Math 101, Spring 2009

Assignment 6

Due at the beginning of the class, Wed., March 25.
No late assignments will be considered.

Show your work!

1. Determine whether the following series converge.

(a) $\sum_{n=2}^{\infty} \frac{(-1)^{n+2}}{\ln n}$: converges

$\frac{1}{\ln n}$ is decreasing and approaching 0 since

$\ln n$ is increasing and approaching to ∞ as $n \to \infty$.

Hence, by the alternating series test, the series converges

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n^2+1}$: diverges:

By using the n^{th} term test,

$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+1}$ does not exist but oscillates between 1 & -1.

Hence, the series diverges

2. Determine whether the following series absolutely converge.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$

Consider $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^n \frac{n!}{n^n}\right| = \sum_{n=1}^{\infty}\frac{n!}{n^n}.$

By using the ratio test,

$$\overline{\lim_{n\to\infty}}\left|\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n}\right| = \overline{\lim_{n\to\infty}}\left|\frac{(n+1)\,n!\,n^n}{(n+1)\,(n+1)^n\cdot n!}\right|$$

$$= \overline{\lim_{n\to\infty}}\left|\left(\frac{n}{n+1}\right)^n\right| = \overline{\lim_{n\to\infty}}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

Thus, the series absolutely converges.

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\cdot 2^n}{\ln n}$

Consider $\displaystyle\sum_{n=2}^{\infty}\left|\frac{(-1)^n\cdot 2^n}{\ln n}\right| = \sum_{n=2}^{\infty}\frac{2^n}{\ln n}.$

By using the ratio test,

$$\overline{\lim_{n\to\infty}}\left|\frac{2^{n+1}/\ln (n+1)}{2^n/\ln n}\right| = \overline{\lim_{n\to\infty}}\left|\frac{2\cdot 2^n \ln n}{2^n\cdot \ln (n+1)}\right| = 2\,\overline{\lim_{n\to\infty}}\left|\frac{\ln n}{\ln (n+1)}\right|$$

$$= 2 > 1.$$

$\uparrow$ by L'Hôpital's Rule

Hence, the series is NOT absolutely convergent.

3. Show $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n(2n-1)}$.

Consider $\frac{1}{n(2n-1)}$ first.

$$\frac{1}{n(2n-1)} = \frac{A}{n} + \frac{B}{2n-1} = \frac{(2A+B)n - A}{n(2n-1)} \Rightarrow \begin{array}{l} A=-1 \\ B=2. \end{array}$$

Thus, $$\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n(2n-1)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-1}{n} + \frac{2}{2n-1}\right)$$

$$= \frac{1}{2}\left\{\left(-1+\frac{2}{1}\right) + \left(-\frac{1}{2}+\frac{2}{3}\right) + \left(-\frac{1}{3}+\frac{2}{5}\right) + \cdots\right\}$$

$$= \left\{\left(1-\frac{1}{2}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \cdots\right\}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} \quad \square.$$

4. Find the radius of convergence of the following series. Check the endpoints of the interval for the convergence of the interval.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n}(x-2)^n$$

By using the ratio test, we need

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}\frac{(n+1)^2}{3^{n+1}}(x-2)^{n+1}}{(-1)^n \frac{n^2}{3^n}(x-2)^n} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{3n^2}|x-2|$$

$$= \frac{1}{3}|x-2| < 1.$$

Thus, $|x-2| < 3 \Leftrightarrow -1 < x < 5$. The series converges for $x \in (-1, 5)$. Check the end points $x=-1$ & $x=5$.

For $x=-1$, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n}(-3)^n = \sum_{n=1}^{\infty} n^2$: diverges

For $x=5$, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n} 3^n = \sum_{n=1}^{\infty} (-1)^n n^2$: diverges.

So, the interval of convergence is $(-1, 5)$.

5. Find the Maclaurin series of $f(x) = \sin(x^3)$ about $x = 0$.

Note that the Maclaurin series of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1!} \quad \text{for all } x.$$

Hence,

$$\sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1!} \quad \text{for all } x.$$

6. Use the Maclaurin series of $f(x) = e^x$ to approximate $\int_0^{0.3} e^{-x^2}\,dx$ with five-place accuracy.

Note that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Thus, $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} \cdots$$

for all x. This is an alternating series.

Hence, we use the alternating series estimate.

$$\int_0^{0.3} e^{-x^2}\,dx = \int_0^{0.3} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} \cdots\right) dx = \left[x - \frac{1}{3}x^3 + \frac{1}{5}\frac{x^5}{2!} - \frac{1}{7}\frac{x^7}{3!} + \cdots\right]_0^{0.3}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (0.3)^{2n+1}}{(2n+1)\, n!}$$

To have the five-place accuracy the remainder $|R_n| < 10^{-5}$.

$\longrightarrow$

So, we need to know how many terms are needed to have that accuracy.

Try $\int_0^{0.3} e^{-x^2}dx = \underbrace{0.3 - \frac{(0.3)^3}{3} + \frac{1}{5\cdot 2!}(0.3)^5}_{S_2} + \underbrace{\text{remainder}}_{R_2}$

Check if $|R_2| < 10^{-5}$.

Since $\int e^{-x^2}dx$ is an alternating series as well for all x,

by the alternating series estimate,

$$|R_2| < a_3 = \frac{(0.3)^7}{7\cdot 3!} = 5.02\times 10^{-6} < 10^{-5}.$$

Thus, we only need three terms to estimate the integral.

Hence, $\int_0^{0.3} e^{-x^2}dx = 0.3 - \frac{(0.3)^3}{3} + \frac{1}{10}(0.3)^5$

$= 0.291243$

≈ 0.29124