Question 1.

Suppose, to the contrary, there exists an algorithm $\mathcal{A}$ that can deterministically find the k^{th} smallest element of S without making n comparisons. Then, some subset T of S will not have been compared to the rest of S. Suppose $\mathcal{A}$ returns some element x as the k^{th} smallest element. However, notice that if any elements of T are less than x then the algorithm will have returned the incorrect answer. Therefore, there cannot exist a deterministic algorithm that computes the k^{th} smallest element without making n comparisons. This implies the lower bound is $\Omega(n)$.

Question 2.

Let x denote the median of medians. Then, at least half the $\lceil \frac{n}{3} \rceil$ medians are greater than x. Note that the ceiling function counts the last group which may not contain three elements. Each group contributes 2 elements except the group containing x and the, possibly, last group. Therefore, the number of elements greater (or less) than x is at least

$$2\left(\frac{1}{2}\left\lceil \frac{n}{3} \right\rceil - 2\right) \geq \frac{n}{3} - 4$$

Therefore, the number of elements we have to recurse on is at most

$$n - \left(\frac{n}{3} - 4\right) = \frac{2n}{3} + 4$$

Thus, the recurrence relation becomes

$$T(n) \leq T\left(\left\lceil \frac{n}{3} \right\rceil\right) + T\left(\frac{2n}{3} + 4\right) + an$$

Assume $T(n) \leq cn \; \forall n > 0$.

$$\begin{aligned}
T(n) &\leq T\left(\left\lceil \frac{n}{3} \right\rceil\right) + T\left(\frac{2n}{3} + 4\right) + an \\
&\leq c\left\lceil \frac{n}{3} \right\rceil + c\left(\frac{2n}{3} + 4\right) + an \\
&\leq \frac{cn}{3} + \frac{2cn}{3} + 4c + an \\
&= cn + 4c + an
\end{aligned}$$

But, $a, c > 0$ so $4c + an > 0 \; \forall n$ which violates our assumption. Therefore, Linear Select with groups of size 3 does not have a linear running time.

Question 3.

(a) We extend the proof given in class by considering the case where there are several heaviest edges in a cycle C of G. Let B be the set of heaviest edges in C. We wish to show that every edge $e \in B$ is not in some MWST for G.

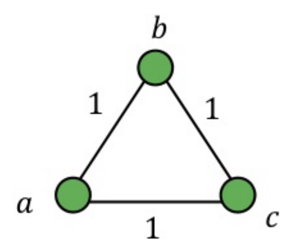
Proof. Let T be a MWST and suppose some edge $e_i \in B$ is present in T. Removing e_i disconnects T into two trees T_1 and T_2. Then, we can merge T_1 and T_2 with some edge $e_j \in B$ where $j \neq i$ to form a tree T'. Then, T' has the same weight as T, but does not contain the edge e_i. □

(b) We extend the proof given in class by considering the case where there are several lightest edges in a cut C of G. Let B be the set of lightest edges in C. We wish to show that every edge $e \in B$ is in some MWST for G.

Proof. Let T be a MWST and suppose some edge $e_i \in B$ is present in T. Then, we cannot have any other edge in B present in T as this would create a cycle. However, we can replace e_i with any other edge $e_j \in B$ such that $j \neq i$ to form a tree T'. Then, T' has the same weight as T. Therefore, we have that T' can contain any edge $e \in B$. □

Question 4.

(a) Consider the following graph:



We see that Boruvka's algorithm can run into problems when there are multiple minimum weight edges to choose from as there is no rule for which edges to pick. Thus, the algorithm may choose edges that result in returning a cycle. In the above graph, a could choose b, b could choose c, and c could choose a.

(b) We can overcome this difficulty by always selecting the lexicographically smallest edge when there are multiple choices where the endpoints of each edge are considered when sorting lexicographically.