All-Pairs Shortest Paths

A dynamic programming approach

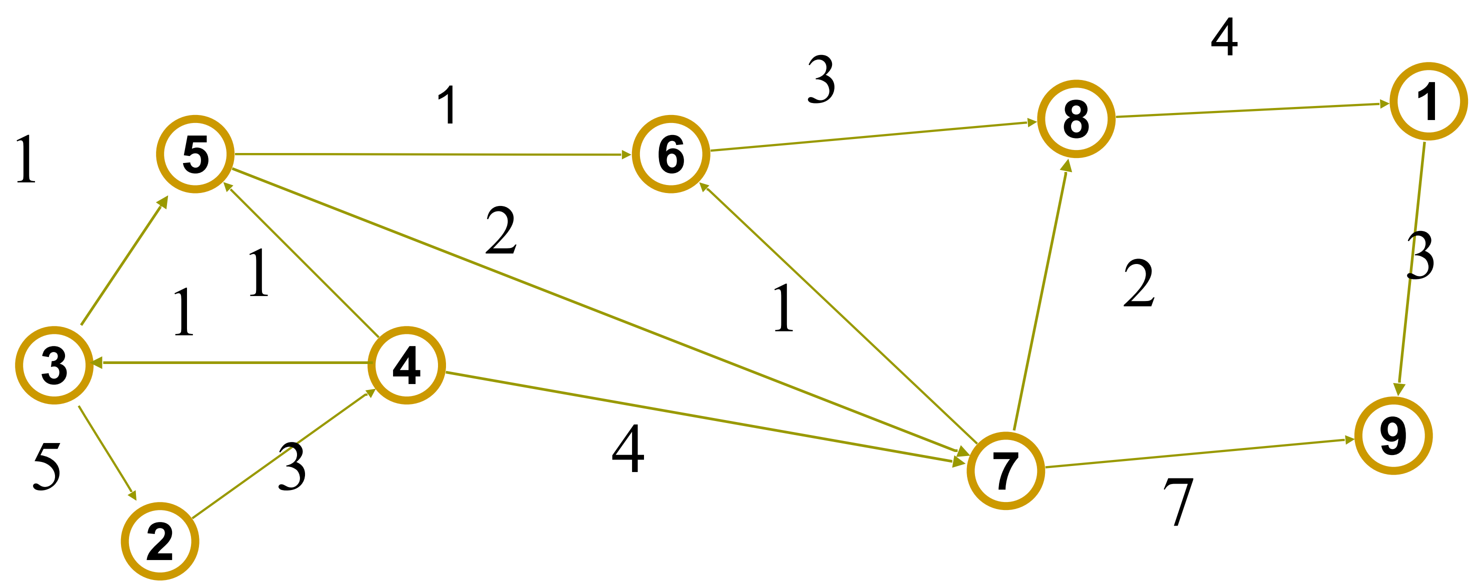
All-Pairs shortest paths in weighted digraphs without negative-weight cycles

- Variation of Floyd-Warshall algorithm
- Algorithm-design technique: dynamic programming

The three (or four) dynamic programming steps

1. Characterize the structure of an optimal solution (**optimal substructure**)
2. Define the value of an optimal solution recursively in terms of the optimal solutions to subproblems (**overlapping subproblems**)
3. Construct an optimal solution with help of a look-up table
4. (optional) Construct an optimal solution from compute information

Algorithm AllPairsShortestPaths(G)



Algorithm AllPairsShortestPaths(G)

Input: An edge-weighted digraph $G=(V,E)$ without negative-weight edges. Further $V=\{1,2,\dots,n\}$

Output: Matrix D s.t. for all $i,j \in V$ $D[i,j]$ denotes the length of a shortest path from i to j

Terminology

- Let $v_1, v_2, \ldots, v_l$ be the vertices of a path p in directed graph G. Then the vertices $v_2, v_3, \ldots, v_{l-1}$ are called *intermediate vertices* of p.
- Let $d_{ij}^{(k)}$ be the length of a shortest path from i to j such that all intermediate vertices that are on the path are members of set $\{1, 2, \ldots, k\}$.
- Then $d_{ij}^{(0)}$ is the weight of the edge from i to j if such an edge exists, $+\infty$ otherwise
- $d_{ij}^{(n)}$ is the shortest path distance from i to j

Observations

- A shortest path does not contain the same vertex twice, since otherwise the path would contain a cycle, and removing the cycle would shorten the path.
- If vertex k is not on a shortest path from i to j with intermediate vertices from $\{1, 2, \ldots, k\}$ then $d_{ij}^{(k)} = d_{ij}^{(k-1)}$
- If vertex k is on a shortest path from i to j with intermediate vertices from $\{1, 2, \ldots, k\}$ then $d_{ij}^{(k)} = d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

We conclude for a shortest path from *i* to *j* with intermediate vertices from {1, 2, …, *k*}:

$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$