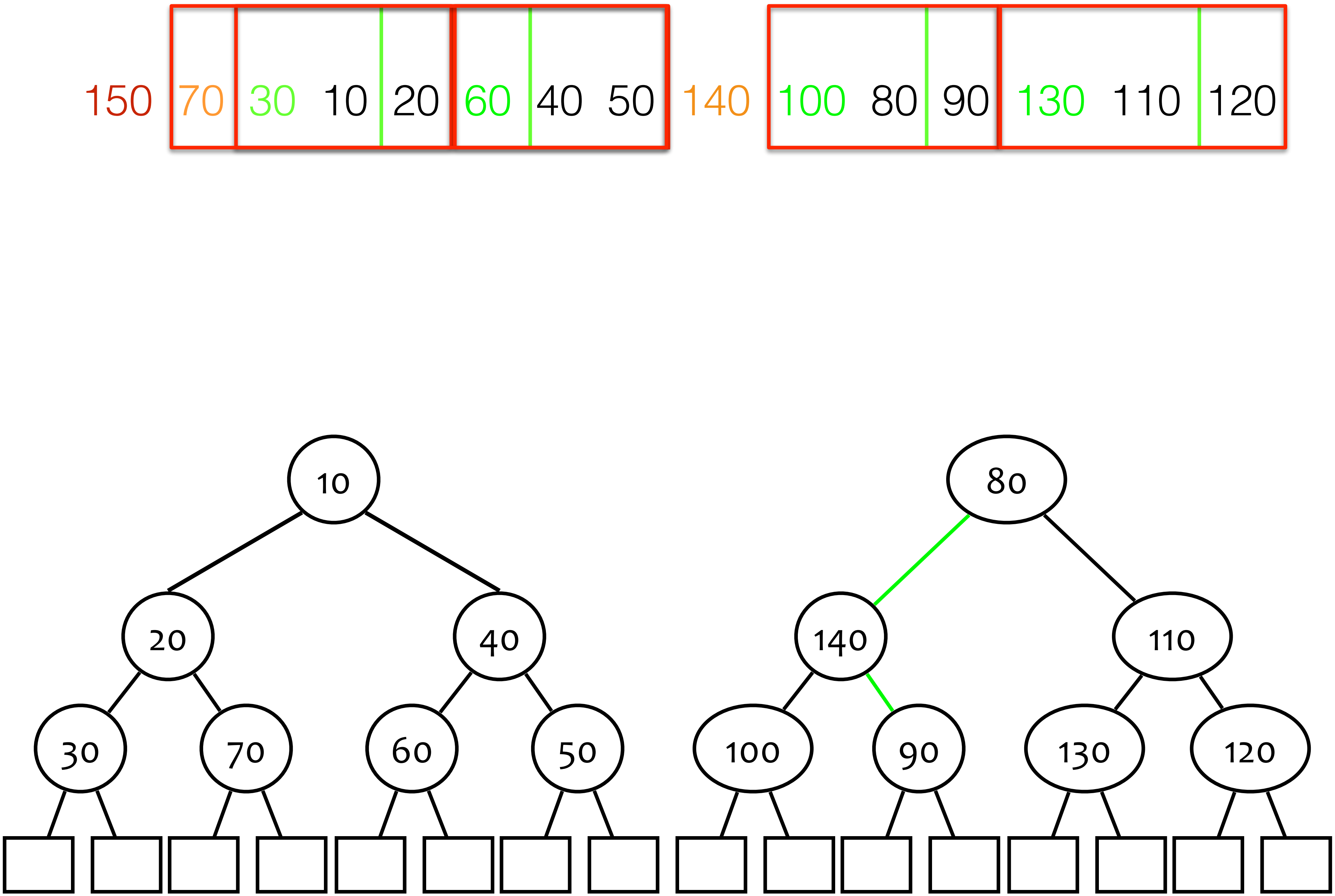
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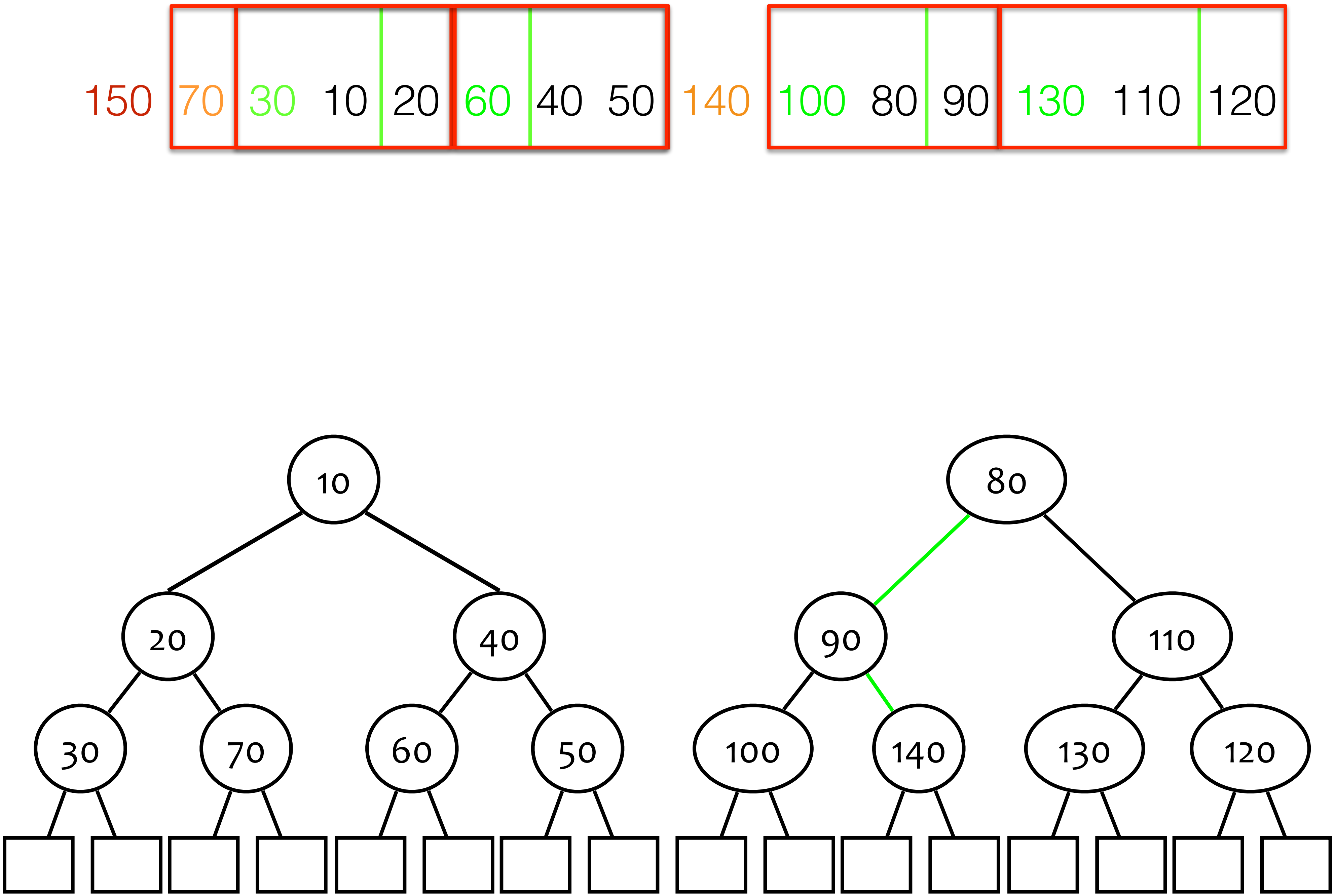
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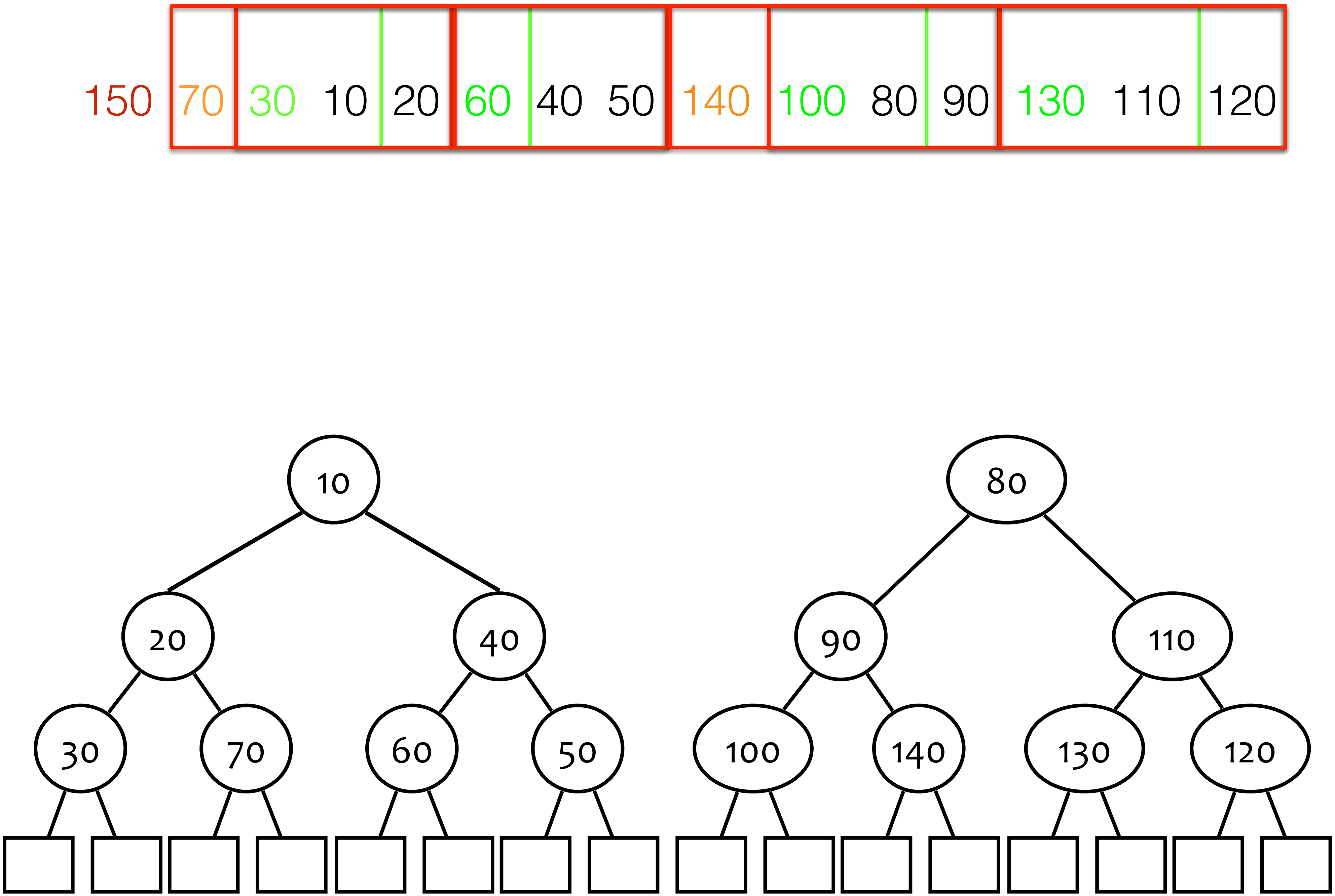
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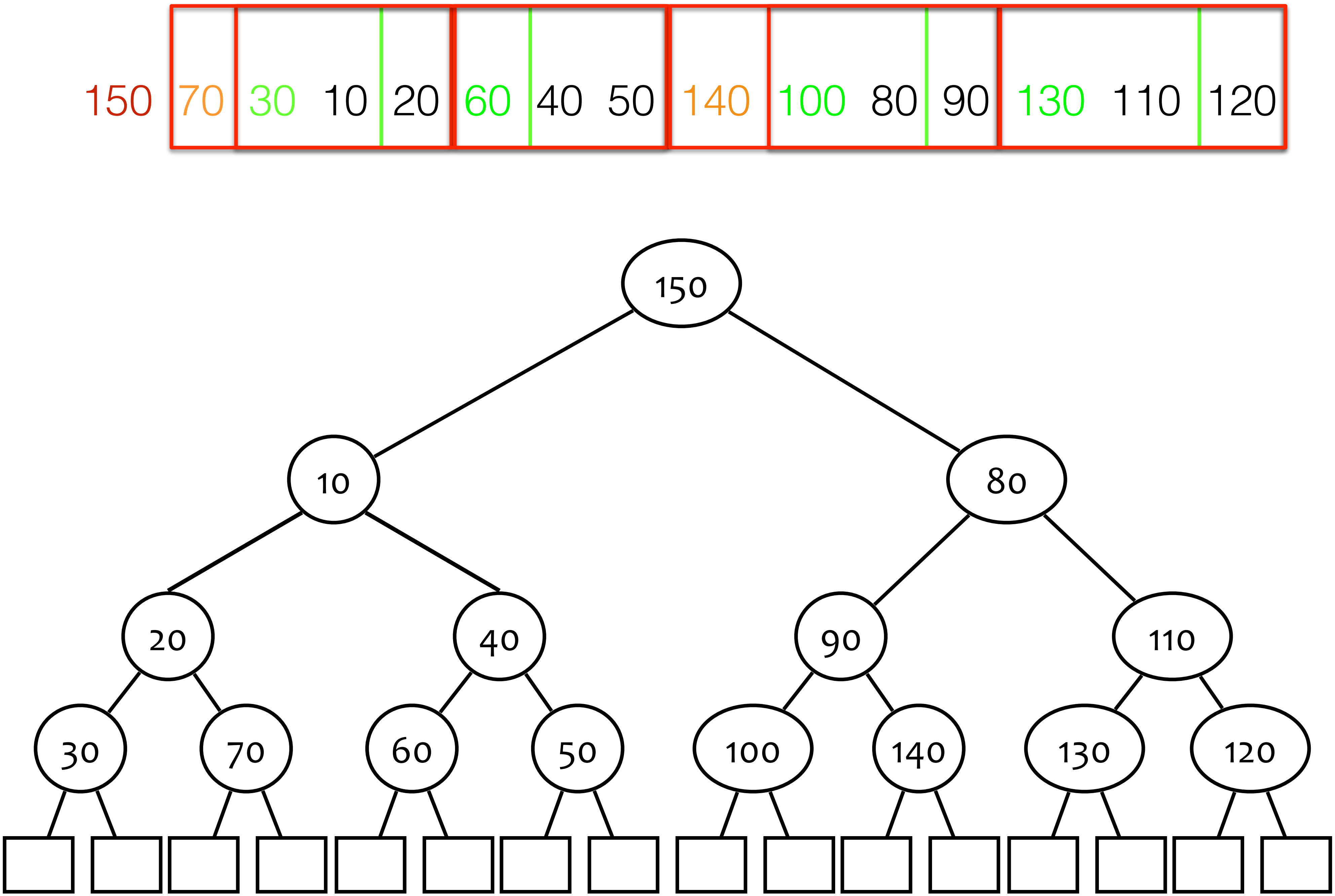

