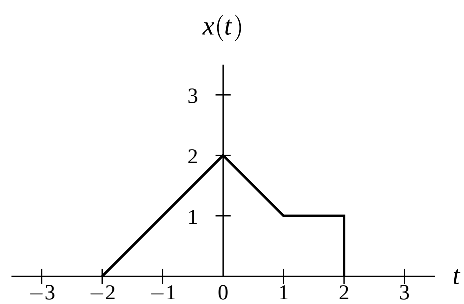
# Chapter 1

# Continuous-Time Signals and Systems (Chapter 2)

**2.11** Suppose that we have the signal $x(t)$ shown in the figure below. Use unit-step functions to find a single expression for $x(t)$ that is valid for all $t$.




**Solution.**

We have

$$\begin{aligned} x(t) &= [t+2][u(t+2)-u(t)] + [-t+2][u(t)-u(t-1)] + [1][u(t-1)-u(t-2)] \\ &= [t+2]u(t+2) + [-t-2-t+2]u(t) + [t-2+1]u(t-1) + [-1]u(t-2) \\ &= (t+2)u(t+2) + (-2t)u(t) + (t-1)u(t-1) + (-1)u(t-2). \end{aligned}$$

**2.12** Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is linear:
(a) $y(t) = \int_{t-1}^{t+1} x(\tau)d\tau$;
(b) $y(t) = e^{x(t)}$;
(c) $y(t) = \text{Even}\{x(t)\}$; and
(d) $y(t) = x^2(t)$.

**Solution.**

(a) Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ denote the responses of the system to the inputs $x_1(t)$, $x_2(t)$, and $a_1x_1(t)+a_2x_2(t)$, respectively, where $a_1$ and $a_2$ are complex constants. If $y_3(t) = a_1y_1(t)+a_2y_2(t)$ for all $x_1(t)$, $x_2(t)$, $a_1$, and $a_2$, then the system is linear. We have

$$\begin{aligned}
y_1(t) &= \int_{t-1}^{t+1} x_1(\tau)d\tau, \\
y_2(t) &= \int_{t-1}^{t+1} x_2(\tau)d\tau, \quad \text{and} \\
y_3(t) &= \int_{t-1}^{t+1} [a_1x_1(\tau)+a_2x_2(\tau)]d\tau \\
&= \int_{t-1}^{t+1} a_1x_1(\tau)d\tau + \int_{t-1}^{t+1} a_2x_2(\tau)d\tau \\
&= a_1\int_{t-1}^{t+1} x_1(\tau)d\tau + a_2\int_{t-1}^{t+1} x_2(\tau)d\tau \\
&= a_1y_1(t)+a_2y_2(t).
\end{aligned}$$

Since $y_3(t) = a_1y_1(t)+a_2y_2(t)$, the system is linear.

(c) Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ denote the responses of the system to the inputs $x_1(t)$, $x_2(t)$, and $a_1x_1(t)+a_2x_2(t)$, respectively, where $a_1$ and $a_2$ are complex constants. If $y_3(t) = a_1y_1(t)+a_2y_2(t)$ for all $x_1(t)$, $x_2(t)$, $a_1$, and $a_2$, then the system is linear. We have

$$\begin{aligned}
y_1(t) &= \tfrac{1}{2}[x_1(t)+x_1(-t)], \\
y_2(t) &= \tfrac{1}{2}[x_2(t)+x_2(-t)], \quad \text{and} \\
y_3(t) &= \tfrac{1}{2}[(a_1x_1(t)+a_2x_2(t))+(a_1x_1(-t)+a_2x_2(-t))] \\
&= \tfrac{1}{2}[a_1x_1(t)+a_1x_1(-t)+a_2x_2(t)+a_2x_2(-t)] \\
&= \tfrac{1}{2}a_1[x_1(t)+x_1(-t)]+\tfrac{1}{2}a_2[x_2(t)+x_2(-t)] \\
&= a_1y_1(t)+a_2y_2(t).
\end{aligned}$$

Since $y_3(t) = a_1y_1(t)+a_2y_2(t)$, the system is linear.

