Flow networks

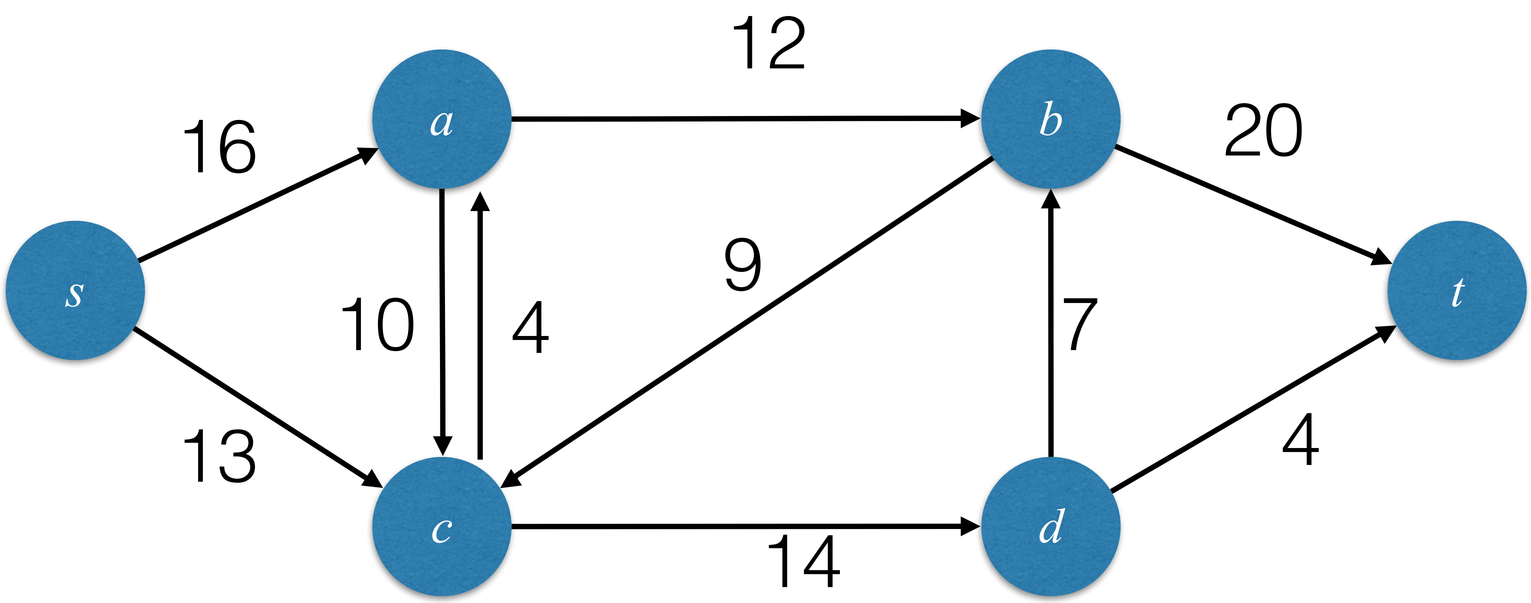
Flow Network (Definitions)

- A *flow network* is an edge-weighted directed graph with positive edge weights, called *capacities* (capacities of non-existing edges are zero)

- An st-*flow network* is a flow network that has two identified vertices, namely source s and sink t

- An st-*flow* in an st-flow network is a set of nonnegative values (*edge flows*) associated with each edge. Further we define
  - *inflow*: total flow of edges into a specific vertex
  - *outflow*: total flow of edges from a specific vertex
  - *netflow*: inflow minus outflow of a specific vertex