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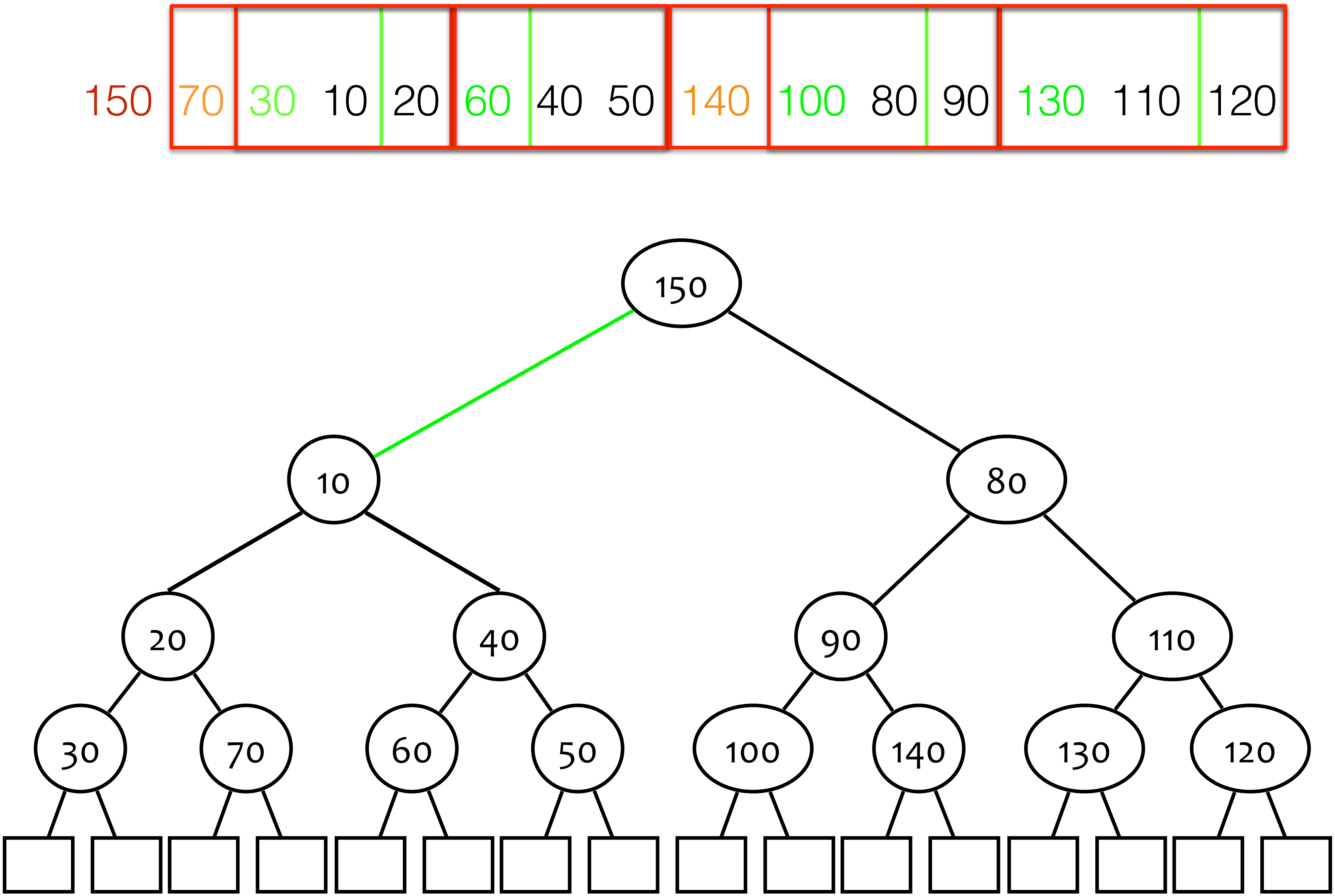
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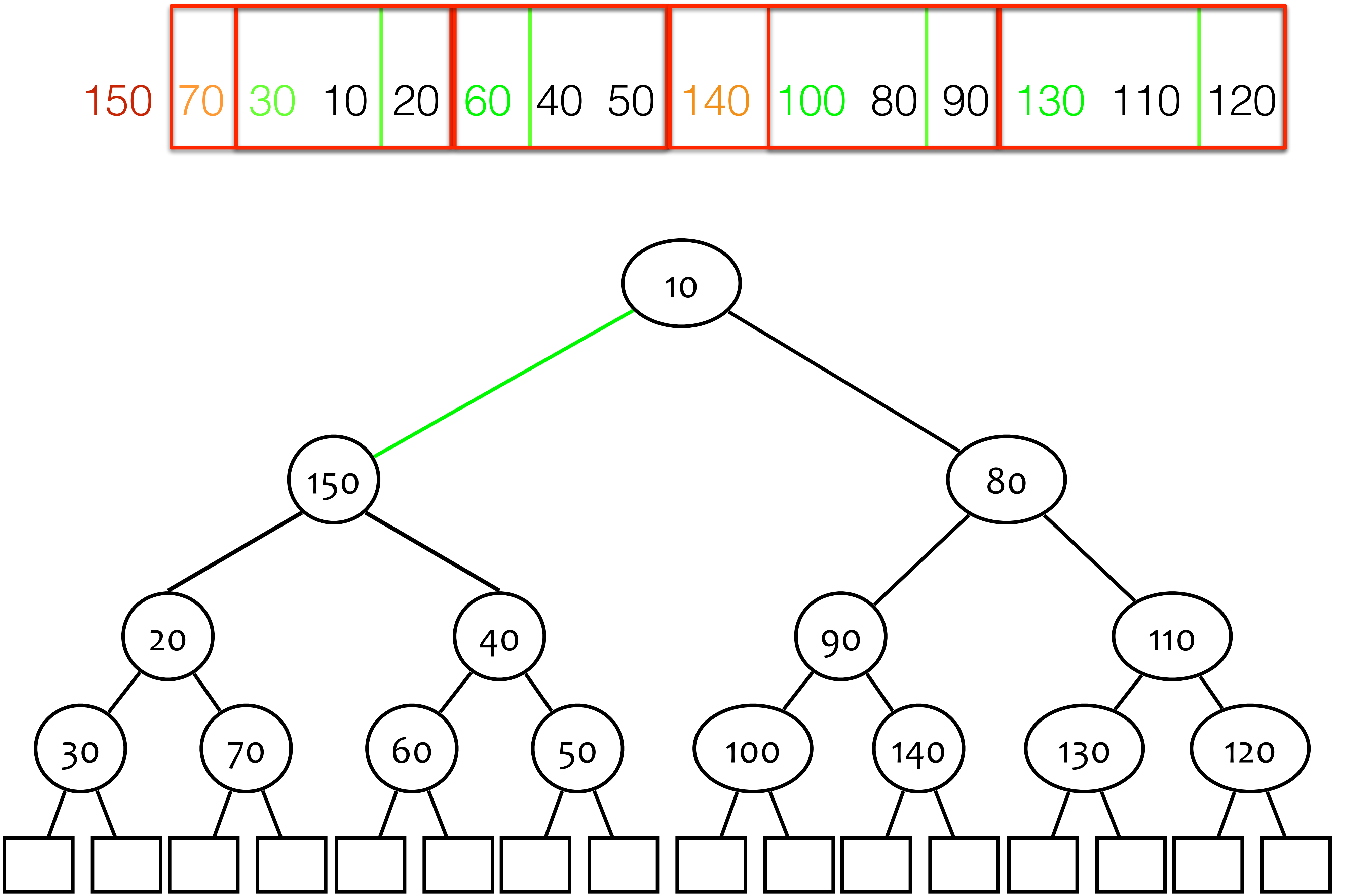
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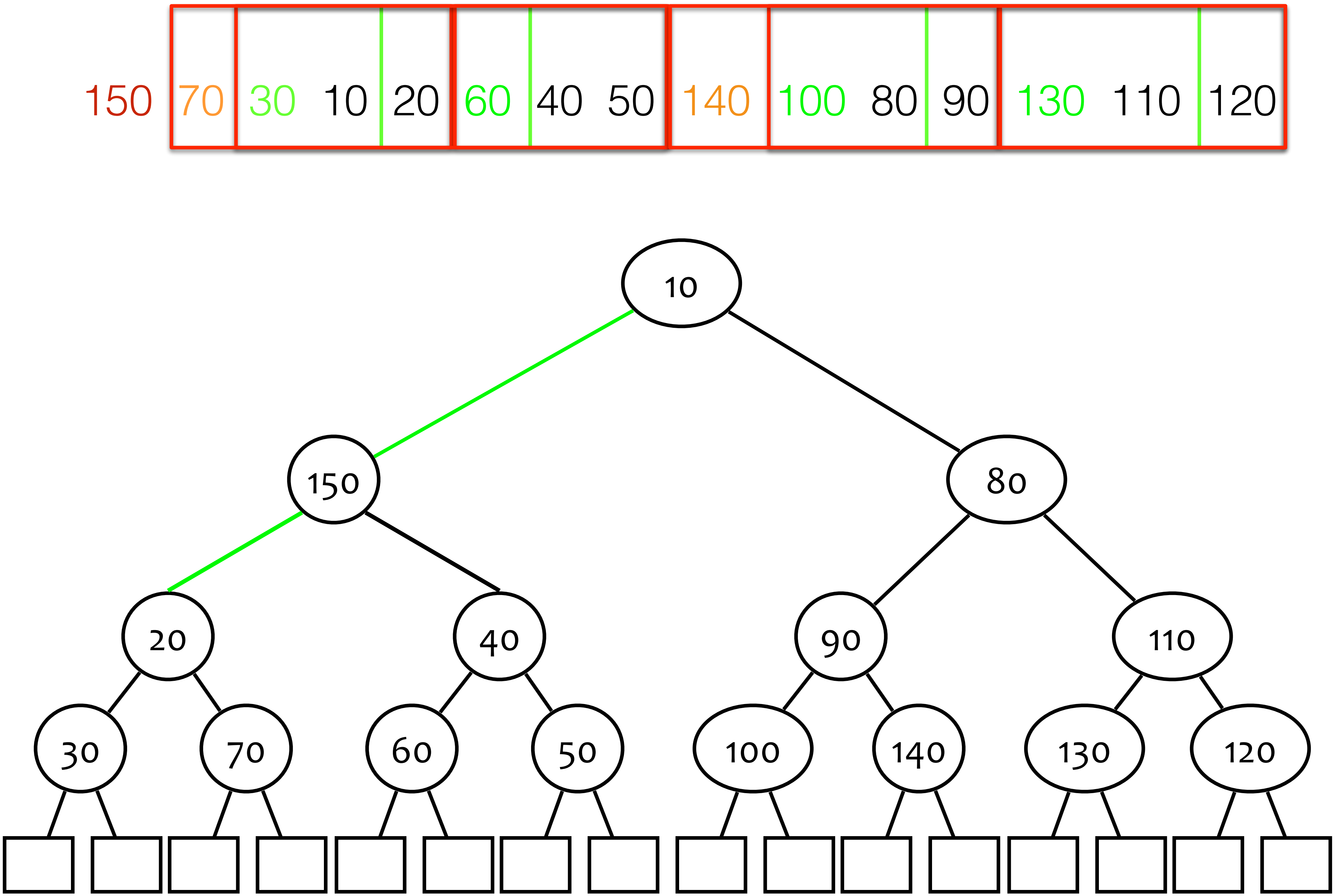
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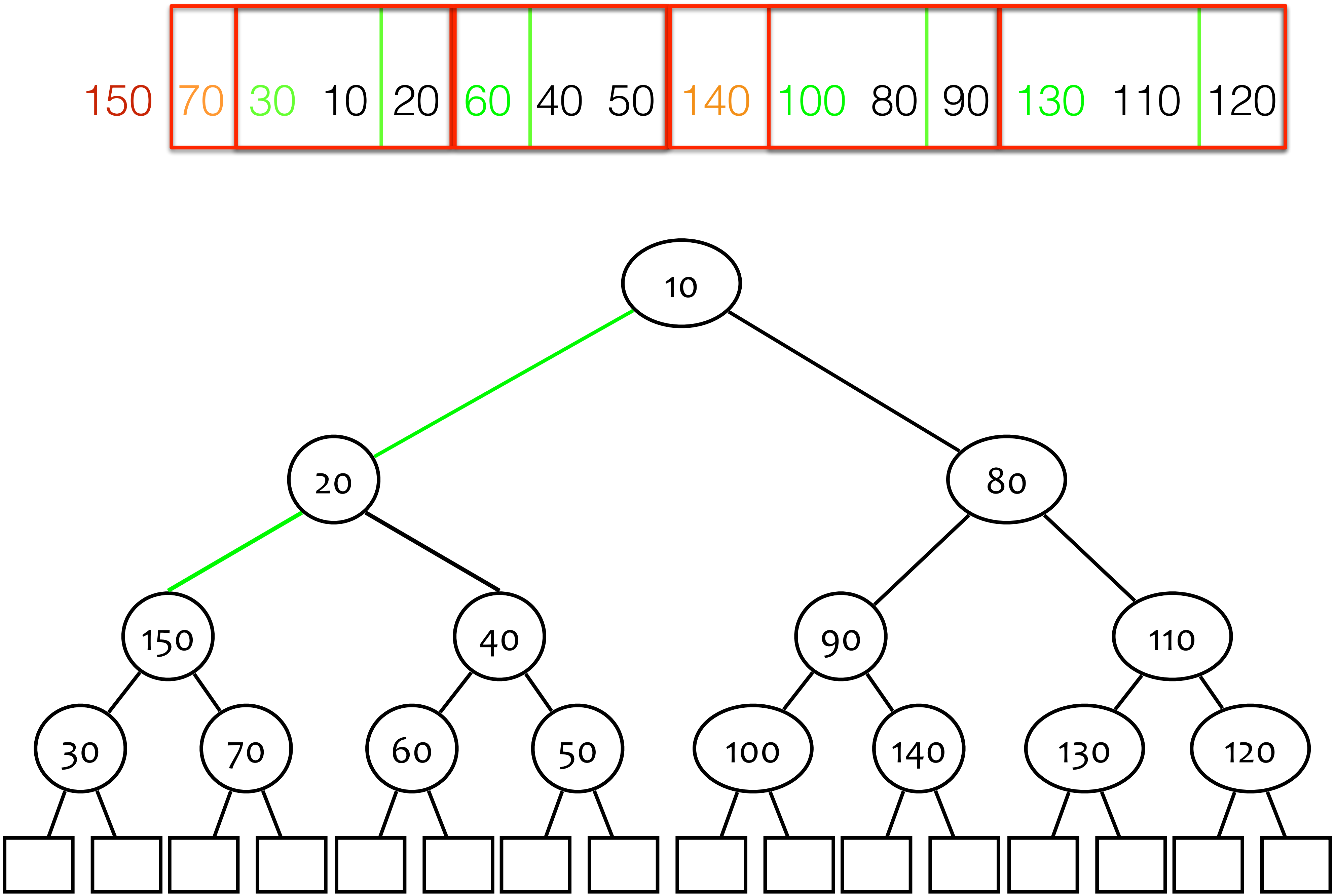
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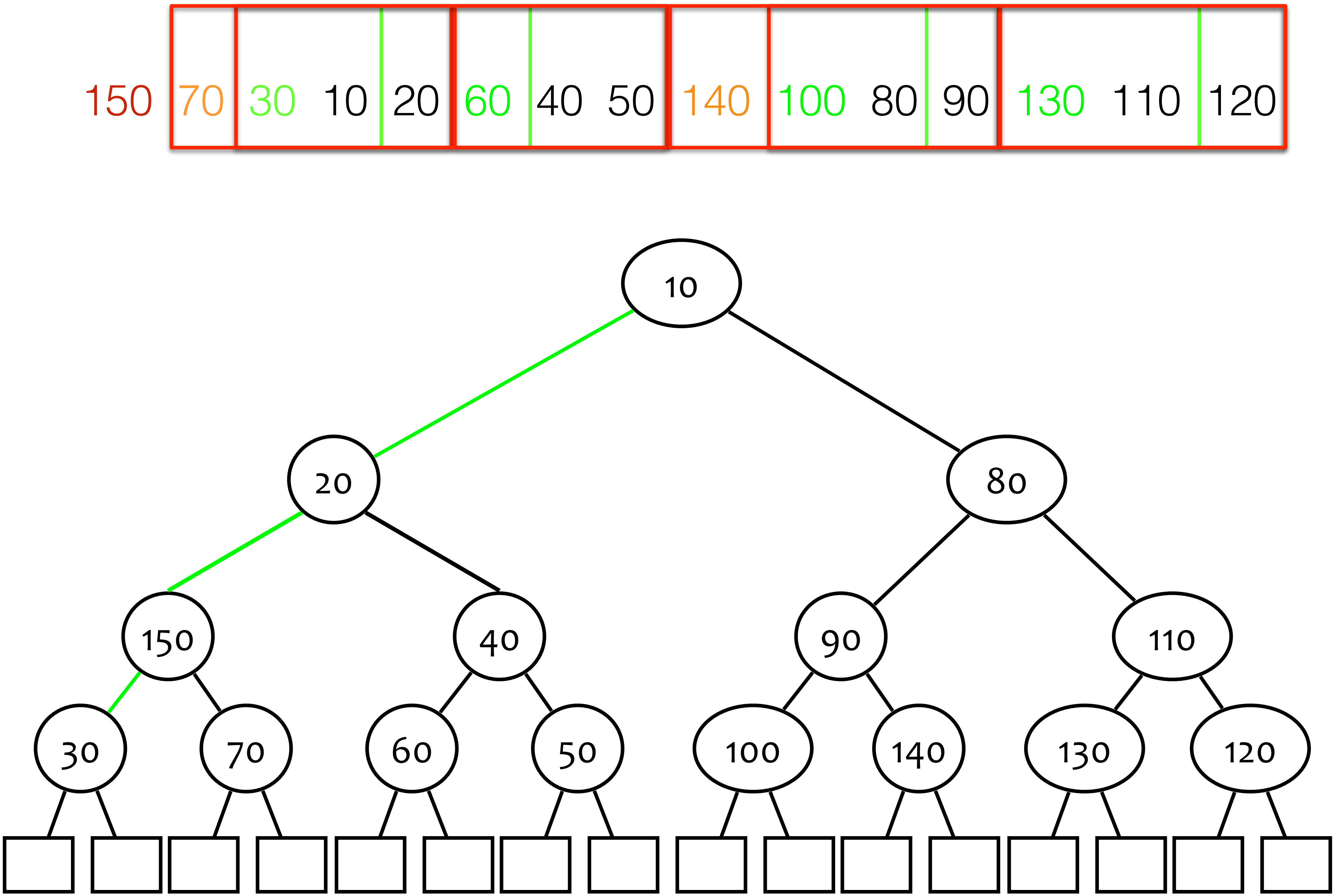

