Iterative Methods for Nonlinear Equations

Nonlinear equations

$f(x) = 0$

$f(x) \Rightarrow$ linear

$x = 2x + 2$

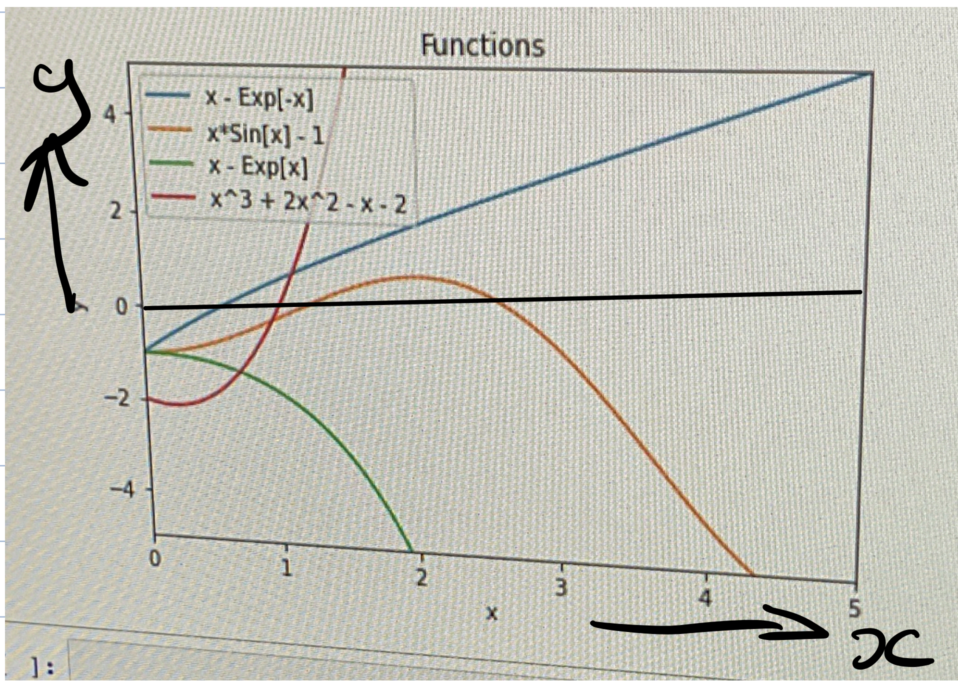
Equations such as

$\Rightarrow x^3 + 2x^2 - x - 2 = 0$ (three roots)

$\Rightarrow x - e^{-x}$ (One root)

$\Rightarrow x - e^{x}$ (No root)

$\Rightarrow x \sin x - 1$ (infinitely many roots)



The number of times the curve cuts across $y=0$ reveals the roots of the equation.

Note that this figure was plotted for positive x-values only.

Rate of Convergence for Iterative Methods

Linear Convergence: Let x^* be a true solution and $x^{(k)}$ the approximate solution at the k^{th} iteration. The error at the k^{th} iteration, denoted $e^{(k)}$ is $e^{(k)} = x^{(k)} - x^*$.
A sequence $x^{(k)}$ with limit x^* is linearly convergent if there exists a constant $c > 0$ such that

$$|e^{(k+1)}| \le c|e^{(k)}|$$

for k sufficiently large such that $x^{(k)}$ is close to x^*.

$$x = \frac{1}{2}(1 + x)$$

Example

Given the iterative scheme;

$$x^{(k+1)} = \frac{1}{2}(1 + x^{(k)})$$

and $x^{(0)} = 2$

Obtain the sequence for $k = 0, 1, 2, 3, 4, \ldots$

Solution

at $k = 0$

$$x^{(0+1)} = \frac{1}{2}(1 + x^{(0)})$$

$$x^{(1)} = \frac{1}{2}(1 + 2) = 3/2 = 1.5$$

at $k = 1$

$$x^{(2)} = \frac{1}{2}(1 + x^{(1)}) = \frac{1}{2}(2.5) = 1.25$$

$$x^{(3)} = \frac{1}{2}(1 + 1.25) = \frac{1}{2}(2.25) = 1.125$$

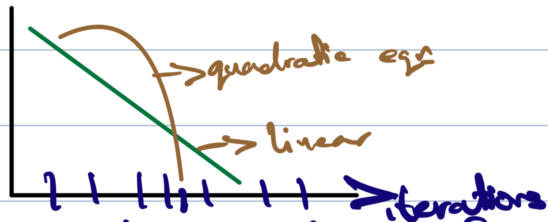
$\vdots$

$$x^{(9)} = 1.001953131250$$

$$x^{(10)} = 1.000976562500$$

Observe that the results are converging to the value $x^*=1$, since

$$|x^{(k+1)}-x^*| = \left|\frac{1}{2}+\frac{1}{2}x^{(k)}-x^*\right| = \frac{1}{2}\left|x^{(k)}-x^*\right|$$



You can prove this by picking any $x^{(k)}$ value.