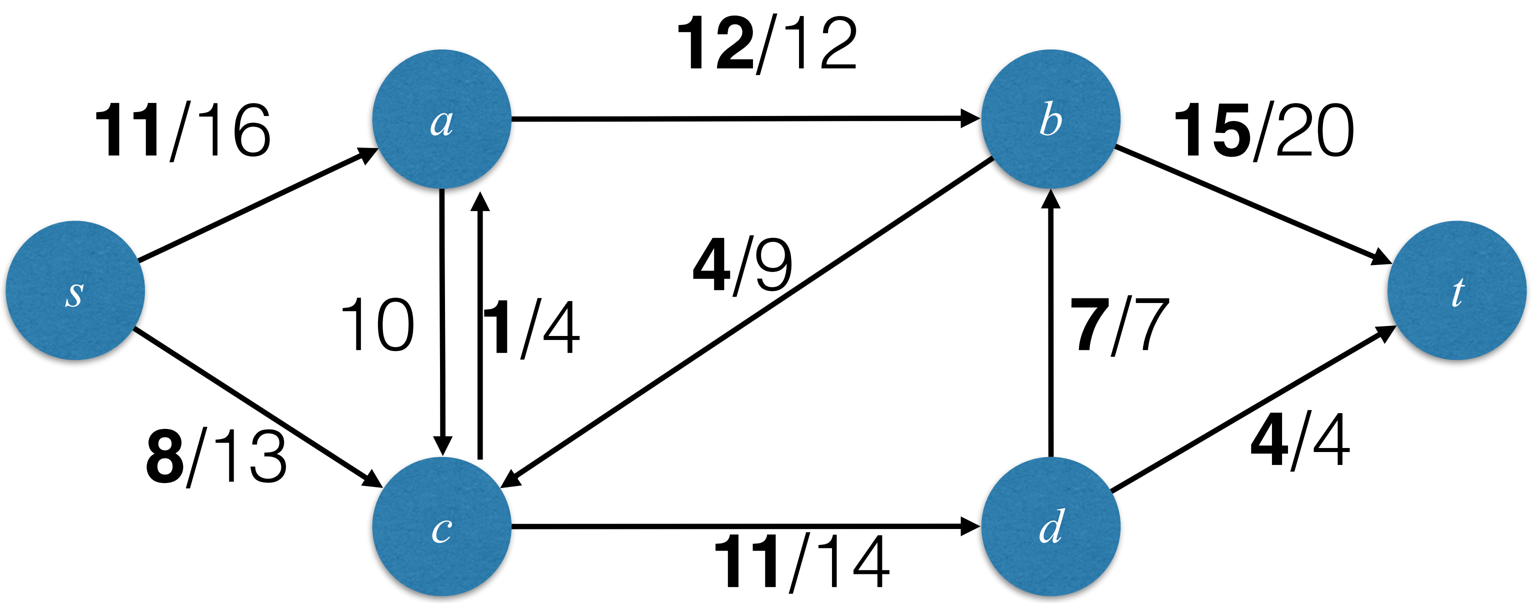
Example of an *st*-flow network



Flow Network (Definitions)

- An st-flow is *feasible* if it satisfies the conditions that
  - no edge's flow is greater than that edge's capacity and
  - the net flow of every vertex (except s and t) in the st-flow network is zero
- st-flow value $|f|$ for st-network G with st-flow f: the sink's inflow
- *Maximum st-flow* (short: *maxflow*): feasible st-flow with maximum st-flow value

