# Markov Chains at Equilibrium



# Classify Markov Chains

- **Class of state**: if i -> j ("i communicates with j" or "j is accessible from i") and j -> i, then i, j are in the same class.
- **Irreducible Markov Chain**: The Markov chain has only one class.
- **Reducible Markov Chain**: The Markov chain has more than one class.

Example: Gambler's Ruin problem



# Periodicity

- Period of a state i:

  k = gcd {n: Pr(X(n)=i|X(0) =i) >0},

  where gcd is greatest common divisor.
  - Example: 1-D Random walk (period = 2)
  - Period not equal to one (named aperiodic) are rare in practice



# Markov Chains

- Theorem: an *irreducible, aperiodic* Markov chain has a unique stationary distribution (or distribution at equilibrium, or distribution at steady state).

  $\lim_{n \to \infty} \boldsymbol{S}(n) = \boldsymbol{S}$
- For *reducible or periodic* Makrov chain, $\boldsymbol{S}(\infty)$ is not unique.
- We only discuss the equilibrium behavior for irreducible, aperiodic Markov chains.



# **s**(n) at equilibrium

We would like to know the distribution vector $\mathbf{s}(n)$ when $n \to \infty$.

The state of the system at equilibrium or steady state can then be used to obtain performance parameters such as throughput, delay, loss probability, etc.



# Find **s**(n)

At steady state as $n \to \infty$ the distribution vector $\mathbf{s}$ settles down to a unique value and satisfies the equation

$$\mathbf{P}\,\mathbf{s} = \mathbf{s}$$

This is because the distribution vector value does not vary from one time instant to another at steady state. We immediately recognize that $\mathbf{s}$ in that case is an eigenvector for $\mathbf{P}$ with corresponding eigenvalue $\lambda = 1$.

## Significance of **s** at "Steady State"

- At steady state the system will not settle down to one particular state, as one might suspect. Steady state means that the probability of being in any state will not change with time. The probabilities, or components, of the vector **s** are the ones that are in steady state.



## Significance of **s** at "Steady State"

Assume a 5-state system whose equilibrium or steady state distribution vector is

$$\begin{aligned} \mathbf{s} &= \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \end{bmatrix}^t \\ &= \begin{bmatrix} 0.2 & 0.1 & 0.4 & 0.1 & 0.2 \end{bmatrix}^t \end{aligned}$$

Which state would you think the system will be in at equilibrium? The answer is: The system is in state $s_1$ **with probability** 20%. Or the system is in state $s_2$ with probability 10%, and so on. However, we can say that at steady state the system is most probably in state $s_3$ since it has the highest probability value.



## Finding Steady State Distribution Vector

- Knowledge of **s** helps us find many performance measures for our system such as packet loss probability, throughput, delay, etc. The technique we choose for finding **s** depends on the size and structure of **P.**



## Finding Steady State Distribution Vector

We saw before that $\mathbf{P}^n$ approaches a special structure for large values of $n$. In this case we find that the columns of $\mathbf{P}^n$, for large values of $n$, will all be identical and equal to the steady state distribution vector $\mathbf{s}$.



Example:

Find the steady state distribution vector for the given transition matrix by:
(a) Calculating higher powers for the matrix $\mathbf{P}^n$.
(b) Calculating the eigenvectors for the matrix.

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.4 & 0.5 \\ 0.8 & 0 & 0.5 \\ 0 & 0.6 & 0 \end{bmatrix}$$



Solution:

The given matrix is column stochastic and hence could describe a Markov chain.