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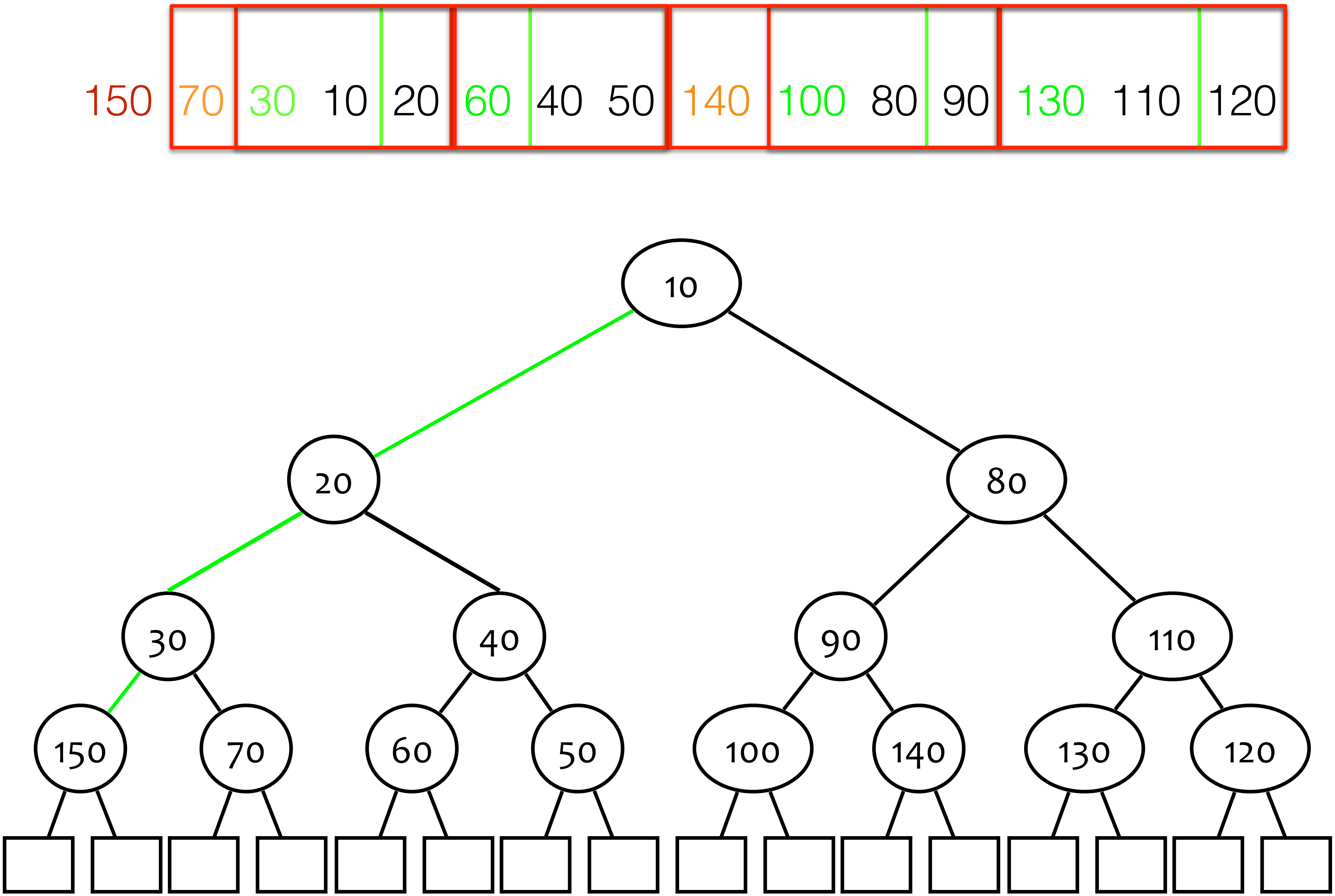
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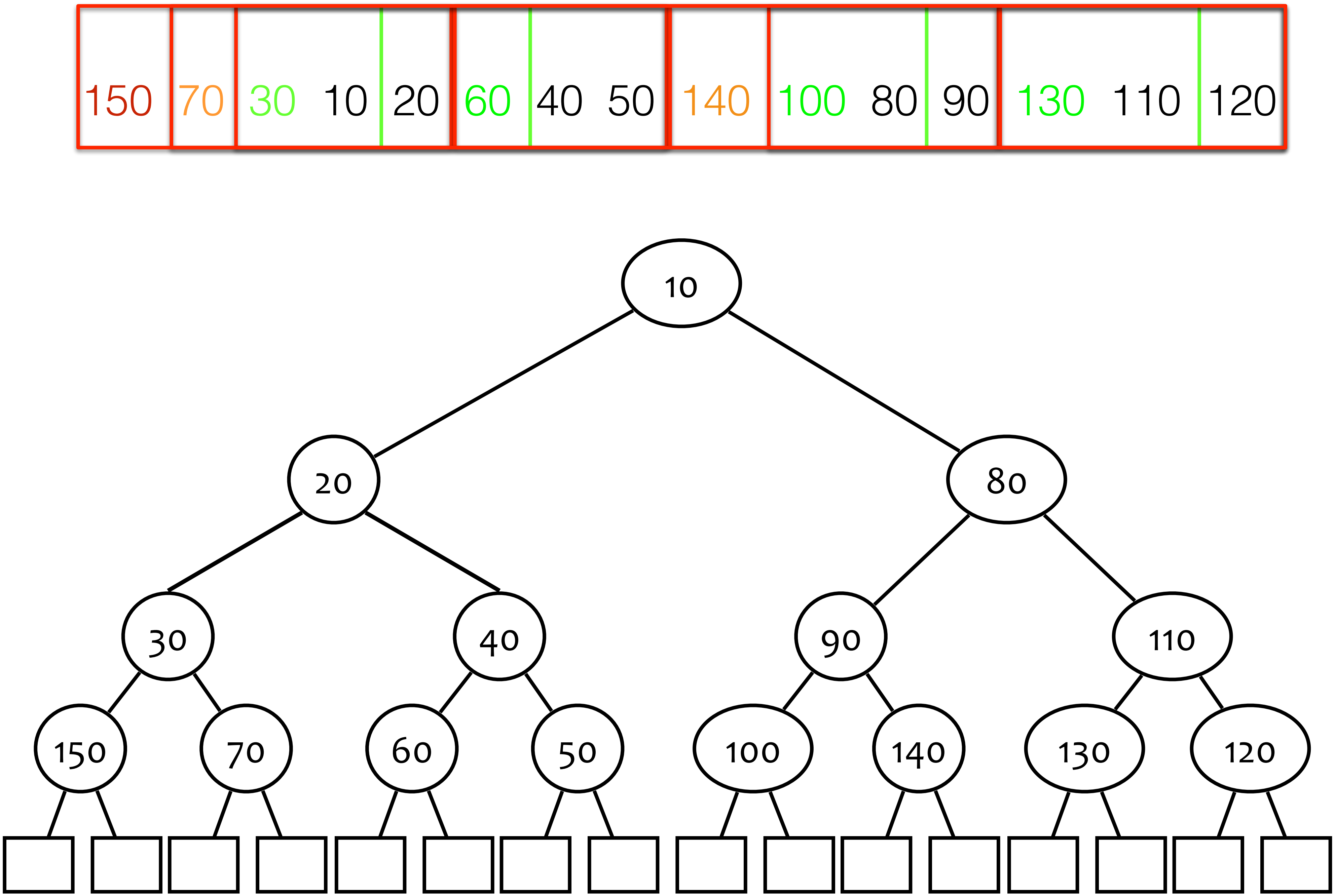

