Independent Set for general and for planar graphs

k-Independent Set Parameterized Decision Problem

- Input: An undirected graph $G = (V,E)$, a positive integer k

- Parameter: k

- Question: Does there exist a independent set V' for G that is of size at least k, that is $|V'| \geq k$?

A brute-force algorithm solving k-Independent Set

Algorithm k-IS(G,k)

if $k = 0$ **then return** yes

if $k > 0$ and $V \neq \varnothing$ **then**

pick vertex x; let $\mathrm{N}(x) = \{a_1, a_2, \ldots, a_d\}$

if k-IS(G-N(x), k-1) returns yes **then**

return k-IS(G-N(x), k-1)

for i **from** 1 **to** d **do**

if k-IS(G-N(a_i), k-1) returns yes **then**

return k-IS(G-N(a_i), k-1)

return no

$O(n^k)$

The parameterized complexity class $\mathrm{W}[1]$

- There exist parameterized decision problems that are likely not fixed-parameter tractable. One possible indicator for the is the property to be $\mathrm{W}[1]$-complete, since it is conjectured that $\mathrm{FPT} \neq \mathrm{W}[1]$.

- It is known that $\mathrm{FPT} \subseteq \mathrm{W}[1]$

- k-Independent Set is $\mathrm{W}[1]$-complete.

More on intractable problems

- When designing algorithms for problems that are generally intractable (such as NP-complete problems and W[1]-complete problems) having knowledge about the type of inputs can be beneficial.

- E.g., k-Independent set is fixed-parameter tractable if the input graph is planar.

Planar graphs

- A graph is *planar* if it can be drawn in the plane without a crossing of edges.
- A crossing-free drawing of a planar graph is also called *plane drawing* or *plane embedding*.
- A plane drawing of a planar graph divides the plane into regions called *faces.* One of these faces is unbounded and is called infinite face.

Properties of planar graphs: Euler's Formula

Let G be a connected planar graph with n vertices, m edges and f faces. Then

$$n - m + f = 1$$

Proof of Euler's Formula

- By induction on m.
- *Base case:* If $m = 0$ then $n = 1$ and $f = 1$ (the only face is the infinite face)
- Thus $n\text{-}m+f = 1\text{-}0+1 = 2$

Proof of Euler's Formula

- *Hypothesis:* Euler's Formula is true for every graph G with less than m edges

- Case 1. G is a tree. A tree with n vertices has $m = n - 1$ edges and one face (the infinite face). Therefore: $n - m + f = n - (n-1) + 1 = 2$

- Case 2. G is not a tree, has n vertices, m edges and f faces. We know that G has a cycle (since it is not a tree). We pick some edge e in a cycle in G. Consider G-e. G-e has n vertices, m-1 edges and f-1 faces. According to the hypothesis, Euler's formula holds for G-e: $n - (m - 1) + f - 1 = 2$.

- Since $n - (m - 1) + f - 1 = n - m + f = 2$, Euler's Formula holds for G.

Observation

- Each face on a connected graph with $n \geq 4$ vertices has at least 3 edges.

A planar graph with n vertices has $O(n)$ edges

Theorem: Let G be a connected planar simple graph with n vertices, where $n \geq 3$, and m edges. Then $m \leq 3n - 6$.

Proof. We first observe:

Every edge belongs to at most two faces.

Every face has at least 3 edges.

Therefore the lower bound of #faces*#edges is $3f$ and the upper bound is $2m$. It follows $2m \geq 3f$

From Euler's Formula it follows $3(m - n + 2) = 3f \leq 2m$ and therefore $m \leq 3n - 6$.

Theorem: A connected planar graph contains at least one vertex of degree 5 or less

Proof. Assume the claim is wrong. The every vertex has degree at least 6. That is the sum of all vertex degrees is at least $6n$: $2m \geq 6n$. Therefore $m \geq 3n$. This contradicts that $m < 3n$ (we know this since $m \leq 3n\text{-}6$).

A bounded search tree for k-Independent Set for planar graphs

- Since we know that every planar graph has at least one vertex of degree 5 or less, we can improve the search tree algorithm above.

A bounded search-tree algorithm solving k-Independent Set for *planar graphs*

Algorithm k-IS(G,k)

if $k = 0$ **then return** yes

if $k > 0$ and $V \neq \varnothing$ **then**

pick vertex x of degree at most 5; let $\mathrm{N}(x) = \{a_1, a_2, \ldots, a_d\}$

if k-IS(G-N(x), k-1) returns yes **then**

return k-IS(G-N(x), k-1)

for i **from** 1 **to** d **do**

if k-IS(G-N(a_i), k-1) returns yes **then**

return k-IS(G-N(a_i), k-1)

return no

$O(6^k n)$