Math 101, Spring 2009
Assignment 3 Solutions

1. (a) (3 points) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx$

   (b) (3 points) $\displaystyle\int \sqrt{\tan x}\sec^4 x\; dx$

Solution:

(a) Using trig substitution: Let $x = \sin\theta$, then $dx = \cos\theta\; d\theta$ and

$$\begin{aligned} \theta(0) &= \sin^{-1} 0 = 0, \\ \theta(1) &= \sin^{-1} 1 = \pi/2. \end{aligned}$$

Substituting, we have:

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\; d\theta &= \int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cos\theta\; d\theta \\ &= \int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta}\cos\theta\; d\theta \\ &= \int_0^{\pi/2} \sin^2\theta\; d\theta \end{aligned}$$

The positive square root is chosen since $\cos\theta > 0$ for $0 \le \theta \le \pi/2$. Using the half-angle formula:

$$\int_0^{\pi/2} \sin^2\theta\; d\theta = \int_0^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

(b) This trigonometric integral can be handled by pulling a $\sec^2$ out and converting the rest into tangents (using $\sec^2 x = 1 + \tan^2 x$).

$$\begin{aligned} \int \sqrt{\tan x}\sec^2 x \sec^2 x\, dx &= \int \sqrt{\tan x}(1+\tan^2 x)\sec^2 x\, dx \\ &= \int \left[(\tan x)^{1/2} + (\tan x)^{5/2}\right]\sec^2 x\; dx \\ &= \int \left[(\tan x)^{1/2} + (\tan x)^{5/2}\right] \underbrace{d[\tan x]}_{= sec^2 x\, dx} \\ &= \frac{2}{3}(\tan x)^{3/2} + \frac{2}{7}(\tan x)^{7/2} + C \end{aligned}$$

2. (a) (4 points) $\int (\sin^{-1} x)^2 \; dx$

(b) (4 points) Using integration by parts, calculate $\int \cos 7x \sin 3x \; dx$

Solution:

(a) With substitution, then integration by parts:

Let $t = \sin^{-1} x$, then $\sin t = x$ and $\cos t \; dt = \; dx$. Substitution yields

$$\int (\sin^{-1} x)^2 \; dx = \int t^2 \cos t \, dt$$

By the LIATE method, let $u = t^2$ and $dv = \cos t \, dt$. Then $du = 2t \, dt$ and $v = \sin t$, and repeated integration by parts yields

$$\begin{aligned} \int t^2 \cos t \, dt &= t^2 \sin t - \int 2t \sin t \, dt \\ &= t^2 \sin t + 2 \int \underbrace{t}_{u} \underbrace{d[\cos t]}_{d[v]} \\ &= t^2 \sin t + 2 \left[t \cos t - \int \cos t \, dt \right] \\ &= t^2 \sin t + 2t \cos t - 2 \sin t + C \\ &= x(\text{arcsin} x)^2 + 2\sqrt{1 - x^2} \arcsin x - 2x + C \end{aligned}$$

With integration by parts directly:

By the LIATE method, let $u = (\arcsin x)^2$ and $dv = dx$. Then $du = \dfrac{2 \arcsin x}{\sqrt{1 - x^2}} dx$ and $v = x$. Repeated integration by parts yields

$$\begin{aligned} \int (\sin^{-1} x)^2 \; dx &= x \, (\arcsin x)^2 - 2 \int \underbrace{\arcsin x}_{u} \underbrace{\frac{x \, dx}{\sqrt{1 - x^2}}}_{dv} \\ &= x \, (\arcsin x)^2 - 2 \int \underbrace{\arcsin x}_{u} \underbrace{d[-\sqrt{1 - x^2}]}_{d[v]} \\ &= x \, (\arcsin x)^2 + 2 \left[\sqrt{1 - x^2} \arcsin x - \int \sqrt{1 - x^2} \underbrace{\frac{dx}{\sqrt{1 - x^2}}}_{du} \right] \\ &= x \, (\arcsin x)^2 + 2\sqrt{1 - x^2} \arcsin x - 2x + C \end{aligned}$$