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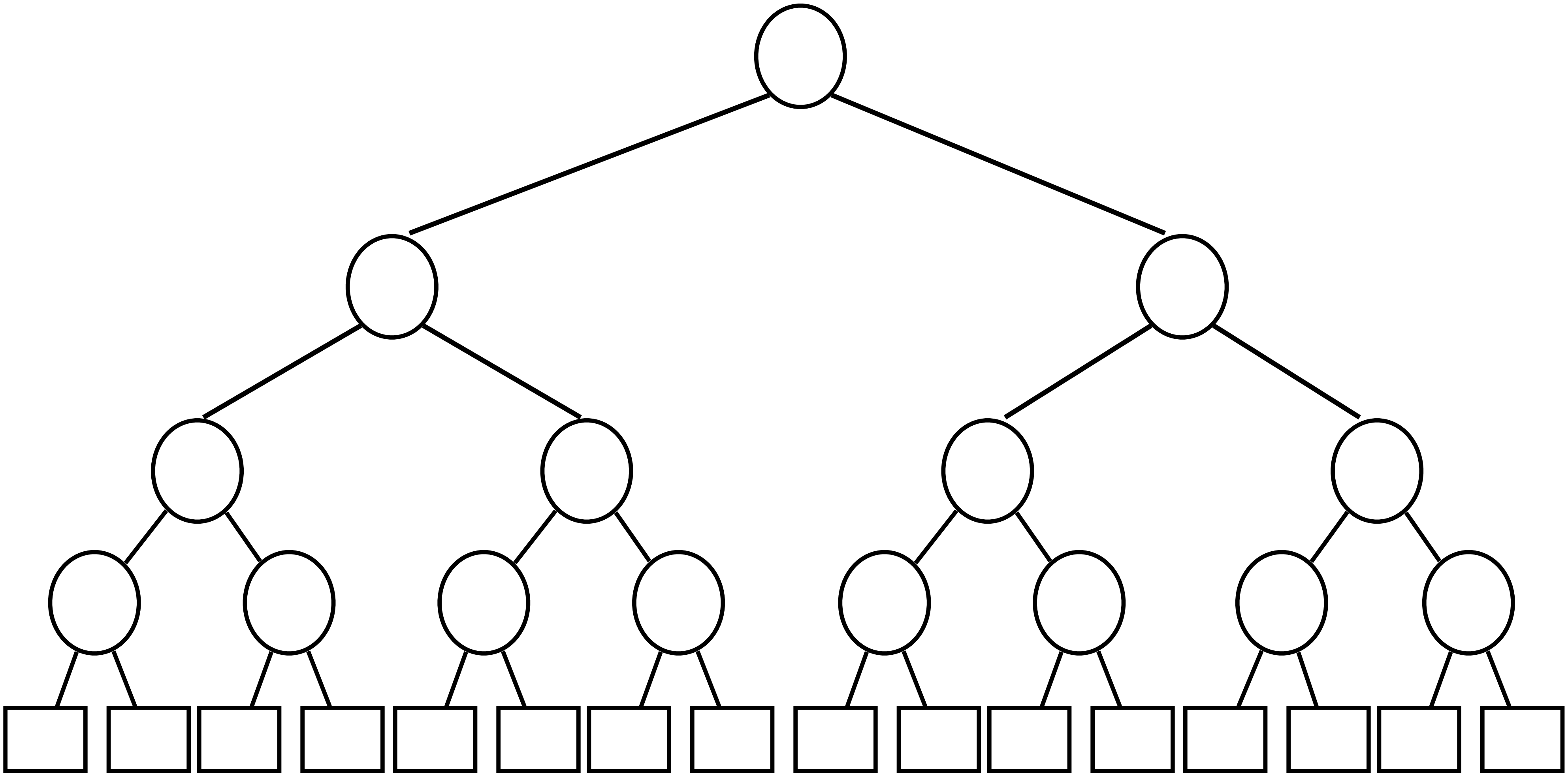
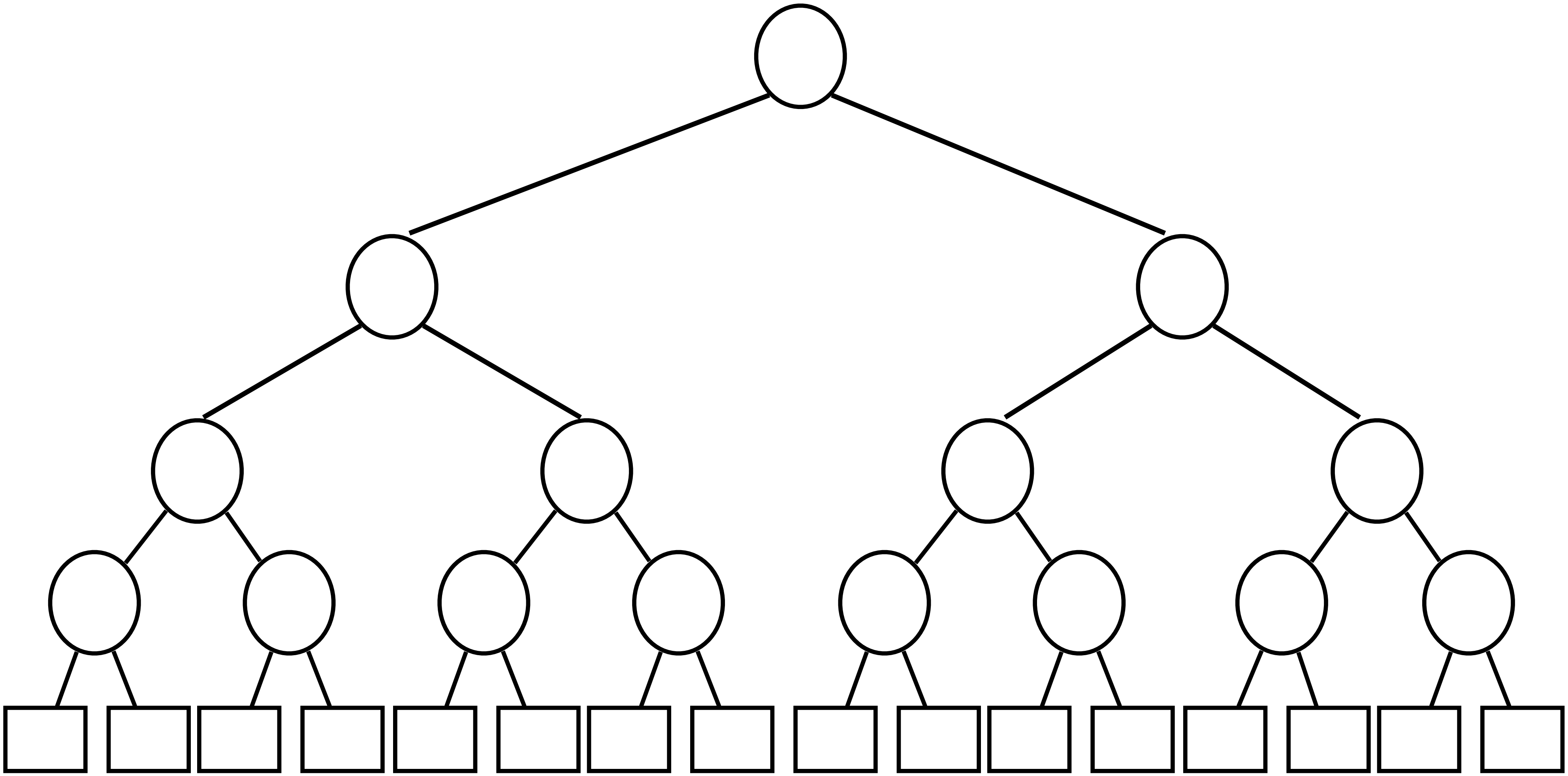
Did we really insert all n elements in $O(n)$ time??



