CENG461/ELEC514 Assignment 1: Solutions

1. A gambler's dispute in 1654 led to the creation of a mathematical theory of probability by two famous French mathematicians, Blaise Pascal and Pierre de Fermat. Antoine Gombaud, Chevalier de Mere, a French nobleman with an interest in gaming and gambling questions, called Pascal's attention to an apparent contradiction concerning a popular dice game. The game consisted in throwing a pair of dice 24 times; the problem was to decide whether or not to bet money on the occurrence of at least one "double six" during the 24 throws. A seemingly well-established gambling rule led de Mere to believe that betting on a double six in 24 throws would be profitable, but his own calculations indicated just the opposite.

   a) What is the probability on the occurrence of at least one "double six" during 24 throws?

   Solution: Let p be the probability that a "double six" occur for a single throw; q be the probability of it does not occur. We have $p = 1/36$ and $q = 35/36$;

   Probability that no "double six" occurs during 24 throws equals $q^2 4$.

   Thus, the probability on the occurrence of at least one "double six" during 24 throws equals $1-q^2 4 = 0.4914$

   b) What is the minimum number of throws such that the probability on the occurrence of at least one "double six" is larger than 0.5?

   Solution: Let k be the number of throws.

   To ensure $1 - q^k > 0.5$, we have $k \geq 25$.

2. In a class, 80% of the students like mathematics, and they belong to group A. The other 20% of them do not like mathematics, and belong to group B. All group A students pass on mathematics, and 50% of the group B students get pass. An instructor sees a (random) student's test report and he observed that this student get pass on mathematics. What is the probability this student is from group A.

   Solution: Let E be the event that the student pass on mathematics.

   We know:

   $p(A) = 0.8$ and $p(B) = 0.2$.

   $p(E|A) = 1$ and $p(E|B) = 0.5$.

   Then, using Bayes' theorem,

$$p(A|E) = \frac{p(E|A)p(A)}{p(E|A)p(A) + p(E|B)p(B)} = \frac{8}{9}.$$

3. Customers call the customer services department of a bank at an average rate of 108 calls per day during 8 am to 5 pm (i.e., 9 business hours per day) with Poisson distribution.

   a) Determine the probability that the customer services department receives no call during 10 minutes.

   Solution:

   $\lambda = 108$ call/day = 12 call/hour, since there are 9 business hours per day.

   $t = 10$ minutes = $1/6$ hour

$\lambda t = 2$

$k = 0$

$p(0) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{2^0 e^{-2}}{1!} = e^{-2}$

b) Determine the probability that the customer services department receives three calls during 10 minutes.

Solution:

$k = 3$

$p(3) = \frac{2^3 e^{-2}}{3!} = \frac{4e^{-2}}{3}$

4. The lifetime of a bulb can be approximated as an exponentially distributed random variable. The average lifetime of a bulb is 1000 hours.

   a) What is the probability that the lifetime of a bulb is greater than 500 hours?

   Solution:

   Let R.V. X be the lifetime of a bulb.

   Since $\mu = 1000$ hours, the PDF of X is $f_X(x) = \frac{1}{1000}\exp(-x/1000)$, for $x \geq 0$.

   $P(X > 500) = \int_{500}^{\infty} f_X(x)dx = \int_{500}^{\infty} \frac{1}{1000}\exp(-x/1000)dx = \exp(-500/1000) - \exp(-\infty) = \exp(-0.5)$.

   b) Given a bulb has lasted 500 hours, what is the probability that the bulb can last at least another 500 hours? (What is your observation comparing the results of a) and b))

   Solution:

   The question is that given X is larger than 500 hours, what is the conditional probability that X is larger than 1000:

   $P(X > 1000|X > 500) = P(X > 1000 \text{ and } X > 500)/P(X > 500) = P(X > 1000)/P(X > 500)$

   Here $P(X > 1000) = \int_{1000}^{\infty} f_X(x)dx = \int_{1000}^{\infty} \frac{1}{1000}\exp(-x/1000)dx = \exp(-1000/1000) - \exp(-\infty) = \exp(-1)$, and $P(X > 500) = \exp(-0.5)$

   Thus, $P(X > 1000|X > 500) = P(X > 1000)/P(X > 500) = \exp(-1 + 0.5) = \exp(-0.5)$.