(b) Let $u = \cos 7x$ and $dv = \sin 3x\ dx$. Then $du = -7\sin 7x$ and $v = -1/3\cos 3x$. Repeated integration by parts yields:

$$\begin{aligned}
\int \cos 7x \sin 3x\ dx &= -\frac{1}{3}\cos 7x\cos 3x - \frac{7}{3}\int \underbrace{\sin 7x}_{u}\ \underbrace{\cos 3x\ dx}_{dv} \\
&= -\frac{1}{3}\cos 7x\cos 3x - \frac{7}{3}\int \underbrace{\sin 7x}_{u}\ \underbrace{d\left[\frac{1}{3}\sin 3x\right]}_{d[v]} \\
&= -\frac{1}{3}\cos 7x\cos 3x - \frac{7}{3}\left[\frac{1}{3}\sin 3x\sin 7x - \int \frac{1}{3}\sin 3x\,\underbrace{(7\cos 7x\,dx)}_{du}\right] \\
&= -\frac{1}{3}\cos 7x\cos 3x - \frac{7}{9}\sin 3x\sin 7x + \frac{49}{9}\int \sin 3x\cos 7x\,dx
\end{aligned}$$

$$\Rightarrow \left(1 - \frac{49}{9}\right)\int \cos 7x\sin 3x\ dx = -\frac{1}{3}\cos 7x\cos 3x - \frac{7}{9}\sin 3x\sin 7x$$

Hence

$$\int \cos 7x\sin 3x\ dx = \frac{3}{40}\cos 7x\cos 3x + \frac{7}{40}\sin 3x\sin 7x + C$$

(When using repeated integration by parts, you need to be careful not to interchange the substitutions in successive applications. Following the LIATE method at every substitution should help you to avoid this. Be also careful when dealing with products of trigonometric functions: successive substitutions should involve the same trigonometric angle.)

3. (10 points) Find $\displaystyle\int \frac{12x^4 + x^3 + 10x^2}{(2x^2+1)^2}\ dx$.

Solution: Degree$[12x^4 + x^3 + 10x^2] \geq$ Degree$[(2x^2+1)^2]$ so the integrand is an improper rational function. By inspection or long division, we find that $(2x^2+1)^2 = 4x^4 + 4x^2 + 1$ goes into $12x^4 + x^3 + 10x^2$ 3 times, with remainder $x^3 - 2x^2 - 3$. So we have

$$\int \frac{12x^4 + x^3 + 10x^2}{(2x^2+1)^2}\ dx = \int \left(3 + \frac{x^3 - 2x^2 - 3}{(2x^2+1)^2}\right) dx$$

The integrand is now expressed as the sum of a polynomial and a proper rational function. The proper rational function may be decomposed into a sum of partial fractions. The polynomial $(2x^2+1)^2$ of the denominator is already in an irreducible form, and so we apply Rule 2 (see guideline, Section 7.5) to the quadratic factor $2x^2+1$ of multiplicity 2. This gives:

$$\frac{x^3-2x^2-3}{(2x^2+1)^2} = \frac{Ax+B}{2x^2+1} + \frac{Cx+D}{(2x^2+1)^2}$$

Finding the least common denominator and equating the numerators yield:

$$\begin{aligned} x^3-2x^2-3 &= (Ax+B)(2x^2+1)+Cx+D \\ &= 2Ax^3+2Bx^2+(A+C)x+(B+D) \end{aligned}$$

Here since there are no linear factors, we cannot use Method 2 (Using the zeros of the linear factors), and we proceed with Method 1 or equating coefficients in front of like powers of x. This yields the following system of algebraic equations:

$$\begin{array}{ccccccccc} 2A & & & & & & & = & 1 \\ & & 2B & & & & & = & -2 \\ A & + & & & C & & & = & 0 \\ & & B & + & & & D & = & -3, \end{array}$$

with solution: $A=-C=\frac{1}{2}$, $B=-1$, and $D=-2$.

So the partial fraction decomposition is:

$$\frac{x^3-2x^2-3}{(2x^2+1)^2} = \frac{1/2x-1}{2x^2+1} + \frac{-1/2x-2}{(2x^2+1)^2}$$

We can break up the original integral into four simpler integrals:

$\displaystyle\int \frac{x^3-2x^2-3}{(2x^2+1)^2}\,dx = \int\left(\frac{1/2x-1}{2x^2+1}+\frac{-1/2x-2}{(2x^2+1)^2}\right)dx = \frac{1}{2}\int\frac{x\,dx}{2x^2+1} - \int\frac{dx}{2x^2+1} - 1/2\int\frac{x}{(2x^2+1)^2}\,dx - \int\frac{2}{(2x^2+1)^2}\,dx$

The first three integrals can be done by u-substitution.

