Dynamic Programming: Finding a Longest Common Subsequence

Terminology

- A *string* $c = c_1\ c_2\ \ldots\ c_n$ is a sequence of characters (or symbols from an alphabet)

- A *substring* $s = s_1\ s_2\ \ldots\ s_m$ of a string $c = c_1\ c_2\ \ldots\ c_n$ is a string with $s_1 = c_i$, $s_2 = c_{i+1}, \ldots, s_m = c_{i+m\text{-}1}$.

- A *subsequence* $x = x_1\ x_2\ \ldots\ x_r$ of a string $s = s_1\ s_2\ \ldots\ s_m$ of a string with $x_1 = c_{i1}$, $x_2 = c_{i2}, \ldots, x_r = c_{ir}$ with $1 \le i1 \le i2 \le \ldots \le i\mathrm{r} \le n$

- Note 1: The characters of a subsequence of a string are not necessarily consecutive in the string.

- Note 2: A substring is also a subsequence, but the reverse is not necessarily true!

- The *length* of a string is the number of its characters

Examples

- The string **lamp** is neither a substring nor a subsequence of the string **examples**.
- The string **ample** is a substring of the string **examples**
- **mps** is a subsequence but not a substring of the string **examples**

More terminology

- Given two *strings* $s = s_1\, s_2\, \ldots\, s_n$ and $t = t_1\, t_2\, \ldots\, t_m$, a common subsequence of s and t is a string that is both: a subsequence of s and a subsequence of t.

- A *longest common subsequence (lcs)* of two string s and t is a common subsequence of maximum length.

Longest Common Subsequence Problem (LCS)

- Input: Strings $x = x_1\, x_2\, \ldots\, x_n$ and $y = y_1\, y_2\, \ldots\, y_m$
- Output: string lcs, where lcs is a longest common subsequence of x and y

Computing an lcs

- Idea:
    - For the shorter of the two given strings create all possible subsequences
    - From the longest to shortest: check if it is also a subsequence of the longer string; output the first one found
- Running time very high

Dynamic programming

- Elements of dynamic programming
  - optimal substructure
  - overlapping subproblems

The three (four) steps of Dynamic Programming

- Characterize the structure of an optimal solution
- Recursively define the value of an optimal solution
- Compute the value of an optimal solution in a bottom-up fashion (e.g., via table)
- [optional] Construct an optimal solution from the computed information

		c	h	i	m	p	a	n	z	e	e
h											
u											
m											
a											
n											

Theorem (Optimal Substructure of lcs)

- Let $x = x_1\, x_2 \ldots x_n$ and $y = y_1\, y_2 \ldots y_m$ be strings, and let $z = z_1\, z_2 \ldots z_k$ be an lcs of x and y. Then

1. If $x_n = \mathrm{y_m}$ then $z_k = x_n = y_m$ and $z_1\, z_2 \ldots z_{k\text{-}1}$ is an lcs of $x_1\, x_2 \ldots x_{n\text{-}1}$ and $y_1\, y_2 \ldots y_{m\text{-}1}$

2. If $x_n \neq y_m$ then $z_k \neq x_n$ implies that z is an lcs of $x_1\, x_2 \ldots x_{n\text{-}1}$ and y

3. If $x_n \neq y_m$ then $z_k \neq y_m$ implies that z is an lcs of $y_1\, y_2 \ldots y_{m\text{-}1}$ and x

Proof of 1. If $x_n = y_m$ then (a) $z_k = x_n = y_m$ and (b) $z_1 z_2 \ldots z_{k\text{-}1}$ is an lcs of $x_1 x_2 \ldots x_{n\text{-}1}$ and $y_1 y_2 \ldots y_{m\text{-}1}$

$x_n = y_m$. We prove claim (a) using contradiction: Assume $z_k \neq x_n$.

But then z could be be made longer by appending $x_n = y_m$ to z. But according to the assumptions of the Theorem, z is an lcs of x and y. Therefore, no longer common subsequence of x and y, including $z\, x_n$, can exist, a contradiction.

This proves (a).

Proof of 1. If $x_n = y_m$ then (a) $z_k = x_n = y_m$ and (b) $z_1 z_2 \ldots z_{k\text{-}1}$ is an lcs of $x_1 x_2 \ldots x_{n\text{-}1}$ and $y_1 y_2 \ldots y_{m\text{-}1}$

$x_n = y_m$ and $z_k = x_n = y_m$. We prove claim (b) contradiction: Assume $z_1 z_2 \ldots z_{k\text{-}1}$ is not an lcs of $x_1 x_2 \ldots x_{n\text{-}1}$ and $y_1 y_2 \ldots y_{m\text{-}1}$

But then there must exist a string w that is of length greater than k-1 and a common subsequence of $x_1 x_2 \ldots x_{n\text{-}1}$ and $y_1 y_2 \ldots y_{m\text{-}1}$.

