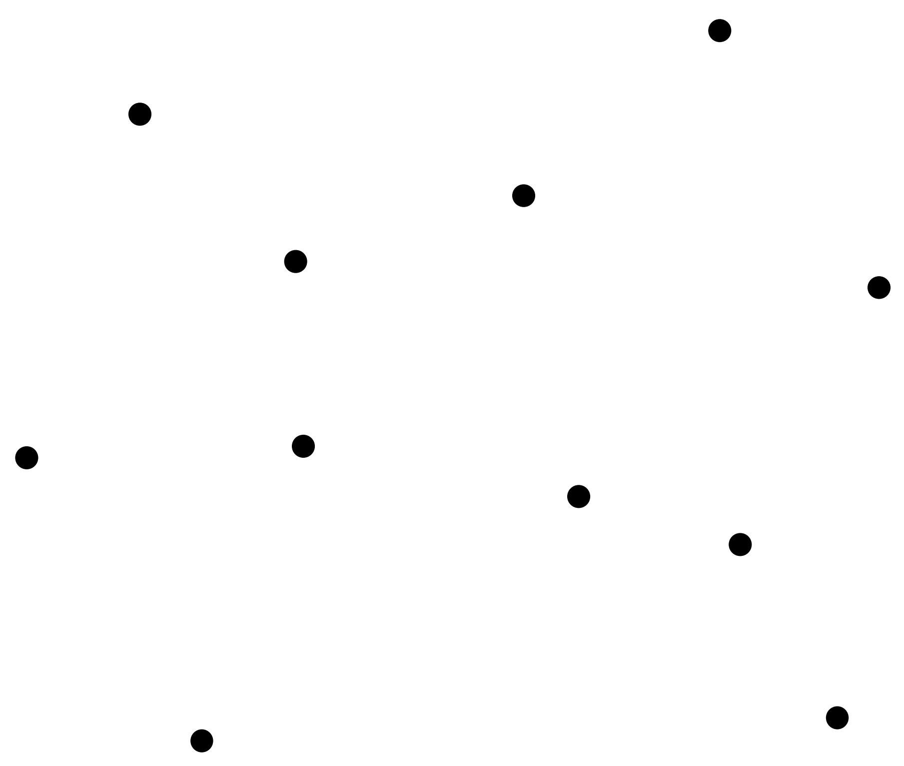
Convex hull computation – Graham's scan

Gabriel Nivasch

Convex hull



Let S be a set of n points in the plane

The ***convex hull*** of S is a convex polygon P, with vertices in S, that contains all the points of S

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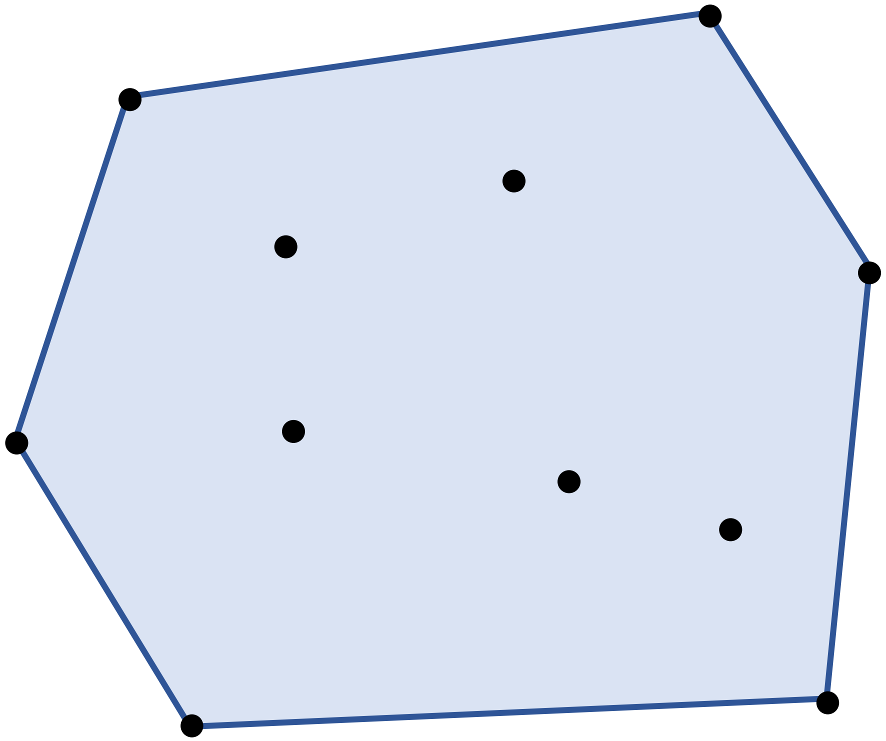

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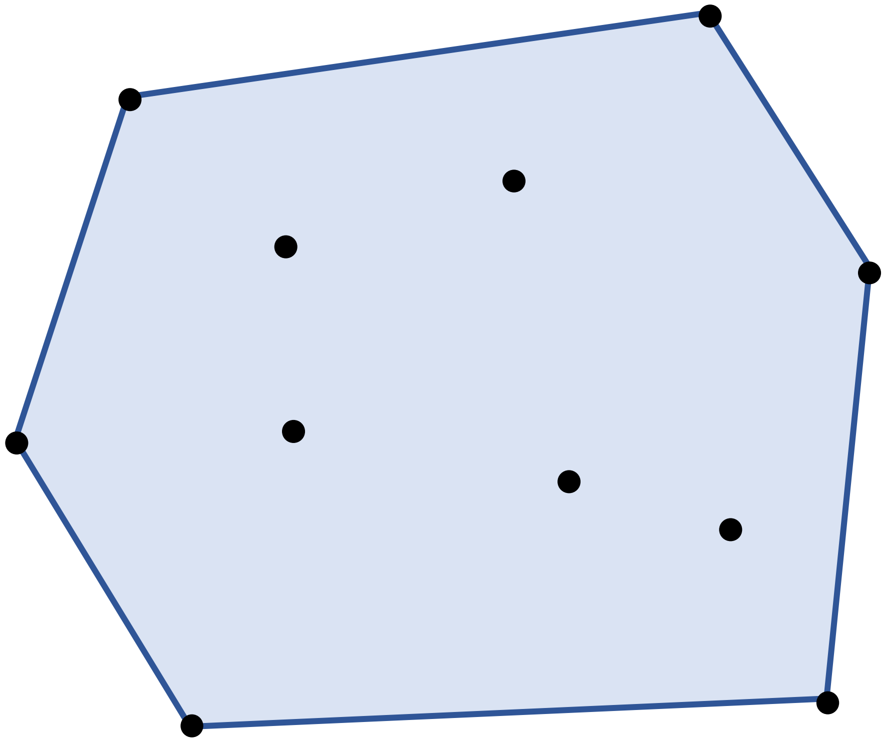

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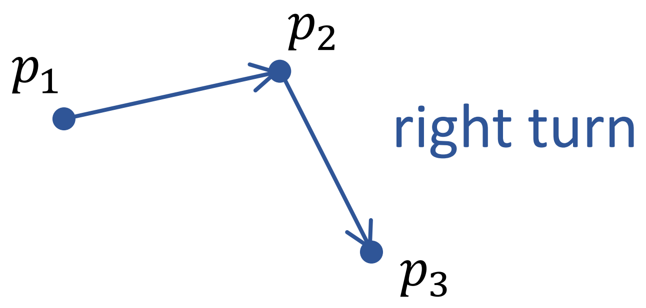
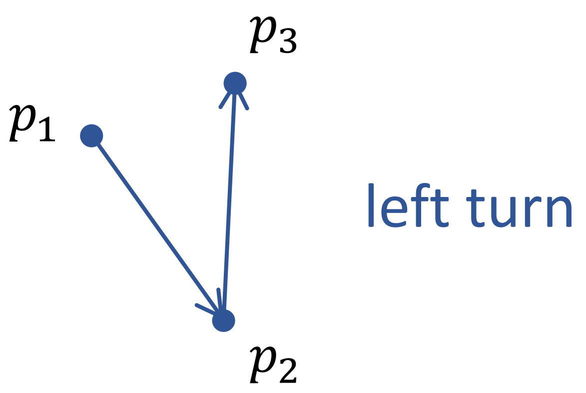
How do we compute the convex hull efficiently?

