Minimum Cuts and Maximum Flows

st-cuts (continued)

- Recall: A *cut* in a (directed) graph is a partition of the vertices into two disjoint subsets.
- The *cut edges* of a graph with a cut are the edges that have one endpoint in each subset of the partition.
- An st-*cut* is a cut that places vertex s in one of its subsets and vertex t in the other.

st-cuts

- *Capacity* of an st-cut in an st-network: sum of the capacities of the cut's edges from the subset containing s to the subset containing t

- *Flow across* an st-cut in an st-network: difference between the sum of the flows of cut's edges from the subset containing s to the subset containing t and the sum of the flows of cut's edges from the subset containing t to the subset containing s

minimum st-cut problem (or *mincat* problem)

- Given an st-network, find an st-cut such that the capacity of no other cut is smaller.

Properties of feasible st-flows in st-flow networks

1. For any st-flow, the flow across each st-cut is equal to the value of the flow

2. The outflow from s is equal to the inflow to t

3. No st-flow's value can exceed the capacity of any st-cut

4. Let f be an st-flow and let (S,T) be an st-cut whose capacity equals $|f|$. Then f is a maximum flow and (S,T) is a minimum cut.

Proof

1. For any st-flow, the flow across each st-cut is equal to the value of the flow

Given an st-flow f in an st-flow network G. We know that the net flow of each vertex (except source and sink) is zero. We show that the flow across each st-cut is $|f|$.

Base case: consider the st-cut with $S = V\backslash\{t\}$ and $T = \{t\}$. Then the cut edges of cut (S,T) are equal to the edges incident to sink T. The edges from S to T are the incoming edges of T. Therefore, the flow across (S,T) is $|f|$.

Hypothesis Assuming an st-cut (S,T) with flow across the cut of value $|f|$ (that is flowAcross(S,T) = $|f|$),

Step pick any vertex x ($x \neq s$) and move it from S to T, resulting in (S', T'). The updated set of cut edges is as follows:

cutEdges(S',T') = cutEdges(S,T) - $\{(x,u) \mid u \in T\}$ - $\{(u,x) \mid u \in T\}$ + $\{(x,u) \mid u \in S\}$ + $\{(x,u) \mid u \in S\}$

How does the flow across the cut change?

flowAcross(S',T') = flowAcross(S,T) - $\sum_{\{(x,u)\in E|u\in T\}} f(x,u)$ + $\sum_{\{(u,x)\in E|u\in T\}} f(u,x)$ + $\sum_{\{(u,x)\in E|u\in S\}} f(u,x)$

- $\sum_{\{(x,u)\in E|u\in S\}} f(x,u)$ = flowAcross(S',T') - outflow(x) + inflow(x) = flowAcross(S,T) + netflow(x) = $|f|$

Proof.

2. The outflow from s is equal to the inflow to t

Consider cut (S,T) with S = {s} and T = V\{s}. Because of 1. we know that flowAcross(S,T) = |f|.

Proof.

3. No st-flow's value can exceed the capacity of any st-cut

To the contrary, assume there are a valid st-flow $|f|$ and an st-cut (S,T) with $|f|$ is larger than (S,T)'s capacity. We know that flowAcross(S,T) = $|f|$. But then flowAcross(S,T) > capacity(S,T). Then there must exist a cut edge (u,v) with $f(u,v) > c(u,v)$. This implies that the flow is not valid, a contradiction.

Proof.

4. Let f be an st-flow and let (S, T) be an st-cut whose capacity equals $|f|$. Then f is a maximum flow and (S, T) is a minimum cut.

Since no st-flow's value can exceed the capacity of any st-cut (Property 3), no flow can be larger than the capacity of a minimum cut. The largest flow therefore cannot be larger than the minimum cut. Since capacity$(S, T) = |f|$, (S, T) must be a minimum cut and f a maximum flow.

Maxflow-Mincut Theorem

- Let f be an st-flow. The following three conditions are equivalent:

    A. there exists an st-cut whose capacity equals $|f|$

    B. f is a maximum flow

    C. there is no augmenting path in G_f with respect to f

Proof.

A ⇒ B. Let f be an st-flow. Further assume that there exists an st-cut (S, T) with capacity$(S, T) = |f|$. Property 4 implies that f is a maximum flow.

B ⇒ C. Let f be a maximum flow. We show that there is no augmenting path in G_f.

If, to the contrary, there is an augmenting path p in G_f then, by Theorem 1, $f + f_p$ is a valid flow in G. Therefore $|f + f_p| > |f|$, a contradiction to f being a maximum flow.

Proof.

C ⇒ A. We know that there is no augmenting path in G_f. We show that then there exists an st-cut (S,T) with capacity(S,T) = $|f|$.

Property 3 yields $|f| \leq$ capacity(S,T). Assume $|f| <$ capacity(S,T) for all cuts (S,T). Then, no matter the cut, the flow across the cut is smaller than its capacity. Therefore, in the residual network there must be path from s to t. This path is augmenting. A contradiction.