(d) Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ denote the responses of the system to the inputs $x_1(t)$, $x_2(t)$, and $a_1x_1(t)+a_2x_2(t)$, respectively, where $a_1$ and $a_2$ are complex constants. If $y_3(t) = a_1y_1(t)+a_2y_2(t)$ for all $x_1(t)$, $x_2(t)$, $a_1$, and $a_2$, then the system is linear. We have

$$\begin{aligned}
y_1(t) &= x_1^2(t) \\
y_2(t) &= x_2^2(t) \\
y_3(t) &= [a_1x_1(t)+a_2x_2(t)]^2 \\
&= a_1^2x_1^2(t)+2a_1a_2x_1(t)x_2(t)+a_2^2x_2^2(t) \\
&\neq a_1y_1(t)+a_2y_2(t).
\end{aligned}$$

Since $y_3(t) \neq a_1y_1(t)+a_2y_2(t)$, the system is not linear.

**2.13** Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is time invariant:
(a) $y(t) = \frac{d}{dt}x(t)$;
(b) $y(t) = \mathrm{Odd}\{x(t)\}$;
(c) $y(t) = \int_t^{t+1} x(\tau-\alpha)d\tau$ where $\alpha$ is a constant;
(d) $y(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$;
(e) $y(t) = x(-t)$; and
(f) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$.

**Solution.**

(a) Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_1(t-t_0)$, respectively, where $t_0$ is a constant. If $y_2(t) = y_1(t-t_0)$ for all $x_1(t)$ and $t_0$, then the system is time invariant. We have

$$\begin{aligned} y_1(t) &= \tfrac{d}{dt}x_1(t), \\ y_2(t) &= \tfrac{d}{dt}x_1(t-t_0), \quad \text{and} \\ y_1(t-t_0) &= \left[\tfrac{d}{dv}x_1(v)\right]\Big|_{v=t-t_0} \\ &= \tfrac{d}{dt}x_1(t-t_0) \\ &= y_2(t). \end{aligned}$$

Since $y_2(t) = y_1(t-t_0)$, the system is time invariant.

(c) Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_1(t-t_0)$, respectively, where $t_0$ is a constant. If $y_2(t) = y_1(t-t_0)$ for all $x_1(t)$ and $t_0$, then the system is time invariant. We have

$$\begin{aligned} y_1(t) &= \int_t^{t+1} x_1(\tau-\alpha)d\tau \quad \text{and} \\ y_2(t) &= \int_t^{t+1} x_1(\tau-\alpha-t_0)d\tau. \end{aligned}$$

Now, we apply a change of variable. Let $\lambda = \tau - t_0$ so that $\tau = \lambda + t_0$ and $d\lambda = d\tau$. Applying this change of variable yields

$$\begin{aligned} y_2(t) &= \int_{t-t_0}^{t+1-t_0} x_1(\lambda-\alpha)d\lambda \\ &= y_1(t-t_0). \end{aligned}$$

Since $y_2(t) = y_1(t-t_0)$, the system is time invariant.

(e) Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_1(t-t_0)$, respectively, where $t_0$ is a constant. If $y_2(t) = y_1(t-t_0)$ for all $x_1(t)$ and $t_0$, then the system is time invariant. We have

$$\begin{aligned} y_1(t) &= x_1(-t), \\ y_2(t) &= x_1(-t-t_0), \quad \text{and} \\ y_1(t-t_0) &= x_1(-[t-t_0]) \\ &= x_1(-t+t_0) \\ &\neq y_2(t). \end{aligned}$$

Since $y_2(t) \neq y_1(t-t_0)$, the system is not time invariant.

**2.14** Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is causal and/or memoryless:
(a) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(b) $y(t) = \operatorname{Odd}\{x(t)\}$;
(c) $y(t) = x(t-1)+1$;
(d) $y(t) = \int_t^{\infty} x(\tau)d\tau$; and
(e) $y(t) = \int_{-\infty}^{t} x(\tau)\delta(\tau)d\tau$.

**Solution.** If the output $y(t)$ at $t = t_0$ for any arbitrary $t_0$ depends only on the input $x(t)$ for $t \leq t_0$, then the system is causal. If the output $y(t)$ at $t = t_0$ for any arbitrary $t_0$ depends only on the input $x(t)$ at $t = t_0$, then the system is memoryless.