Appending $x_n = y_m$ to w creates a common subsequence of x and y. But w is longer than z, a contradiction.

This proves (b).

Proof of 2. If $x_n \neq y_m$ then $z_k \neq x_n$ implies that z is an lcs of $x_1\, x_2 \ldots x_{n-1}$ and y

- Assume $x_n \neq y_m$ and $z_k \neq x_n$. Then z is common subsequence of $x_1\, x_2 \ldots x_{n-1}$ and y (since otherwise z would not be a subsequence of x and y).

- Assume that z is not a longest common subsequence. Then there exists a longer common subsequence, w, of $x_1\, x_2 \ldots x_{n-1}$ and y. But then w is of length greater than k, which does not exist according to the assumptions of the Theorem. A contradiction. This proves that z is an lcs of $x_1\, x_2 \ldots x_{n-1}$ and y and therefore (2)

Proof of (3)

- Works analogous to the proof of (2)

What can we conclude from the Theorem?

- Let $x = x_1\, x_2 \ldots x_n$ and $y = y_1\, y_2 \ldots y_m$ be strings, and let $z = z_1\, z_2 \ldots z_k$ be an lcs of x and y. Then

  1. If $x_n = y_m$ then $z_k = x_n = y_m$ and $z_1\, z_2 \ldots z_{k-1}$ is an lcs of $x_1\, x_2 \ldots x_{n-1}$ and $y_1\, y_2 \ldots y_{m-1}$

  2. If $x_n \neq y_m$ then $z_k \neq x_n$ implies that z is an lcs of $x_1\, x_2 \ldots x_{n-1}$ and y

  3. If $x_n \neq y_m$ then $z_k \neq y_m$ implies that z is an lcs of $y_1\, y_2 \ldots y_{m-1}$ and x

What can we conclude from the Theorem?

- Let $x = x_1\, x_2\, \ldots\, x_n$ and $y = y_1\, y_2\, \ldots\, y_m$ be strings, and let $z = z_1\, z_2\, \ldots\, z_k$ be an lcs of x and y. Let llcd denote the length of an lcs. Then

  - If $x_n = y_m$ then $\mathrm{llcs}(x_1\, x_2\, \ldots\, x_n, y_1\, y_2\, \ldots\, y_m) =$
    $\mathrm{llcs}(x_1\, x_2\, \ldots\, x_{n\text{-}1}, y_1\, y_2\, \ldots\, y_{m\text{-}1}) + 1$

  - If $x_n \neq y_m$ then $\mathrm{llcs}(x_1\, x_2\, \ldots\, x_n, y_1\, y_2\, \ldots\, y_m) =$
    $\max\{\mathrm{llcs}(x_1\, x_2\, \ldots\, x_{n\text{-}1}, y_1\, y_2\, \ldots\, y_m),$
    $\mathrm{llcs}(x_1\, x_2\, \ldots\, x_n, y_1\, y_2\, \ldots\, y_{m\text{-}1})\}$