$\displaystyle 1/2\int\frac{x}{2x^2+1}\,dx = \frac{1}{8}\ln(2x^2+1)$

$\displaystyle \int\frac{1}{2x^2+1}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}x$

$\displaystyle 1/2\int\frac{x}{(2x^2+1)^2}\,dx = -1/8\frac{1}{2x^2+1}$

We use trig substitution for the fourth integral:

The form of the denominator, a sum of squares, suggests a tangent substitution.

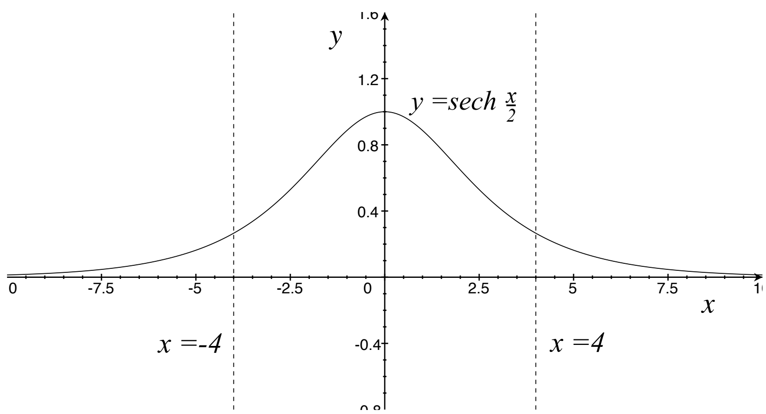
$\displaystyle x=\frac{1}{\sqrt{2}}\tan\theta,\ dx=\frac{1}{\sqrt{2}}\sec^2\theta\ d\theta$

$$\int \frac{2}{(2x^2+1)^2}\,dx = \int \frac{2}{(\tan^2\theta+1)^2}\frac{1}{\sqrt{2}}\sec^2\theta\,d\theta = \int \frac{2}{(\sec^2 x)^2}\frac{1}{\sqrt{2}}\sec^2\theta\,d\theta = \int \sqrt{2}\cos^2\theta\,d\theta =$$
$$\frac{1}{\sqrt{2}}\int (1+\cos 2\theta)d\theta = \frac{1}{\sqrt{2}}(\theta + 1/2\sin 2\theta) = \frac{1}{\sqrt{2}}(\theta+\sin\theta\cos\theta) = \frac{1}{\sqrt{2}}\left(\tan^{-1}\sqrt{2}x + \left(\frac{\sqrt{2}x}{\sqrt{1+2x^2}}\right)\left(\frac{1}{\sqrt{1+2x^2}}\right)\right)$$

The final answer is:

$$3x + \frac{1}{8}\ln(2x^2+1) - \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}x + \frac{1}{8}\frac{1}{2x^2+1} - \frac{1}{\sqrt{2}}\left(\tan^{-1}\sqrt{2}x + \frac{\sqrt{2}x}{1+2x^2}\right) + C$$

4. (6 points) Find the area of the region bounded by the function $y = \text{sech}\dfrac{x}{2}$, the x-axis and the lines $x = -4$ and $x = 4$.



Solution:

Using the parity of the function — $\text{sech}\,\frac{x}{2}$ is an even function — :

With the definition of $\operatorname{sech} x$:

$$\begin{aligned} A = \int_{-4}^{4} \operatorname{sech} \frac{x}{2}\, dx = 2\int_{0}^{4} \operatorname{sech} \frac{x}{2}\, dx &= 2\int_0^4 \frac{2}{e^{x/2}+e^{-x/2}} dx \\ &= 4\int_0^4 \frac{1}{e^{x/2}+e^{-x/2}} \cdot \frac{e^{x/2}}{e^{x/2}}\, dx \\ &= 4\int_0^4 \frac{e^{x/2}}{e^x+1}\, dx \\ &= 8\int_1^{e^2} \frac{du}{u^2+1} \\ &= 8\big[\arctan u\big]_1^{e^2} = 8\big(\arctan e^2 - \arctan 1\big) \\ &\qquad\qquad = 8\big(\arctan e^2 - \pi/4\big) \approx 5.207 \end{aligned}$$

With hyperbolic identities:

$$\begin{aligned} A = 2\int_0^4 \operatorname{sech} \frac{x}{2}\, dx &= 2\int_0^4 \frac{1}{\cosh x/2} \cdot \frac{\cosh x/2}{\cosh x/2}\, dx \\ &= 2\int_0^4 \frac{\cosh x/2}{\cosh^2 x/2}\, dx \\ &= 2\int_0^4 \frac{\cosh x/2}{1+\sinh^2 x/2}\, dx \\ &= 4\int_0^{\sinh 2} \frac{du}{1+u^2} \\ &= 4\big[\arctan u\big]_0^{\sinh 2} = 4\arctan(\sinh 2) \approx 5.207 \end{aligned}$$

5. (8 points) Find a reduction formula for the integral $\int e^x \sin^n x\, dx$, expressing it in terms of the integral $\int e^x \sin^{n-2} x\, dx$. Here n is an integer greater than 1.

