Math 101, Spring 2009

Assignment 4

Due at the beginning of the class, Wed., March 04.
No late assignments will be considered.

Show your work!

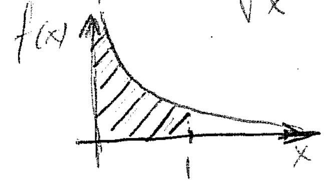
1. In each part, determine all values of p for which the integral is improper.

[3] (a) $\int_0^1 \frac{dx}{x^p}$. For this integral to be improper, $f(x)=x^{-p}$ has to have discont. on $[0,1]$.

Therefore, if $-p \geq 0$, $f(x)=x^{-p}$ is cont. (or if $p \leq 0$)

If $p>0$, then $f(x)=x^{-p}=\frac{1}{x^p}$ has discont. at $x=0$. For example, see $f(x)=\frac{1}{\sqrt{x}}$

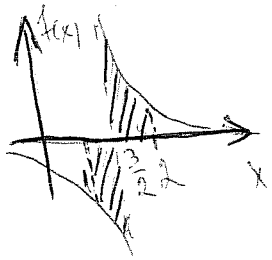
$\rightarrow$ If $p>0 \Rightarrow$ integr. is improper



[3] (b) $\int_1^2 \frac{dx}{x-p}$ $\leftarrow$ If this integral is improper, then $f(x)=\frac{1}{x-p}$ has disc. on $[1,2]$

Therefore, if $1 \leq p \leq 2$, $f(x)$ will have discont. on $[1,2]$. For example, see

$f(x)=\frac{1}{x-\frac{3}{2}}$.



$\rightarrow$ If $1 \leq p \leq 2$, then integr. is improper.

2. Determine if given integrals converge or diverges:

[2] (a) $\int_{e}^{+\infty} \frac{1}{x \ln^3 x}\, dx$ $= I$; Let $u = \ln x \rightarrow du = \frac{dx}{x}$

if $x = e \rightarrow u = 1$

as $x \rightarrow +\infty$, $u \rightarrow +\infty$

$$I = \int_1^{+\infty} \frac{du}{u^3} = \lim_{t\to+\infty} \int_1^t \frac{du}{u^3} = \lim_{t\to+\infty} \left(-\frac{1}{2u^2}\Big|_1^t \right) =$$

$$= \lim_{t\to+\infty} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) = \frac{1}{2} \leftarrow \text{converge}$$

[2] (b) $\int_{-\infty}^{3} \frac{dx}{x^2+9}$ $$= \int_{-\infty}^{3} \frac{dx}{x^2+3^2} = \lim_{t\to-\infty} \int_t^3 \frac{dx}{x^2+3^2} =$$

$$= \lim_{t\to-\infty} \frac{1}{3} \arctan \frac{x}{3} \Big|_t^3 = \lim_{t\to-\infty} \left(\frac{1}{3} \arctan 1 - \right.$$

$$\left. - \frac{1}{3} \arctan \frac{t}{3} \right) = \frac{1}{3} \cdot \frac{\pi}{4} - \frac{1}{3} \cdot \left(-\frac{\pi}{2} \right) =$$

$$= \left(\frac{1}{3} \right) \cdot \left(\frac{\pi}{4} + \frac{\pi}{2} \right) = \frac{1}{3} \cdot \frac{3\pi}{4} = \frac{\pi}{4} \leftarrow \text{converge}$$

