A Term Encoding to Analyze Runtime Complexity via Innermost Runtime Complexity

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Outline

Introduction

- A term rewrite system(TRS) is a finite set of rules

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Example 1.

$$
\begin{array}{lcl}
\alpha_1 : \mathsf{add}(x, 0) & \to & x \\
\alpha_2 : \mathsf{add}(x, \mathsf{s}(y)) & \to & \mathsf{add}(\mathsf{s}(x), y)
\end{array}
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- A term rewrite system(TRS) is a finite set of rules

Example 1.

$$\begin{aligned} \alpha_1 : \mathsf{add}(x, 0) &\rightarrow x \\ \alpha_2 : \mathsf{add}(x, \mathsf{s}(y)) &\rightarrow \mathsf{add}(\mathsf{s}(x), y) \end{aligned}$$

$$\begin{array}{ccccccc} 3+2 & = & 4+1 & = & 5+0 & = & 5 \\ \mathsf{add}(\mathsf{s}^3(0), \mathsf{s}^2(0)) & \rightarrow_{\mathcal{R}_1} & \mathsf{add}(\mathsf{s}^4(0), \mathsf{s}(0)) & \rightarrow_{\mathcal{R}_1} & \mathsf{add}(\mathsf{s}^5(0), 0) & \rightarrow_{\mathcal{R}_1} & \mathsf{s}^5(0) \end{array}$$

Preliminaries

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\alpha_1 : \mathsf{add}(x, 0) &\quad\rightarrow\quad x \\
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- Signature \Sigma and variables \matchal V

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- A term over \Sigma

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- Positions of a term

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- Reducible expression and rewrite steps

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- Positions of a term
- Rewrite rule
- Defined and constructor symbols
- Substitution
- Reducible expression and rewrite steps
- Runtime complexity

Motivation

- Automatic complexity analysis

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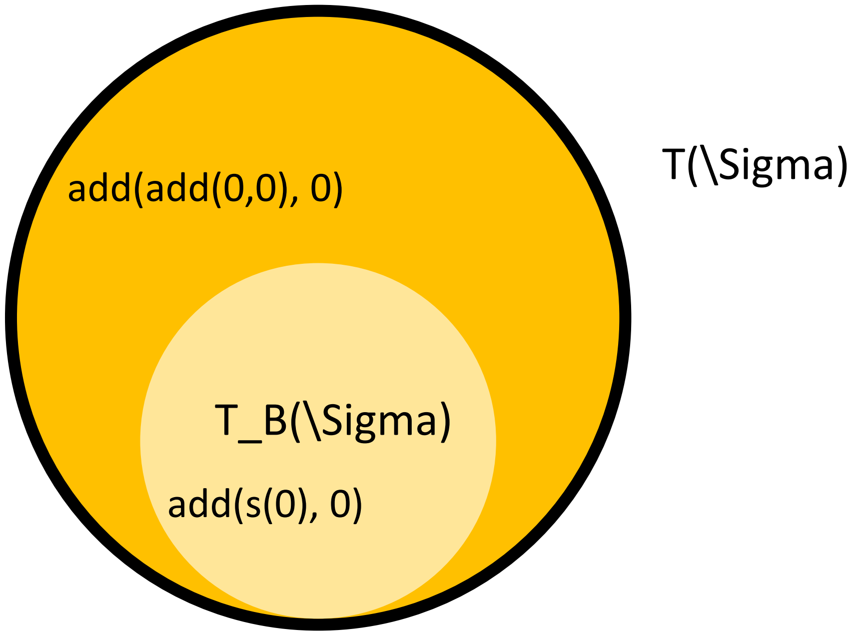
- Derivational complexity vs Runtime complexity



Motivation

- Automatic complexity analysis
- Derivational complexity vs Runtime complexity
- Full runtime vs Innermost runtime complexity

irc \leq rc

Non-Dup-Generalized

- An ndg TRS is one for which all rewrite steps are ndg

Non-Dup-Generalized

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  - All proper subterms of the redex are in normal form

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- For an ndg TRS R it holds that

rc_R = irc_R

Non-ndg

- A non-ndg rewrite step means either
  - the redex has a proper subterm not in normal form or
  - for a duplicated variable it holds that \sigma(x) is not in normal form

- A non-ndg TRS has at least one non-ndg rewrite step

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Example 2.