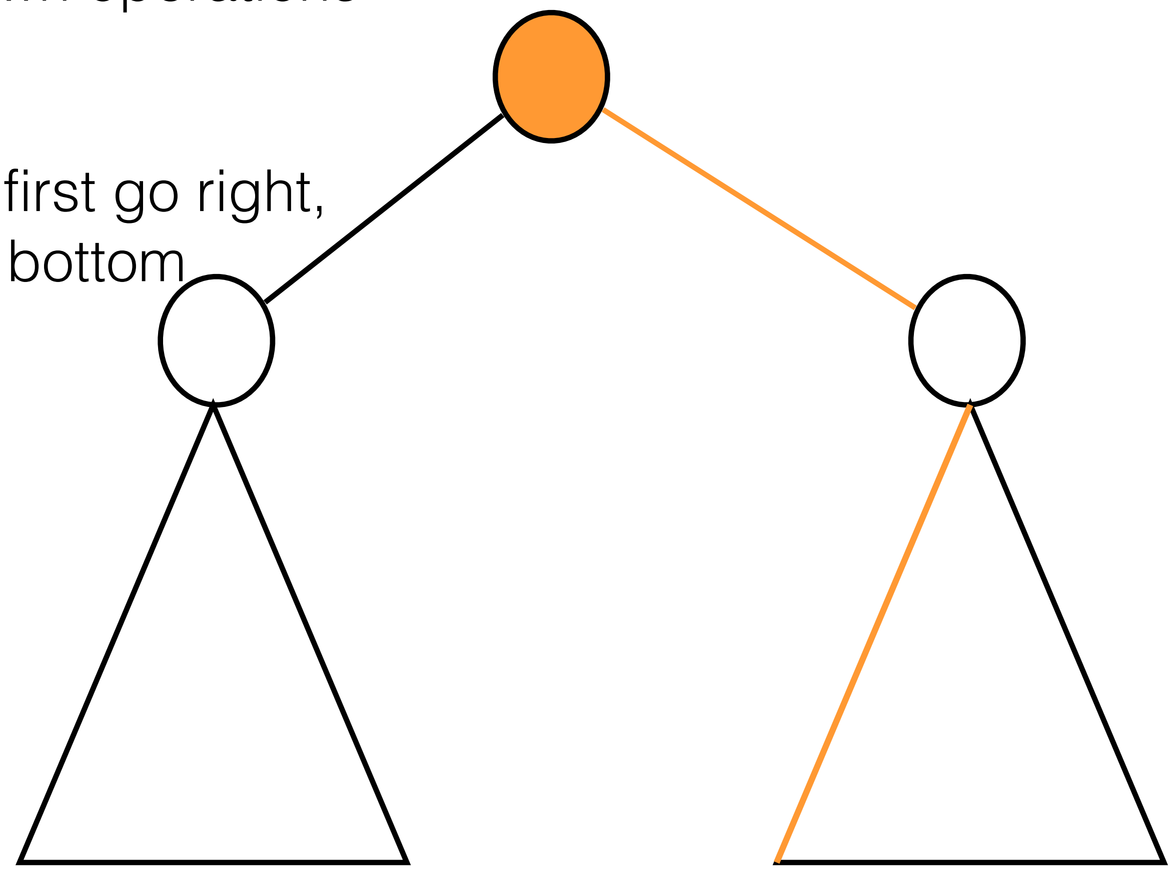
We show: max# bubble down ops $\leq$ # heap edges



Proof idea

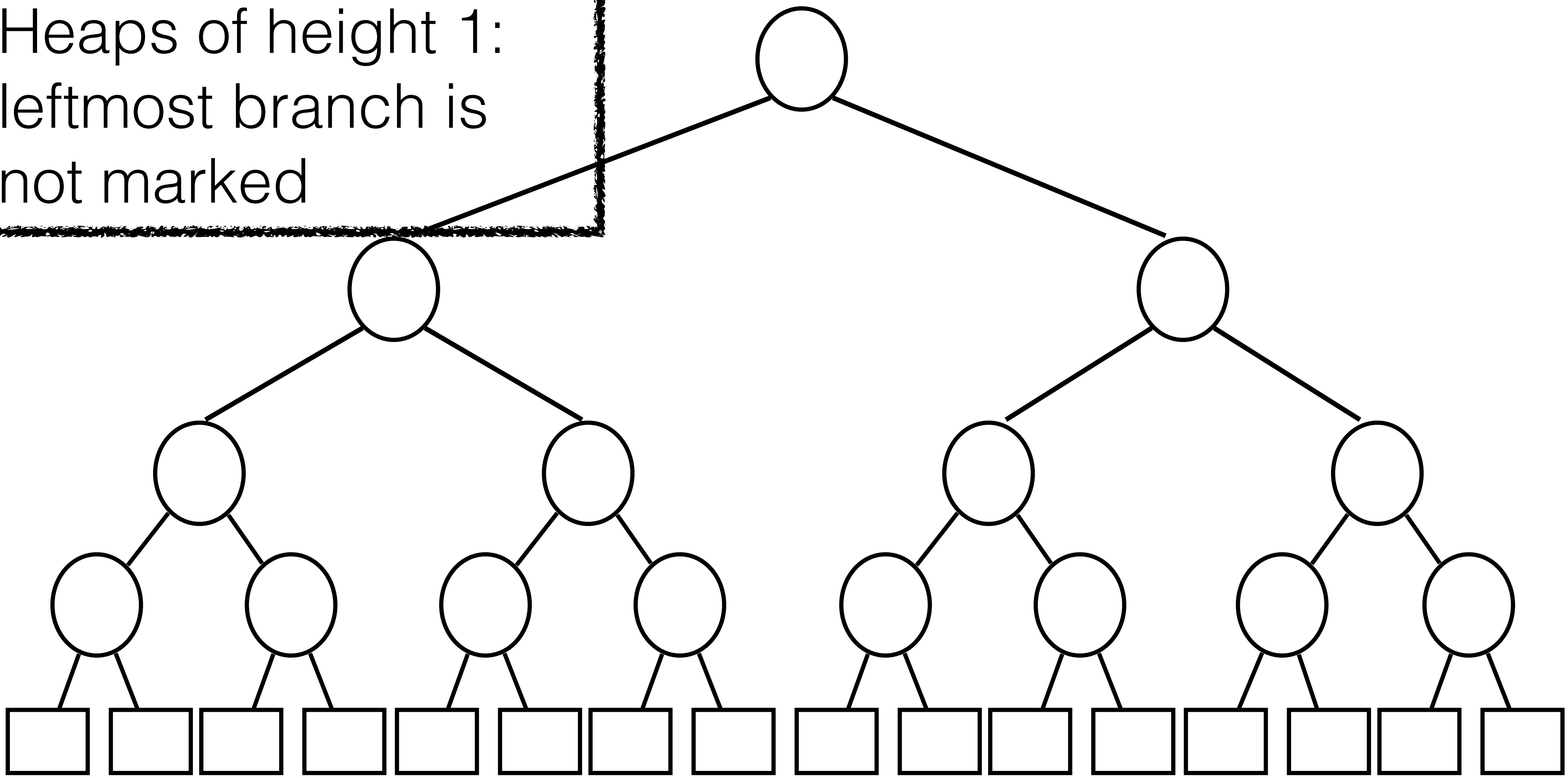
For each new node joining two heaps: mark path of maximum number of bubble-down operations

For marking: first go right, then left until bottom



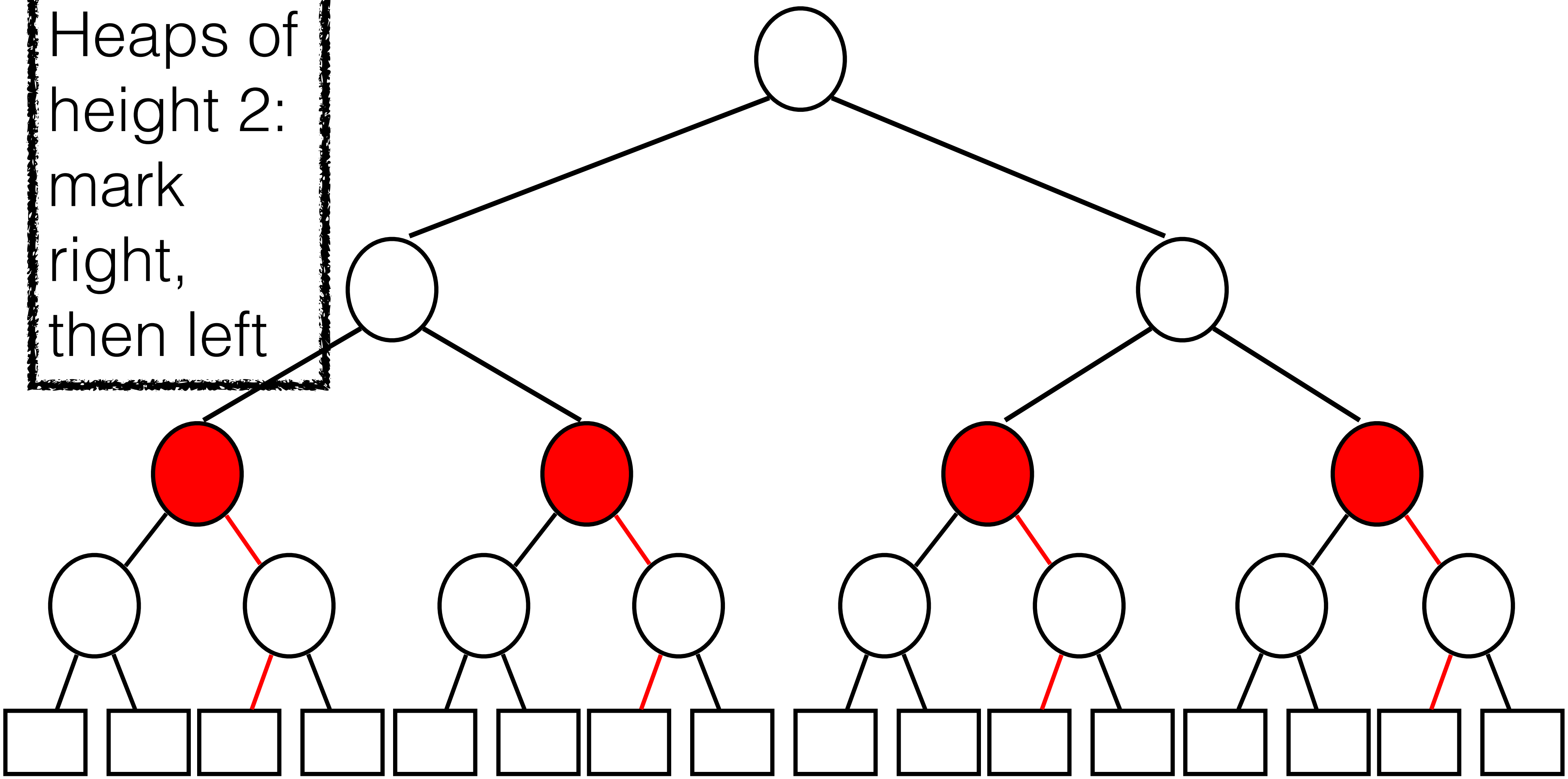
For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 1: leftmost branch is not marked