Example of a feasible ***st*-flow** in an *st*-flow network



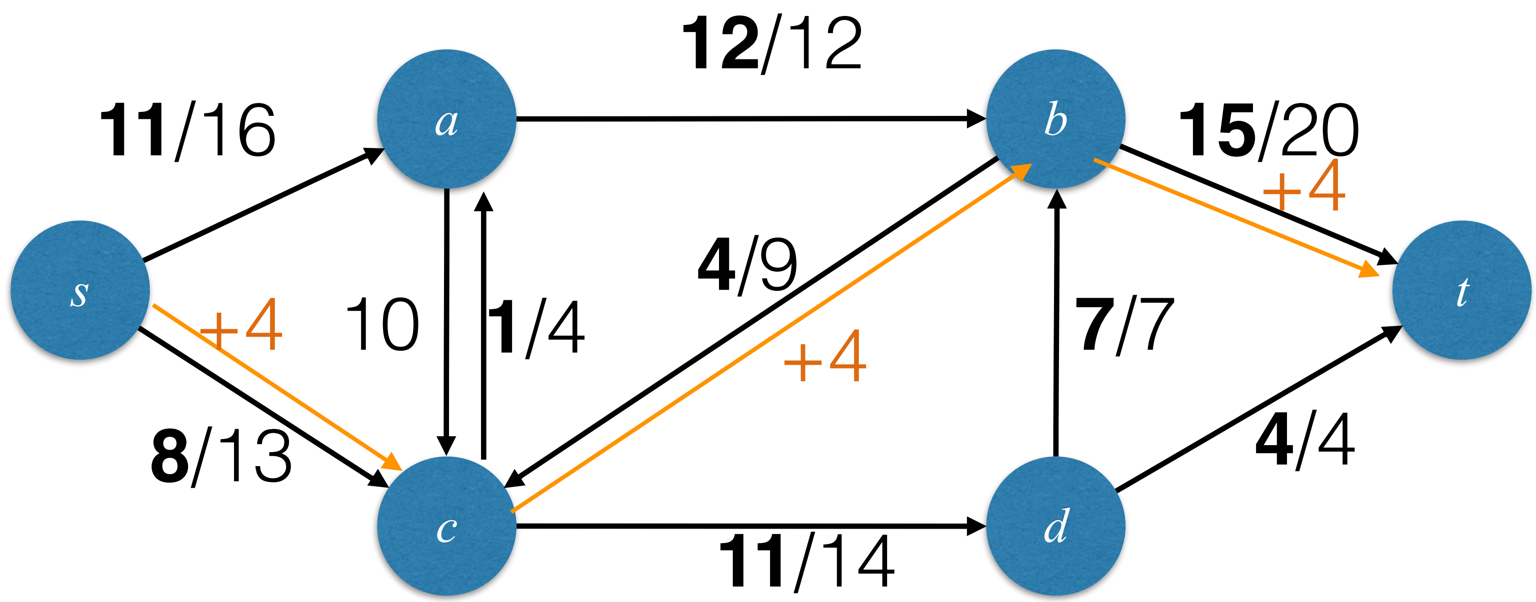
Maximum Flow Problem (short: *maxflow problem*)

- Input: An st-flow network
- Output: A maximum st-flow

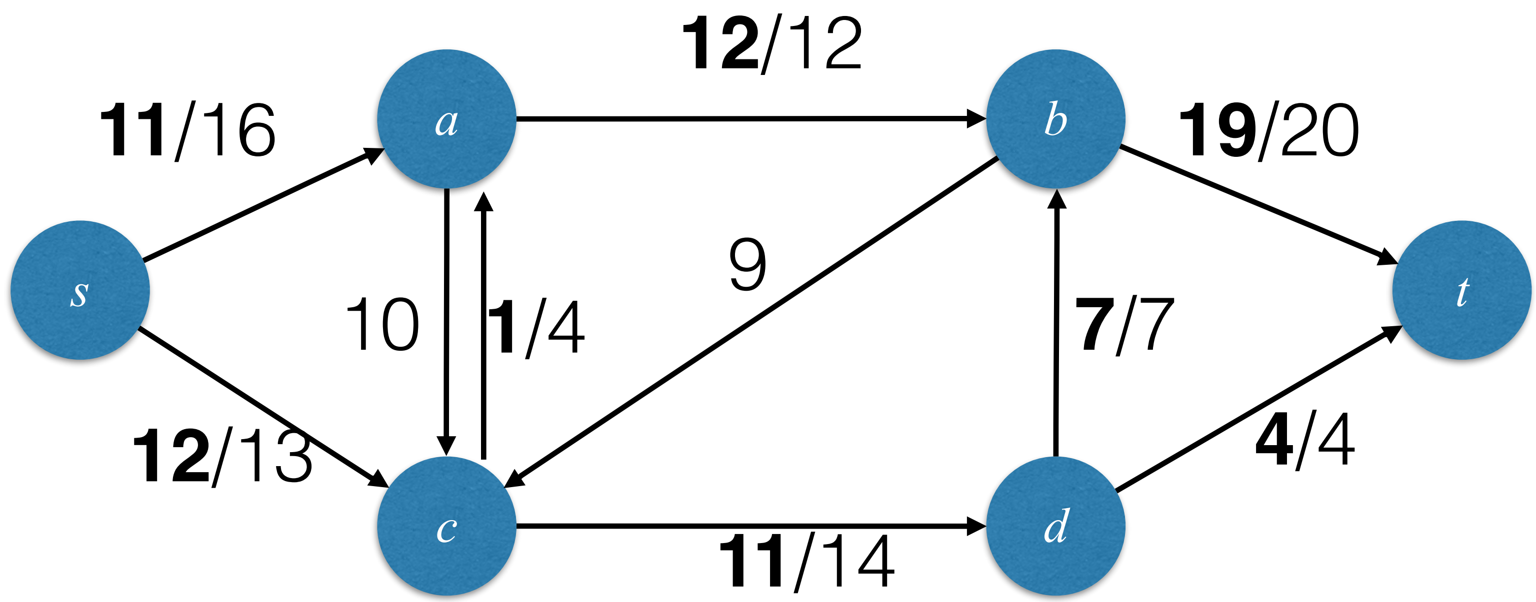
Augmenting paths in st-flow networks—Idea