$$\alpha_1 : \mathsf{f}(x) \to \mathsf{g}(\mathsf{a}(x)) \qquad \alpha_2 : \mathsf{a}(x) \to \mathsf{b}(\mathsf{x})$$
$$\alpha_3 : \mathsf{g}(\mathsf{b}(x)) \to \mathsf{lin}(x) \qquad \alpha_4 : \mathsf{g}(\mathsf{a}(x)) \to \mathsf{quad}(x)$$
$$\alpha_5 : \mathsf{lin}(\mathsf{s}(x)) \to \mathsf{lin}(x) \qquad \alpha_6 : \mathsf{quad}(\mathsf{s}(x)) \to \mathsf{c}(\mathsf{lin}(x), \mathsf{quad}(x))$$

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f(s^n(0)) → g(a(s^n(0)))

g(a(s^n(0))) → g(b(s^n(0))) → lin(s^n(0)) → …

g(a(s^n(0))) → quad(s^n(0)) → e(lin(s^n-1(0)), quad(s^n-1 (0))) → …

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rc_R \in O(n^2)

irc_R \in O(n)

Encoding

Concept of the Encoding

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g(b(s^n(0))) → lin(s^n(0)) → …

f(s^n(0)) → g(a(s^n(0)))

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i(l_a(x)) -> i(a(x))

i(x) -> x

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g(b(s^n(0))) ⟶ lin(s^n(0)) ⟶ …

f(s^n(0)) ⟶ g(a(s^n(0)))

quad(s^n(0)) ⟶ e(lin(s^n-1(0)), quad(s^n-1 (0))) ⟶ …

- irc = rc is not necessarily true. Consider

  g(l_a(0)) -> quad(0)

- Therefore analyze irc’ , which is irc limited on the starting terms of the original TRS

Introduction

Encoding

Expanded Encoding

Conclusion

Non-ndg Locations

$$\alpha_1 : \mathsf{f}(x) \to \mathsf{g}(\mathsf{a}(x)) \qquad \alpha_2 : \mathsf{a}(x) \to \mathsf{b}(\mathsf{x})$$
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- Locations in a TRS in order to avoid confusion with positions in terms

(\alpha_1, L, \eps)

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  - Remove locations with constructor symbols

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- Refining the set
  - Remove locations of variables on the lhs
  - Remove locations with constructor symbols
  - Remove locations where no redexes flow

Refining the set of non-ndg locations

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- Set of these locations is the smallest set that includes
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  - All locations where locations from the set flow

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- Set of these locations is the smallest set that includes
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  - All locations where locations from the set flow (CAP-function)

CAP(g(a(x))) = g(y)

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- Intersection of X_R and Y_R

First look at an encoded TRS

$\alpha'_1 : \mathsf{f}(x) \to \mathsf{g}(\mathsf{i}(l_\mathsf{a}(x)))$

$\alpha'_2 : \mathsf{a}(x) \to \mathsf{b}(\mathsf{x})$

$\alpha'_3 : \mathsf{g}(\mathsf{b}(x)) \to \mathsf{lin}(x)$

$\alpha'_4 : \mathsf{g}(l_\mathsf{a}(x)) \to \mathsf{quad}(\mathsf{i}(x))$

$\alpha'_5 : \mathsf{lin}(\mathsf{s}(x)) \to \mathsf{lin}(x)$

$\alpha'_6 : \mathsf{quad}(\mathsf{s}(x)) \to \mathsf{c}(\mathsf{lin}(\mathsf{i}(x)), \mathsf{quad}(\mathsf{i}(x)))$

$\beta_1 : \mathsf{i}(x) \xrightarrow{0} x$

$\beta_2 : \mathsf{i}(l_\mathsf{a}(x)) \xrightarrow{0} \mathsf{i}(\mathsf{a}(x))$

$\beta_3 : \mathsf{i}(l_\mathsf{a}(x)) \xrightarrow{0} \mathsf{i}(l_\mathsf{a}(\mathsf{i}(x)))$

- 3 types of added 0-cost rules
  - Omission rule \beta1
  - Execution rules \beta2
  - Propagation rules \beta3

