CSC 225 - Summer 2014
Sample Identities for Induction Proofs

Example 1

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \quad \forall n \geq 0$$

Basis

When $n = 0$, $\sum_{i=0}^{n} i = 0$, and $\frac{n(n+1)}{2} = 0$, so the identity holds.

Induction Hypothesis

Suppose $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ for some $n \geq 0$.

Induction Step

Consider $n+1$.

$$\begin{aligned}
\sum_{i=0}^{n+1} i &= (n+1) + \sum_{i=0}^{n} i \\
&= (n+1) + \frac{n(n+1)}{2} && \text{(By the induction hypothesis)} \\
&= \frac{2n+2+n^2+n}{2} \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

Therefore, the identity holds for $n+1$, and by induction, the identity holds for all $n \geq 0$.

Example 2

$$\sum_{i=1}^{n} 2i - 1 = n^2 \quad \forall n \geq 0$$

The 'obvious' solution is the following proof:

Basis

When $n = 1$, $\sum_{i=1}^{n} 2i - 1 = 2 - 1 = 1 = 1^2$, so the identity holds.

Induction Hypothesis

Suppose the identity holds for some $n \geq 1$.

Induction Step

Consider $n+1$.

$$\begin{aligned}
\sum_{i=1}^{n+1} 2i - 1 &= (2\cdot 1 - 1) + (2 \cdot 2 - 1) + \ldots + (2 \cdot n - 1) + (2 \cdot (n+1) - 1) \\
&= \left[\sum_{i=1}^{n} 2i - 1\right] + (2(n+1) - 1) \\
&= n^2 + (2(n+1) - 1) && \text{(By the induction hypothesis)} \\
&= n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$

Therefore, the identity holds for $n+1$, and, by induction, for all $n \geq 1$.

An alternative proof technique is to rearrange the identity algebraically. Each line below is equivalent to the original identity. Note that arithmetic operations do not change the underlying statement to be proven, nor do they prove anything about the identity itself (that is, if the identity is false, all of the equivalent forms are also false).

$$\begin{aligned}
\sum_{i=1}^{n} 2i - 1 &= n^2 \\
\left(\sum_{i=1}^{n} 2i\right) - \left(\sum_{i=1}^{n} 1\right) &= n^2 \\
2\left(\sum_{i=1}^{n} i\right) - (n) &= n^2 \\
2\left(\sum_{i=1}^{n} i\right) &= n^2 + n \\
\sum_{i=1}^{n} i &= \frac{n^2+n}{2} \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}
\end{aligned}$$

The last version is equivalent to the identity in example 1, which might be easier to prove. To prove this particular identity on an assignment or test, it is probably easier to use the obvious proof above. However, for some identities, manipulating the identity algebraically before proving it can make the proof significantly easier.

Example 3

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \quad \forall n \geq 0$$

Basis

When $n = 0$, $\sum_{i=0}^{n} 2^i = 2^0 = 1 = 2^1 - 1 = 2^{n+1} - 1$, so the identity holds.

Induction Hypothesis

Suppose the identity holds for some $n \geq 0$.

Induction Step

Consider $n+1$.

$$\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= 2^0 + 2^1 + 2^2 + \ldots + 2^n + 2^{n+1} \\
&= 2^{n+1} + \sum_{i=0}^{n} 2^i \\
&= 2^{n+1} + \left(2^{n+1} - 1\right) && \text{(By the induction hypothesis)} \\
&= 2 \cdot 2^{n+1} - 1 \\
&= 2^{n+2} - 1
\end{aligned}$$

Therefore, the identity holds for $n+1$, and, by induction, it holds for all $n \geq 0$.

Example 4

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \geq 0$$

Basis

When $n = 0$, $\sum_{i=0}^{n} i^2 = 0^2 = 0 = \frac{n(n+1)(2n+1)}{6}$, so the identity holds.

Induction Hypothesis

Suppose the identity holds for some $n \geq 0$.

