1. (4 points) Use integration by parts to evaluate $\displaystyle\int_1^{e^\pi} \cos(\ln x)\,dx$.

Solution: Here the integrand is a single function, so we have no choice but to let:

$$\begin{aligned} u &= \cos(\ln x) & dv &= dx \\ du &= -\underbrace{\sin(\ln x)\frac{1}{x}}_{\text{chain rule here!}}\,dx & v &= x. \end{aligned}$$

Repeated (indefinite) integration by parts yields

$$\begin{aligned} \int \cos(\ln x)\,dx &= x\cos(\ln x) + \int \sin(\ln x)\,dx \\ &= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx. \end{aligned}$$

$$\begin{aligned} \therefore \qquad \int_1^{e^\pi} \cos(\ln x)\,dx &= \left[\frac{x}{2}\Big(\cos(\ln x) + \sin(\ln x)\Big)\right]_1^{e^\pi} \\ &= \frac{e^\pi}{2}\cos\pi - \frac{1}{2}\cos 0 = -\frac{(e^\pi+1)}{2}. \end{aligned}$$

<u>Alternatively:</u> With the prior substitution $w = \ln x$, then $dw = \frac{1}{x}dx$, and

$$\int_1^{e^\pi} \cos(\ln x)dx = \int_0^\pi e^w \cos w\,dw.$$

LIATE then suggests to let $u = e^w$ and $dv = \cos w\,dw$, although here one could also use $u = \cos w$ and $dv = e^w\,dw$. Opting for the second substitution, one gets

$$\begin{aligned} \int e^w \cos w\,dw = \int \underbrace{\cos w}_{u}\,\underbrace{d[e^w]}_{dv} &= e^w\cos w - \int \underbrace{e^w}_{v}\,\underbrace{d[\cos w]}_{du} \\ &= e^w \cos w + \int e^w \sin w\,dw \\ &= e^w\big(\cos w + \sin w\big) - \int e^w \cos w\,dw \end{aligned}$$

$$\therefore \qquad \int_0^\pi e^w \cos w\,dw = \left[\frac{e^w}{2}\big(\cos w + \sin w\big)\right]_0^\pi = -\frac{(e^\pi+1)}{2}$$

2. (4 points) Evaluate $\displaystyle\int_0^{\pi/4} \sin^2 x \cos^2 x \, dx$.

Solution:

Alternative 1: Since here the integrand is the product of even powers of sine and cosine, we use the half-angle formula repeatedly:

$$\begin{aligned}
\int_0^{\pi/4} \sin^2 x \cos^2 x \, dx &= \int_0^{\pi/4} \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\
&= \frac{1}{4}\int_0^{\pi/4} \big(1-\cos^2(2x)\big)\, dx \qquad (1) \\
&= \frac{1}{4}\int_0^{\pi/4} \left(1-\frac{(1+\cos 4x)}{2}\right) dx \\
&= \frac{1}{8}\int_0^{\pi/4} (1-\cos 4x)\, dx = \frac{1}{8}\left[x-\frac{\sin 4x}{4}\right]_0^{\pi/4} = \frac{\pi}{32}
\end{aligned}$$

Alternative 2: The integral in (1) could have been done using integration by parts:

$$\begin{aligned}
\frac{1}{4}\int_0^{\pi/4} \big(1-\cos^2 2x\big) dx &= \frac{\pi}{16} - \frac{1}{4}\int_0^{\pi/4} \cos^2 2x \, dx \qquad (2) \\
&= \frac{1}{4}\int_0^{\pi/4} \sin^2 2x \, dx \\
&= \frac{1}{4}\int_0^{\pi/4} \sin 2x \sin 2x \, dx \\
&= -\frac{1}{8}\int_0^{\pi/4} \sin 2x \, d[\cos 2x] \\
&= -\frac{1}{8}\big[\sin 2x \cos 2x\big]_0^{\pi/4} + \frac{1}{4}\int_0^{\pi/4} \cos^2 2x \, dx \qquad (3)
\end{aligned}$$

Combining the integrals on the left and right hand sides (lines (2) and (3), respectively):

$$\therefore \qquad \frac{\pi}{16} - \frac{1}{2}\int_0^{\pi/4} \cos^2 x \, dx = 0 \qquad \text{or} \qquad \int_0^{\pi/4} \cos^2 x \, dx = \frac{\pi}{8}$$

Alternative 3: Using the double-angle formula, followed by the half-angle formula:

$$\int_0^{\pi/4} \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int_0^{\pi/4} \sin^2 2x \, dx = \frac{1}{8}\int_0^{\pi/4} (1-\cos 4x) dx = \frac{\pi}{32}$$

Clearly, **alternative 3** was the best choice!

3. (a) (4 points) Find the general ***explicit*** solution $y = f(x)$ of the differential equation $dy/dx = y - y^2$. (Use partial fractions.)