We have therefore, shown linear convergence with $c=1/2$, implying the error is decreasing by a factor of $1/2$ as we iterate.

Quadratic Convergence:

Let x^* be a true solution and $x^{(k)}$ the approximate solution at the k^{th} iteration. The error at the k^{th} iteration, denoted $e^{(k)}=x^{(k)}-x^*$.
A sequence $x^{(k)}$ with limit x^* is quadratically convergent if there exists a constant $c>0$ such that

$$|e^{(k+1)}| \leq c\,|e^{(k)}|^2$$

for k sufficiently large such that $x^{(k)}$ is close to x^*.

Example: $x^{(k+1)} = (x^{(k)})^2 - 2x^{(k)} + 2$

using $x^{(0)} = 1.5$

$x^{(1)} = (1.5)^2 - 2(1.5) + 2 = 1.25$

$x^{(2)} = 1.0625$

$x^{(3)} = 1.00390625$

$x^{(4)} = 1.00001525878906$

We observe a quadratic convergence here

$$|x^{(k+1)} - x^*| = |(x^{(k)})^2 - 2x^{(k)} + 2 - x^*| = |x^{(k)} - x^*|^2$$

Choose any x-value to show this.

at $c = 1$, we have shown quadratic convergence.

Two types of iterative method to consider;

1. Bracketing methods (Interval methods)
2. Fixed point methods

Bracketing (Interval) Methods

(i) Choose an initial interval that guarantees a root (solution) exists.

(ii) Reduce the width of the interval iteratively until the root is enclosed to an acceptable accuracy level.

We consider two interval methods;

(a) Bisection method

(b) False position method (Regula Falsi)

Bisection method

Algorithm

(i) Given $f(x) = 0$

(ii) Bracket a root in the interval (a, b)

(iii) Continually halve the intervals.

(iv) When interval is sufficiently small enough, stop.

Note that if there is a root in the interval (a, b), then $f(a)$ and $f(b)$ always have opposite signs, i.e.

(i) $f(a)$ is positive and $f(b)$ is negative,

(ii) or $f(a)$ is negative and $f(b)$ is positive.

This follows the Intermediate Value Theorem;

If $f(x)$ is real and continuous in an interval $[a, b]$ and $f(a)f(b) < 0$, then there exists a point $c \in (a,b)$ such that $f(c)=0$.

→ The idea is to choose an interval (a,b) such that when you compute $f(a) \times f(b)$, it should give a value that is less than 0.

e.g.

$$f(x) = x^2 - 1$$

if $x = 0$, $f(x) = -1$

if $x = 3$, $f(x) = 8$

$\therefore (0,3)$ is an appropriate start interval.

We can proceed to calculate C_1.

1st $C_1 = \frac{a+b}{2} = \frac{0+3}{2} = \frac{3}{2}$

$$f(C_1) = \left(\frac{3}{2}\right)^2 - 1 = \frac{9}{4} - 1 = \frac{5}{4}$$

$\therefore f(C_1) > 0$ and $f(b) > 0$. Discard $f(b)$

New interval $(0, 3/2)$

2nd $C_2 = \frac{a+C_1}{2} = \frac{3}{2} \div 2 = 3/2 \times 1/2 = 3/4$

$$f(3/4) = (3/4)^2 - 1 = \frac{9}{16} - 1 = -7/16$$

$\therefore f(a)$ is discarded.

3rd $C_3 = \frac{C_2 + C_1}{2} = \frac{3/4 + 3/2}{2} = 9/8$

$=$

$$f(C_3) = \left(\frac{13}{32}\right)^2 - 1 = -\frac{855}{1024}$$

$(9/8, 3/2)$

$\therefore C_2$ is discarded

⇒ The process is continued.

As observed in the example above, we need to define a stopping criteria.