Repeated multiplication shows that the entries for $\mathbf{P}^n$ settle down to their stable values.

$$\mathbf{P}^2 = \begin{bmatrix} 0.36 & 0.38 & 0.30 \\ 0.16 & 0.62 & 0.40 \\ 0.48 & 0 & 0.30 \end{bmatrix}$$

$$\mathbf{P}^5 = \begin{bmatrix} 0.3648 & 0.3438 & 0.3534 \\ 0.4259 & 0.3891 & 0.3970 \\ 0.2093 & 0.2671 & 0.2496 \end{bmatrix}$$



$$\mathbf{P}^{10} = \begin{bmatrix} 0.3535 & 0.3536 & 0.3536 \\ 0.4042 & 0.4039 & 0.4041 \\ 0.2424 & 0.2426 & 0.2423 \end{bmatrix}$$

$$\mathbf{P}^{20} = \begin{bmatrix} 0.3535 & 0.3535 & 0.3535 \\ 0.4040 & 0.4040 & 0.4040 \\ 0.2424 & 0.2424 & 0.2424 \end{bmatrix}$$



The entries for $\mathbf{P}^{20}$ all reached their stable values. Since all the columns of $\mathbf{P}^{20}$ are identical, the stable distribution vector is independent of the initial distribution vector (could you prove that? It is rather simple). Furthermore, any column of $\mathbf{P}^{20}$ gives us the value of the equilibrium distribution vector.



The eigenvector corresponding to unity eigenvalue is found to be

$$\mathbf{s} = \begin{bmatrix} 0.3535 & 0.4040 & 0.2424 \end{bmatrix}^t$$

Notice that the equilibrium distribution vector is identical to the columns of the transition matrix $\mathbf{P}^{20}$.



# Techniques for finding s

We can use one of the following approaches for finding the steady state distribution vector $\mathbf{s}$. Some approaches are algebraic while the others rely on numerical techniques.

1. Eigenvector corresponding to eigenvalue $\lambda = 1$ for $\mathbf{P}$.
2. Difference equations.
3. Z-transform (generating functions).
4. Direct numerical techniques to solve a system of linear equations.
5. Iterative numerical techniques to solve a system of linear equations.

Which technique is easier depends on the structure of $\mathbf{P}$.



# Finding s Using Eigenvector Approach

In this case we are interested in finding the eigenvector $\mathbf{s}$ which satisfies the condition

$$\mathbf{P}\,\mathbf{s} = \mathbf{s}$$

MATLAB gives us the command `[X,D] = eig(P)` to find the eigenvectors and eigenvalues of $\mathbf{P}$.

Having found a numeric or symbolic answer, we must normalize $\mathbf{s}$ to ensure that

$$\sum_i s_i = 1$$

Example

Find the steady state distribution vector for the following transition matrix.

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.7 & 0.5 \\ 0.15 & 0.2 & 0.3 \\ 0.05 & 0.1 & 0.2 \end{bmatrix}$$



Solution:

We find the eigenvalues and eigenvectors of **P**

$$\mathbf{s}_1 = \begin{bmatrix} 0.9726 & 0.2153 & .0877 \end{bmatrix}^t \qquad \leftrightarrow \lambda_1 = 1$$

$$\mathbf{s}_2 = \begin{bmatrix} 0.8165 & -0.4882 & -0.4082 \end{bmatrix}^t \qquad \leftrightarrow \lambda_2 = 0.2$$

$$\mathbf{s}_3 = \begin{bmatrix} 0.5345 & -0.8018 & 0.2673 \end{bmatrix}^t \qquad \leftrightarrow \lambda_3 = 0$$



The steady state distribution vector $\mathbf{s}$ corresponds to $\mathbf{s}_1$ and we have to normalize it. We have

$$\sum_i s_i = 1.2756$$

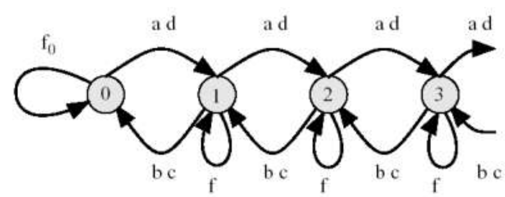
Dividing $\mathbf{s}_1$ by this value we get the steady state distribution vector as

$$\mathbf{s}_1 = \begin{bmatrix} 0.7625 & 0.1688 & 0.0687 \end{bmatrix}^t$$



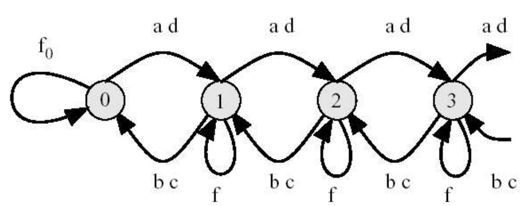
# Finding **s** Using Difference Equations

- This technique for finding **s** is useful only when the state transition matrix **P** is banded. Consider the Markov chain representing a simple discrete-time birth-death process whose state transition diagram is shown in figure below.





