Some extra notes on countability

Theorem 1: If there is a 1-1 mapping $g: A \to \mathbb{N}$, then A is countable

Proof: If A is finite we are done. Otherwise, we will use g to define a 1-1 and onto mapping $f: A \to \mathbb{N}$. Let $f(x) = |\{y : g(y) < g(x)\}|$ (i.e. $f(x)$ is the number of "predecessors" of $g(x)$). We need to show that f is 1-1 and onto.

We first show that f is 1-1. Suppose $x \neq x'$. Then $g(x) \neq g(x')$. Suppose $g(x) < g(x')$ (the case $g(x') < g(x)$ follows the same argument.) Then for any k, $k < g(x) \Rightarrow k < g(x')$, so $f(x) \leq f(x')$. But then since $g(x) < g(x')$, we in fact have $f(x) < f(x')$, so $f(x) \neq f(x')$.

Showing that f is onto is a bit trickier. We have to use induction, and also use the fact that any set of natural numbers has a smallest element. If we take x such that $g(x) = \min \{g(y) : y \in A\}$, then $f(x) = 0$. Now suppose we have an x such that $f(x) = k$. Take x' such that

$$g(x') = \min\{g(y): y \in A \text{ } and \text{ } g(y) > g(x)\}$$

Note that there will always be such an x' if A is infinite and g is 1-1. Then $f(x') = k + 1$. So by induction, for every $k \in \mathbb{N}$, there is some x such that $f(x) = k$.

Corollary 1: If A is countable and there is a 1-1 mapping $g: B \to A$, then B is countable.

Proof: Just use the fact that the composition of 1-1 mappings it itself 1-1.

Theorem 2: For any finite alphabet Σ, Σ^* is countable.

Proof: We will describe a 1-1 mapping $g: \Sigma^* \to \{0,1\}^*$. First of all, suppose $\Sigma = \{s_1, s_2, \ldots, s_k\}$. Let $\overline{s_i}$ denote the binary representation of i. Define $d(0) = 00$, $d(1) = 01$. For $w = w_1 w_2 \ldots w_n \in \{0,1\}^*$, define $D(w) = d(w_1)d(w_2) \ldots d(w_n)$. Then, for $u = u_1 u_2 \ldots u_m \in \Sigma^*$, define

$$g(u) = D(\overline{u_1})11D(\overline{u_2})11 \ldots 11D(\overline{u_m})$$

It is not hard to see that this is 1-1. (Exercise: think about how to determine u given $g(u)$. What is the purpose of the d mapping and the 11's?)

Example 1: Suppose $\Sigma = \{(,), a, b\}$. So $\overline{(} = 1$, $\overline{)} = 10$, $\overline{a} = 11$, and $\overline{b} = 100$. Then

$$g(a(b)a) = 010111011101000011010011010 1$$

Theorem 3: The set of all Turing-recognizable languages over $\{0,1\}$ is countable.

Proof: Let $\mathfrak{M}(0,1)$ represent the set of all TMs with tape alphabet $\{0,1\}$. In class we argued that there will be an alphabet Σ and a 1-1 mapping $\langle\cdot\rangle: \mathfrak{M}(0,1) \rightarrow \Sigma^*$. By Theorem 2 Σ^* is countable, so by Corollary 1, $\mathfrak{M}(0,1)$ is countable. Now a language L is Turing-recognizable iff there is a TM that recognizes it, so there is a 1-1 and onto mapping between the Turing-recognizable languages over $\{0,1\}$ and $\mathfrak{M}(0,1)$. So the set of Turing-recognizable is countable.