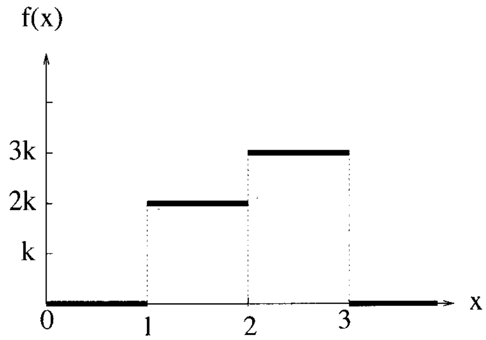
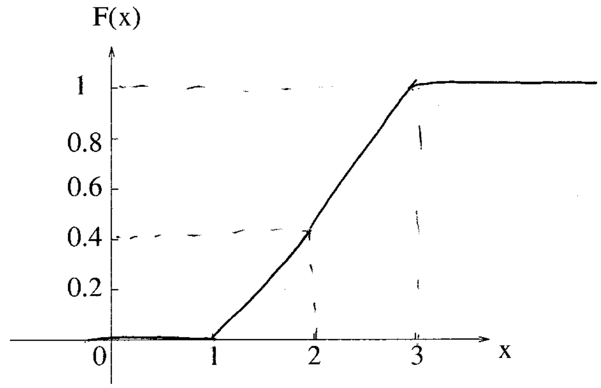
Question 1



(a) PDF of X



(b) CDF of X

The PDF function of R.V. X is shown in the above figure (a).

(1) Find out k. [2 marks]

(2) What is the CDF of X? Scratch it in the figure (b). [3 marks]

(3) What is the probability of $2 \leq x \leq 6$? [3 marks]

(1) $f(x) = \begin{cases} 2k & 1 \leq x < 2 \\ 3k & 2 \leq x < 3 \\ 0 & \text{o.w} \end{cases}$ $\int_{-\infty}^{\infty} f(x)\,dx = \int_1^2 2k\,dx + \int_2^3 3k\,dx = 5k = 1$

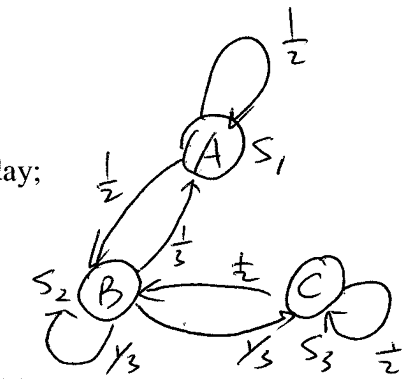
$\Rightarrow k = \frac{1}{5}$

(2) $F(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 0 & x < 1 \\ \int_1^x \frac{2}{5}\,dy = \frac{2}{5}(x-1) & 1 \leq x < 2 \\ \int_1^2 \frac{2}{5}\,dy + \int_2^x \frac{3}{5}\,dy = \frac{2}{5} + \frac{3}{5}(x-2) = \frac{3}{5}x - \frac{4}{5} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$

(3) $P(2 \leq x \leq 6) = F(6) - F(2^-) = 1 - \frac{2}{5} = \frac{3}{5}$

Question 2 A truck travels between three places, A, B, and C.

If it is in place A, it will be in place A or B with equal probability the next day;
if it is in place B, it will be in place A, B, or C with equal probability the next day;
if it is in place C, it will be in place B or C with equal probability the next day.



(1) The truck is in place A today. What is the probability that the truck will be in place C the day after tomorrow? [5 marks]

(2) In the long term, what is the percentage of time that the truck is in place A, B, and C respectively? [5 marks]

(1) $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$

$$\vec{S}(2) = P^2 \vec{S}(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{24} & \times & \times \\ \frac{10}{24} & \times & \times \\ \frac{1}{6} & \times & \times \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{12} \\ \frac{5}{12} \\ \frac{1}{6} \end{bmatrix}$$

$\therefore$ P (truck in C two days later) $= \frac{1}{6}$

(2) $\begin{cases} S_1 \cdot \frac{1}{2} = S_2 \cdot \frac{1}{3} \\ S_2 \cdot \frac{1}{3} = S_3 \cdot \frac{1}{2} \\ S_1 + S_2 + S_3 = 1 \end{cases} \Rightarrow \begin{cases} S_1 = S_1 \\ S_2 = \frac{3}{2} S_1 \\ (+) \; S_3 = \frac{2}{3} S_2 = S_1 \end{cases}$

$$1 = \frac{7}{2} S_1$$

$\therefore \; S_1 = \frac{2}{7}, \; S_2 = \frac{3}{7}, \; S_3 = \frac{2}{7}$

The percentages of time that the truck is in place A, B and C are $\frac{2}{7}, \frac{3}{7}, \frac{2}{7}$, respectively.