Basics: Orientation of 3 points

Given points p_1, p_2, p_3, not on the same line,





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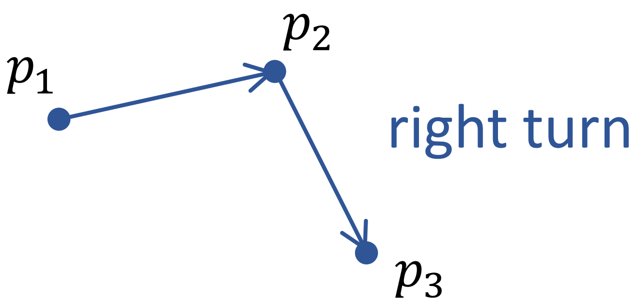

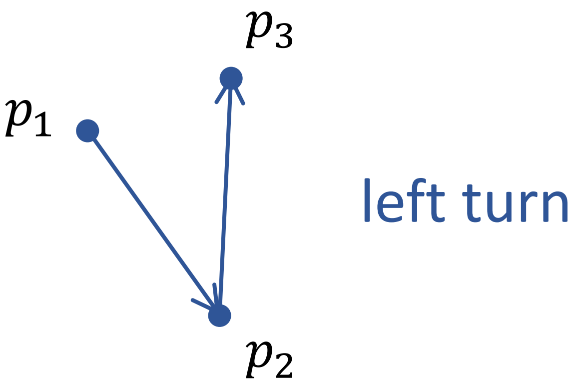

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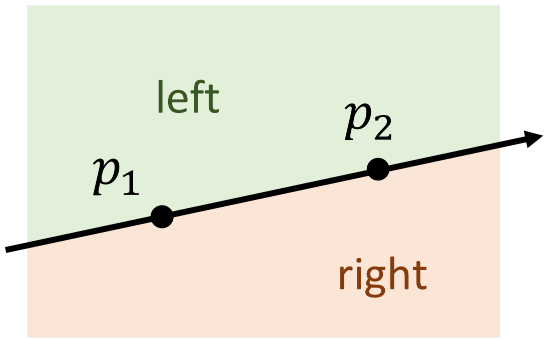

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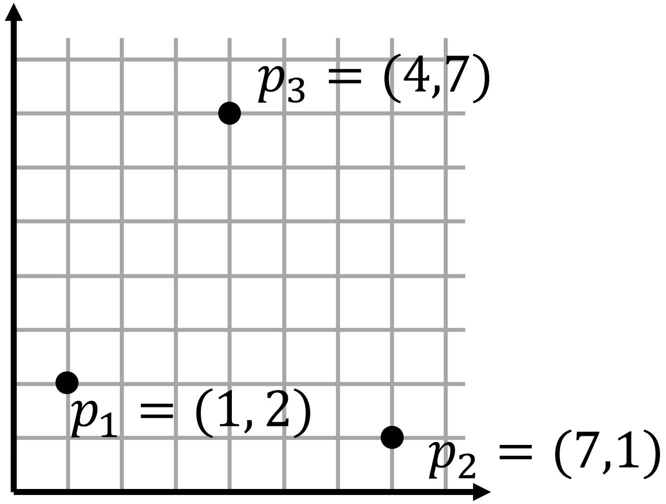
Example:



$$\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 7 & 4 \\ 2 & 1 & 7 \end{bmatrix} = 33$$

Left turn

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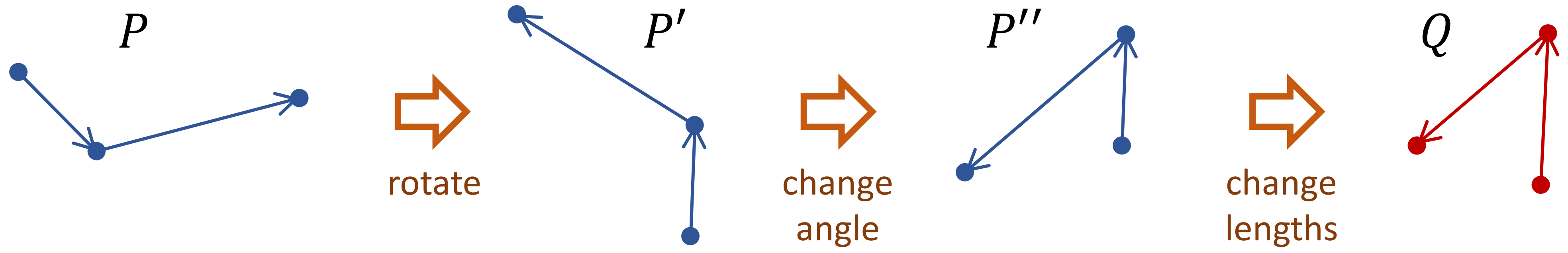
(Also $y_1 \neq y_2$)

Proof (cont.):

Suppose $P = (p_1, p_2, p_3)$ make a left turn, and $Q = (q_1, q_2, q_3)$ also

We can transform P into Q continuously, without the points ever becoming collinear



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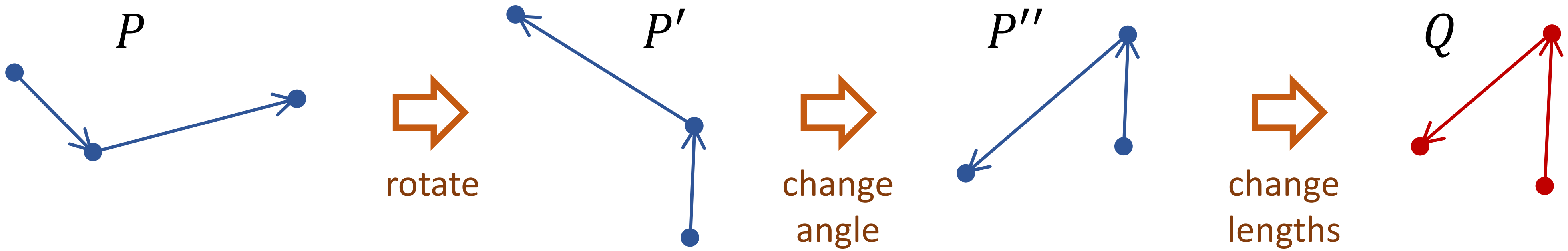

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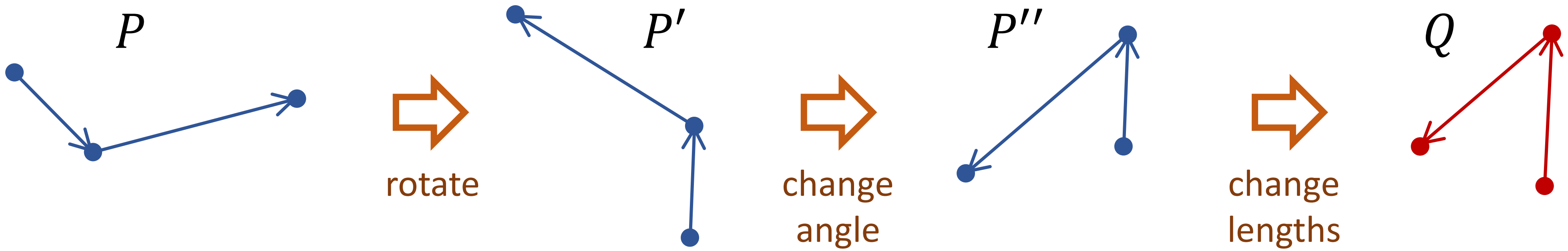

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➔ P and Q have the same determinant sign

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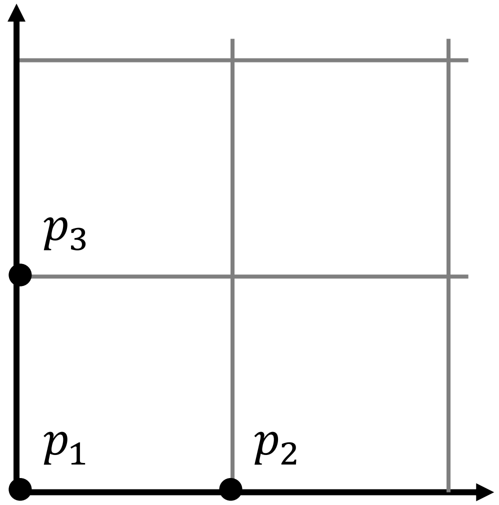
Verify an easy “left turn” case: $p_1 = (0,0),\ p_2 = (1,0),\ p_3 = (0,1)$



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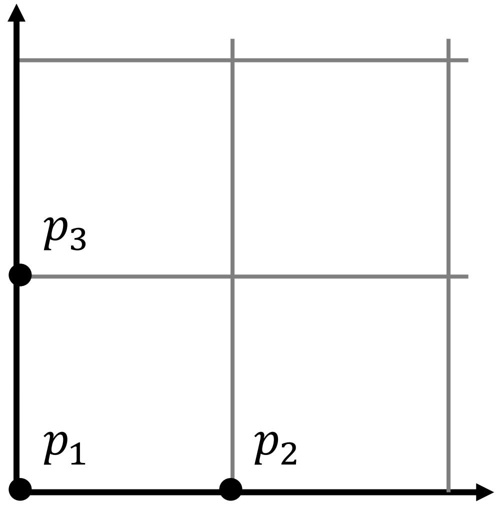