Solution: The differential equation $y' = y - y^2$ is separable:

$$\frac{dy}{dx} = y - y^2 \qquad \Rightarrow \qquad \frac{dy}{y(1-y)} = dx \tag{4}$$

The rational function $\frac{1}{y(1-y)}$ is proper, we may reduce it in a sum of partial fractions:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

Multiplying through by $y(1-y)$, we obtain the identity $1 = A(1-y) + By = (B-A)y + A$ which is true if $B - A = 0$ and $A = 1$ or $B = A = 1$. Antidifferentiating Equation (4) now gives

$$\begin{aligned} \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dx &= \int dx \\ \ln|y| - \ln|1-y| &= x + C \\ \ln\left|\frac{y}{1-y}\right| &= x + C \qquad \text{(general implicit solution)} \end{aligned} \tag{5}$$

The general explicit solution is found by isolating y in (5):

$$\begin{aligned} \left|\frac{y}{1-y}\right| &= e^{x+C} \\ \frac{y}{1-y} &= Ce^x \\ y &= Ce^x(1-y) \end{aligned}$$

$$\therefore \qquad y(x) = \frac{Ce^x}{1+Ce^x} \quad (C \text{ an arbitrary constant}) \tag{6}$$

(b) (1 point) Verify by direct substitution that the function $f(x)$ is a solution of the given differential equation.

Solution: $f'(x) = \dfrac{(1+Ce^x)Ce^x - Ce^x(Ce^x)}{(1+Ce^x)^2} = \dfrac{Ce^x}{(1+Ce^x)^2}$.

Also: $f(x) - f^2(x) = \dfrac{Ce^x}{1+Ce^x} - \dfrac{Ce^{2x}}{(1+Ce^x)^2} = \dfrac{Ce^x}{(1+Ce^x)^2}$.

$\therefore \qquad f'(x) = f(x) - f^2(x),$ and so f is a solution of the given DE.

4. (4 points) Write the partial fraction decomposition for

$$\frac{4x^5 - 6x^3 + x - 1}{(2x^2-1)(x^2+2x+3)^3}.$$

Do NOT solve for any of the constants.

Solution: The rational fraction $\frac{4x^5-6x^3+x-1}{(2x^2-1)(x^2+2x+3)^3}$ is proper. The denominator is already factorized into the product of the two quadratic terms $2x^2-1$ and x^2+2x+3. x^2+2x+3 has no *real* roots and hence is irreducible. $2x^2-1$ has the two real roots $x = \pm\frac{1}{\sqrt{2}}$ or $\sqrt{2}x = \pm 1$. So the complete factorization is

$$\frac{4x^5 - 6x^3 + x - 1}{(2x^2-1)(x^2+2x+3)^3} = \frac{4x^5 - 6x^3 + x - 1}{(\sqrt{2}x-1)(\sqrt{2}x+1)(x^2+2x+3)^3}$$

The partial fraction decomposition is

$$\frac{A}{\sqrt{2}x-1} + \frac{B}{\sqrt{2}x+1} + \frac{Cx+D}{x^2+2x+3} + \frac{Ex+F}{(x^2+2x+3)^2} + \frac{Gx+H}{(x^2+2x+3)^3}$$

5. (5 points) Determine whether the improper integral

$$\int_1^\infty \frac{dx}{(x-1)^2}$$

converges or diverges. If it converges, find its value.

Solution: This is a doubly improper integral of type I & II since the interval of integration is infinite (type I) and the integrand has an infinite discontinuity at $x = 1$ (type II). Split the integral at an arbitrary value $1 < c < \infty$. We choose $c = 2$. Then

$$\int_1^\infty \frac{dx}{(x-1)^2} = \underbrace{\int_1^2 \frac{dx}{(x-1)^2}}_{(1)} + \underbrace{\int_2^\infty \frac{dx}{(x-1)^2}}_{(2)}$$

Applying the definition to each of the improper integrals (1) and (2):

$$\begin{aligned}\int_1^2 \frac{dx}{(x-1)^2} = \lim_{a\to1^+}\int_a^2 \frac{dx}{(x-1)^2} = \lim_{a\to1^+}\left[\frac{-1}{(x-1)}\right]_a^2 &= -1 + \lim_{a\to1^+}\frac{1}{(a-1)} \\ &= -1 + \frac{1}{0^+} = +\infty \ \therefore \text{ diverges}\end{aligned}$$

Integral (2):

$$\lim_{a\to+\infty}\int_2^{+\infty}\frac{dx}{(x-1)^2}=\lim_{a\to+\infty}\left[\frac{-1}{(x-1)}\right]_2^a = \lim_{a\to+\infty}\frac{-1}{(a-1)}+1$$
$$= \frac{-1}{+\infty}+1=0+1=1 \quad \therefore \text{ converges}$$

Since for the integral $\int_1^{\infty}\frac{dx}{(x-1)^2}$ to converge, both the integrals (1) and (2) need to converge, $\int_1^{\infty}\frac{dx}{(x-1)^2}$ diverges.

6. Determine whether the following series are convergent or divergent. If the series converges, find its sum. State any theorem or formula used.

