Algorithm for finding ***non-ndg*** locations
(commentary)

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INPUT: $TRS\ \mathfrak{R}$

OUTPUT: Finite set of locations $F(\mathfrak{R})$

We follow the creation of the set $F(\mathfrak{R})$ in two main sets which encompass the marked locations on the left- and on the right-hand sides of rules:

$LEFT_0(\mathfrak{R})$ is initialized to contain all location with a nested defined symbol on the left.

$$LEFT_0(\mathfrak{R}) := \left\{ (\alpha, L, \tau) \mid \alpha = l \to r \in \mathfrak{R},\ \tau \in pos(l) \setminus \{\varepsilon\},\ l|_\tau \in D \right\}$$

$RIGHT_0(\mathfrak{R})$ is initialized to contain all location with a duplicated variable on the right.

$$RIGHT_0(\mathfrak{R}) := \left\{ (\alpha, R, \tau) \mid \alpha = l \to r \in \mathfrak{R},\ \tau, \pi \in pos(r),\ \tau \neq \pi,\ r|_\tau = r|_\pi \in V \right\}$$

//Sadly, *Slides* don't offer inserting equations, so I have to copy with screenshots

Next up, we define $\mathrm{LEFT}_i(\mathfrak{R})$ for all $i > 0$:

$$\begin{aligned} LEFT_i(\mathfrak{R}) \quad := \{ \ (\alpha, L, \tau) \mid \alpha = l \rightarrow r \in \mathfrak{R},\ \tau \in pos(l),\ \pi \in pos(r), \\ l|_\tau = r|_\pi \in V\ ,\ (\alpha, R, \pi) \in RIGHT_{i-1}(\mathfrak{R}) \ \} \\ \cup\ LEFT_{i-1}(\mathfrak{R}) \end{aligned}$$

This simply adds to the set locations of variables on the left, which now flow directly into a marked location on the right-hand side of the same rule. Note that for $\mathrm{LEFT}_1(\mathfrak{R})$ we only require $\mathrm{RIGHT}_0(\mathfrak{R})$, which is defined on its own. It also holds:

$$\mathrm{LEFT}_0(\mathfrak{R}) \subseteq \mathrm{LEFT}_1(\mathfrak{R}) \subseteq \mathrm{LEFT}_2(\mathfrak{R}) \subseteq \ldots$$

$$LEFT_i(\mathfrak{R}) \quad := \{ (\alpha, L, \tau) \mid \alpha = l \to r \in \mathfrak{R},\ \tau \in pos(l),\ \pi \in pos(r),$$
$$l|_\tau = r|_\pi \in V,\ (\alpha, R, \pi) \in RIGHT_{i-1}(\mathfrak{R}) \}$$
$$\cup\ LEFT_{i-1}(\mathfrak{R})$$

Because of

$$\mathrm{LEFT}_0(\mathfrak{R}) \subseteq \mathrm{LEFT}_1(\mathfrak{R}) \subseteq \mathrm{LEFT}_2(\mathfrak{R}) \subseteq \ldots$$

and because there are finite many locations on the left, it can be concluded that for some $n \in N$:

$$(1) \qquad \ldots \subseteq \mathrm{LEFT}_{n-1}(\mathfrak{R}) \subseteq \mathrm{LEFT}_n(\mathfrak{R}) = \mathrm{LEFT}_{n+1}(\mathfrak{R}) = \ldots$$

Next up, we define $\text{RIGHT}_i(\mathfrak{R})$ for all $i > 0$:

$$
\begin{aligned}
RIGHT_i(\mathfrak{R}) \quad := \; \{ \; (\alpha, R, \tau) \mid \alpha = l \rightarrow r \in \mathfrak{R},\; \tau \in pos(r),\; \tau = \pi.\upsilon \\
\alpha \neq \beta = s \rightarrow t \in \mathfrak{R},\; \exists\, MGU \; for \; \mathfrak{R}|_{(\alpha,R,\pi)} \; and \; \mathfrak{R}|_{(\beta,L,\varepsilon)}, \\
\upsilon \nparallel \omega \in pos(s),\; \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\mathfrak{R}) \; \} \\
\cup \; \{ (l \rightarrow r, R, \tau) \mid \exists (\beta, P, \pi) \in M_{i-1}(\mathfrak{R}),\; root(l) = root(\mathfrak{R}|_{(\beta,P,\pi)}) \in D \,,\; \tau \in pos(r) \} \\
\cup \; RIGHT_{i-1}(\mathfrak{R})
\end{aligned}
$$

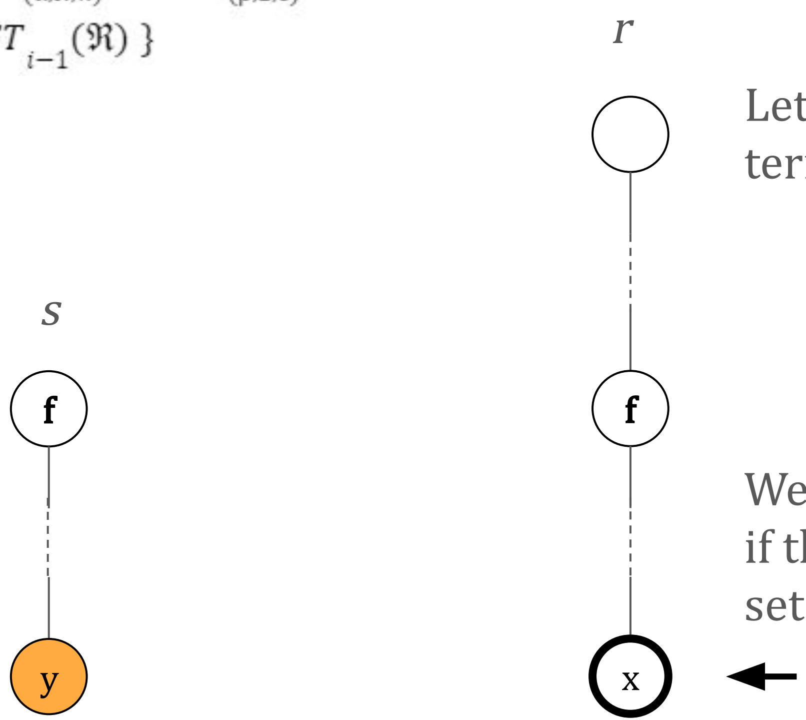
We are going to take a look at the first subset in more detail next.

