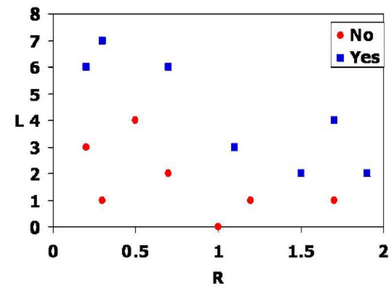
Linear Classifiers I: Perceptron

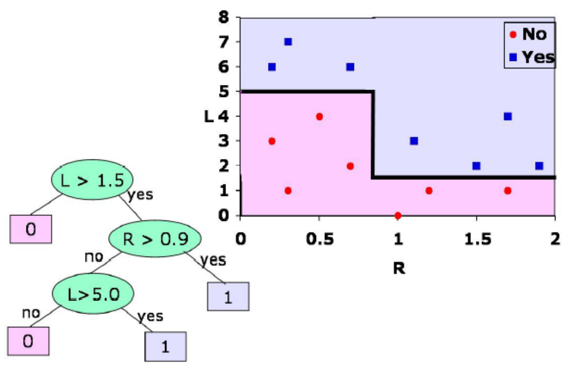
Bankruptcy example

R is the ratio of earnings to expenses
L is the number of late payments on credit cards over the past year.

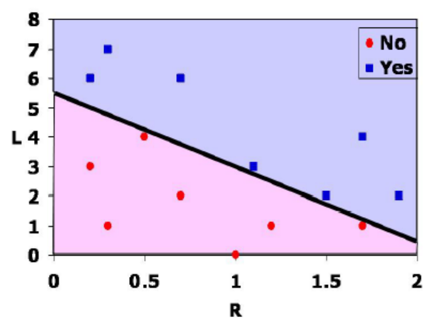
We would like to draw a **linear separator**, and thus get a classifier.



Classification as Boundary: E.g. Decision Tree Boundary



Simple Linear Boundary



Linear Hypothesis Class

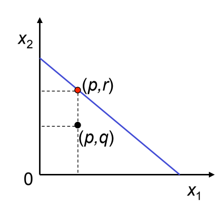
- Line equation (assume 2D first): $w_2x_2+w_1x_1+b=0$
- **Fact1**: All points (x_1, x_2) lying on the line make the equation true.
- **Fact2**: The line separates the plane in two half-planes.
- **Fact3**: The points (x_1, x_2) in one half-plane give us an inequality with respect to 0, which has the same direction for each of the points in the half-plane.
- **Fact4**: The points (x_1, x_2) in the other half-plane give us the reverse inequality with respect to 0.

Fact 3 proof

$w_2x_2+w_1x_1+b=0$

We can write it as:

$$x_2 = -\frac{w_1}{w_2}x_1 - \frac{b}{w_2}$$



(p,r) is on the line so:

$$r = -\frac{w_1}{w_2}p - \frac{b}{w_2}$$

For $q<r$, so we have: $q < r = -\frac{w_1}{w_2}p - \frac{b}{w_2}$ *i.e.*

$$w_2q + w_1p + b < 0 \qquad \text{if } w_2 > 0$$
$$w_2q + w_1p + b > 0 \qquad \text{if } w_2 < 0$$

Since (p,q) was an arbitrary point in the half-plane, we say that the same direction of inequality holds for any other point of the half-plane. Fact 4 is similar.

Linear classifier

$h(\mathbf{x}, \mathbf{w}, b) = sign(\mathbf{w} \cdot \mathbf{x} + b)$

Which outputs +1 or –1.

Say:
+1 corresponds to blue, and
-1 to red, or vice versa.

One small change

$$h(\mathbf{x}, \mathbf{w}, b) = \text{sign}\left(\left(\sum_{i=1}^{m} w_i x_i\right) + b\right) \qquad h(\mathbf{x}, \mathbf{w}) = \text{sign}\left(\left(\sum_{i=1}^{m} w_i x_i\right) + w_0\right)$$

$w_0 = b$

$\mathbf{x} = [1, x_1, .., x_m]$ $\quad$ $\mathbf{w} = [w_0, w_1, .., w_m]$

$$h(\mathbf{x}, \mathbf{w}) = \text{sign}\left(\sum_{i=0}^{m} w_i x_i\right) = \text{sign}(\mathbf{w} \cdot \mathbf{x}) = \text{sign}(\mathbf{w}^T \mathbf{x})$$

Learning Algorithm

Start with random w's $\qquad h(\mathbf{x}, \mathbf{w}) = \text{sign}(\mathbf{w}^T \mathbf{x})$

Training tuples
$\mathbf{x}^1, y^1$
$\mathbf{x}^2, y^2$
...
$\mathbf{x}^n, y^n$

For each misclassified training tuple, e.g.

$\text{sign}(\mathbf{w}^T \mathbf{x}^k) \neq y^k$

Update **w**

$\mathbf{w} = \mathbf{w} + \eta \cdot y^k \mathbf{x}^k$

Well, why is this a good rule?

It can be shown that if the data is linearly separable, and we repeat this procedure many times, we will get a line that separates the training tuples.

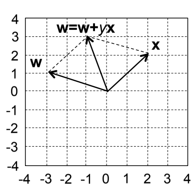
Recall the dot product

$$A \cdot B = \sum_i A_i B_i$$
$$= ||A|| * ||B|| * cos(\theta)$$

- ||A|| and ||B|| can only be positive
- cos(θ) is negative when the angle is between ½ pi and 3/2 pi
  - that is, when the angle is obtuse

Sign of dot product and **misclassification**

y=+1
w.x<0
y.**w.x**<0



y=-1
w.x>0
y.**w.x**<0