(a) From the equation

$$y(t) = \int_{-\infty}^{2t} x(\tau)d\tau,$$

we can see that $y(t)|_{t=t_0}$ depends on $x(t)$ for $-\infty < t \leq 2t_0$. Therefore, the system is not causal (since $2t_0 > t_0$ for any positive $t_0$) and the system is not memoryless.

(c) We have

$$y(t) = x(t-1) + 1.$$

Consider $y(t)|_{t=t_0}$. This quantity depends on $x(t)$ for $t = t_0 - 1$. Therefore, the system is causal (since $t_0 - 1 < t_0$) and the system is not memoryless (since $t_0 - 1 \neq t_0$).

(e) We have

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(\tau)\delta(\tau)d\tau \\ &= \int_{-\infty}^{t} x(0)\delta(\tau)d\tau \\ &= x(0)\int_{-\infty}^{t} \delta(\tau)d\tau \\ &= x(0)u(t). \end{aligned}$$

Consider $y(t)|_{t=t_0}$. If $t_0 > 0$, this quantity depends on $x(t)$ for $t = 0$. Otherwise, this quantity does not depend on $x(t)$ at all. Therefore, the system is causal, since $y(t_0)$ only depends on $x(t)$ for $t \leq t_0$). Also, the system is not memoryless (since for $t_0 > 0$, $y(t_0)$ depends on $x(t)$ for $t \neq t_0$).

**2.15** Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the equations given below is invertible. If the system is invertible, specify its inverse.
(a) $y(t) = x(at - b)$ where $a$ and $b$ are real constants and $a \neq 0$;
(b) $y(t) = e^{x(t)}$;
(c) $y(t) = \text{Even}\{x(t)\} - \text{Odd}\{x(t)\}$; and
(d) $y(t) = \frac{d}{dt}x(t)$.

**Solution.** A system is invertible if any two distinct inputs always produce distinct outputs.

(a) We have

$$y(t) = x(at - b).$$

Now, we employ a change of variable. Let $\lambda = at - b$ so that $t = \frac{1}{a}[\lambda + b]$. Since we are told that $a \neq 0$, we do not need to worry about division by zero. Applying the change of variable yields

$$y(\tfrac{1}{a}[\lambda + b]) = x(\lambda),$$

or alternatively,

$$x(t) = y(\tfrac{1}{a}[t + b]).$$

Thus, we have found the inverse system. Therefore, the system is invertible (since we have just found its inverse).

(c) We have

$$\begin{aligned} y(t) &= \text{Even}\{x(t)\} - \text{Odd}\{x(t)\} \\ &= \tfrac{1}{2}[x(t) + x(-t)] - \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) - \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) \\ &= x(-t). \end{aligned}$$

So, $y(t) = x(-t)$. Thus, we have $x(t) = y(-t)$. Therefore, the system is invertible (since we have just found the inverse, namely $x(t) = y(-t)$).

(d) Consider an input $x_1(t)$ of the form

$$x_1(t) = A$$

where $A$ is a constant. Such an input will always yield the output $y_1(t)$ given by

$$\begin{aligned} y_1(t) &= \tfrac{d}{dt}x_1(t) \\ &= \tfrac{d}{dt}A \\ &= 0. \end{aligned}$$

Therefore, any constant input will produce the same output (namely, an output of zero). Since distinct inputs yield the same output, the system is not invertible.

**2.16** Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the equations given below is BIBO stable.
(a) $y(t) = \int_t^{t+1} x(\tau)d\tau$;
(b) $y(t) = \frac{1}{2}x^2(t) + x(t)$; and
(c) $y(t) = 1/x(t)$.
[Hint for part (a): For any function $f(x)$, $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx$.]

**Solution.** A system is BIBO stable if its response $y(t)$ to any bounded input $x(t)$ is always bounded. That is,

$$|x(t)| \leq A \Rightarrow |y(t)| \leq B,$$

where $A$ and $B$ are finite constants.