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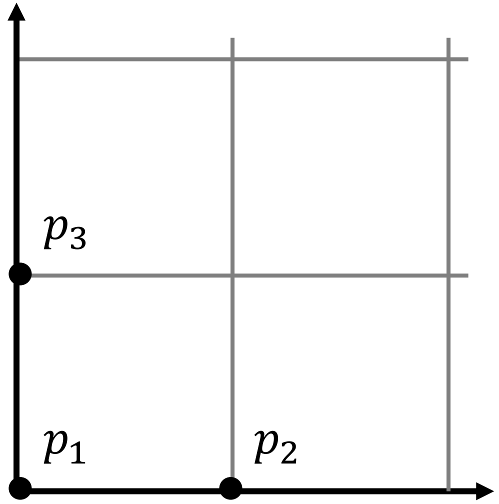

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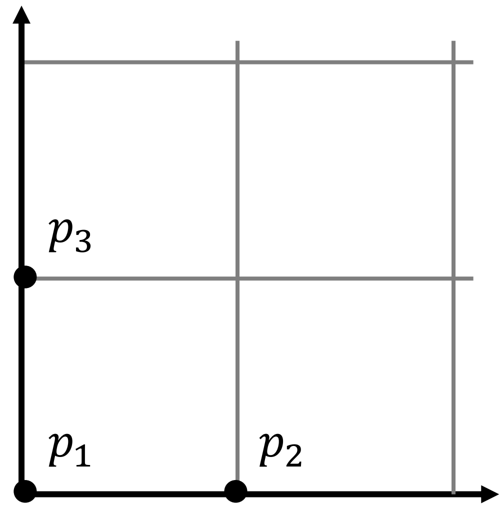

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"Right turn": Similarly **QED**

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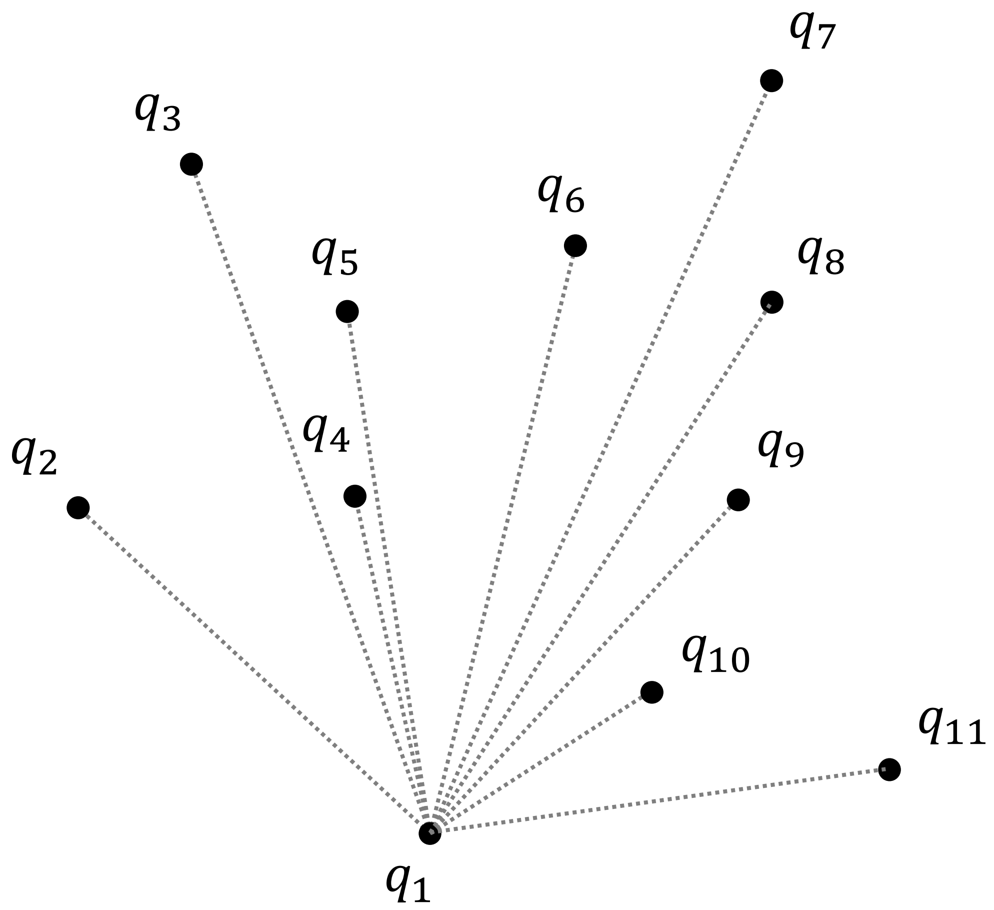
Step 1: Find a point q_1 that is certainly a CH vertex, say point with lowest y-coordinate

Time: $O(n)$

Step 2: Sort the remaining points clockwise w.r.t. q_1



Whenever the sorting algorithm asks "is $r < s$?" we check whether r, q_1, s make a left turn

Time: $O(n \log n)$

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- Initialize a stack to $S = (q_1, q_2)$
- For $i = 3$ to n:
  - While $S[-2], S[-1], q_i$ make a left turn
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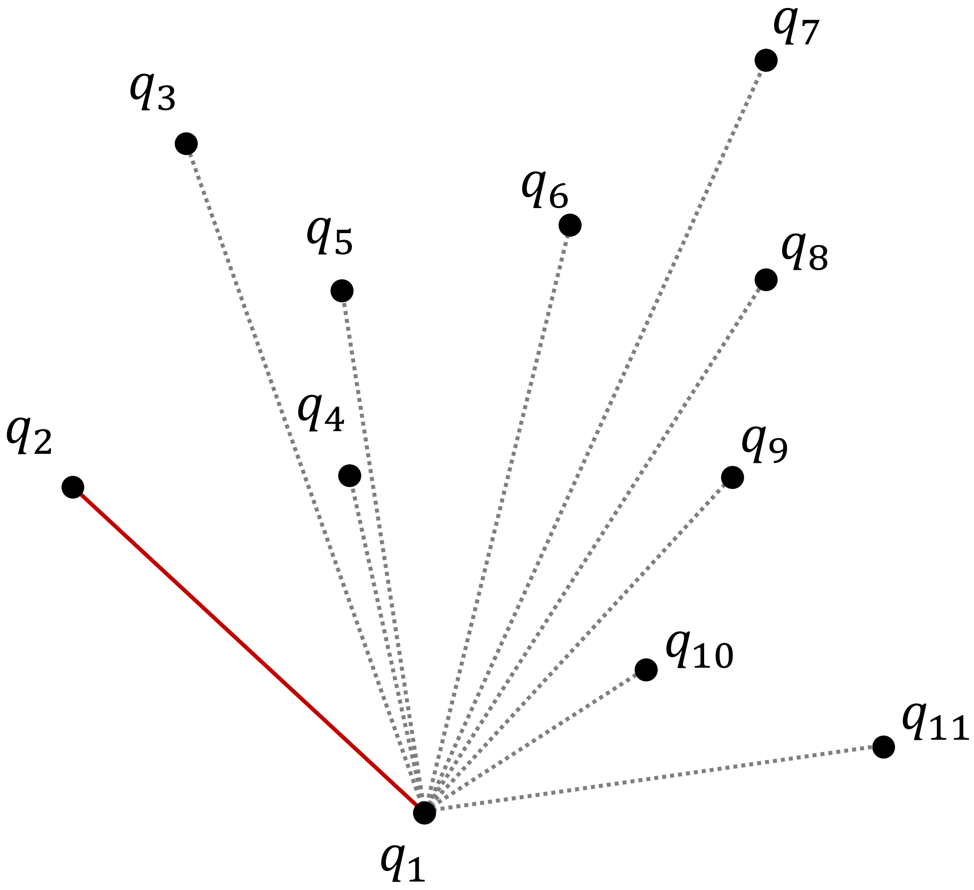