The three (four) steps of Dynamic Programming

- Characterize the structure of an optimal solution
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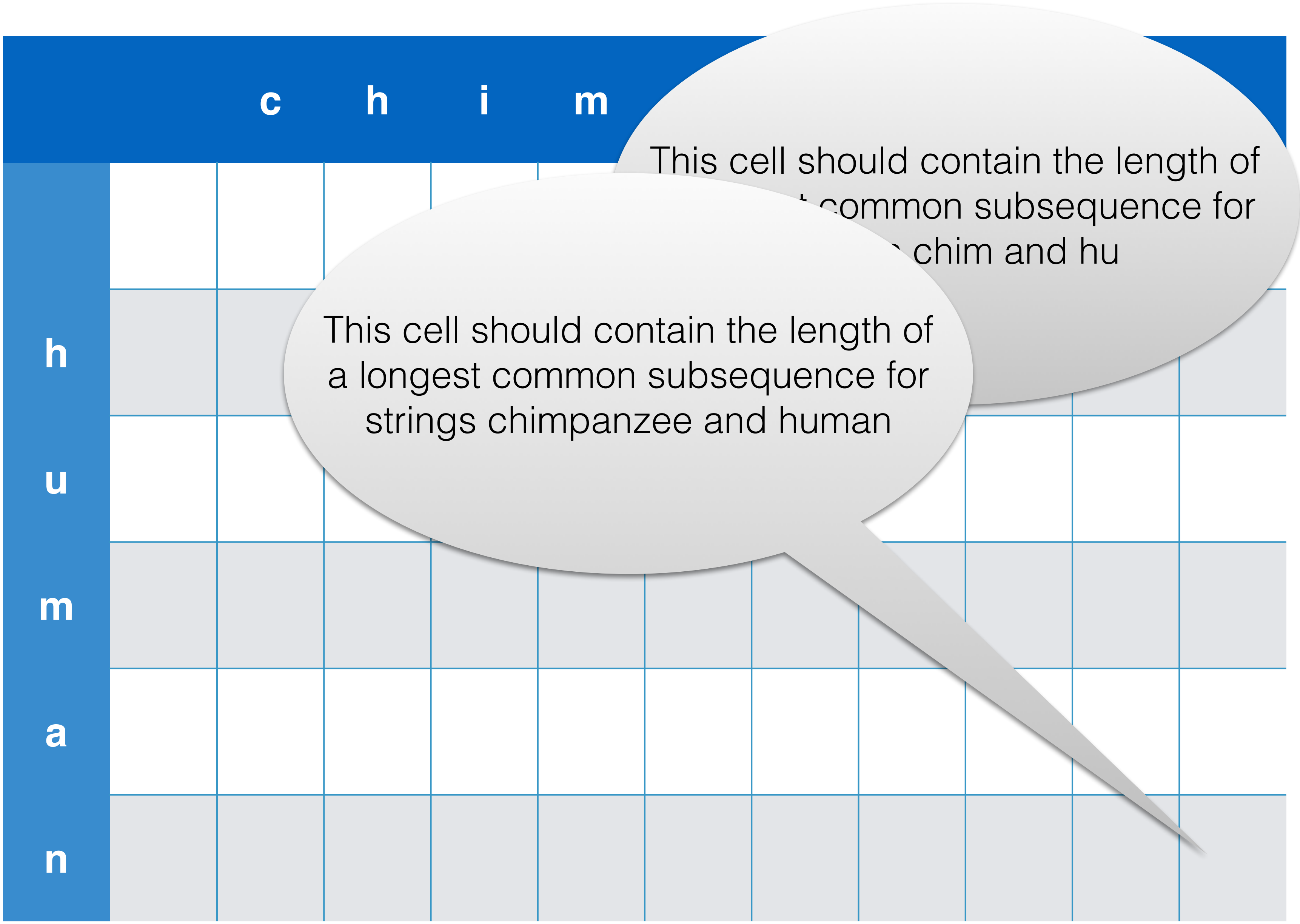
c h i m
h u m a n
This cell should contain the length of
common subsequence for
chim and hu
This cell should contain the length of
a longest common subsequence for
strings chimpanzee and human

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0										
u	0										
m	0										
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0									
u	0										
m	0										
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1								
u	0										
m	0										
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1							
u	0										
m	0										
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0										
m	0										
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0									
m	0										
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1								
m	0										
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0										
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0									
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1								
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1							
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2						
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2					
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0										
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0									
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1								
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1							
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2						
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2					
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3				
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3			
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0										

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0									

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1								

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3				

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4			

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4		

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4	4	4

We know the length llcs, but not an lcs

- Each time when computing the content of a new cell, remember where you came from!

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0										
u	0										
m	0										
a	0										
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0 ←	0									
u	0										
m	0										
a	0										
n	0										

If x[i] = y[j]: llcs[i,j] = llcs[i-1,j -1]+1

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1								
u	0										
m	0										
a	0										
n	0										

If $x[i] \neq y[j]$: $llcs[i,j] = \max\{llcs[i,j-1], llcs[i-1,j]\}$

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1							
u	0										
m	0										
a	0										
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0										
m	0										
a	0										
n	0										

If $x[i] \neq y[j]$: $llcs[i,j] = \max\{llcs[i,j-1], llcs[i-1,j]\}$

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0									
m	0										
a	0										
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1								
m	0										
a	0										
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0										
a	0										
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0									
a	0										
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1								
a	0										
n	0										

If $x[i] \neq y[j]$: $llcs[i,j] = \max\{llcs[i,j-1], llcs[i-1,j]\}$

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1							
a	0										
n	0										

If x[i] = y[j]: llcs[i,j] = llcs[i-1,j -1]+1

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2						
a	0										
n	0										

If $x[i] \neq y[j]$: $llcs[i,j] = \max\{llcs[i,j-1], llcs[i-1,j]\}$

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2					
a	0										
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0										
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0									
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1								
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1							
n	0										