(a) Suppose that $|x(t)| \leq A < \infty$ (i.e., $x(t)$ is bounded by $A$). Taking the absolute value of both sides of the input-output equation for the system, we obtain

$$|y(t)| = \left|\int_t^{t+1} x(\tau)d\tau\right|.$$

Using the fact that $\left|\int_\alpha^\beta f(x)dx\right| \leq \int_\alpha^\beta |f(x)|\,dx$, we can write

$$\begin{aligned} |y(t)| &\leq \int_t^{t+1} |x(\tau)|\,d\tau \\ &\leq \int_t^{t+1} A\,d\tau \\ &= [A\tau]|_t^{t+1} \\ &= A(t+1) - At \\ &= A \\ &< \infty. \end{aligned}$$

Thus, we have that

$$|x(t)| \leq A < \infty \Rightarrow |y(t)| \leq A < \infty.$$

Therefore, the system is BIBO stable.

(b) Suppose that $x(t)$ is bounded as

$$|x(t)| \leq A.$$

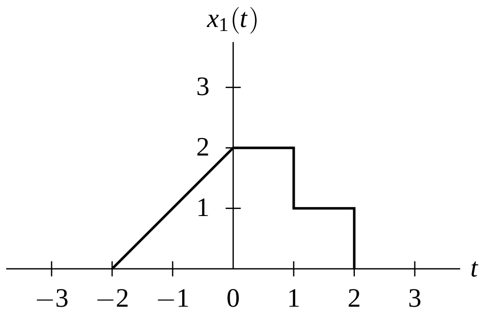
Then, we have

$$\begin{aligned} y(t) &\le \tfrac{1}{2}A^2 + A \\ &< \infty. \end{aligned}$$

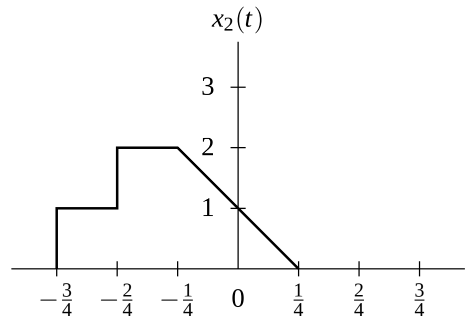
Therefore, a bounded input always yields a bounded output, and the system is BIBO stable.

(c) Consider the input $x(t) = 0$. This will produce the output $y(t) = \frac{1}{0} = \infty$. Since a bounded input produces an unbounded output, the system is not BIBO stable.

**2.19** Given the signals $x_1(t)$ and $x_2(t)$ shown in the figures below, express $x_2(t)$ in terms of $x_1(t)$.




(a)




(b)

**Solution.**

We observe that $x_2(t)$ is simply a time-shifted, time-scaled, and time-reversed version of $x_1(t)$. More specifically, $x_2(t)$ is generated from $x_1(t)$ through the following transformations (in order): 1) time shifting by 1, 2) time scaling by 4, and 3) time reversal. Thus, we have

$$x_2(t) = x_1(-4t - 1).$$

# Chapter 12

# MATLAB (Appendix E)

**E.107** (a) Write a function called `unitstep` that takes a single real argument $t$ and returns $u(t)$, where

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Modify the function from part (a) so that it takes a single vector argument $t = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}^T$ (where $n \geq 1$ and $t_1, t_2, \ldots, t_n$ are real) and returns the vector $\begin{bmatrix} u(t_1) & u(t_2) & \cdots & u(t_n) \end{bmatrix}^T$. Your solution must employ a looping construct (e.g., a for loop).
(c) With some ingenuity, part (b) of this problem can be solved using only two lines of code, without the need for any looping construct. Find such a solution. [Hint: In MATLAB, to what value does an expression like "`[-2 -1 0 1 2] >= 0`" evaluate?]

**Solution.**

(a) This problem can be solved with code such as that shown below.

```
function x = unitstep(t)

if t >= 0
    x = 1;
else
    x = 0;
end
```

(b) This problem can be solved with code such as that shown below.

```
function x = unitstep(t)

% Create a vector of zeros with the same size as the input vector.
x = zeros(size(t));

% Correctly set the elements in the result vector that should be one.
m = length(x);
for i = 1 : m

    if t(i) >= 0
        x(i) = 1;
    end

end
```

(c) This problem can be solved with code such as that shown below.

```
function x = unitstep(t)

x = (t >= 0);
```