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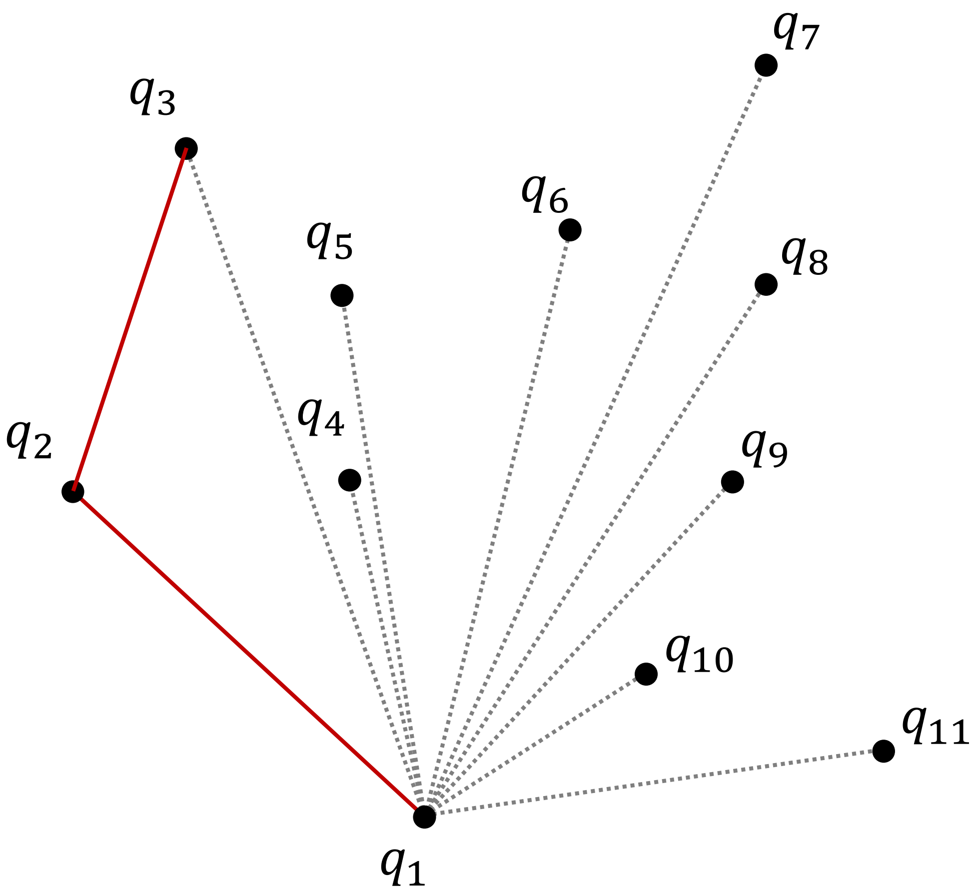

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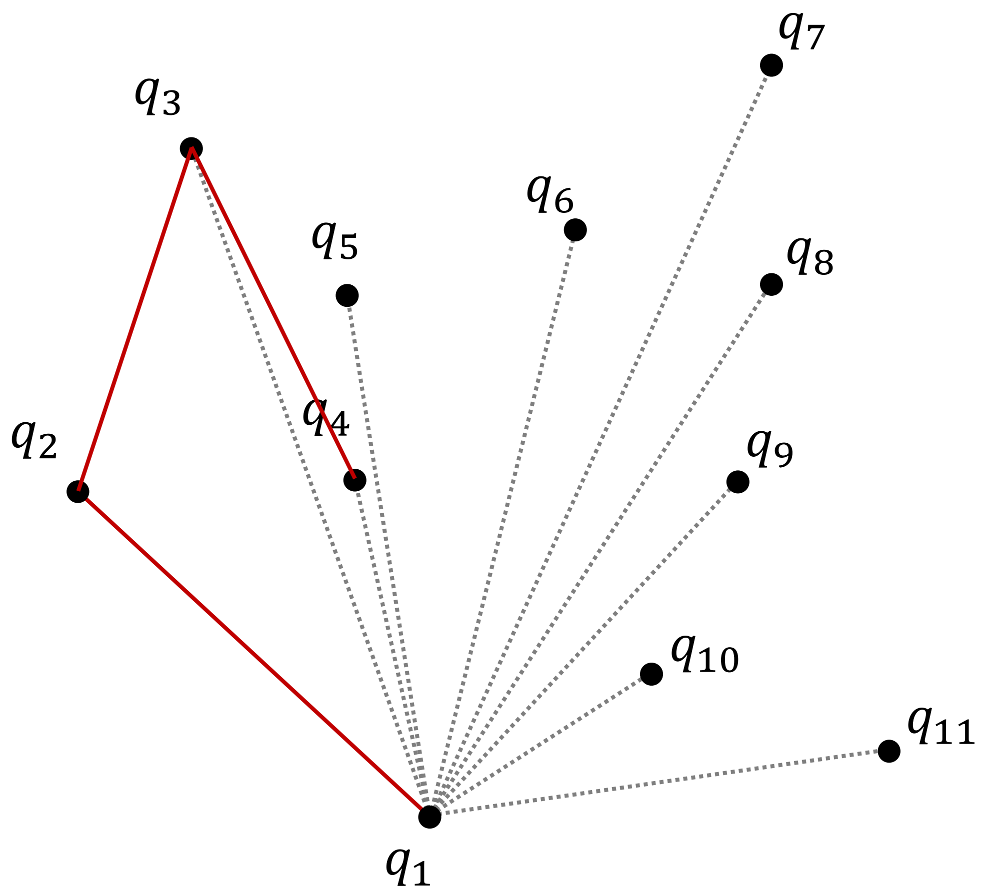

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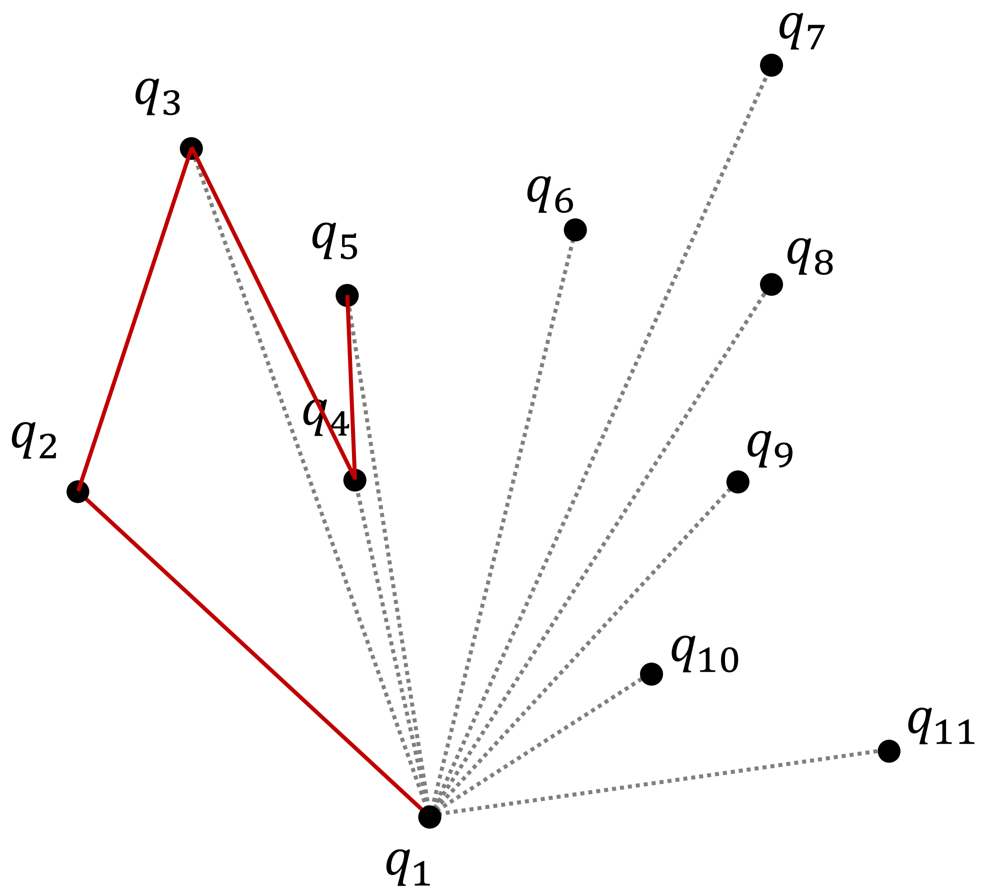

