MATH 101
Assignment #5

1. Find the following limits if they exist, or show that they do not exist.

(a) $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n-1} = \lim_{n\to\infty} \exp\left[\ln\left(\left(1+\frac{1}{2n}\right)^{n-1}\right)\right]$

$= \exp\left[\lim_{n\to\infty} \ln\left(\left(1+\frac{1}{2n}\right)^{n-1}\right)\right] = \exp\left[\lim_{n\to\infty} (n-1)\ln\left(1+\frac{1}{2n}\right)\right]$

$= \exp\left[\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{2n}\right)}{\left(\frac{1}{n-1}\right)}\right] = \exp\left[\lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{1}{2n}\right)}\cdot\left(\frac{-1}{2n^2}\right)}{\left(-1/(n-1)^2\right)}\right]$

by L'Hôpital's Rule

$= \exp\left[\lim_{n\to\infty} \frac{(n-1)^2}{2n^2+n}\right] = \exp\left[\lim_{n\to\infty} \frac{n^2-2n+1}{2n^2+n}\right] = \exp\left[\lim_{n\to\infty} \frac{1-2/n+1/n^2}{2+1/n}\right]$

$\cancel{= \exp \lim_{n\to\infty}}$ $= \exp\left[\frac{1}{2}\right] = \sqrt{e}$

(b) $\lim_{n\to\infty}\left(\frac{\ln(n+\cos(n))}{\ln(n^2)}\right).$

$$\frac{\ln(n-1)}{\ln(n^2)} \leq \frac{\ln(n+\cos(n))}{\ln(n^2)} \leq \frac{\ln(n+1)}{\ln(n^2)}$$

$\lim_{n\to\infty} \frac{\ln(n\pm1)}{\ln(n^2)} = \lim_{n\to\infty} \frac{\ln(n\pm1)}{2\ln n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n\pm1}\right)}{\left(\frac{2}{n}\right)}$ by L'Hôpital's Rule

$= \lim_{n\to\infty} \frac{n}{2(n\pm1)} = \lim_{n\to\infty} \frac{1}{2(1\pm1/n)} = \frac{1}{2}$

Thus, $\frac{\ln(n+\cos(n))}{\ln(n^2)}$ has the same limit by the Squeeze Law:

$\frac{1}{2}$

2. Given that $\lim_{n\to\infty} a_n$ exists, when $a_1 = 1$ and $a_{n+1} = \sqrt{1+\frac{6}{a_n}}$ for $n \geq 1$, find the limit. [**Hint:** if you have trouble solving for the limit, try calculating the first 6 terms of the sequence, make a guess, and see if you can complete your solution using this guess.]

Suppose $\lim_{n\to\infty} a_n = L$. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1+\frac{6}{a_n}} = \sqrt{1+\frac{6}{(\lim_{n\to\infty} a_n)}}$

So $L = \sqrt{1+\frac{6}{L}}$. Try to solve for L:

$\Rightarrow L^2 = 1+\frac{6}{L} \Rightarrow L^3 = L+6 \Rightarrow L^3 - L - 6 = 0$ cubic

If you can't guess a solution, try numerical experiment:

$a_1 = 1$

$a_2 = \sqrt{1+\frac{6}{1}} = \sqrt{7} \simeq 2.646$

$a_3 \simeq 1.808$

$a_4 \simeq 2.078$

$a_5 \simeq 1.972$ ... guess that $L = 2$?

Check: $2^3 - 2 - 6 = 0$ so $L=2$ is, in fact, a root of the cubic, and $\therefore$ the desired limit.

Could there be other limits L (other roots of the cubic)?

$$
\begin{array}{r}
L^2 + 2L + 3 \\
(L-2)\overline{\big)\, L^3 + 0L^2 - L - 6} \\
\underline{L^3 - 2L^2 \quad\quad\quad} \\
2L^2 - L \quad\; \\
\underline{2L^2 - 4L \quad\;} \\
3L - 6 \\
\underline{3L - 6} \\
0
\end{array}
$$

So $L^3 - L - 6 = (L-2)(L^2+2L+3)$

This last quadratic has roots

$L = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm \frac{1}{2}\sqrt{-8}$

$= -1 \pm i\sqrt{2}$ Not real.

So $L=2$ is the only possible limit.

3. Find the sum of the following series if they converge or show that they diverge:

(a) $\sum_{n=0}^{\infty} \frac{2^{3n}-1}{3^{2n}}$. $= \sum_{n=0}^{\infty} \left[\left(\frac{8}{9}\right)^n - \left(\frac{1}{9}\right)^n\right] = \frac{1}{1-8/9} - \frac{1}{1-1/9}$

$= \frac{9}{9-8} - \frac{9}{9-1} = 9 - \frac{9}{8} = \frac{72-9}{8} = \frac{63}{8}$

(b) $\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n}\right)$. $= \sum_{n=1}^{\infty} \left[\ln(n+2) - \ln(n)\right]$

$S_n = [\ln(3) - \ln(1)] + [\ln(4) - \ln(2)] + [\ln(5) - \ln(3)] + \dots$ &

$+ [\ln(n+1) - \ln(n-1)] + [\ln(n+2) - \ln(n)]$

$= -\ln(1) - \ln(2) + \ln(n+1) + \ln(n+2)$

$\lim_{n\to\infty} S_n = -\ln 2 + \infty = \infty$ diverges

4. Find the first 4 terms (up to the cubic term) in the Maclaurin Series for $f(x) = \ln(1 + e^x)$.

$f(x) = \ln(1+e^x)$ $\qquad f(0) = \ln 2$

$f'(x) = \frac{1}{1+e^x} \cdot e^x = \frac{e^x}{1+e^x}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$, $f''(0) = \frac{1}{4}$

$f'''(x) = \frac{(1+e^x)^2 e^x - e^x \cdot 2(1+e^x)e^x}{(1+e^x)^4} = \frac{e^x + e^{2x} - 2e^{2x}}{(1+e^x)^3} = \frac{e^x - e^{2x}}{(1+e^x)^3}$, $f'''(0) = 0$

$f(x) = (\ln 2) + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{4}\right)}{2!}x^2 + \frac{(0)}{3!}x^3 + \ldots$

$= \ln 2 + \frac{x}{2} + \frac{x^2}{8} + 0x^3 + \ldots$

5. Show that $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$.

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$\frac{e^{ix} + e^{-ix}}{2} = \frac{(\cos x + i\sin x) + (\cos x - i\sin x)}{2} = \frac{2\cos x}{2} = \cos x$

6. Find the real part of $z = e^{(1-i\frac{\pi}{3})}$.

$$z = e^{(1 - i\frac{\pi}{3})} = e^1 e^{-i\pi/3} = e\left[\cos(\pi/3) - i\sin(\pi/3)\right]$$

$$= e\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

$$= \frac{e}{2} - i\frac{\sqrt{3}e}{2}$$

$$\mathrm{Re}(z) = \frac{e}{2}$$