- An *augmenting path* in an st-flow network with feasible st-flow is a directed path from source to sink along which we can push more flow, obtaining an st-flow with higher st-flow value

Example of a path that improves the flow



Improved flow



Ford-Fulkerson's maxflow method

1. Initialize with a flow with st-flow value zero
2. Increase the flow along any augmenting path from s to t
3. Repeat step 2 as long as an augmenting path exists

Finding Augmenting Paths: the residual network

- Consider an st-flow f in st-flow network G and vertices u and v for directed edge (u,v) in G

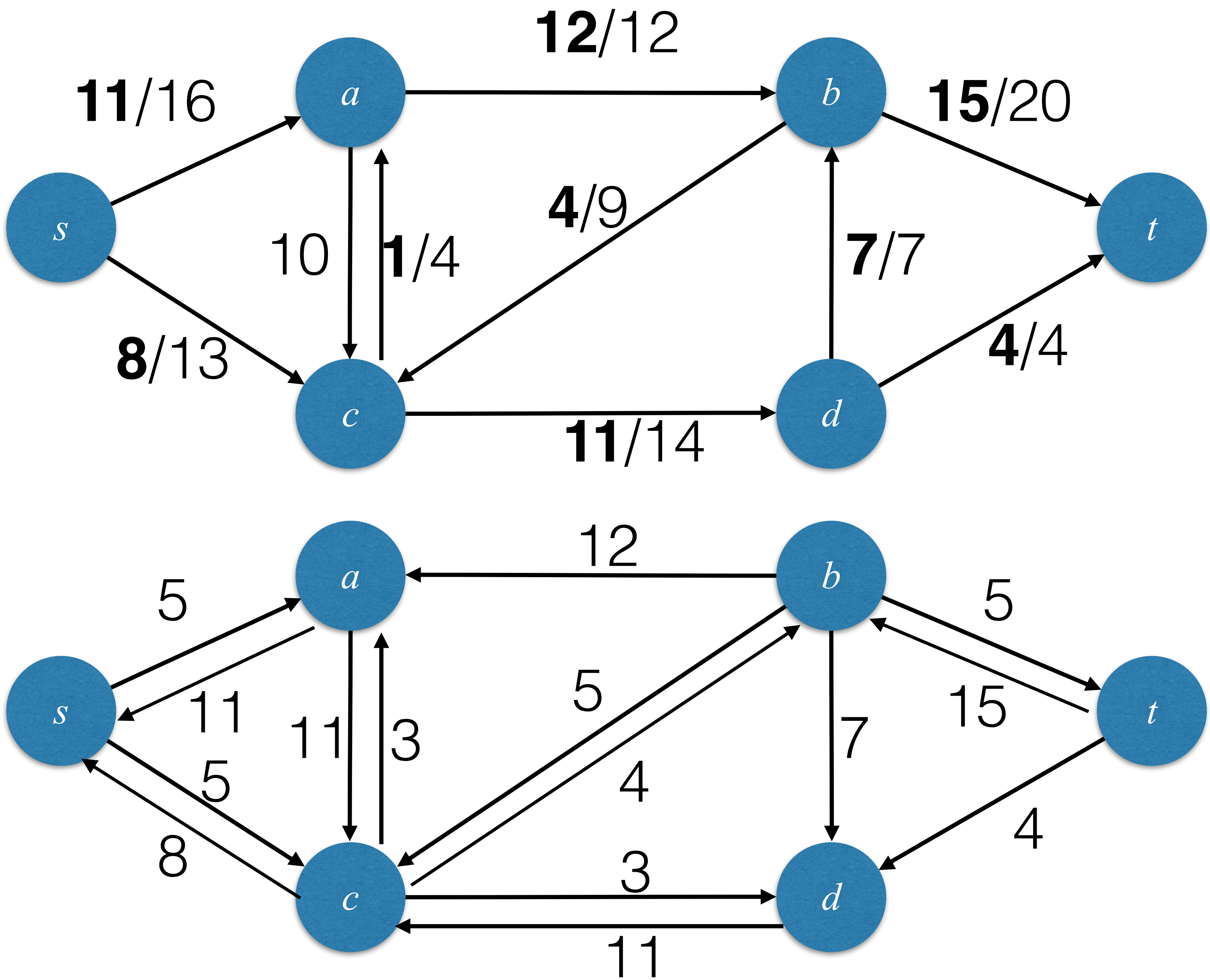
- The amount of additional netflow that we can push from u to v along (u,v) is called *residual capacity* of (u,v)