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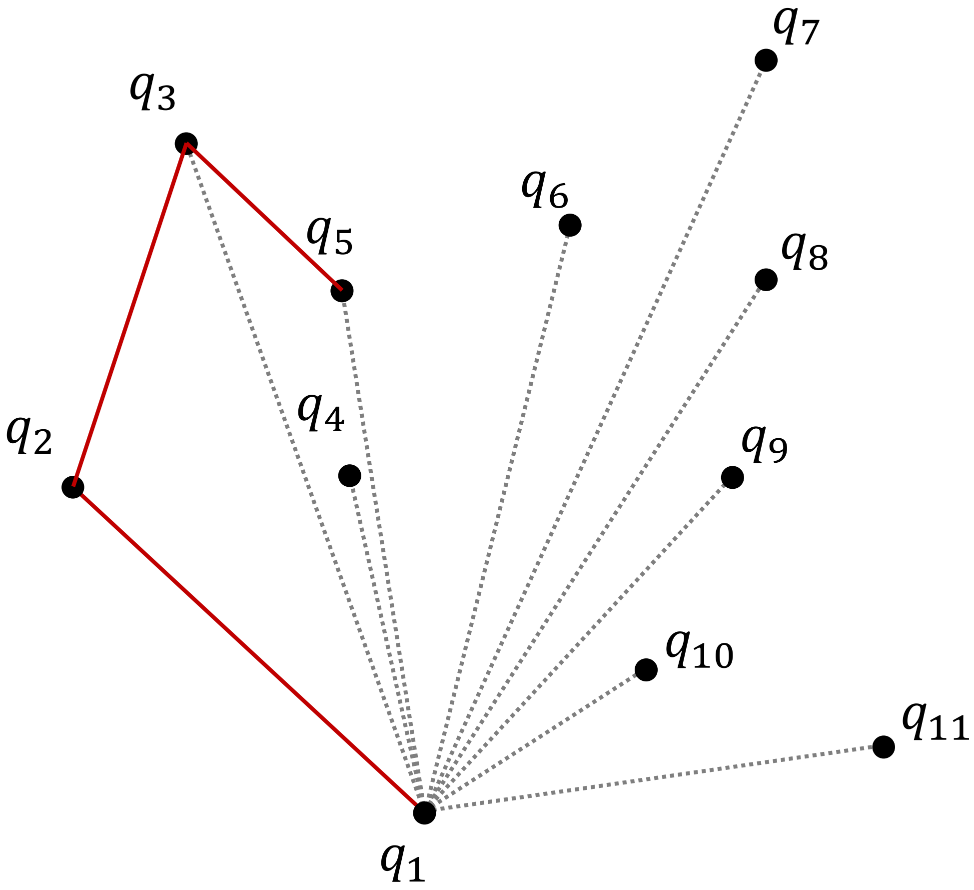

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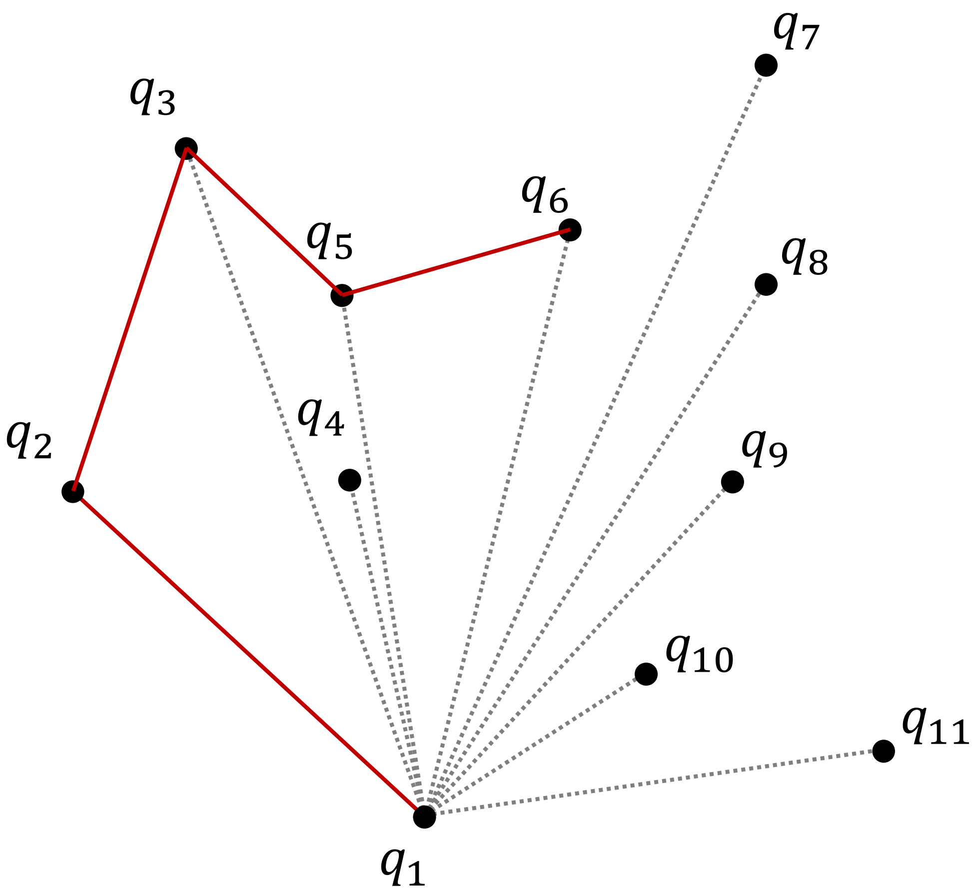

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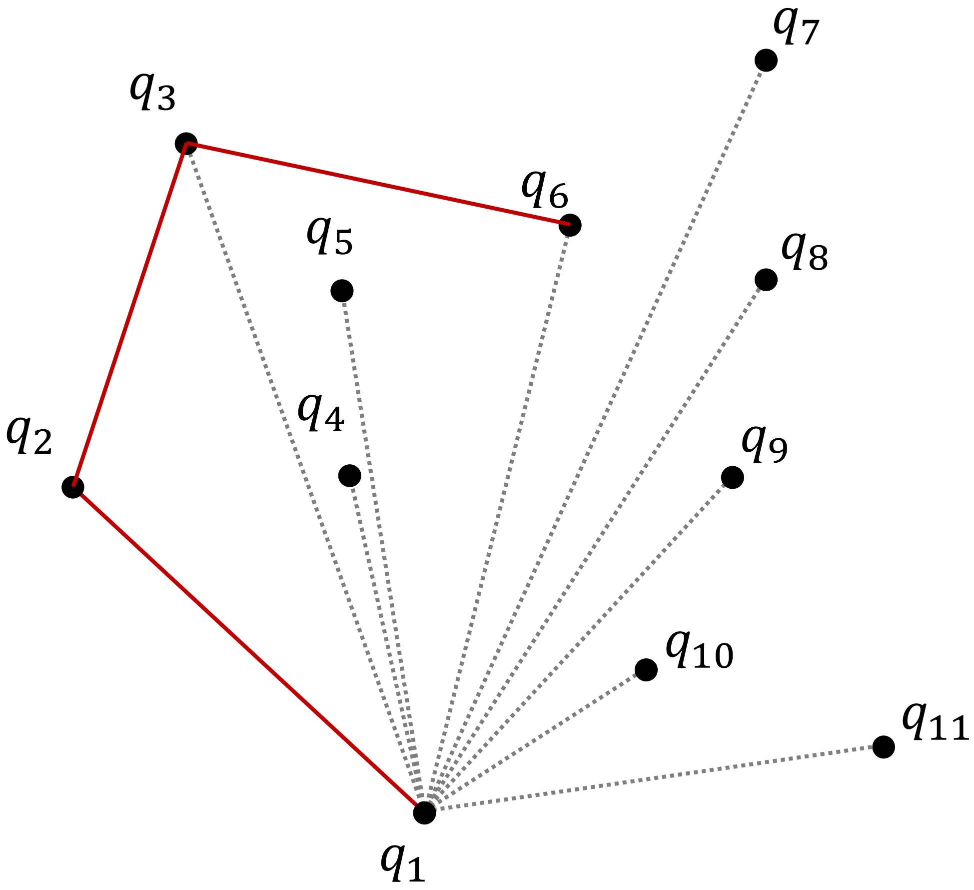

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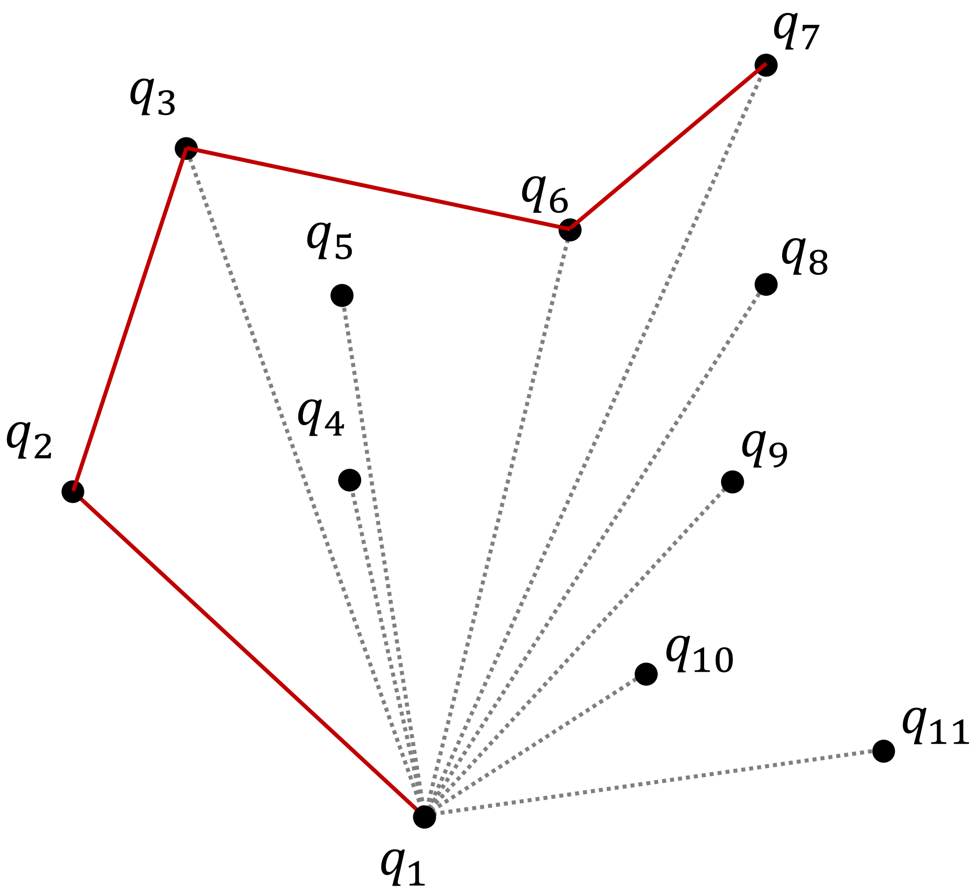