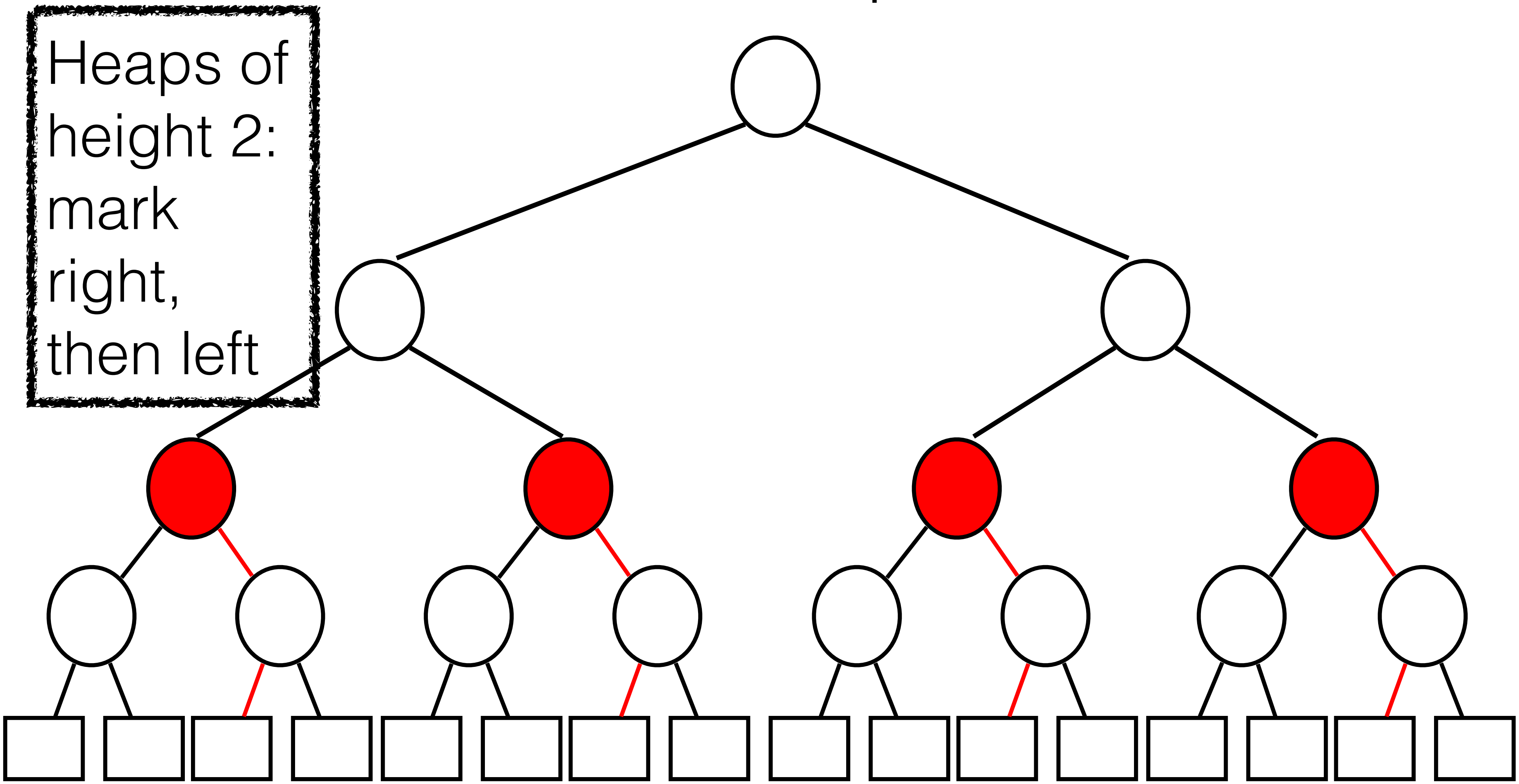


For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 2: mark right, then left



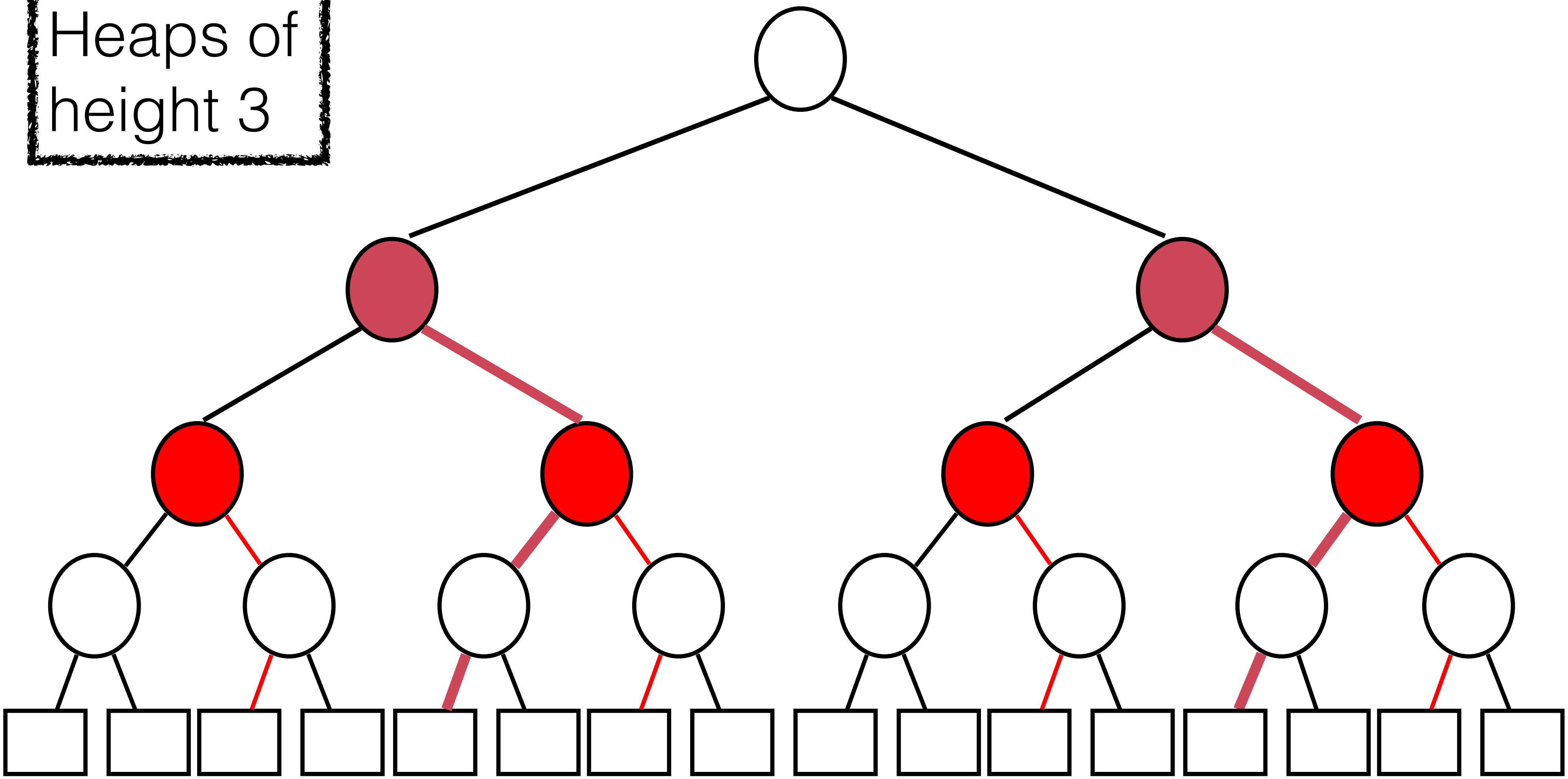
For each new node joining two heaps: mark path with maximum number of bubble-down operations



For each height-2 heap, leftmost branch not marked

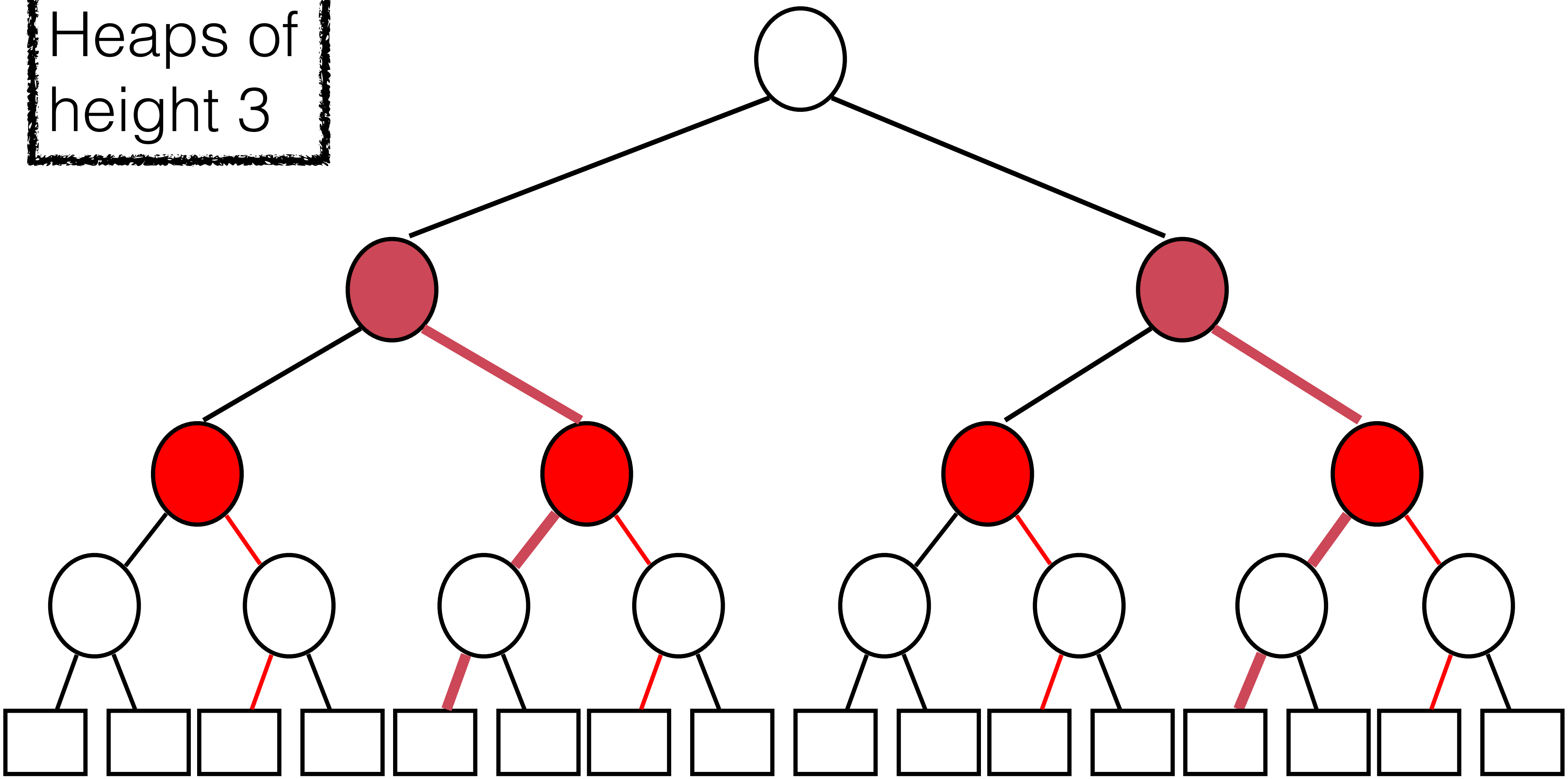
For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 3



For each new node joining two heaps: mark path with maximum number of bubble-down operations