- That is: for edge (u,v) with capacity $c(u,v)$ and flow value $f(u,v)$ from u to v we have the residual capacity $c_f(u,v) = c(u,v) - f(u,v)$

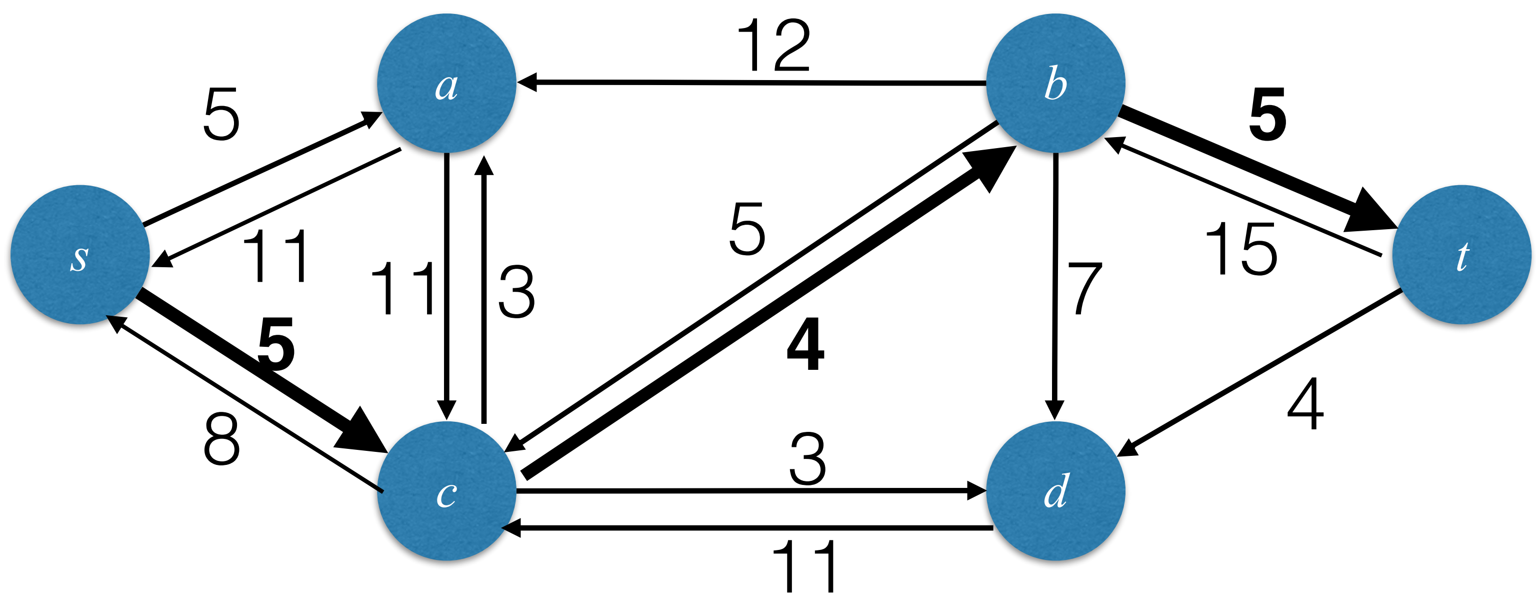
Residual Network

- Given an st-flow network $G = (V,E)$ and a flow f, the *residual network* of G *induced by* f is $G_f = (V, E_f)$ where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$