* Choose a tolerance $\varepsilon > 0$, and generate sequence of midpoints $c_1, c_2, \dots, c_n$ of intervals $[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]$ and use any of the stopping criteria

① $|c_n - c_{n-1}| < \varepsilon$

② $\dfrac{|c_n - c_{n-1}|}{|c_n|} < \varepsilon$

③ $f(c_n) < \varepsilon$

④ $|b_n - a_n| < \varepsilon$

Note! Given n is the final interval, the number of steps (interval halving) required is given by $n-1$.

Example: Find the root of $f(x) = \sin(x) - 0.5$ between 0 and 1. Iterate until the interval is of length $1/2^3$. $\left[\frac{1}{2^{n-1}} \Rightarrow \frac{1}{2^{4-1}} = \frac{1}{2^3}\right]$

Solution: From the notes, we find that $n = 4$ $\therefore n-1 = 3$. Hence we need to compute a_4, b_4, and c_4

* $a = a_1 = 0$ & $b = b_1 = 1$

$f(0) = \sin(0) - 0.5$ & $f(1) = \sin(1) - 0.5$

$f(0) = -0.5$ & $f(1) = 0.341471$

line with the x-axis is the point c.

* The equation of the line through $(a, f(a))$ and $(b, f(b))$ is

$$y = f(a) + \frac{x-a}{b-a}\left(f(b) - f(a)\right)$$

The point c where $y = f(c) = 0$ is required.

$$* \quad f(c) = f(a) + \frac{c-a}{b-a}\left(f(b) - f(a)\right) = 0.$$

To solve for c;

$$c - a = \frac{-f(a)}{f(b)-f(a)} \cdot b-a$$

$$c = \frac{-b f(a) + a f(a)}{f(b) - f(a)} + a$$

$$c = \frac{-b f(a) + \cancel{a f(a)} + a f(b) - \cancel{a f(a)}}{f(b) - f(a)}$$

$$\therefore \quad c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

If $f(c)$ and $f(a)$ have the same sign, discard a. Else, discard b.

Example

Perform two iterations of the false position method on the function $f(x) = x^2 - 1$, using $[0, 3]$ as your initial interval. Compare answers to what was obtained with the bisection method

Solution

$(0,2)$ $f(b_1) = 3$

$b_1 = 2$

$a_1 = 0$ $b_1 = 3$

$f(a_1) = -1$ $f(b_1) = 8$

$$C_1 = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$$

$$= \frac{0 \cdot 8 - 3 \cdot (-1)}{8 - (-1)} = \frac{3}{9} = \frac{1}{3}$$

$$f(c_1) = \left(\frac{1}{3}\right)^2 - 1 = \frac{1}{9} - 1 = -\frac{8}{9}$$

$\therefore$ a_1 is discarded $\cancel{\phi}$ (c_1, b_1)

$$C_2 = \frac{c_1 \cdot f(b_1) - b_1 \cdot f(c_1)}{f(b_1) - f(c_1)}$$

$$= \frac{\frac{1}{3} \cdot 8 - 3\left(-\frac{8}{9}\right)}{8 - \left(-\frac{8}{9}\right)}$$

$= 3/5$

$$f(C_2) = (3/5)^2 - 1 = 9/25 - 1$$

$$= -\frac{16}{25}$$

$$= -0.64$$

C_1 is discarded.

$\therefore$ new interval is $(0.6, 3)$

* Notice that the interval is decreasing

from the left hand-side only.

* Choose the stopping criteria

$$|c - c^*| < \varepsilon$$

where c^* is the value of c calculated from the previous step.

Exercise

Use the Bisection method and the false position method to find the roots of $f(x) = x^2 - x - 2$ that lies in the interval $[1, 4]$

Solution: From the notes, we find that $n = 4$ $\therefore n - 1 = 3$. Hence we need to compute a_4, b_4, and c_4

* $a = a_1 = 0$ & $b = b_1 = 1$

$f(0) = \sin(0) - 0.5$ & $f(1) = \sin(1) - 0.5$

$f(0) = -0.5$ & $f(1) = 0.341471$

$$c_1 = \frac{0+1}{2} = 1/2$$

$$f(c_1) = \sin(0.5) - 0.5 \approx -0.020574$$

Replace a_1 with c_1

$(c_1, b_1) = (0.5, 1)$

$$c_2 = \frac{c_1 + b_1}{2} = \frac{0.5 + 1}{2} = 0.75$$

$$f(c_2) = \sin(0.75) - 0.5 \simeq \underline{\underline{0.1816}}$$

b_1 is replaced by c_2.

(c_1, c_2)

→ Exercise!!!