Necessity of propagation rules

f -> dbl(a(a(0)))   dbl(x) -> d(x,x)   a(x) -> b(x)

f -> dbl(i(l_a(l_a(0))))   dbl(x) -> d(i(x),i(x))   a(x) -> b(x)

i(x) -> x   i(l_a(x)) -> i(a(x))

f -> dbl(a(a(0))) -> d(a(a(0)), _) -> d(a(b(0)), _) -> d(b2(0), _) ->2 d(b20,b20)

f -> dbl(i(l_a(l_a(0)))) ->0 dbl(l_a(l_a(0))) -> d(i(l_a(l_a(0))), _) ->0 d(i(a(l_a(0))), _) -> d(i(b(l_a(0))) , _) ->

Necessity of propagation rules

f -> dbl(a(a(0)))   dbl(x) -> d(x,x)   a(x) -> b(x)

f -> dbl(i(l_a(i(l_a(0)))))   dbl(x) -> d(i(x),i(x))   a(x) -> b(x)

i(x) -> x   i(l_a(x)) -> i(a(x))

f -> dbl(a(a(0))) -> d(a(a(0)), _) -> d(a(b(0)), _) -> d(b2(0), _) ->2 d(b20,b20)

f -> dbl(i(l_a(i(l_a(0))))) ->0 dbl(l_a(i(l_a(0)))) ->0 dbl(l_a(l_a(0))) -> - - -

Necessity of propagation rules

f -> dbl(a(a(0)))   dbl(x) -> d(x,x)   a(x) -> b(x)

f -> dbl(i(l_a(l_a(0))))   dbl(x) -> d(i(x),i(x))   a(x) -> b(x)

i(x) -> x   i(l_a(x)) -> i(a(x))
i(l_a(x)) -> i(l_a(i(x)))

f -> dbl(a(a(0))) -> d(a(a(0)), _) -> d(a(b(0)), _) -> d(b2(0), _) ->2 d(b20,b20)

f -> dbl(i(l_a(l_a(0)))) ->0   dbl(l_a(l_a(0))) -> d(i(l_a(l_a(0))), _) ->0 d(i(l_a(i(l_a(0)))), _) -> - - -

Formal definition of the Encoding

- Locations on the lhs get encoded like this
  - g(a(x)) -> quad(x)   =>   g(l_a(x)) -> q

- Locations on the rhs get encoded like this
  - f(x) -> g(a(x))   =>   f(x) -> g(i(l_a(x)))

- 0-cost rules are added
  - The omission rule
  - The execution rule for encoded defined symbols
  - The propagation rule for encoded defined symbols with arity > 0

Outline

Infinite 0-cost rewriting

- An infinite rewrite sequence of 0-cost rewriting can occur in an encoded TRS

f -> dbl(a(a(0)))   dbl(x) -> d(x,x)   a(x) -> b(x)

f -> dbl(i(l_a(l_a(0))))   dbl(x) -> d(i(x),i(x))   a(x) -> b(x)

i(x) -> x   i(l_a(x)) -> i(a(x))
i(l_a(x)) -> i(l_a(i(x)))

f -> dbl(i(l_a(l_a(0)))) ->0 dbl(i(l_a(i(l_a(0))))) ->0 dbl(i(l_a(l_a(0)))) ->0 - - -

Infinite 0-cost rewriting

- An infinite rewrite sequence of 0-cost rewriting can occur in an encoded TRS

f -> dbl(a(a(0)))   dbl(x) -> d(x,x)   a(x) -> b(x)

f -> dbl(i(l_a(l_a(0))))   dbl(x) -> d(i(x),i(x))   a(x) -> b(x)

i(x) -> x   i(l_a(x)) -> i(a(x))
i(l_a(x)) -> l_a(i(x))

f -> dbl(i(l_a(l_a(0)))) ->0 dbl(l_a(i(l_a(0)))) -> - - -

Infinite 0-cost rewriting

- An infinite rewrite sequence of 0-cost rewriting can occur in an encoded TRS

f -> dbl(a(a(0)))   dbl(x) -> d(x,x)   a(x) -> b(x)

f -> dbl(i^2(l_a(l_a(0))))   dbl(x) -> d(i^2(x),i^2(x))   a(x) -> b(x)

i(x) -> x   i(l_a(x)) -> i(a(x))
i(l_a(x)) -> l_a(i(x))

f -> dbl(i^2(l_a(l_a(0)))) ->0 dbl(i(l_a(i(l_a(0))))) -> - - -

Even more content

- Mooore content

Outline