Example of residual network

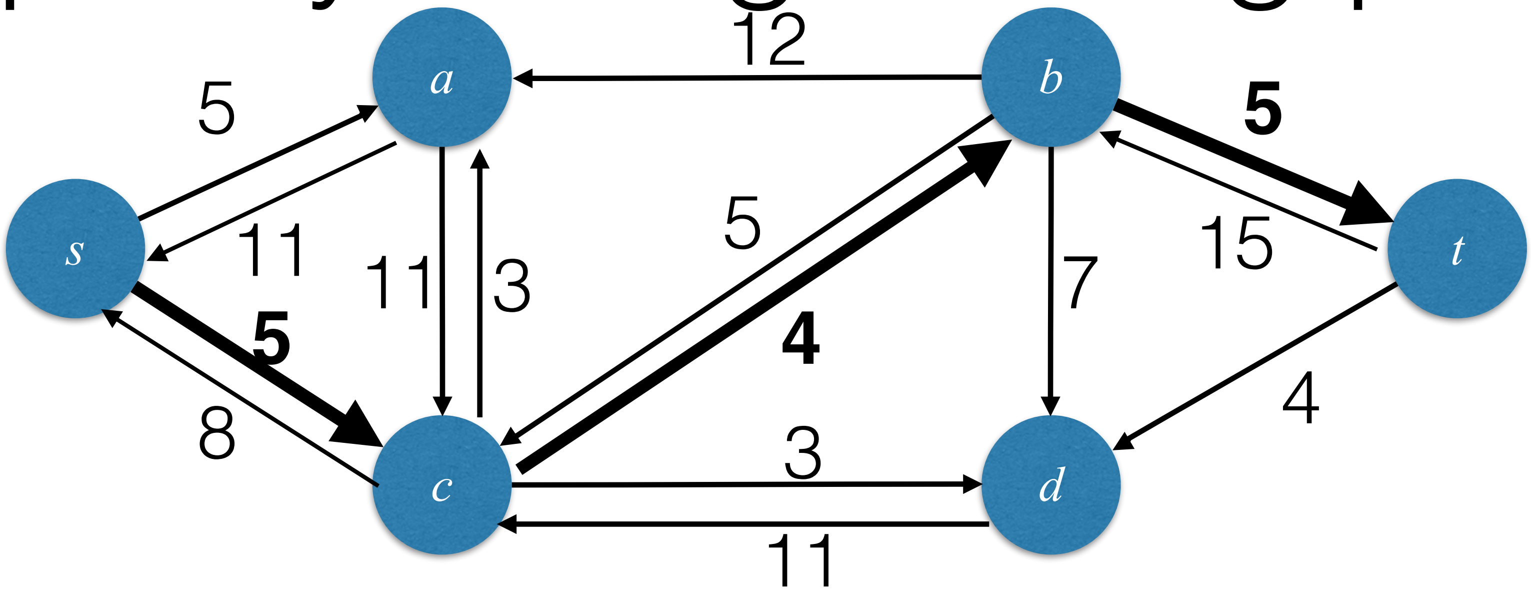


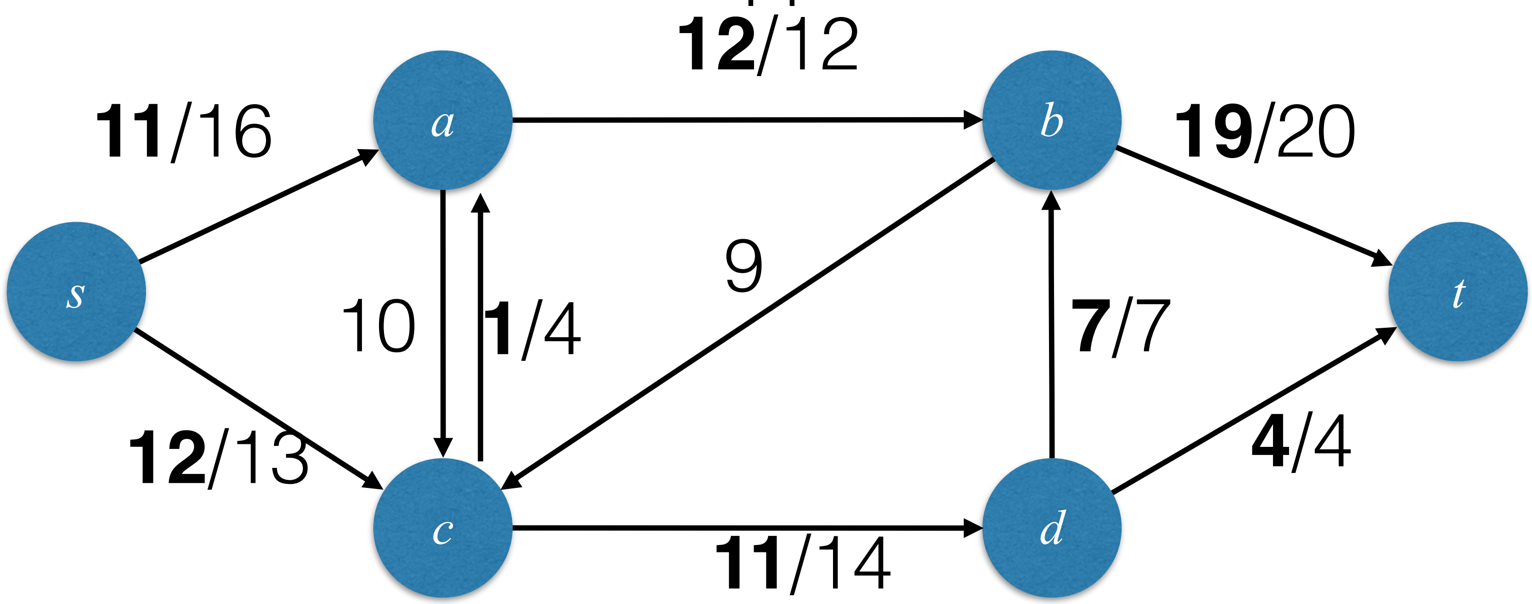
Augmenting path from s to t in residual network



Residual capacity of augmenting path is 4

Increasing flow by residual capacity of augmenting path





Properties of residual network

- $|E_f| \leq 2|E|$
- The residual network G_f with capacities c_f of st-flow network G is an st-flow network

Definition of Augmenting Path

Given an st-flow f in st-flow network $G = (V, E)$ an augmenting path p is a path from s to t in the residual network G_f.

Theorem 1. Let $G = (V, E)$ be an st-flow network, and let f be an st-flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then the flow sum $f+f'$, with $(f+f')(u,v) = f(u,v) + f'(u,v)$ for all $u,v \in V$, is a flow in G with st-flow value $|f+f'| = |f|+|f'|$.

Theorem 2. Let $G = (V, E)$ be an st-flow network, and let f be an st-flow in G. Let p be an augmenting path in G_f. Further f_p: $V \times V \rightarrow IR$ is defined as follows: $f_p(u,v) = c_f(p)$ if (u, v) is on p, $f_p(u,v) = -c_f(p)$ if (v, u) is on p, and $f_p(u,v) = 0$ otherwise. Then f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Algorithm Ford-Fulkerson(G, s, t)

for each edge $(u,v) \in E$ **do**

$f(u,v) \leftarrow 0; f(v,u) \leftarrow 0$

while there exists a path p from s to t in G_f **do**

compute $c_f(p)$

for each edge (u,v) in p **do**

if (u,v) $\in E$ **then** $f(u, v) \leftarrow f(u,v) + c_f(p)$

else $f(v, u) \leftarrow -f(u,v)$