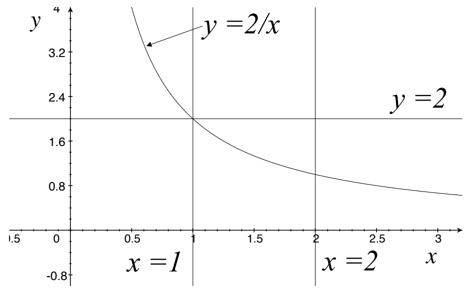
1. (6 points) Find the volume of the solid generated by revolving the region bounded by the graph of the equation $y = 2/x$, the lines $y = 0$, $x = 1$, and $x = 2$, around the line $y = 2$. Express your result as a simplified fraction.



Solution:

Method of cross sections:

The resulting solid of revolution has cross sections perpendicular to the x-axis that are washers of inner radius $2 - y = 2 - 2/x$ and outer radius 2. The cross-sectional area function is $A(x) = \pi {r_{out}}^2 - \pi {r_{in}}^2 = \pi 2^2 - \pi\big(2 - (\frac{2}{x})\big)^2 = \pi\big(\frac{8}{x} - \frac{4}{x^2}\big)$.

$$\Rightarrow V = \int_1^2 A(x)dx = \int_1^2 \pi\left(\frac{8}{x} - \frac{4}{x^2}\right)dx \quad = \quad \pi\left[8\ln|x| - \frac{4}{x}\right]_1^2 = \pi(8\ln 2 - 2)$$

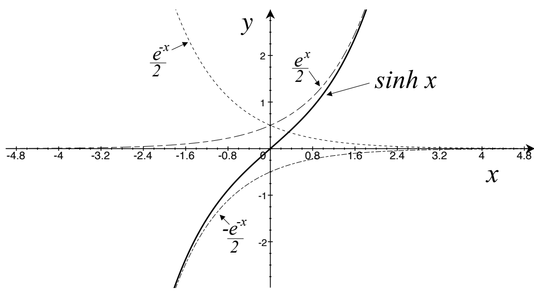
Method of cylindrical shells:

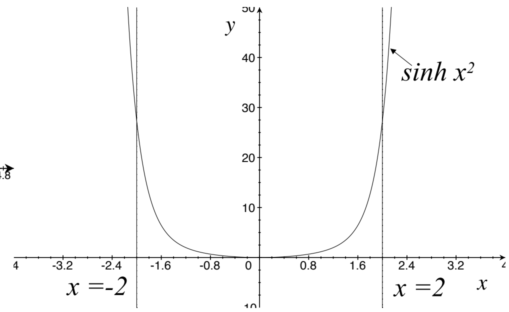
We consider cylindrical shells of infinitesimal volume $dV = 2\pi r h dy$, where $r = 2 - y$. The equation for the height of the cylindrical shells is not uniform on the y-interval $[1, 2]$: Their height is constant on $[0, 1]$ $(h = 1)$, but they have variable height $h = \frac{2}{y} - 1$ on $[1, 2]$.

$$\begin{aligned}
\Rightarrow V &= \int_0^1 2\pi(2-y)dy + \int_1^2 2\pi(2-y)\Big(\frac{2}{y} - 1\Big)dy \\
&= 2\pi\left[2y - \frac{y}{2}\right]_0^1 + 2\pi\int_1^2 \Big(\frac{4}{y} + y - 4\Big)dy \\
&= 3\pi + 2\pi\left[4\ln|y| + \frac{y^2}{2} - 4y\right]_1^2 \\
&= 3\pi + 2\pi\left[4\ln 2 - \frac{5}{2}\right] = \pi(8\ln 2 - 2)
\end{aligned}$$

The first integral corresponds to the volume of a hollowed cylinder of height 1, inner radius 1, and outer radius 2.