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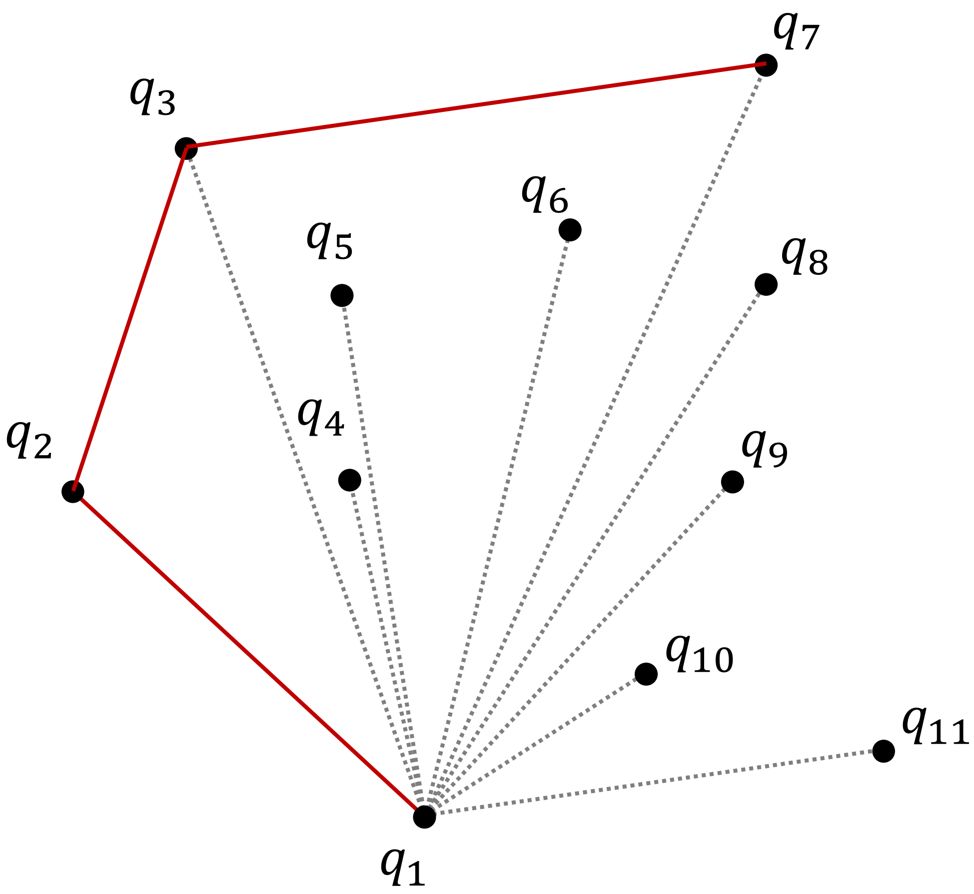

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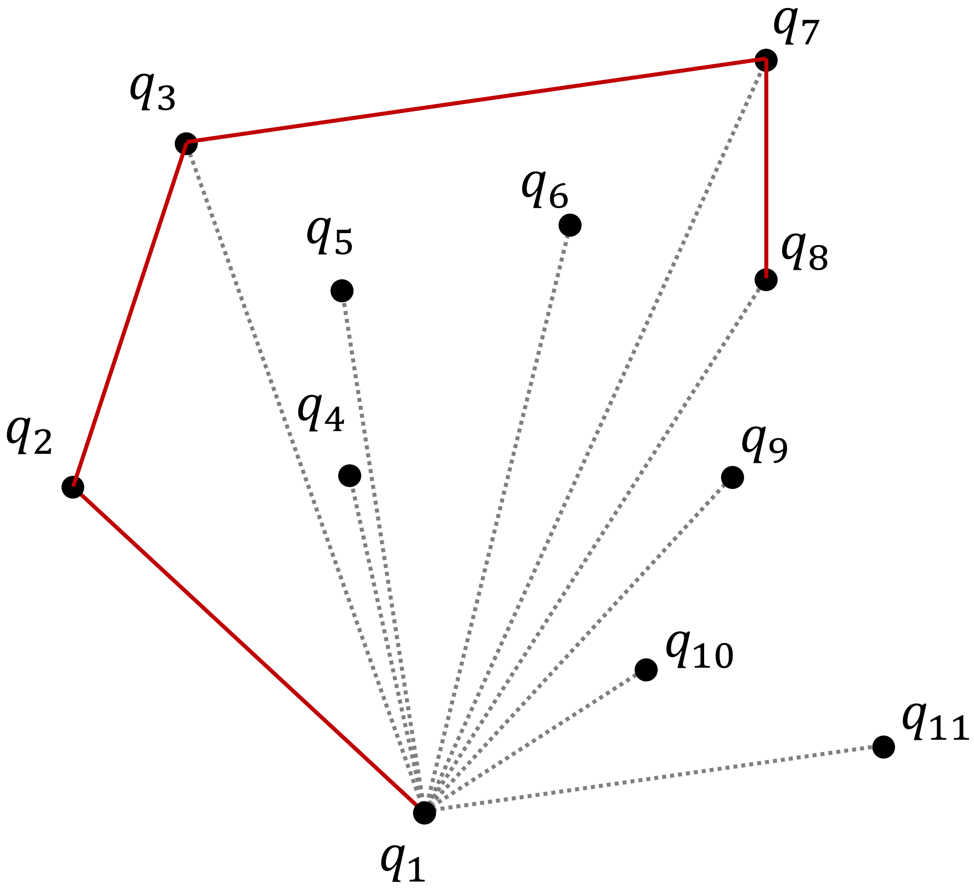

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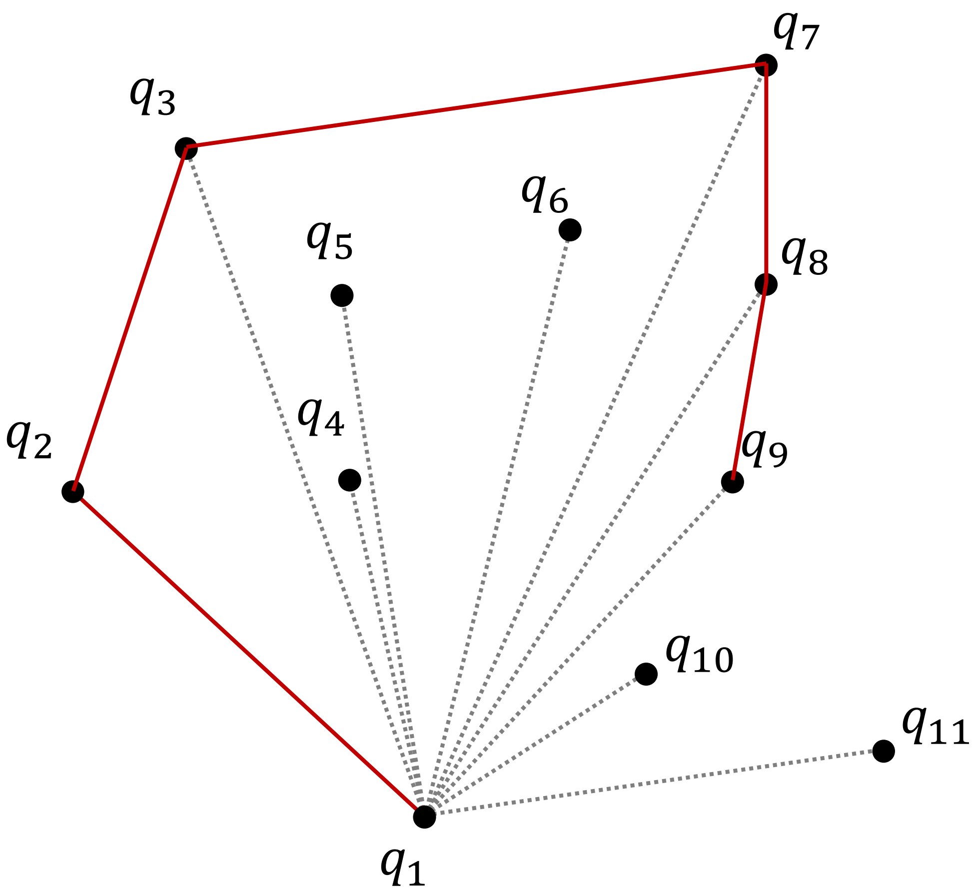