Question 3 Consider the queue of a bank ATM machine. Customers arrival process can be modeled as a Markovian process that every minute, the probability of one customer arrival is $a = 0.8$. The customer departure process can also be approximated as a Markovian process that every minute, the probability of the customer (currently using the ATM machine) finishing his/her transition and leaving is $c = 0.5$. Half of the customers are female and half are male. For a female customer, when she arrives at the ATM machine and finds that there are two or more other customers in the queue system, she will decide not to use the machine (being blocked); for a male customer, when he finds that there are three or more other customers in the queue system, he will decide not to use the machine (being blocked).

(1) Draw the state transition diagram using the total number of customers in the system as the state and obtain the state transition probabilities. [3 marks]

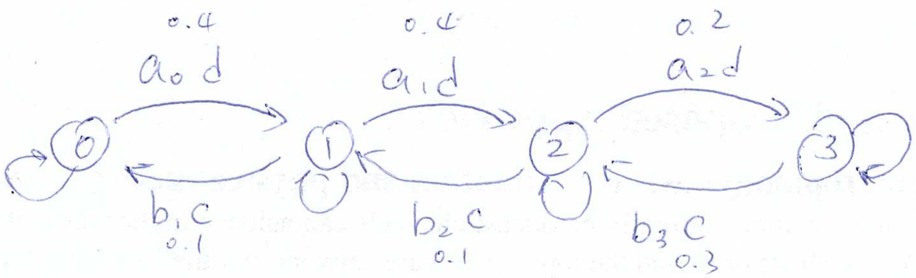
(2) Find out the steady state distribution of the system (s). [3 marks]

(3) What is the average number of customers in the queue? [3 marks]

(4) What is the percentage of *female* customers being blocked, L_1? [Hint: under what conditions a female customer will be blocked?] [3 marks]

$a = a = 0.8$ $\quad b_i = 0.2$

(1)



$a_0 = a_1 = 0.8$, $a_2 = \frac{a}{2} = 0.4$, $d = 0.5$, $c = 0.5$

$P_{01} = b_1 c = P_r\{\text{No arrival, one dept. when currently the queue length is } 1\}$

$= 0.2 \cdot 0.5 = 0.1$

$P_{12} = b_2 c = P_r\{\text{No arrival, one dept. when } Q = 2\} = 0.2 \cdot 0.5 = 0.1$

$P_{23} = b_3 c = P_r\{\text{No man arrival, one dept. when } Q = 3\} = 0.6 \cdot 0.5 = 0.3$

$$P = \begin{bmatrix} 0.6 & 0.1 & 0 & 0 \\ 0.4 & 0.5 & 0.1 & 0 \\ 0 & 0.4 & 0.7 & 0.3 \\ 0 & 0 & 0.2 & 0.7 \end{bmatrix}$$

(2) $\left.\begin{array}{l} S_0 \cdot 0.4 = S_1 \cdot 0.1 \\ S_1 \cdot 0.4 = S_2 \cdot 0.1 \\ S_2 \cdot 0.2 = S_3 \cdot 0.3 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} S_0 = S_0 \\ S_1 = 4S_0 \\ S_2 = 16 S_0 \\ S_3 = \frac{32}{3} S_0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} S_0 = \frac{3}{95} \\ S_1 = \frac{12}{95} \\ S_2 = \frac{48}{95} \\ S_4 = \frac{32}{95} \end{array}\right.$

(3) $Q_a = S_1 + 2S_2 + 3S_3 = \frac{204}{95}$

(4) $L_1 = S_2 d + S_3 = \frac{56}{95}$