Assignment 2 – Solution Key

1. (a) $\displaystyle\int_1^2 \frac{2^{-1/t}}{t^2}\,dt = \int_1^2 \frac{e^{-\ln 2/t}}{t^2}\,dt = \int_{-\ln 2}^{-\ln 2/2} e^u \frac{du}{\ln 2} = \frac{1}{\ln 2}\Big[e^u\Big]_{-\ln 2}^{-\ln 2/2}$

$\begin{pmatrix} u = -\ln 2/t \\ du = \frac{\ln 2}{t^2}\,dt \end{pmatrix}$

$\displaystyle = \frac{1}{\ln 2}\left[e^{-\ln 2/2} - e^{-\ln 2}\right]$

$\displaystyle = \frac{1}{\ln 2}\left[2^{-1/2} - 2^{-1}\right] = \frac{\sqrt{2}-1}{\ln 4} = 0.298$

(b) $\displaystyle\int_1^2 \frac{\operatorname{csch}(1/x)\coth(1/x)\,dx}{x^2} = \int_1^{1/2} -\operatorname{csch} u \cdot \coth u\,du$ $\quad\begin{pmatrix} u = 1/x \\ du = -dx/x^2 \end{pmatrix}$

$\displaystyle = -\int_1^{1/2} \frac{1}{\sinh u}\cdot\frac{\cosh u}{\sinh u}\,du$

$\displaystyle = -\int_1^{1/2} \frac{\cosh u}{\sinh^2 u}\,du$

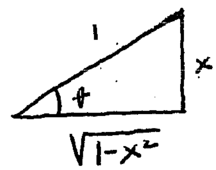
$\displaystyle = -\int_1^{1/2} \frac{d[\sinh u]}{\sinh^2 u} = \left[\frac{1}{\sinh u}\right]_1^{1/2} = \Big[\operatorname{csch} u\Big]_1^{1/2}$

$\displaystyle = \frac{2}{e^{1/2}-e^{-1/2}} - \frac{2}{e-e^{-1}}$

$= 1.0681$

2. (a) $y = \tan(\arcsin x)$

$\theta = \arcsin x$, $\sin\theta = x$, $\tan\theta = \dfrac{x}{\sqrt{1-x^2}}$



$\displaystyle y = \tan\theta = \frac{x}{\sqrt{1-x^2}}, \quad y'(x) = \frac{(1-x^2)^{1/2} - x\cdot\frac{1}{2}\cdot(1-x^2)^{-1/2}(-2x)}{1-x^2}$ $\quad\otimes\left(\frac{1-x^2}{1-x^2}\right)$

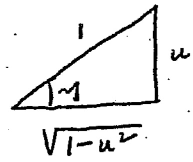
$\displaystyle = \frac{1-x^2+x^2}{(1-x^2)^{3/2}} = \frac{1}{(1-x^2)^{3/2}}$

(b) $y = \arctan\sqrt{t}$, $\tan y = \sqrt{t}$

$\displaystyle[\tan y = \sqrt{t}\,]' = \sec^2 y\; y' = \frac{1}{2}\frac{1}{\sqrt{t}}$

$\displaystyle y' = \frac{1}{2\sqrt{t}\cdot\sec^2 y} = \frac{1}{2\sqrt{t}\,(1+\tan^2 y)} = \frac{1}{2\sqrt{t}\,(1+t)}$

(c) $y = \arcsin u$, $\sin y = u$, for $|u| \le 1$



$\tan y = \dfrac{u}{\sqrt{1-u^2}}$, $|u| \neq 1$

$$y = \arcsin u = \arctan\left(\frac{u}{\sqrt{1-u^2}}\right), \text{ for } |u| < 1$$

(d) $y = \operatorname{arccosh} 3x$, $\cosh y = 3x$

$[\cosh y = 3x]'$

$\sinh y \cdot y' = 3$, $y' = \dfrac{3}{\sinh y} = \dfrac{3}{\pm\sqrt{\cosh^2 y - 1}}$

$y' = \dfrac{3}{\sqrt{9x^2-1}}$ (choose positive branch since $y = \operatorname{arccosh} 3x \ge 0 \Rightarrow \sinh y \ge 0$)

3. $\tanh y = x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

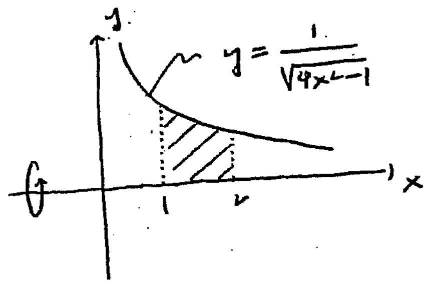
$(e^y + e^{-y})x = e^y - e^{-y}$

$e^{-y}(x+1) = e^y(1-x)$

$e^{2y} = \dfrac{x+1}{1-x}$, $2y = \ln\left(\dfrac{x+1}{1-x}\right)$, for $\dfrac{x-1}{1-x} \ge 0$ or $|x| < 1$

$$y = \operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right), \ |x| < 1$$

4.



$A(x) = \pi y^2 = \pi \cdot \dfrac{1}{4x^2-1}$

$$V = \int_1^2 \pi \cdot \frac{1}{4x^2-1}\,dx \quad \left(\begin{array}{l} u = 2x \\ du = 2dx \end{array}\right)$$

$$= -\frac{\pi}{2}\int_2^4 \frac{1}{1-u^2}\,du$$

$$= -\frac{\pi}{2}\left[\operatorname{arccoth} u\right]_2^4 = -\frac{\pi}{2}\left[\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)\right]_2^4$$

$$= -\frac{\pi}{4}\left(\ln(5/3) - \ln 3\right) = \frac{\pi}{4}\ln\left(\frac{9}{5}\right) = 0.1469\,\pi$$