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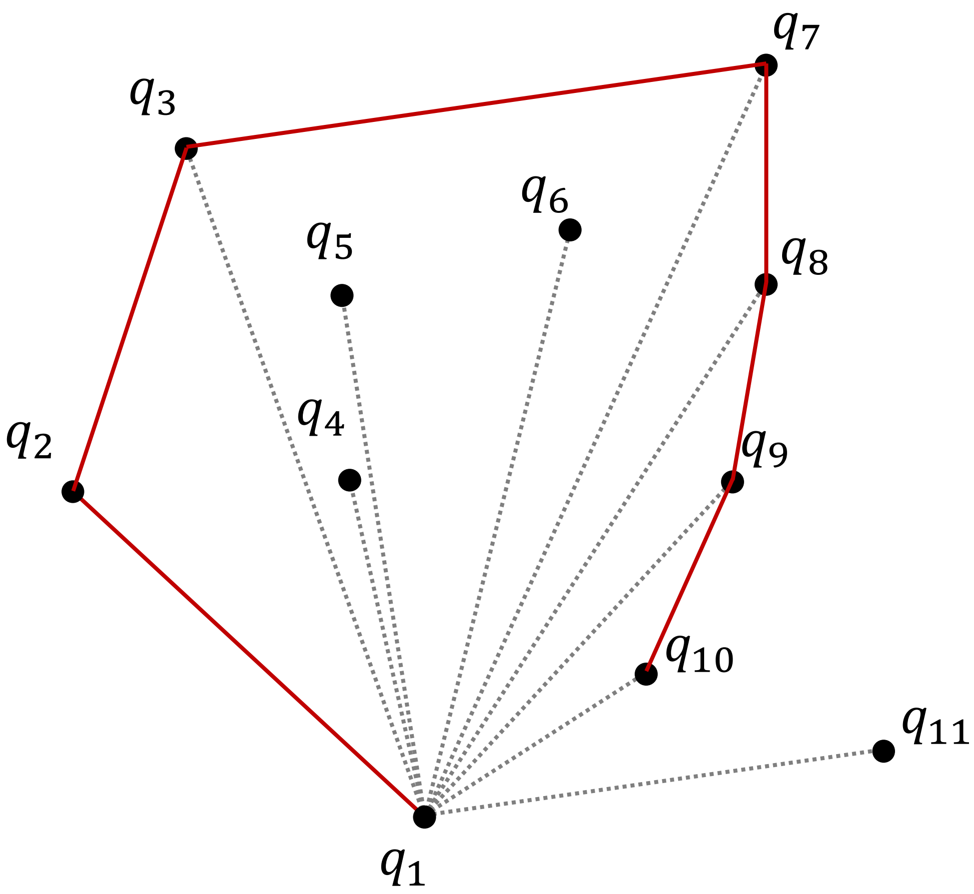

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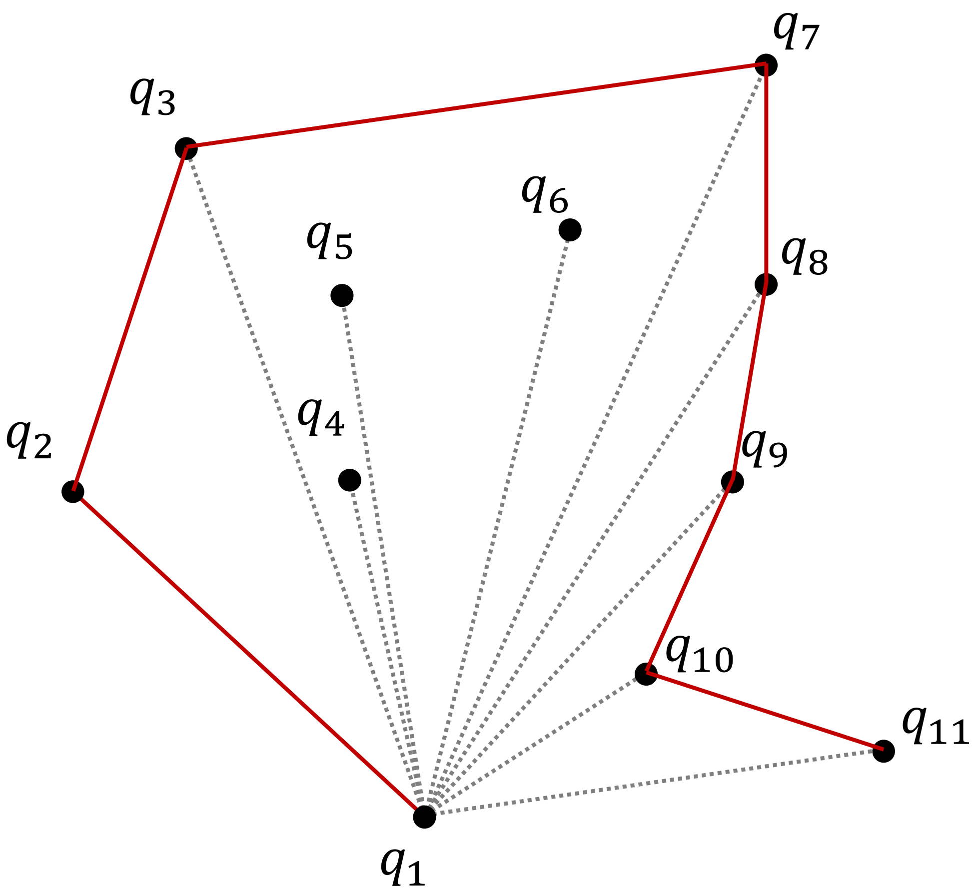

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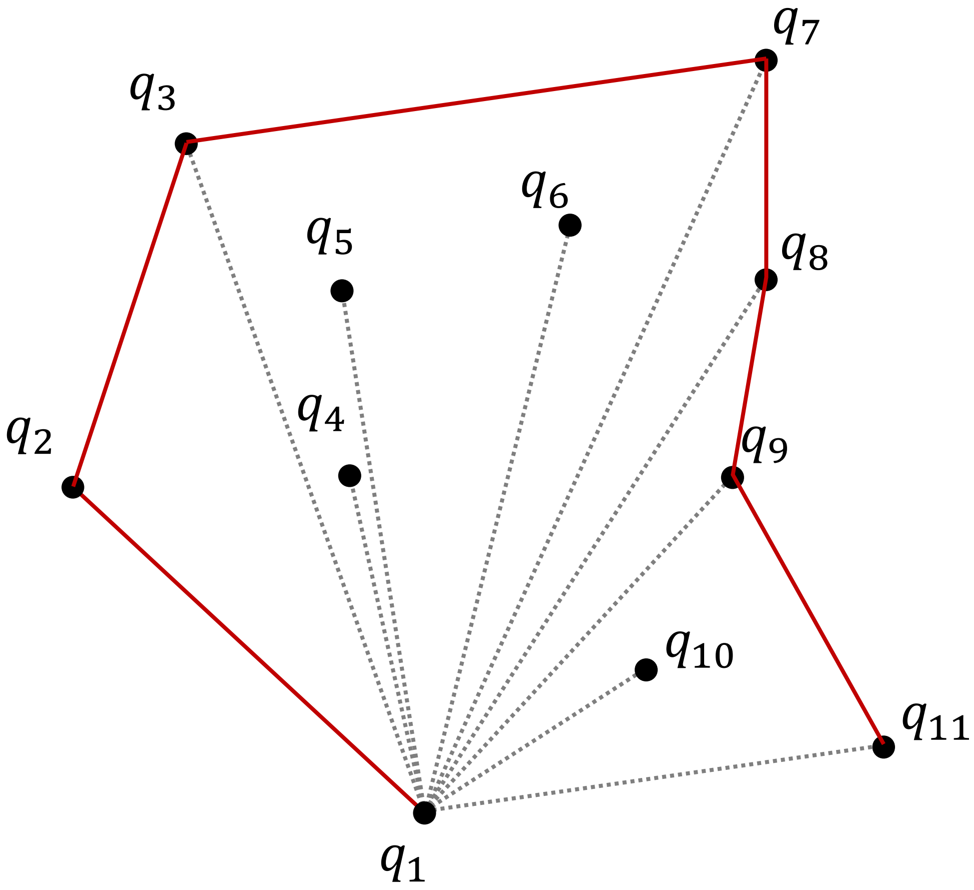