(a) (4 points) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-1}{n+1}\right)^n$

Solution: The series is divergent by the n^{th}-Term Test, since

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{1-\frac{1}{n}}{1+\frac{1}{n}}\right)^n$$
$$= \frac{\lim_{n\to\infty}(1-\frac{1}{n})^n}{\lim_{n\to\infty}(1+\frac{1}{n})^n}=\frac{e^{-1}}{e}=e^{-2}\neq 0 \qquad (7)$$

Remember the general limit expression $e^x=\lim_{x\to\infty}\left(1+\frac{x}{n}\right)^n$ (Math 100: p. 305 in text., and done in class for the case $x=1$.)

The formal calculation of (7) is as follows:

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n = \lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x$$

Let $y=\left(\frac{x-1}{x+1}\right)^x$. Then $\ln y = x\ln\left(\frac{x-1}{x+1}\right)$ and

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\left(\frac{x-1}{x+1}\right) \qquad \text{(indeterminate case } \infty\cdot 0)$$
$$= \lim_{x\to\infty}\frac{\ln\left(\frac{x-1}{x+1}\right)}{1/x} \qquad \text{(indeterminate case } 0/0)$$
$$\overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{x+1}{x-1}\cdot\frac{2}{(x+1)^2}}{-1/x^2}=\lim_{x\to\infty}\frac{-2x^2}{x^2-1}=\lim_{x\to\infty}\frac{-2}{1-\frac{1}{x^2}}=-2$$

$$\therefore \quad \lim_{x\to\infty} y = e^{-2} \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^n=e^{-2}$$

(b) (4 points) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}+1}{2}\right)^n 5^{-n/2}$

Solution: The series

$$\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}+1}{2}\right)^n 5^{-n/2} = \sum_{n=1}^{\infty}\left(\frac{\sqrt{5}+1}{2}\right)^n\left(\frac{1}{\sqrt{5}}\right)^n = \sum_{n=1}^{\infty}\left(\frac{\sqrt{5}+1}{2}\cdot\frac{1}{\sqrt{5}}\right)^n$$

is a geometric series of ratio $r = \frac{\sqrt{5}+1}{2}\cdot\frac{1}{\sqrt{5}} = \frac{1+\frac{1}{\sqrt{5}}}{2} < 1$ and hence, converges. The sum of the series is

$$\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}+1}{2}\right)^n 5^{-n/2} = \frac{1}{1-\left(\frac{1+\frac{1}{\sqrt{5}}}{2}\right)} - 1 = \frac{2}{2-\left(1+1/\sqrt{5}\right)} - 1 = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

(Note that the sum of the series is $\frac{1}{1-r}-1$, since the term $n=1$ is absent. What if the series index starts at $n=2$? $n=3$? $n=10$?)

7. (4 points) Determine whether the sequence $\left\{\dfrac{\tan^{-1} n}{\ln(n+1)}\right\}$ converges or diverges, and find its limit if it does converge. Justify your answer.

Solution: Since $-\frac{\pi}{2} < \tan^{-1} n < \frac{\pi}{2}$, then

$$\frac{-\pi/2}{\ln(n+1)} < \frac{\tan^{-1} n}{\ln(n+1)} < \frac{\pi/2}{\ln(n+1)}.$$

By the Squeeze Theorem, since $\displaystyle\lim_{n\to\infty}\frac{-\pi/2}{\ln(n+1)} = \lim_{n\to\infty}\frac{\pi/2}{\ln(n+1)} = 0$, then

$$\lim_{n\to\infty}\frac{\tan^{-1} n}{\ln(n+1)} = 0.$$

8. (6 points) Evaluate $\displaystyle\int \frac{\sqrt{4x^2-25}}{x}\,dx$.

Solution: $\displaystyle\int \frac{\sqrt{4x^2-25}}{x}\,dx = 5\int \frac{\sqrt{\frac{4x^2}{25}-1}}{x}\,dx = 5\int \frac{\sqrt{(\frac{2x}{5})^2-1}}{x}\,dx$

Let $\frac{2x}{5} = \sec\theta$, then $\frac{2}{5}dx = \sec\theta\tan\theta\,d\theta$, $\sqrt{(\frac{2x}{5})^2-1} = \sqrt{\sec^2\theta-1} = \sqrt{\tan^2\theta} = \tan\theta$, and

$$\begin{aligned}
\int \frac{\sqrt{4x^2-25}}{x}\,dx &= 5\int \frac{\tan\theta}{\frac{5}{2}\sec\theta}\cdot\frac{5}{2}\sec\theta\tan\theta\,d\theta \\
&= 5\int \tan^2\theta\,d\theta \\
&= 5\int (\sec^2\theta-1)d\theta = 5(\tan\theta-\theta) \\
&= 5\Big(\sqrt{(4x^2/25)-1} - \operatorname{arcsec}(2x/5)\Big) \\
&= \sqrt{4x^2-25} - 5\operatorname{arcsec}(2x/5) + C
\end{aligned}$$