Induction Step

Consider $n+1$.

$$\begin{aligned}
\sum_{i=0}^{n+1} i^2 &= 0^2 + 1^2 + 2^2 + \ldots + n^2 + (n+1)^2 \\
&= (n+1)^2 + \sum_{i=0}^{n} i^2 \\
&= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} && \text{(By the induction hypothesis)} \\
&= \frac{6(n+1)^2 + n(n+1)(2n+1)}{6} \\
&= \frac{(n+1)\left(6(n+1) + n(2n+1)\right)}{6} \\
&= \frac{(n+1)\left(6n + 6 + 2n^2 + n\right)}{6} \\
&= \frac{(n+1)\left(2n^2 + 7n + 6\right)}{6}
\end{aligned}$$

To finish the proof, we expand the right hand side of the identity until it matches the form above. Although it is possible to continue the algebra above to obtain the right hand side, doing so requires factoring, while starting with the right hand side only requires expanding. As usual, as long as the proof is concise and mathematically correct, it will receive full marks on an assignment or test (so there is no requirement that it be done a certain way).

For $n+1$, the right hand side of the identity is

$$\begin{aligned}
\frac{(n+1)(n+2)(2(n+1)+1)}{6} &= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)(2n^2+7n+6)}{6}
\end{aligned}$$

We have shown that, for $n+1$, the two sides of the identity can be transformed into the same formula using only algebra and the induction hypothesis. Therefore, the identity holds for $n+1$, completing the proof. As a result, by induction, the identity holds for all $n \geq 0$.

Example 5

$$\sum_{i=1}^{n} 3i^2 - 3i + 1 = n^3 \quad \forall n \geq 0$$

Basis

When $n = 0$, $\sum_{i=0}^{n} 3i^2 - 3i + 1 = 0 = 0^3$, so the identity holds.

Induction Hypothesis

Suppose the identity holds for some $n \geq 0$.

Induction Step

Consider $n+1$. As with example 4, the proof is easier if we manipulate both sides of the equation until they match. First we expand the right hand side:

$$\begin{aligned}
(n+1)^3 &= n^3 + 3n^2 + 3n + 1 \\
&= \left(\sum_{i=0}^{n} 3i^2 - 3i + 1\right) + 3n^2 + 3n + 1 \qquad \text{(By the induction hypothesis)}
\end{aligned}$$

Next, we expand the sum on the left hand side. Note that the induction hypothesis is not used in this part of the proof.

$$\begin{aligned}
\sum_{i=0}^{n+1} 3i^2 - 3i + 1 &= \left(\sum_{i=0}^{n} 3i^2 - 3i + 1\right) + 3(n+1)^2 - 3(n+1) + 1 \\
&= \left(\sum_{i=0}^{n} 3i^2 - 3i + 1\right) + 3(n^2 + 2n + 1) - 3n - 3 + 1 \\
&= \left(\sum_{i=0}^{n} 3i^2 - 3i + 1\right) + 3n^2 + 6n + 3 - 3n - 3 + 1 \\
&= \left(\sum_{i=0}^{n} 3i^2 - 3i + 1\right) + 3n^2 + 3n + 1
\end{aligned}$$

Therefore, it is possible to transform the left hand side of the identity into the right hand side using only algebra and the induction hypothesis, so the identity holds for $n+1$, and by induction, it holds for all $n \geq 0$.

Example 6

Given some constant real number $x \neq 1$,

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x} \quad \forall n \geq 0$$

Basis

When $n = 0$, $\sum_{i=0}^{n} x^i = x^0 = 1 = \frac{1-x^1}{1-x}$, so the identity holds.

Induction Hypothesis

Suppose the identity holds for some $n \geq 0$.

Induction Step

Consider $n+1$.

$$\begin{aligned}
\sum_{i=0}^{n+1} x^i &= \left(\sum_{i=0}^{n} x^i\right) + x^{n+1} \\
&= \frac{1-x^{n+1}}{1-x} + x^{n+1} && \text{(By the induction hypothesis)} \\
&= \frac{1-x^{n+1}}{1-x} + \frac{(1-x)x^{n+1}}{1-x} \\
&= \frac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x} \\
&= \frac{1-x^{n+2}}{1-x}
\end{aligned}$$

Therefore, the identity holds for $n+1$, and, by induction, it holds for all $n \geq 0$.