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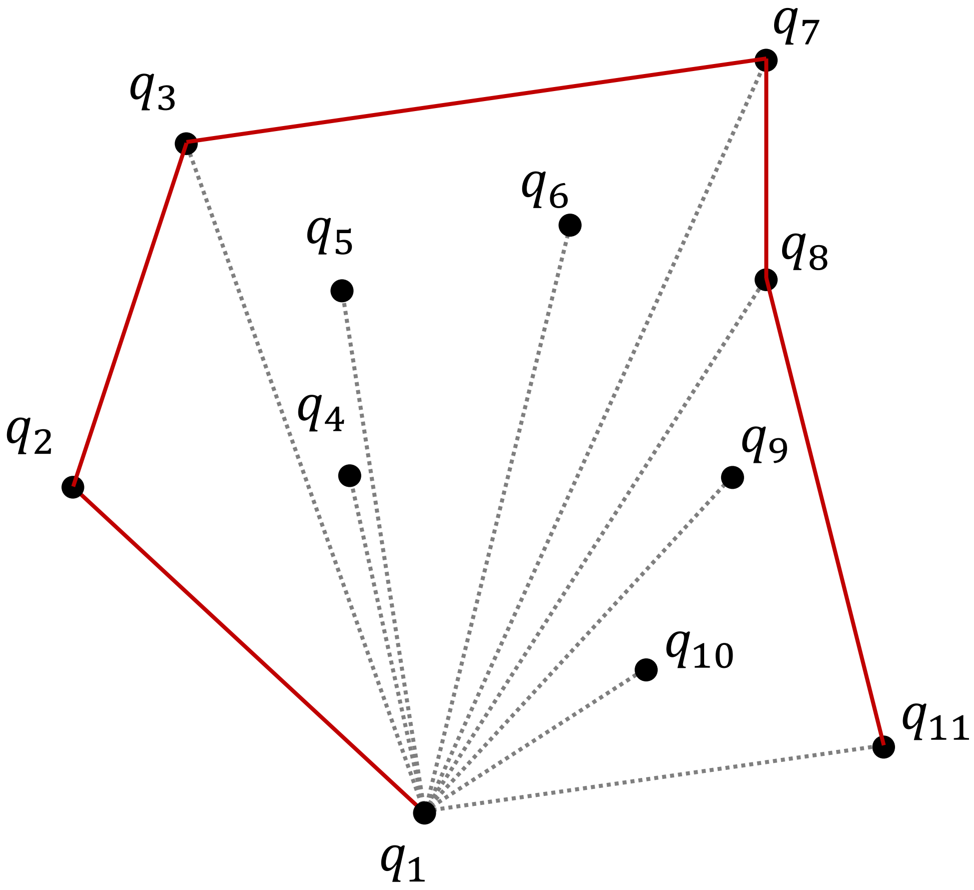

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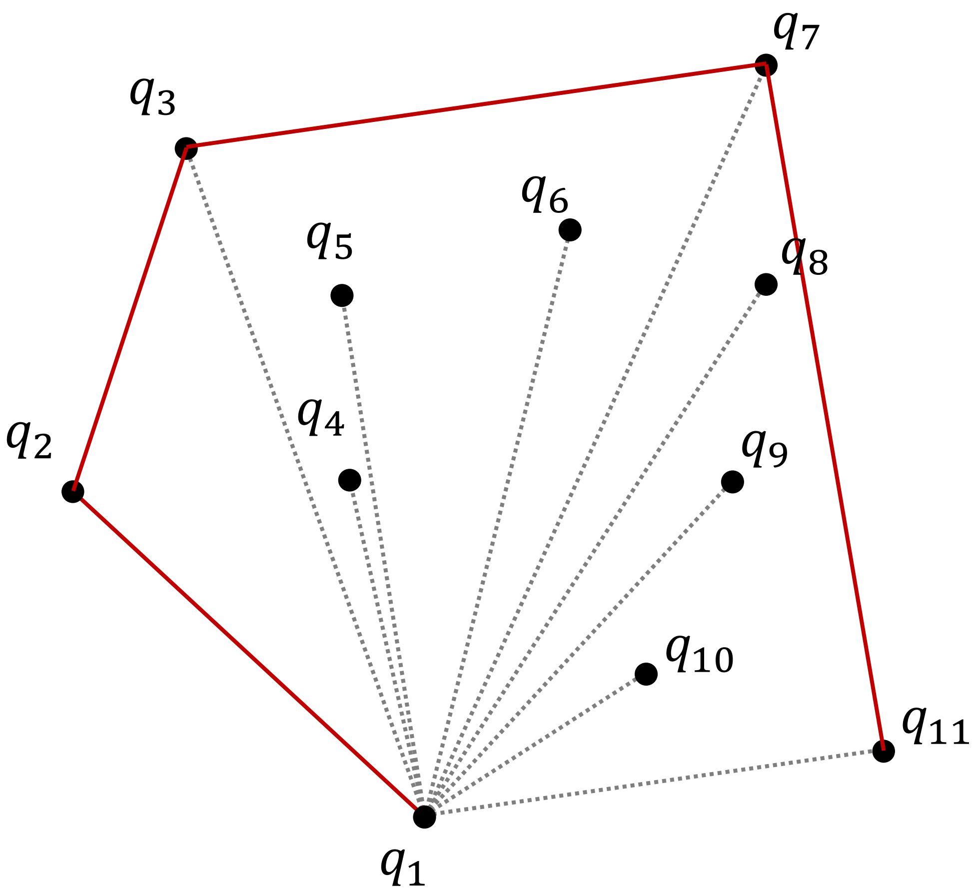

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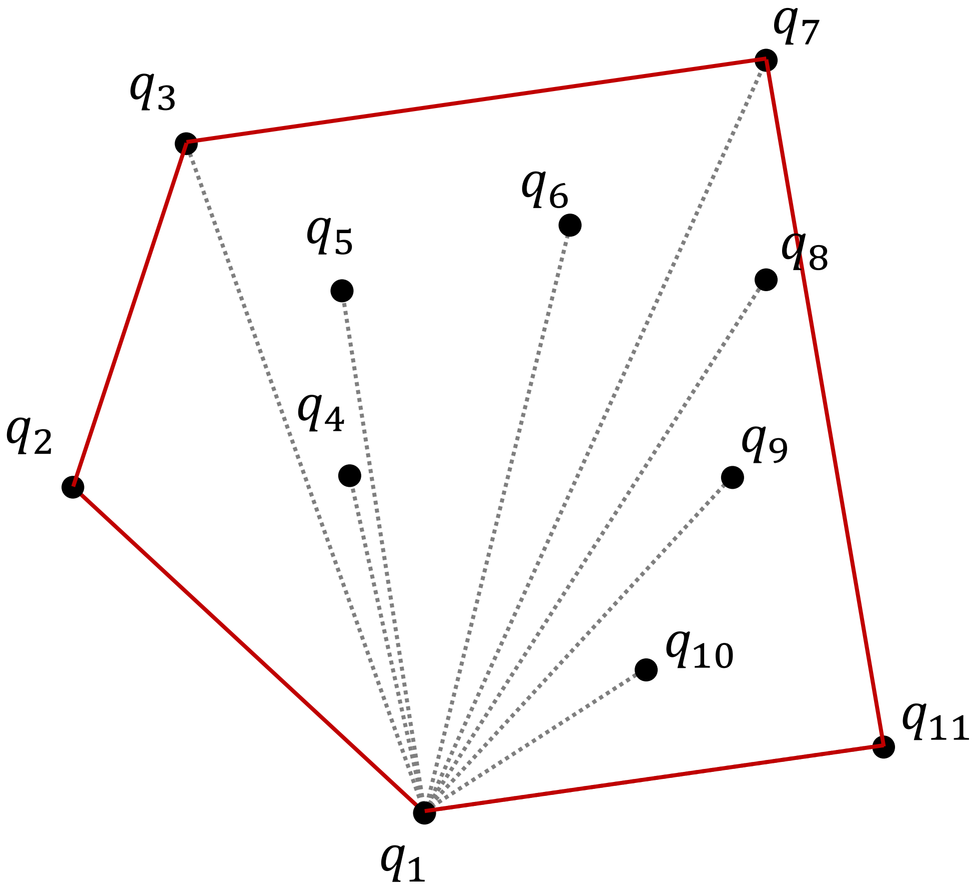

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Step 3 running time: $O(n)$

This is called ***amortized analysis***

Time to handle each point q_i in step 3:

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$$\boxed{\textbf{Overall running time of Graham's scan: } O(n \log n)}$$