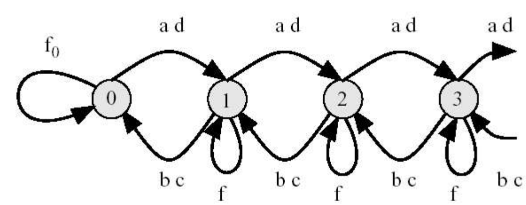

# Finding **s** Using Difference Equations




a probability of one packet arrives in a time step
b=1-a probability of no arrival in a time step
c probability of one packet leaves in a time step
d=1-c probability of no departure in a time step



# Finding **s** Using Difference Equations




We make the following assumptions for the Markov chain.

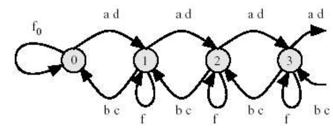
1. The state of the Markov chain corresponds to the number of packets in the buffer or queue. $s_i$ is the probability that $i$ packets are in the buffer.
2. There are $\infty$ states since the size of the buffer or queue is assumed unrestricted.

# Finding **s** Using Difference Equations

3. The probability of a packet arriving to the system is $a$ at a particular time; and the probability that a packet does not arrive is $b = 1 - a$.
4. The probability of a packet departing the system is $c$ at a particular time; and the probability that a packet does not depart is $d = 1 - c$.
5. When a packet arrives, it could be serviced at the same time step and it could leave the queue, at that time step, with probability $c$.



# Finding **s** Using Difference Equations




From the transition diagram, we write the state transition matrix as

$$\mathbf{P} = \begin{bmatrix} f_0 & bc & 0 & 0 & \cdots \\ ad & f & bc & 0 & \cdots \\ 0 & ad & f & bc & \cdots \\ 0 & 0 & ad & f & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1}$$

where $f_0 = 1 - ad$ and $f = ac + bd$.



# Finding **s** Using Difference Equations

We can write

$$\mathbf{P}\,\mathbf{s} = \mathbf{s}$$

Equate corresponding elements on both sides

$$\begin{aligned} ad\,s_0 - bc\,s_1 &= 0 \\ ad\,s_0 - g\,s_1 + bc\,s_2 &= 0 \\ ad\,s_{i-1} - g\,s_i + bc\,s_{i+1} &= 0 \qquad i > 0 \end{aligned}$$

where $g = 1 - f$ and $s_i$ is the $i^{\text{th}}$ component of the state vector $\mathbf{s}$ which is equal to the probability that the system is in state $i$.



# Finding **s** Using Difference Equations

We can write the expressions

$$s_1 = \left(\frac{a\,d}{b\,c}\right) s_0$$

$$s_2 = \left(\frac{a\,d}{b\,c}\right)^2 s_0$$

$$s_3 = \left(\frac{a\,d}{b\,c}\right)^3 s_0$$



# Finding **s** Using Difference Equations

and in general

$$s_i = \left(\frac{ad}{bc}\right)^i s_0 \qquad i \geq 0$$



# Finding **s** Using Difference Equations

It is more convenient to write $s_i$ in the form

$$s_i = \rho^i\, s_0 \qquad i \geq 0$$

where

$$\rho = \frac{a\,d}{b\,c} < 1$$

In that sense $\rho$ can be thought of as **distribution index** that dictates the magnitude of the distribution vector components.