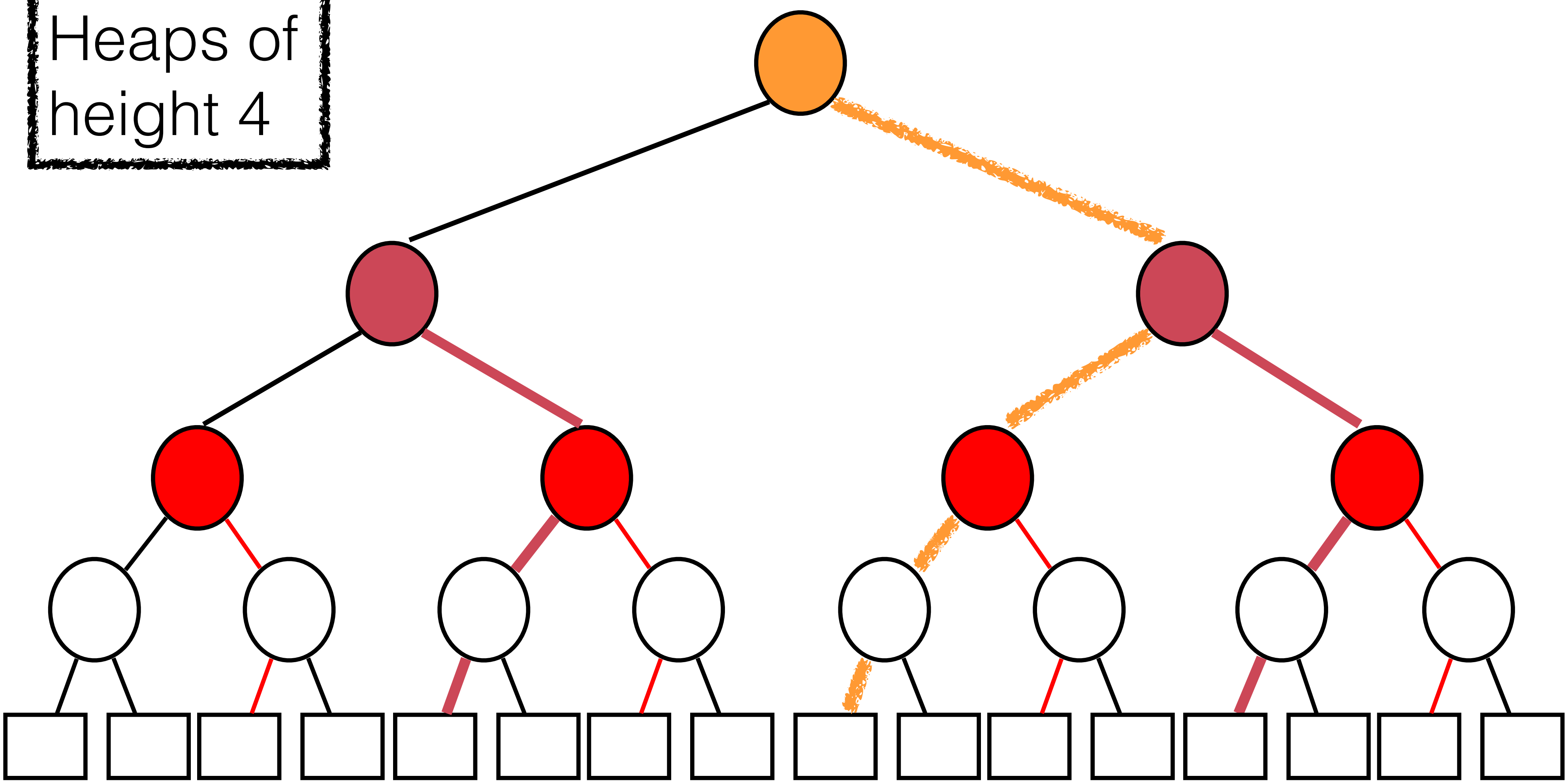
Heaps of height 3



For each height-3 heap, leftmost branch not marked

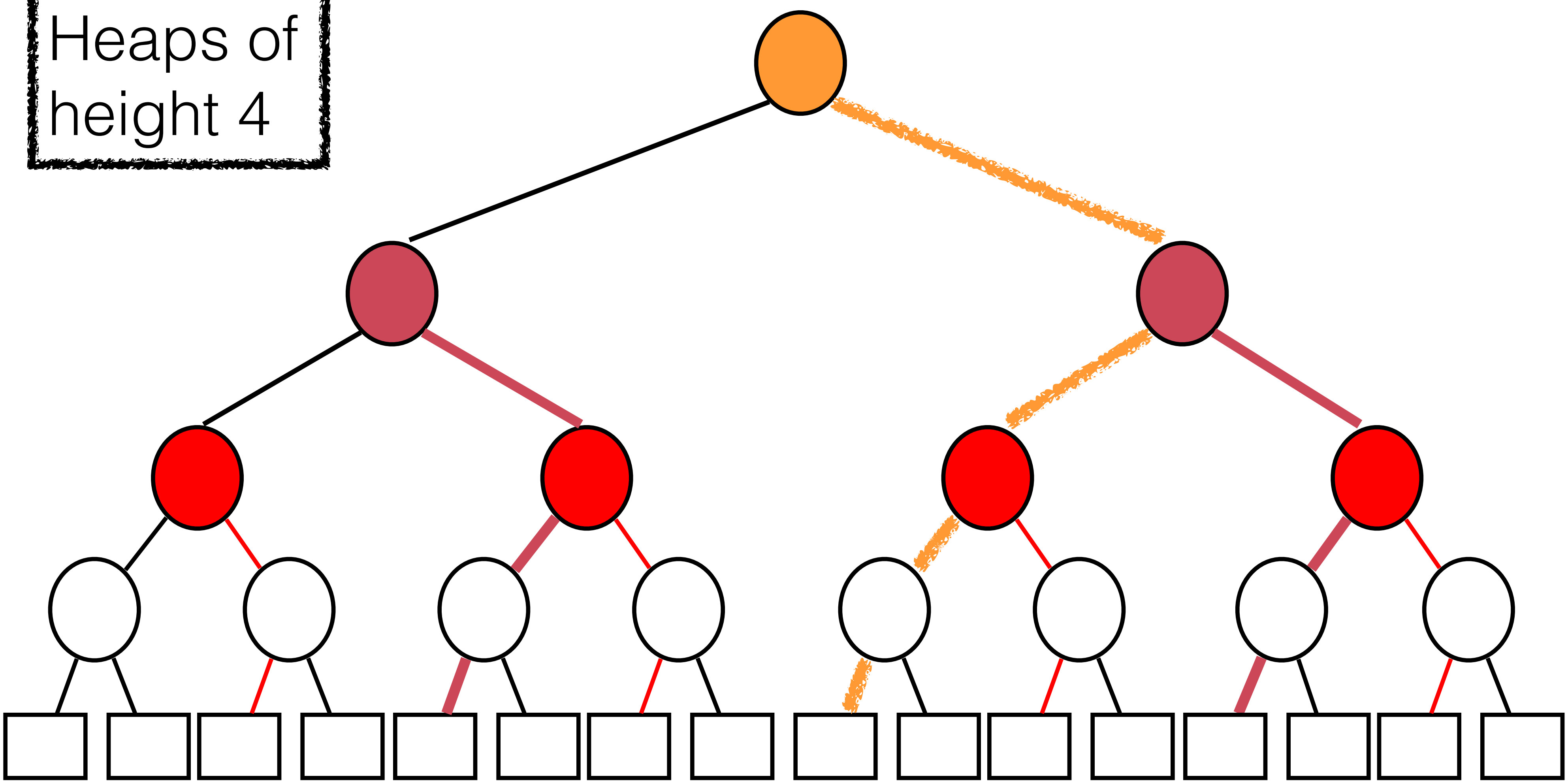
For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 4



For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 4



For height-4 heap, leftmost branch not marked

Inductive argument: marking procedure will never mark all edges in heap, since the leftmost branch is never marked

- Note: leftmost branch in height-h heap: not marked
- When joining 2 heaps of height h to heap of height $h + 1$: new edges to be marked are
  - edge joining new node and right heap of height h, and
  - edges on left path in the right heap of height h
- We conclude: leftmost branch in height-(h+1) heap is not marked

More efficient version (Kruskal's algorithm)

- union-find data structure

Kruskal's Algorithm

Algorithm Kruskal
Input: a weighted connected graph $G = (V, E)$
Output: an MST T for G
Data structure: Disjoint sets (lists or union-find) *DS*;
sorted weights priority queue A; and tree T

for each $v \in V$ **do** C(v) $\leftarrow$ DS.insert(v) **end //** one cluster per vertex
for each (u,*v)* $\in E$ **do** A.insert((u,v)) **end //** sort edges by weight
$T \leftarrow \varnothing$
while T has fewer than n-1 edges **do**
  (u, v) $\leftarrow$ A.deleteMin() // edge with smallest weight
  $C(u) \leftarrow$ DS.findCluster(u);
  $C(v) \leftarrow$ DS.findCluster(v);
  if $C(u) \neq C(v)$ **then**
    add edge (u, v) to T;
    DS.insert(DS.union($C(v)$, $C(u)$)); // merge two clusters
  end
end
return T

Kruskal's Algorithm

Time complexity analysis

Algorithm Kruskal
Input: a weighted connected graph $G = (V, E)$
Output: an MST T for G
Data structure: Disjoint sets (lists or union-find) DS;
    sorted weights A[]; and tree T

for each vertex v in V of G **do** C(v) ← DS.insert(v) **end** — $O(n)$
A[] ← sort all edges by edge weight; // sorted array priority queue
$T \leftarrow \emptyset$; $k \leftarrow 0$; — $O(m \log m)$ or O(m) with heap
while T has fewer than n-1 edges **do**
  $(u, v) \leftarrow$ A[k]; $k \leftarrow k + 1$; // next edge with smallest weight; deleteMin()
  $C(v) \leftarrow$ DS.findCluster(v);
  $C(u) \leftarrow$ DS.findCluster(u);
  if $C(v) \neq C(u)$ **then**
    add edge (v, u) to T;
    DS.insert(DS.union($C(v)$, $C(u)$)); // merge two clusters — Total merging $O(m \log n)$
  end
end
return T

Total $O(m \log n)$

Union-find (or disjoint set) data structure

Disjoint Set Data Structure

- Given a set of elements, it is often useful to break them up or partition them into a number of separate, nonoverlapping sets
- A ***disjoint-set data structure*** is a data structure that keeps track of such a partitioning
- A ***union-find algorithm*** is an algorithm that performs two useful operations on such a data structure:
  - ***Find***: Determine which set a particular element is in. Also useful for determining if two elements are in the same set.
  - ***Union***: Combine or merge two sets into a single set.
  - ***MakeSet***: Creates a set containing only a given element

http://en.wikipedia.org/wiki/Disjoint-set_data_structure

Disjoint Set Data Structure

- The universe consists of n elements, named $1, 2, \ldots, n$
- The ADT is a collection of sets of elements
- Each element is in exactly one set
  - Sets are disjoint
  - To start, each set contains one element
- Each set has a name
  - Which is the name of one of its elements
  - Any name of one of its elements will do

Disjoint Set Operations

- **find(elementName)**
  - Returns the name of the unique set that contains the given element
- **union(setName1, setName2)**
  - Merges two sets and replaces them with one

- **Time complexity analysis**
  - Involves analyzing the *amortized* worst-case running time over a sequence of f find and u union operations

Disjoint Set Implementation I

- Create a linked list for each set and choose the element at the head of the list as the representative
- ***MakeSet*** creates a list of one element
- ***Union*** simply appends two lists, a constant-time operation
- ***Find*** requires linear time (i.e., may search entire list)
- A sequence of m union-find operations takes time $O(mn)$.The amortized time per operation is $O(n)$.

Disjoint Set Implementation II

- Two complementary techniques, *union by rank* and *path compression,* reduce the amortized time complexity per operation to $O(\alpha(n))$, where $\alpha(n)$ is the *inverse* of function $f(n) = A(n,n)$, and A is the extremely quickly-growing Ackermann function
- Since $\alpha(n)$ is its inverse it grows extremely slowly: it is less than 5 for all remotely practical values of n
- Thus, the amortized running time per operation is effectively a small constant
- Thus, n union and find operations require time $O(n\alpha(n))$

Union-find operations

- possible implementation
- Representing of sets in partition

Union-Find Operations: union(a, b)

- Given a partition, let a and b be *canonical elements* of sets A and B, respectively
- If $a \neq b$:
  - Form a new set that is the union of the two sets
  - Destroy the two old sets
  - Select & return canonical element for new set

How should we represent the sets?

- Each set is rooted tree
- Each element of set corresponds to node in tree
- *Canonical element* (= name of set) is root of tree
- Each node e has reference e.parent to its parent in tree, the root points to itself (r.parent = r)

Implementation I
Naive union-find algorithm

- makeset(e): e.parent $\leftarrow e$
  canonical element is root

- find(e):
  - Follow parent pointers from e to root of tree that contains e
  - Return root

- union(a,b)
  - a.parent $\leftarrow b$
  - Return b as canonical element of new set

Union-Find Operations: makeSet(e)

Create singleton set containing single element e
[we do this if e is encountered but is previously in no set]

e.parent $\leftarrow e$

canonical element is root, i.e., e.