[2] (c) $\int_0^{\pi/2} \tan x \, dx = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \tan x \, dx = I$

$$\int_0^t \tan x \, dx = \int_0^t \frac{\sin x}{\cos x} \, dx = \left| \begin{array}{l} \text{Let } u = \cos x \\ \Rightarrow du = -\sin x \, dx \end{array} \right| =$$

$$= -\int_{x=0}^{x=t} \frac{du}{u} = -\ln|u| \Big|_{x=0}^{x=t} = -\ln|\cos x| \Big|_{x=0}^{x=t} =$$

$$= \ln|\sec x| \Big|_{x=0}^{x=t} = \ln|\sec t| - \ln 1 = \ln|\sec t|$$

as $t \to +\infty$ $\lim_{t \to \infty} \sec t$ DNE

diverge

[2] (d) $\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}} = I = \lim_{t \to +\infty} \int_1^t \frac{dx}{x\sqrt{x^2-1}}$

$$\int_1^t \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| \Big|_1^t = \operatorname{arcsec}|t| - \operatorname{arcsec} 1 =$$

$$= \operatorname{arcsec}|t|$$

$$I = \lim_{t \to +\infty} \operatorname{arcsec}|t| = \frac{\pi}{2} \leftarrow \text{converge}$$

[4] 3. Make the u-substitution $u = \sqrt{x}$ and evaluate the resulting definite integral:

$$\int_{12}^{+\infty} \frac{dx}{\sqrt{x}(x+4)} = I$$

Let $u = \sqrt{x} \rightarrow du = \frac{1}{2\sqrt{x}}\,dx \rightarrow \boxed{\frac{dx}{\sqrt{x}} = 2\,du}$

Then $x = u^2$. If $x = 12 \Rightarrow u = \sqrt{12}$,

if $x \to +\infty \Rightarrow u \to +\infty$

$$I = \int_{\sqrt{12}}^{+\infty} \frac{2\,du}{u^2+4} = 2\int_{\sqrt{12}}^{+\infty} \frac{du}{u^2+2^2} = 2\cdot \lim_{t\to+\infty} \int_{\sqrt{12}}^{t} \frac{du}{u^2+2^2} =$$

$$= 2\cdot \lim_{t\to+\infty} \frac{1}{2}\tan^{-1}\frac{u}{2}\Big|_{\sqrt{12}}^{t} = \lim_{t\to+\infty} \tan^{-1}\frac{u}{2}\Big|_{\sqrt{12}}^{t} =$$

$$= \lim_{t\to+\infty}\left(\tan^{-1}\frac{t}{2} - \tan^{-1}\frac{\sqrt{12}}{2}\right) = \frac{\pi}{2} - \tan^{-1}\frac{2\sqrt{3}}{2} =$$

$$= \frac{\pi}{2} - \tan^{-1}\sqrt{3} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

[3] + [3]

4. (a) Sketch some typical integral curves of the differential equation $y' = y/(2x)$.

(b) Find an equation for the integral curve that passes through the point $(2,1)$.

(a) $\frac{dy}{dx} = \frac{y}{2x} \rightarrow \frac{dy}{y} = \frac{dx}{2x} \rightarrow \int \frac{dy}{y} = \frac{1}{2}\int \frac{dx}{x}$

$\ln|y| = \frac{1}{2}\ln|x| + C \rightarrow \ln|y| = \ln\sqrt{|x|} + C$

$e^{\ln|y|} = e^{\ln\sqrt{|x|} + C} = e^{\ln\sqrt{|x|}} \cdot e^{C} = \sqrt{|x|} \cdot C_1,$

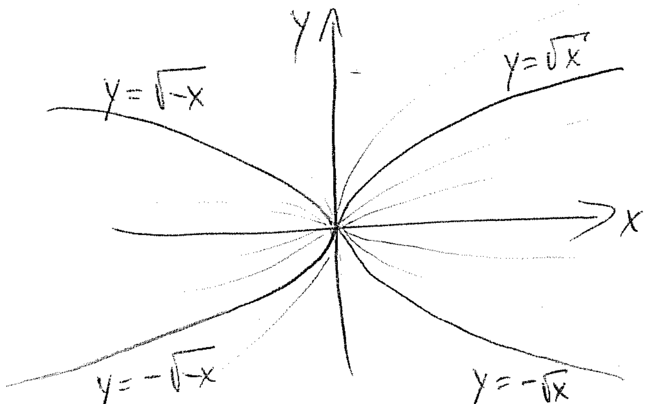
where $C_1 = e^{C} > 0$

Therefore, $|y| = \sqrt{|x|} \cdot C_1$, so

$y = \sqrt{|x|} \cdot C_1$ or $y = -\sqrt{|x|} \cdot C_1$.