## Finding **s** Using Difference Equations

The complete solution is obtained from the above equations, plus the condition

$$\sum_{i=0}^{\infty} s_i = 1$$

Substituting the expressions for each $s_i$ in the above equation, we get

$$s_0 \sum_{i=0}^{\infty} \rho^i = 1$$



## Finding **s** Using Difference Equations

Thus we obtain

$$\frac{s_0}{1-\rho} = 1 \qquad \text{for } \rho < 1$$

from which we obtain the probability that the system is in state 0 as

$$s_0 = 1 - \rho$$



## Finding **s** Using Difference Equations

The components of the equilibrium distribution vector is given from by

$$s_i = (1-\rho)\rho^i \qquad i \geq 0$$

For the system to be stable we must have $\rho < 1$.



## Finding **s** Using Difference Equations

### Example

Consider the transition matrix for the discrete-time birth-death process that describes single-arrival, single-departure queue with the following parameters a=0.4, c=0.6. Construct the transition matrix and find the equilibrium distribution vector.



Solution:

The transition matrix becomes

$$\mathbf{P} = \begin{bmatrix} 0.84 & 0.36 & 0 & 0 & \cdots \\ 0.16 & 0.48 & 0.36 & 0 & \cdots \\ 0 & 0.16 & 0.48 & 0.36 & \cdots \\ 0 & 0 & 0.16 & 0.48 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$



Probability that queue is empty is

$$s_0 = 0.5556$$

and probability that queue has $i$ customers is

$$s_i = (1-\rho)\rho^i = (0.5556) \times 0.4444^i$$

The distribution vector at steady-state is

$$\mathbf{s} = \begin{bmatrix} 0.5556 & 0.2469 & 0.1097 & 0.0488 & 0.0217 & \cdots \end{bmatrix}^t$$



## Finding **s** Using Forward- or Back Substitution

This technique is useful when the transition matrix $\mathbf{P}$ a is a lower or upper Hessenberg matrix and the elements in each diagonal are not equal.

$$\mathbf{P} = \begin{bmatrix} h_{11} & h_{12} & 0 & 0 & 0 & 0 \\ h_{21} & h_{22} & h_{23} & 0 & 0 & 0 \\ h_{31} & h_{32} & h_{33} & h_{34} & 0 & 0 \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & 0 \\ h_{51} & h_{52} & h_{53} & h_{54} & h_{55} & h_{56} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} \end{bmatrix}$$



## Finding **s** Using Forward- or Back Substitution

At steady state the distribution vector $\mathbf{s}$ satisfies

$$\mathbf{P}\,\mathbf{s} = \mathbf{s}$$

and when $\mathbf{P}$ is lower Hessenberg we have

$$\begin{bmatrix} h_{11} & h_{12} & 0 & 0 & \cdots \\ h_{21} & h_{22} & h_{23} & 0 & \cdots \\ h_{31} & h_{32} & h_{33} & h_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \end{bmatrix}$$



## Finding **s** Using Forward- or Back Substitution

The first row gives $s_1 = h_{11}s_1 + h_{12}s_2$

We can get $s_2 = [(1 - h_{11})/h_{12}]s_1$

The second row gives
$s_2 = h_{21}s_1 + h_{22}s_2 + h_{23}s_3 - h_{21}s_1$

Substituting $s_2$, we get
$s_3 = [(1 - h_{11})(1 - h_{22}/h_{12}h_{23} - h_{12}/h_{23}]s_1$



## Finding **s** Using Forward- or Back Substitution

Continuing in this fashion, we can get all the states $s_i$ as a function of $s_1$.

Since $\sum_{i=1}^{m} s_i = 1$, we can obtain $s_1$. Thereafter, $s_i$ can be obtained for $i > 1$.