Union-Find Operations: makeSet(e)

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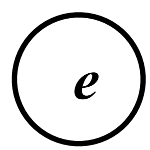

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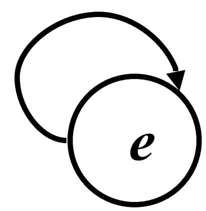

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Union-Find Operations: find(e)

Return canonical element of the set containing e

- Follow parent pointers from e to root r of tree that contains e
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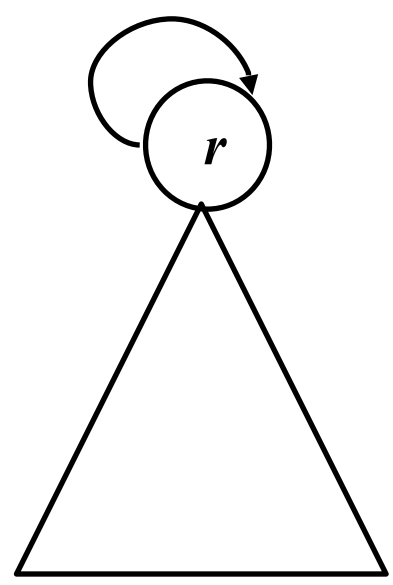

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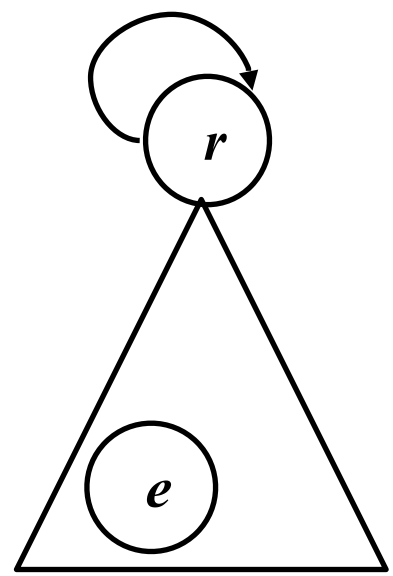

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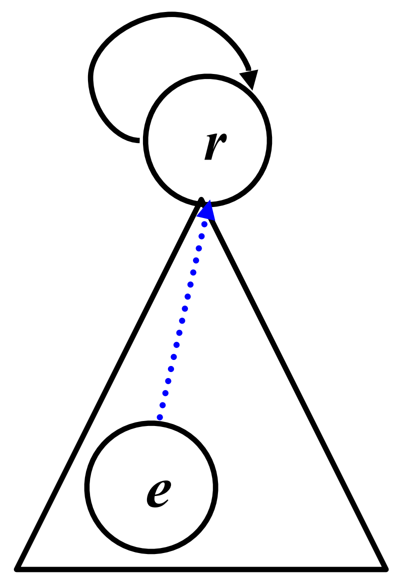
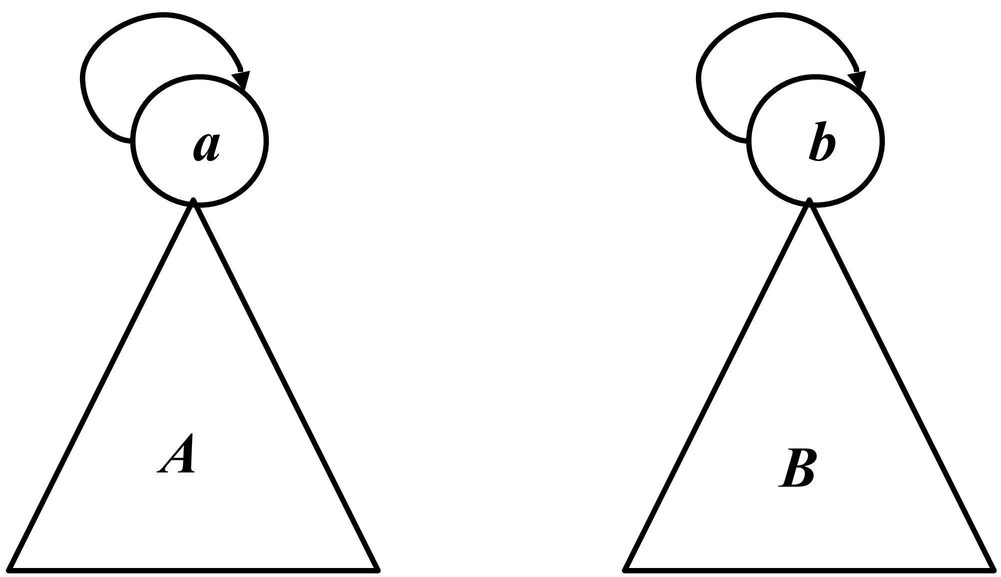


Union-Find Operations: union(a, b)

- Let a and b be canonical elements of sets A and B, respectively
- If $a \neq b$ then form a new set: union of two sets whose canonical elements are a and b
- Destroy two old sets
- Select and return canonical element for new set: a.parent $\leftarrow b$
- Return b as canonical element of new set



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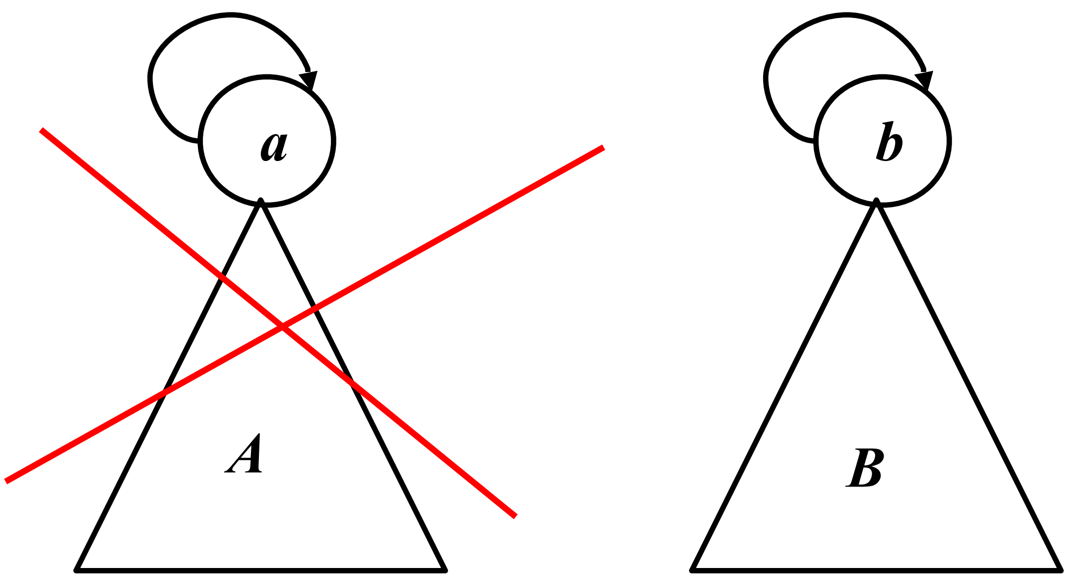

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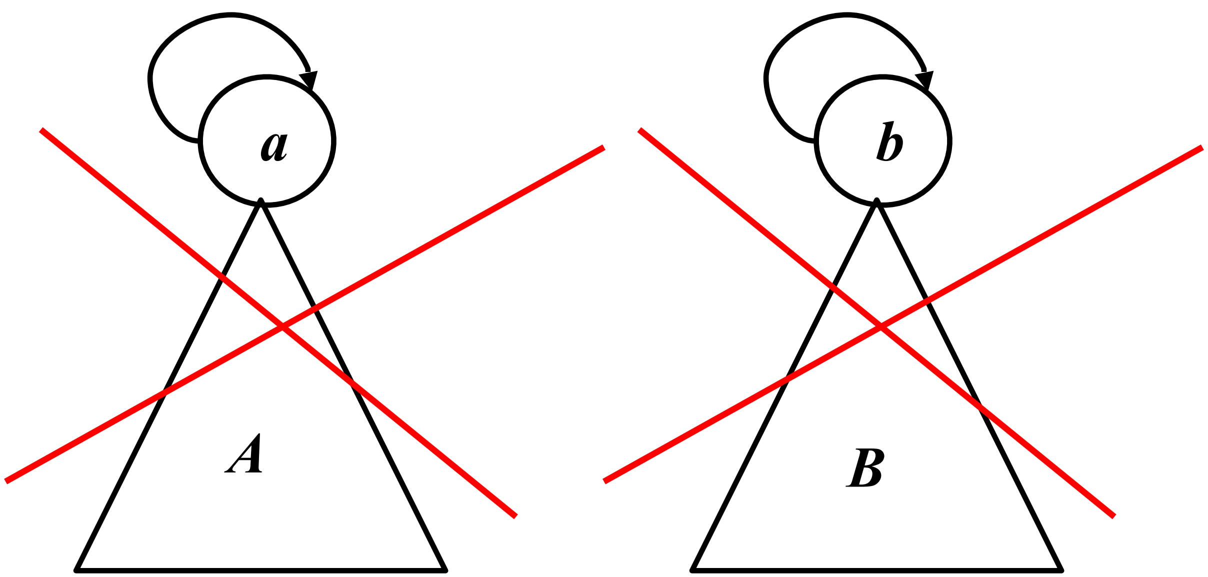

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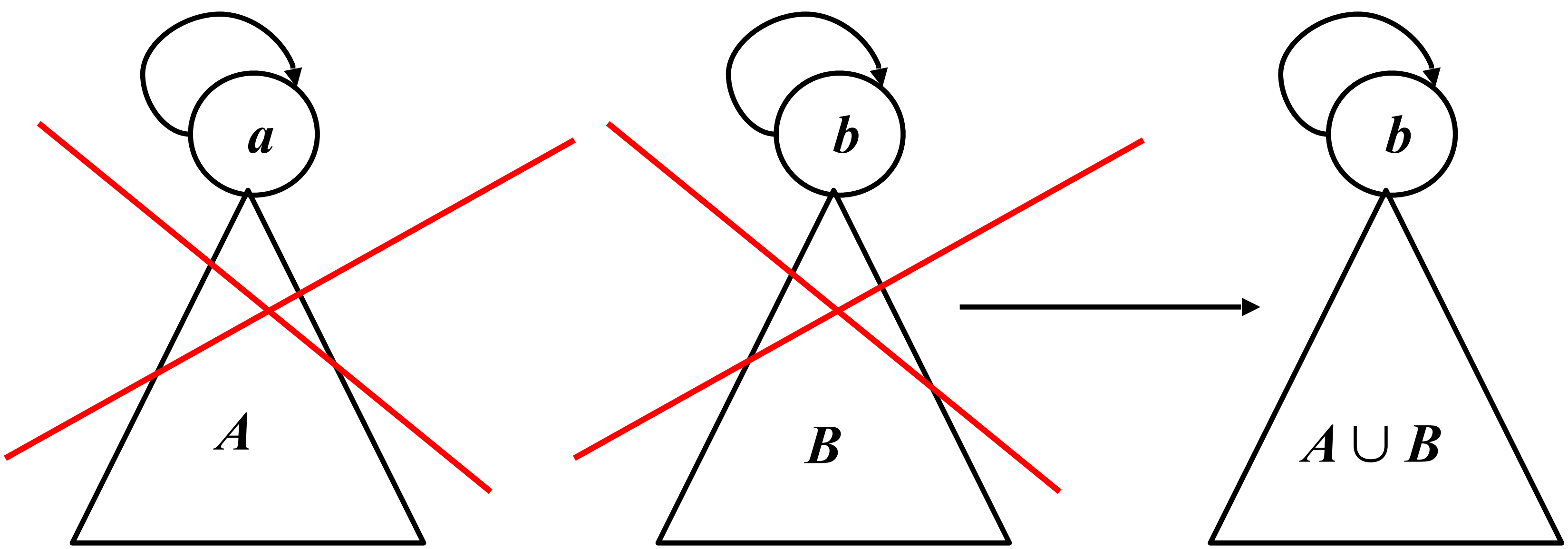

The naive union-find algorithm

- makeset(e): e.parent $\leftarrow e$ (canonical element is root)

- find(e):
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- union(a,b): a.parent $\leftarrow b$ (Return b as canonical element of new set)

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  $O(1)$

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The naive union-find algorithm

- makeset(e): e.parent $\leftarrow e$ (canonical element is root) O(1)
- find(e):
  - Follow parent pointers from e to root of tree containing e
  - Return root

  O(n)
- union(a,b): a.parent $\leftarrow b$ (Return b as canonical element of new set) O(1)

An *improved* union-find algorithm

- makeset(e): e.parent $\leftarrow e$, the canonical element is the root.

- find(e): Follow the parent pointers from e to the root of the tree containing e. Return the root.

- **union(a,b): a.parent $\leftarrow$ canonical element of *larger* of two sets**

Why is this better?

Improved Union-Find: Union by Rank

Idea: Store with each node v the size of the subtree (*rank*) rooted at y. In a union operation, make the tree of the smaller set/rank a subtree of the other tree, and update the size field of the root of the resulting tree.

Union by Rank

- With each node x we store a nonnegative integer x.rank that is an upper bound on the height of x.

- When carrying out makeSet(x), we set x.rank $\leftarrow$ 0.
- To carry out union(x,y), we compare x.rank and y.rank.
  - If x.rank < y.rank then x.parent $\leftarrow y$.
  - If y.rank < x .rank then y.parent $\leftarrow x$.
  - If x.rank = y.rank then x .parent $\leftarrow y$ and y.rank $\leftarrow$ y.rank + 1.

Amortized Time Complexity

- A series of n makeSet, (modified) union, and find operations starting from an initially empty partition take
  O(n log n) time.

- The *amortized* running time per operation is O(log n) time.

A series of n makeSet, (modified) union, and find operations starting from an initially empty partition take O(n log n) time

Lemma: Starting with sets of size 1, using the modified union algorithm, any tree of m nodes has height of at most log m .