For $y = \sqrt{|x|} \cdot C_1 = \begin{cases} \sqrt{x} \cdot C_1, & \text{if } x \geq 0 \\ \sqrt{-x} \cdot C_1, & \text{if } x < 0 \end{cases}$

For $y = -\sqrt{|x|} \cdot C_1 = \begin{cases} -\sqrt{x} \cdot C_1, & \text{if } x \geq 0 \\ -\sqrt{-x} \cdot C_1, & \text{if } x < 0 \end{cases}$



(b) If $(x,y) = (2,1)$ then $|1| = \sqrt{|2|} \cdot C_1 \rightarrow$

$1 = \sqrt{2} \cdot C_1 \rightarrow C_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \rightarrow |y| = \sqrt{|x|} \cdot \frac{\sqrt{2}}{2}$

but since $x = 2 \rightarrow x > 0 \rightarrow |y| = \frac{\sqrt{2}}{2}\sqrt{x} = \sqrt{\frac{x}{2}}$

Since $y = 1 \rightarrow y > 0$, or $y = \sqrt{\frac{x}{2}}$

5. A cup of water with a temperature of 95°C is placed in a room with a constant temperature of 21°C.

(a) Assuming that Newton's Law of Cooling applies, find the temperature of the water t minutes after it is placed in the room.

(b) How many minutes will it take for the water to reach a temperature of 51°C if it cools to 85°C in 1 minute?

Newton's Law of Cooling: $u(t)$ = temperature of water

A = room temperature

(a) $\frac{du}{dt} = -k(u-A)$

$\frac{du}{dt} = -k(u-21) \rightarrow \int \frac{du}{u-21} = -k\int dt$

$\ln|u-21| = -kt + c \rightarrow |u-21| = e^{-kt} \cdot c_1$, where $c_1 = e^{c}$

$|u-21| = \begin{cases} u-21, & \text{if } u \geq 21 \\ -u+21, & \text{if } u < 21. \end{cases}$ This is not possible! temperature of the water can not get lower than room temperature!

$\rightarrow u-21 = e^{-kt} \cdot c_1 \rightarrow u = 21 + c_1 e^{-kt}$

At the time $t=0$, $u = 95°C$ therefore $95 = 21 + c_1 \cdot e^{0}$

$\rightarrow 95 = 21 + c_1 \rightarrow c_1 = 74 \Rightarrow \boxed{u(t) = 21 + 74e^{-kt}}$

(b) $u - 21 = 74e^{-kt} \rightarrow \frac{u-21}{74} = e^{-kt} \rightarrow \ln\frac{u-21}{74} = -kt$

$\rightarrow t = -\frac{1}{k}\ln\frac{u-21}{74}$, So we need to find k.

Observe that at $t=1$, $u=85$, therefore

$$85 = 21 + 74e^{-k \cdot 1} \rightarrow \frac{64}{74} = e^{-k} \rightarrow k = -\ln\frac{64}{74}$$

or $k = \ln\frac{74}{64} = \ln\frac{37}{32}$

Using formula for t, when $u = 51°C$:

$$t = -\frac{1}{k}\ln\frac{u-21}{74} = -\frac{1}{\ln\frac{37}{32}} \cdot \ln\frac{51-21}{74} =$$

$$= -\frac{\ln\frac{30}{74}}{\ln\frac{37}{32}} = \frac{\ln\frac{74}{30}}{\ln\frac{37}{32}} = \frac{0.9029}{0.1452} \approx 6.22 \text{ min}$$