## Example

Use forward-substitution to find the equilibrium distribution vector $\mathbf{s}$ for the Markov chain with transition matrix given by

$$\mathbf{s} = \begin{bmatrix} 0.4 & 0.2 & 0 & 0 & 0 & 0 \\ 0.3 & 0.35 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.25 & 0.35 & 0.2 & 0 & 0 \\ 0.1 & 0.15 & 0.25 & 0.35 & 0.2 & 0 \\ 0 & 0.05 & 0.15 & 0.25 & 0.35 & 0.2 \\ 0 & 0 & 0.05 & 0.2 & 0.45 & 0.8 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 & 0 & 0 \\ 0.3 & 0.35 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.25 & 0.35 & 0.2 & 0 & 0 \\ 0.1 & 0.15 & 0.25 & 0.35 & 0.2 & 0 \\ 0 & 0.05 & 0.15 & 0.25 & 0.35 & 0.2 \\ 0 & 0 & 0.05 & 0.2 & 0.7 & 0.8 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix}$$



The first row gives us for $s_2 = 3s_1$. Continuing with successive rows, we get

$$\begin{aligned} s_3 &= 8.25 s_1 \\ s_4 &= 22.0625 s_1 \\ s_5 &= 58.6406 s_1 \\ s_6 &= 156.0664 s_1 \end{aligned}$$



Summing the values of all o the components, we get

$$\sum_{i=1}^{6} s_i = 249.0195 s_1 = 1$$

Thus, the distribution vector is

$$[0.0040 \quad -0.0120 \quad 0.0331 \quad 0.0886 \quad 0.2355 \quad 0.6267]^t$$



# Finding **s** Using Direct Techniques

- Direct techniques are useful when the transition matrix **P** has no special structure but its size is small.



# Finding **s** Using Direct Techniques

In that case we start with the equilibrium equation

$$\mathbf{P}\,\mathbf{s} = \mathbf{s}$$

where $\mathbf{s}$ is the unknown $n$-component distribution vector. This can be written as

$$\begin{aligned} (\mathbf{P} - \mathbf{I})\,\mathbf{s} &= \mathbf{0} \\ \mathbf{A}\,\mathbf{s} &= \mathbf{0} \end{aligned}$$



# Finding **s** Using Direct Techniques

$$\begin{aligned} (\mathbf{P} - \mathbf{I})\,\mathbf{s} &= \mathbf{0} \\ \mathbf{A}\,\mathbf{s} &= \mathbf{0} \end{aligned}$$

which describes a **homogeneous system of linear equations** with $\mathbf{A} = \mathbf{P} - \mathbf{I}$. The rank of $\mathbf{A}$ is $n-1$ since the sum of the columns must be zero. Thus, there are many possible solutions to the system and we need an extra equation to get a unique solution.

# Finding **s** Using Direct Techniques

The extra equation that is required is the normalizing condition

$$\sum_{i=1}^{m} s_i = 1$$

where we assumed our states are indexed as $s_1, s_2, \cdots, s_m$. We can delete any row matrix $\mathbf{A}$ in the equation $\mathbf{A}\,\mathbf{s} = \mathbf{0}$ and replace it with the normalizing condition.



# Finding **s** Using Direct Techniques

In that case we have the system of linear equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,m} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_{m-1} \\ s_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

This gives us a system of linear equations whose solution is the desired steady state distribution vector.



## Example

Find the steady state distribution vector for the state transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0 \\ 0.1 & 0.5 & 0.6 \\ 0.5 & 0.3 & 0.4 \end{bmatrix}$$



## Solution:

First, we have to obtain matrix $\mathbf{A} = \mathbf{P} - \mathbf{I}$

$$\mathbf{A} = \begin{bmatrix} -0.6 & 0.2 & 0 \\ 0.1 & -0.5 & 0.6 \\ 0.5 & 0.3 & -0.6 \end{bmatrix}$$

Now we replace the last row in $\mathbf{A}$ with all ones to get

$$\mathbf{A} = \begin{bmatrix} -0.6 & 0.2 & 0 \\ 0.1 & -0.5 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}$$



The system of linear equations we have to solve is

$$\begin{bmatrix} -0.6 & 0.2 & 0 \\ 0.1 & -0.5 & 0.6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The solution for $\mathbf{s}$ is

$$\mathbf{s} = \begin{bmatrix} 0.1579 & 0.4737 & 0.3684 \end{bmatrix}^t$$