   The results show that if we observe the bulb at the time instant it has been manufactured and observe the bulb any time later so long as the bulb is still alive, the remaining lifetime will follow the same distribution. Counter-intuitive, isn't it? This is the **memoryless** property of exponential distribution. Imagine what happens if the bus inter-arrival time is exponentially distributed. When you wait at a bus stop, the probability of the next bus will come in x minutes will not change regardless of how long you have been waiting there! Luckily, in reality, the bus inter-arrival time is not exponentially distributed.

5. In a wireless network with N independent channels, at time instant t, there are K users competing for the N channels. Each user will randomly (uniformly) choose one channel with equal priority, independent of other users. If more than one users choose the same channel, a collision will happen.

A tagged user can successfully access a channel and transmit a packet if nobody else choose that channel simultaneously.

a) Obtain an expression for p_s (the probability that a tagged user successfully accesses a channel) as a function of K and N.

Solution:

p_s for a tagged user equals the probability that other $N-1$ users choose channels different from the one chosen by the tagged user. The probability of another user chooses a different channel equals $(N-1)/N$.

Therefore, $p_s = (\frac{N-1}{N})^{K-1}$.

b) Calculate this probability when $N = 5$ and $K = 4$.

Solution:

$p_s = (\frac{5-1}{5})^3 = 0.512$

c) Assume that at time instants $t + nT$, there are always 4 users competing for the 5 channels, where $n = 0, 1, 2, ...$ For a tagged user, what is the expected number of attempts needed before a success? (Hint: for geometric distribution, $E[X] = (1-p)/p$.)

Solution:

Let R.V. X be the number of failed tranmissions before a success.

$p(X = n) = p_s(1-p_s)^n$ for $n \geq 0$. X is a geometric R.V.

$E[X] = (1-p_s)/p_s = 0.488/0.512 = 0.953$

d) Assume $T = 10$ms, and the time to transmit a packet $T_x = 5$ms. What is the average (expected) delay for a packet, from the time instant it is transmitted the first time till the time instant it is successfully transmitted?

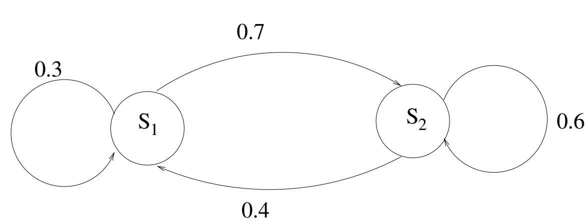
Solution:

Let R.V. D be the delay for a packet, which equals $XT + T_s$.

$E[D] = E[X]T + T_s = 9.53 + 5 = 14.53$ ms.

6. The weather of each day can be modeled by a Markov chain with two states, rain or no-rain. If we have rain today, the probability of rain tomorrow is 0.3; if no-rain today, the probability of no-rain tomorrow is 0.6. Given that we have rain today, what is the probability of no-rain three days later?

   Solution:



Let s_1 be the state of rain, and s_2 be the state of no-rain.

Given $p_{11} = 0.3$, $p_{21} = 1 - p_{11} = 0.7$.

Given $p_{22} = 0.6$, $p_{12} = 1 - p_{22} = 0.4$.

The state transition matrix $\mathbf{P} = \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}$

$\mathbf{P}^3 = \begin{bmatrix} 0.328 & 0.336 \\ 0.672 & 0.674 \end{bmatrix}$.

The initial distribution vector $\mathbf{s}(0) = [1 \;\; 0]^t$.

$$\begin{aligned} \mathbf{s}(3) &= \mathbf{P}^3\mathbf{s}(0) \\ &= [0.363 \;\; 0.637]^t \end{aligned}$$

Therefore, the probability of no-rain three days later is $s_2(3) = 0.637$.