2. (6 points) Use the method of *cylindrical shells* to find the volume of the solid that is generated when the region enclosed by the graph of $y = \sinh(x^2)$, the lines $|x| = 2$ and the x-axis is revolved about the y-axis. **Express your result as a simplified fraction. Do not use your calculator**.





Solution: The function $\sinh x^2$ is even and hence symmetrical with respect to the y-axis. The solid is generated by an 180° revolution of the region in the plane around y-axis. We consider cylindrical shells of infinitesimal volume $dV = 2\pi r h dx$, where $r = x$ and the height h is given by $y = \sinh x^2$.

$$\begin{aligned} \Rightarrow V = \int_0^2 dV = \int_0^2 2\pi x \sinh x^2 dx &= \pi\big[\cosh x^2\big]_0^2 \\ &= \pi\big(\cosh 4 - \cosh 0\big) \\ &= \frac{\pi}{2}\big(e^4 - e^{-4} - 2\big) \end{aligned}$$

3. (a) (4 points) Find $\dfrac{d}{dx}\operatorname{arsinh} x$. [Hint: Use the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$, and the fact that $\cosh x > 0$ for all x.]

Solution: We let $y = \operatorname{arsinh} x$ and rewrite the identity as $\sinh y = x$. Taking the derivative of the latter with respect to x yields

$$\cosh y \cdot \frac{dy}{dx} = 1 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\cosh y}$$

From the elementary hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$ and the fact that $\cosh x > 0$ for all x we have that $\cosh x = \sqrt{1 + \sinh^2 x}$. Hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

(b) (4 points) Use your result in (a) to find

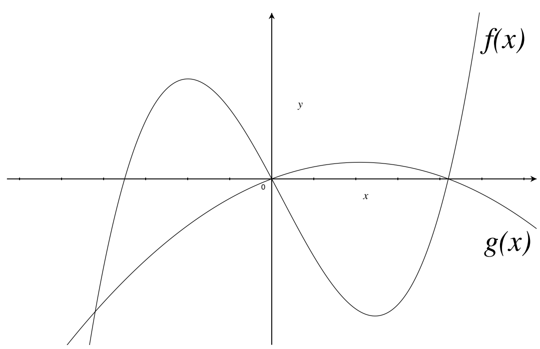
$$\int_0^2 \frac{5x}{\sqrt{x^4+1}}dx.$$

Use your calculator to express the result as a number which is accurate to at least three decimal places.

Solution:

$$\begin{aligned}\int_0^2 \frac{5x}{\sqrt{x^4+1}}dx = \frac{5}{2}\int_0^2 \frac{d[x^2]}{\sqrt{[x^2]^2+1}} = \frac{5}{2}\big[\operatorname{arsinh} x^2\big]_0^2 &= \frac{5}{2}\big(\operatorname{arsinh} 4 - \operatorname{arsinh} 0\big)\\ &= \frac{5}{2}\operatorname{arsinh} 4 \approx 5.237\end{aligned}$$

4. (4 points) Find the area of the region bounded by the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$. Use your calculator to express the result as a number which is accurate to at least three decimal places.



Solution: The points of intersection of the two graphs satisfy both $f(x)$ and $g(x)$ simultaneously. Solving for x in $3x^3 - x^2 - 10x = -x^2 + 2x \Rightarrow x(3x^2 - 12) = 0$ gives $x = 0$, -2, and 2.

$$A = \int_{-2}^{2} \big|f(x) - g(x)\big|dx = \int_{-2}^{0} \big[f(x) - g(x)\big]dx + \int_{0}^{2} \big[g(x) - f(x)\big]dx$$

Since $f(x) - g(x) = 3x^3 - 12x = -\big[g(x) - f(x)\big]$, then

$$\begin{aligned}A &= \int_{-2}^{0} \big[3x^3 - 12x\big]dx + \int_0^2 \big[12x - 3x^3\big]dx\\ &= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[6x^2 - \frac{3x^4}{4}\right]_0^2 = -(12-24) + (-12+24) = 24\end{aligned}$$

5. (a) (3 points) Write the expression $y = \csc\left(\arctan \dfrac{x}{\sqrt{2}}\right)$ in algebraic form. [Hint: Use a right triangle.]