If $x[i] \neq y[j]$: $llcs[i,j] = \max\{llcs[i,j-1], llcs[i-1,j]\}$

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2						
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2					
n	0										

If $x[i] = y[j]$: $llcs[i,j] = llcs[i-1,j-1]+1$

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3				
n	0										

If $x[i] \neq y[j]$: $llcs[i,j] = \max\{llcs[i,j-1], llcs[i-1,j]\}$

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3			
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0										

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0									

$$\text{If } x[i] \neq y[j]\text{: } llcs[i,j] = \max\{llcs[i,j-1], llcs[i-1,j]\}$$

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1								

If x[i] ≠ y[j]: llcs[i,j] = max{llcs[i,j-1],llcs[i-1,j]}

		c	**h**	**i**	**m**	**p**	**a**	**n**	**z**	**e**	**e**
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3				

If x[i] = y[j]: llcs[i,j] = llcs[i-1,j -1]+1

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4			

If x[i] = y[j]: llcs[i,j] = llcs[i-1,j -1]+1

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4		

If x[i] = y[j]: llcs[i,j] = llcs[i-1,j -1]+1

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4	4	4

Step 4: Extracting the path to obtain the lcd

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4	4	4

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4	4	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4	4	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	4	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	4	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	3	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	2	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	**2**	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	<u>a</u>	<u>n</u>	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	2	**2**	2	2	2	2	2
<u>a</u>	0	0	1	1	2	2	**3**	3	3	3	3
<u>n</u>	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	**2**	**2**	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	1	1	1	1	1	1	1	1
m	0	0	1	1	**2**	**2**	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	**1**	1	1	1	1	1	1	1
m	0	0	1	1	**2**	**2**	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	<u>m</u>	p	<u>a</u>	<u>n</u>	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	1	**1**	1	1	1	1	1	1	1
<u>m</u>	0	0	1	1	**2**	**2**	2	2	2	2	2
<u>a</u>	0	0	1	1	2	2	**3**	3	3	3	3
<u>n</u>	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	**1**	**1**	1	1	1	1	1	1	1
m	0	0	1	1	**2**	**2**	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	<u>m</u>	p	<u>a</u>	<u>n</u>	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	1	1	1	1	1	1	1	1	1
u	0	0	**1**	**1**	1	1	1	1	1	1	1
<u>m</u>	0	0	1	1	**2**	**2**	2	2	2	2	2
<u>a</u>	0	0	1	1	2	2	**3**	3	3	3	3
<u>n</u>	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	**1**	1	1	1	1	1	1	1	1
u	0	0	**1**	**1**	1	1	1	1	1	1	1
m	0	0	1	1	**2**	**2**	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	**1**	1	1	1	1	1	1	1	1
u	0	0	**1**	**1**	1	1	1	1	1	1	1
m	0	0	1	1	**2**	**2**	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

		c	h	i	m	p	a	n	z	e	e
		0	0	0	0	0	0	0	0	0	0
h	0	0	**1**	1	1	1	1	1	1	1	1
u	0	0	**1**	**1**	1	1	1	1	1	1	1
m	0	0	1	1	**2**	**2**	2	2	2	2	2
a	0	0	1	1	2	2	**3**	3	3	3	3
n	0	0	1	1	2	2	3	**4**	**4**	**4**	**4**

Algorithm **LCS-length**(string x, string y)

$n \leftarrow x$.length()

$m \leftarrow x$.length()

for i **from** 1 **to** n **do** llcs[i,0] $\leftarrow$ 0

for j **from** 1 **to** m **do** llcs[i,0] $\leftarrow$ 0

for i **from** 1 **to** n **do**

  for j **from** 1 **to** m **do**

    if $x[i] = y[j]$ **then**

      llcs[i, j] $\leftarrow$ llcs[i-1, j-1] +1; path[i, j] $\leftarrow$ "↖"

    else if llcs[i-1, j] > llcs[i, j-1] **then**

      llcs[i, j] $\leftarrow$ llcs[i-1, j]; path[i, j] $\leftarrow$ "↑"

    else llcs[i, j] $\leftarrow$ llcs[i, j-1]; path[i, j] $\leftarrow$ "←"

return llcs & path

Algorithm **Print-LCS**(matrix **path**, string x, positive integers i, j)

if $i = 0$ or $j = 0$ **then return**

if path[i, j] = "↖" **then**

  print-LCS(path, x, i-1, j-1)

  print $x[i]$

else if path[i, j] = "↑" **then**

  print-LCS(path, x, i-1, j)

else print-LCS(path, x, i, j-1)

Running times

- LCS-length: O(nm)
- print-LCS: O(n+m)