CENG461/ELEC514 Assignment 2: Solutions

1. Consider the queue of an ATM machine. Customers arrival process can be modeled as a Markovian process that every minute, the probability of one customer arrive is a. The customer departure process can also be approximated as a Markovian process that every minute, the probability of the customer (currently using the ATM machine) finishing his/her transition and leaving is c. We can use a discrete time M/M/1 queue to analyze the system with $a = 0.1$ and $c = 0.2$.

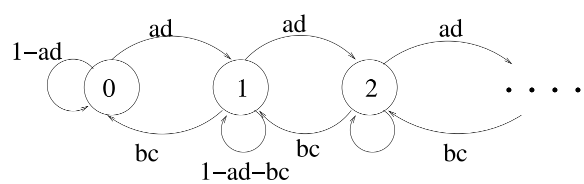
   a) What is the probability that when a customer arrives, nobody is using the ATM machine, i.e., queue is empty. (Hint: find out s_0)

   b) What is the average waiting time of the customers using the ATM machine?

   c) When a customer arrives the ATM machine, he/she will be unhappy if there are more than 4 people in the queue (including the one currently using the machine). What is the probability that the customer will be unhappy?

   d) We assume that all customers will decide not to use the ATM machine if there are 4 in the queue (including the one currently using the machine). What is the average waiting time of the customers using the ATM machine? (Hint: using M/M/1/B queue)

   Solution: a)



   For M/M/1 queue, we have $s_i = s_0\rho^i$. Here $\rho = ad/bc = (0.1 \times 0.8)/(0.2 \times 0.9) = 4/9$.

   Since $\sum_{i=0}^{\infty} s_i = 1$, $s_0(1 + \rho + \rho^2 + ...) = 1$.

   $s_0 = 1 - \rho = 5/9$.

   Customer will find empty queue with probability $5/9$.

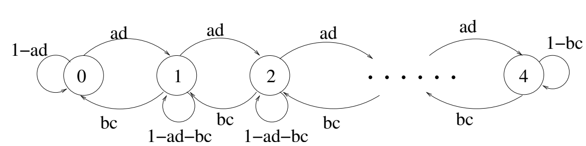
   b) $W = Q_a/Th$, where $Q_a = \sum_{i=0}^{\infty} i s_i = \frac{\rho}{1-\rho} = 4/5$ and $Th = a = 0.1$.

   Thus, the average waiting time W equal to 8 minutes.

   c) $\sum_{i=5}^{\infty} s_i = \sum_{i=5}^{\infty}(1-\rho)\rho^i = (1-\rho)\rho^5 \sum_{k=0}^{\infty} \rho^k = \rho^5 = (4/9)^5 = 0.0173$.

   The probability of unhappy is 0.0173.

   d) Since the queue length will not exceed 4, we have a M/M/1/B queue with $B = 4$.



   $\rho = 4/9$

   $s_0 = \frac{1-\rho}{1-\rho^{B+1}} = 0.565$

$Th = c(1 - bs_0) = 0.2(1 - 0.9 \times 0.565) = 0.098$

$Q_a = \frac{\rho[1-(B+1)\rho^B + B\rho^{B+1}]}{(1-\rho)(1-\rho^{B+1})} = 0.712$

The average waiting time, W, equal to $Q_a/Th = 0.712/0.098 = 7.265$ minutes.

2. One packet will arrive at a router with probability $a = 0.4$ per time step. The probability of one packet being served and leaving the router per time step is $c = 0.5$. What is the appropriate buffer size (B) such that the packet loss ratio (L) is less than 1%.

   Solution: This is a M/M/1/B queuing system.

   $\rho = ad/bc = 2/3$

   $s_B = \frac{(1-\rho)\rho^B}{1-\rho^{B+1}}$

   $L = s_B d < 0.01$

   $\Rightarrow \frac{(1-\rho)\rho^B}{1-\rho^{B+1}} < 0.02$

   $\Rightarrow \rho^B < 0.0577$

   $\Rightarrow B > 7.035$

   The buffer size should be no less than 8.

3. At the Markov Coffee Shop, there is only one cashier. Due to the limited space, she allows at most 3 customers to line before her at any time. In other words, if a customer finds there are 3 customers there (including the one currently being served by the cashier), the new arriving customer will leave the Shop immediately (be blocked).

   Every minute, the following may occur:

   - one new customer arrives with probability $\frac{0.8}{2^k}$, where k is the number of customers in the Shop (including the one currently being served by the cashier);
   - one customer leaves the Shop with probability 0.5 after being served;

   (a) This problem can be modeled as an M/M/1/B system. Define appropriate states, find the state transition probabilities, and draw the state transition graph.

   (b) After the Coffee Shop has been open for a long time (in steady state), what is the probability that the cashier is idle, i.e., no customer in the Shop.

   (c) In steady state, calculate on the average how many customers are in the Shop.

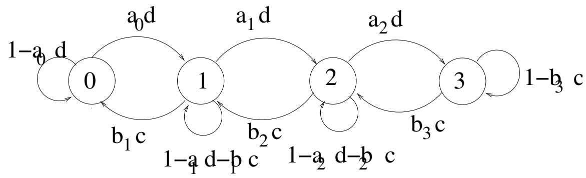
   Solution:

   (a) Let state k represent that there are k customers in the Shop, where $0 \le k \le 3$.

   The probability of one custom arrival per time step when the system is in state k is $a_k = \frac{0.8}{2^k}$ for $0 \le k \le 3$.

   The probability of no custom arrival per time step when the system is in state k is $b_k = 1 - \frac{0.8}{2^k}$ for $0 \le i \le 3$.

   The probability of one custom depart per time step is $c = 0.5$. The probability of no custom depart per time step is $d = 0.5$.



(b) From balance equations:

$s_1 = s_0 \frac{a_0 d}{b_1 c} = \frac{4}{3} s_0$
$s_2 = s_0 \frac{a_0 d a_1 d}{b_1 c b_2 c} = \frac{2}{3} s_0$
$s_3 = s_0 \frac{a_0 d a_1 d a_2 d}{b_1 c b_2 c b_3 c} = \frac{4}{27} s_0$

Since $\sum_{i=0}^{3} s_i = 1$, we get $s_0 = \frac{27}{85} \approx 0.318$.

(c) $Q_a = \sum_{i=0}^{3} i s_i = s_1 + 2s_2 + 3s_3 = \frac{84}{85} \approx 0.988$

(d) $L = s_3 a_3 d / N_a(in)$ where $N_a(in) = \sum_{i=0}^{3} s_i a_i$. $L = 0.005$.