Solution:

We use integration by parts, as we often do for such formulae. Pick $u = \sin^n x$ and

$dv = e^x dx$. Then $du = n\sin^{n-1}x\cos x\; dx$ and $v = e^x$.

$$\begin{aligned}
\int e^x \sin^n x\, dx &= e^x sin^n x - n\int e^x sin^{n-1}x\cos x\, dx \\
&= e^x sin^n x - n\, e^x \sin^{n-1}x\cos x \\
&\qquad + n\int e^x\Big[(n-1)\sin^{n-2}x\cos^2 x - \sin^n x\Big]dx \\
&= e^x sin^n x - n\, e^x \sin^{n-1}x\cos x \\
&\qquad + n\int e^x\Big[(n-1)\sin^{n-2}x(1-\sin^2 x) - \sin^n x\Big]dx \\
&= e^x sin^n x - n\, e^x \sin^{n-1}x\cos x \\
&\qquad + n(n-1)\int e^x \sin^{n-2}x\, dx - n^2\int e^x\sin^n x\, dx
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int e^x \sin^n x\, dx &= \frac{1}{1+n^2}\, e^x sin^n x - \frac{n}{1+n^2}\, e^x\sin^{n-1}x\cos x \\
&\qquad + \frac{n(n-1)}{1+n^2}\int e^x \sin^{n-2}x\, dx.
\end{aligned}$$

6. (7 points) Find $\displaystyle\int \sqrt{x^2-2x}\, dx$.

Solution: Given any quadratic equation of the form x^2+bx+c, it is possible to form a square by expressing it as

$$x^2+bx+c = x^2+bx+C+\left(\tfrac{b}{2}\right)^2 - \left(\tfrac{b}{2}\right)^2 = \left(x+\tfrac{b}{2}\right)^2 + C - \left(\tfrac{b}{2}\right)^2.$$

Using this, or by inspection, complete the square of the integrand: $x^2-2x = x^2-2x+1-1 = (x-1)^2-1$. Then

$$\int \sqrt{x^2-2x}\, dx = \int \sqrt{(x-1)^2-1}\, dx$$

With the trig substitution $x-1=\sec\theta$ ($dx = \sec\theta\tan\theta\, d\theta$), the integrand reduces to $\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \pm\tan\theta$

$$\begin{aligned}
\int \sqrt{(x-1)^2-1}\,dx &= \int \sec\theta\tan^2\theta\,d\theta \\
&= \int \sec\theta(\sec^2\theta - 1)\,d\theta \\
&= \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta \\
&= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| - \ln\left|\sec\theta+\tan\theta\right| + C \\
&= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| + C
\end{aligned}$$

where the second integral is solved by parts (see Section 7.4, done in class). Since $\sec\theta = \frac{hyp}{adj} = \frac{x-1}{1}$, then $\tan\theta = \sqrt{x^2-2x}$, and

$$\int \sqrt{(x-1)^2-1}\,dx = \frac{1}{2}(x-1)\sqrt{x^2-2x} - \frac{1}{2}\ln\left|x-1+\sqrt{x^2-2x}\right| + C$$

Bonus problem:

$$\int_{-1}^{1} \frac{\tan^7 x^3}{\sqrt[4]{x^4+2x^2+\cos x}}\,dx$$

This definite integral is 0. The trick is to notice this is the integral of an odd function over an interval of the form $[-a, a]$. Noticing symmetries like this is often very useful for evaluating integrals in applications.

Recall a function f is called *odd* if $f(-x) = -f(x)$, and called *even* if $f(x) = f(-x)$. For example x^3 and $\tan x$ are odd, while x^2 and $\cos x$ are even. Over the interval $[-a, a]$, if you graph an odd function, you will see the positive area cancels with the negative area (integral from $[-a, 0]$ cancels the integral from $[0, a]$).

To notice the given function is odd, it helps to use the following rules: an odd function divided by an even function is odd, the composition of odd functions is odd, a composition with "inside" function even is always even, the addition of even functions is even, and the addition of odd functions is odd.