Solution: $\theta = \arctan \dfrac{x}{\sqrt{2}} \Rightarrow \tan\theta = \dfrac{x}{\sqrt{2}}$ $\left(= \frac{\text{opp}}{\text{adj}}\right)$.

Hence $y = \csc\theta = \dfrac{\sqrt{x^2+2}}{x}$ $\left(= \frac{\text{hyp}}{\text{opp}}\right)$.

Using trigonometric identities: $1 + \cot^2\theta = \csc^2\theta$

$$\csc\theta = \pm\sqrt{1 + \frac{1}{\tan^2\theta}} = \pm\sqrt{1 + \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2}} = \pm\sqrt{1 + \frac{2}{x^2}}$$

$$\Rightarrow y = \csc\theta = \frac{\sqrt{x^2+2}}{x}$$

The positive square root is chosen for we have: $x \geq 0 \Rightarrow 0 \leq \theta \leq \pi/2 \Rightarrow \csc\theta > 0$ and $x \leq 0 \Rightarrow -\pi/2 \leq \theta \leq 0 \Rightarrow \csc\theta < 0$.

(b) (3 points) Find $\dfrac{dy}{dx}$ if $y = 3\arccos(x^2 + 0.5)$

Solution: $y = 3\arccos(x^2+0.5) \Rightarrow \cos\left(\dfrac{y}{3}\right) = x^2 + 0.5$. By the chain rule:

$$-\frac{1}{3}\sin(y/3)\cdot y'(x) = 2x$$

$$y'(x) = \frac{-6x}{\sin(y/3)}$$

$$y'(x) = \frac{-6x}{\sqrt{1-\cos^2(y/3)}}$$

$$y'(x) = \frac{-6x}{\sqrt{1-(x^2+0.5)^2}}$$

$$y'(x) = \frac{-6x}{\sqrt{3/4 - x^2 - x^4}}$$

Positive square root is chosen since $0 \leq \frac{y}{3} \leq \pi \Rightarrow 0 \leq \sin\frac{y}{3} \leq 1$.

6. (5 points) The base of a solid is bounded by $y = x^3$, $y = 0$, and $x = 1$. Find the volume of the solid if the cross sections perpendicular to the y-axis are equilateral triangles.

Solution: Cross sections are equilateral triangles of side length s. Use pythagorean theorem to find that their height h equals $\sqrt{s^2 - s^2/4} = \frac{\sqrt{3}}{2}s$. The cross sectional area function is then

$$A(y) = \frac{s \cdot h}{2} = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(1 - \sqrt[3]{y})^2 = \frac{\sqrt{3}}{4}(1 - 2y^{1/3} + y^{2/3})$$

so

$$V = \int_0^1 A(y)dy = \int_0^1 \frac{\sqrt{3}}{4}(1 - 2y^{1/3} + y^{2/3})dy = \frac{\sqrt{3}}{4}\Big[y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3}\Big]_0^1$$

$$= \frac{\sqrt{3}}{4}\Big(1 - \frac{3}{2} + \frac{3}{5}\Big) = \frac{\sqrt{3}}{4}$$

7. (5 points) Find the arc length of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $1 \leq x \leq 2$.

Solution: With $\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$ and the fact that $s = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$, we have:

$$= \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx = \int_1^2 \sqrt{1 + \frac{x^4}{4} + \frac{1}{4x^4} - \frac{1}{2}}\, dx = \int_1^2 \sqrt{\frac{1}{2} + \frac{x^4}{4} + \frac{1}{4x^4}} dx$$

$$= \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} dx = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \Big[\frac{x^3}{6} - \frac{1}{2x}\Big]_1^2 = \frac{7}{6} + \frac{1}{4} = \frac{17}{12}$$