$$\{\ (\alpha, R, \tau) \mid \alpha = l \to r \in \Re,\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \to t \in \Re,\ \exists\, MGU\ for\ \Re|_{(\alpha,R,\pi)}\ and\ \Re|_{(\beta,L,\varepsilon)},$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\Re)\ \}$$

This set includes locations on the right that flow into *non-ndg* locations on the left. Let's look at an example.

$$
\begin{aligned}
\{ (\alpha, R, \tau) \mid\ & \alpha = l \to r \in \mathfrak{R},\ \tau \in pos(r),\ \tau = \pi.\upsilon \\
& \alpha \neq \beta = s \to t \in \mathfrak{R},\ \exists\, MGU\ for\ \mathfrak{R}|_{(\alpha,R,\pi)}\ and\ \mathfrak{R}|_{(\beta,L,\varepsilon)}, \\
& \upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\mathfrak{R}) \}
\end{aligned}
$$

Let this be the right term of α, i.e. *r*



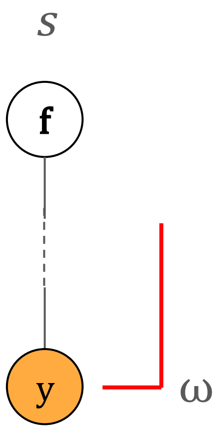
Let this be the left term of β, i.e. *s*

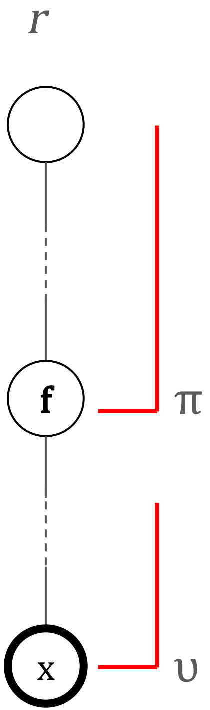
We want to determine if this location is in the set

x is marked here

//Other branches of these terms are irrelevant

$$\{\ (\alpha, R, \tau)\ |\ \alpha = l \rightarrow r \in \Re,\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \rightarrow t \in \Re,\ \exists\ MGU\ for\ \Re|_{(\alpha,R,\pi)}\ and\ \Re|_{(\beta,L,\varepsilon)},$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\ (\beta, L, \omega) \in LEFT_{i-1}(\Re)\ \}$$





π marks the start of the term, which we try to match

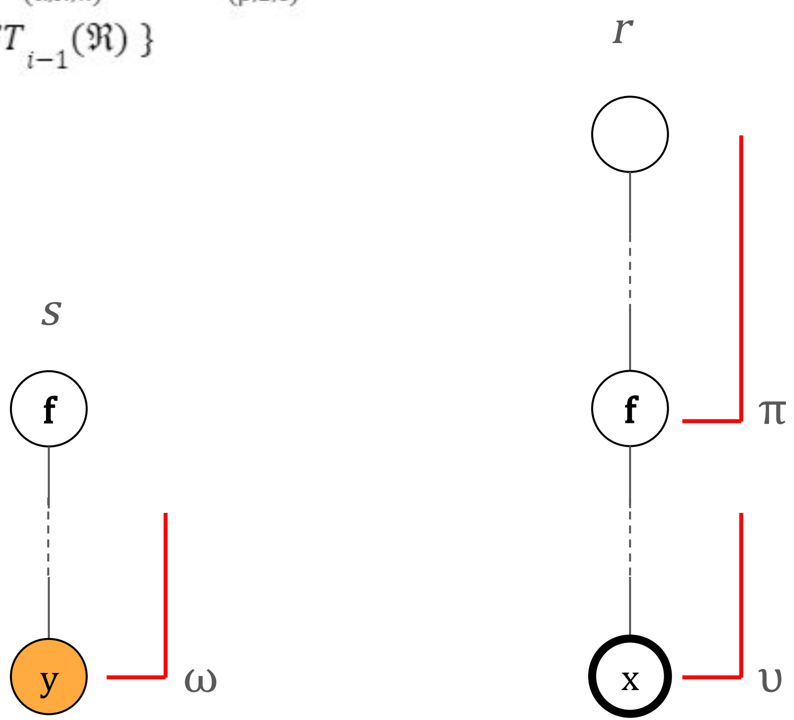
Ideally the terms match perfectly and we have:

$$\omega \geq \upsilon\ ,\ \upsilon \geq \omega$$

$$\{\ (\alpha, R, \tau) \mid \alpha = l \rightarrow r \in \Re,\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \rightarrow t \in \Re,\ \exists\, MGU\ for\ \Re|_{(\alpha,R,\pi)}\ and\ \Re|_{(\beta,L,\varepsilon)}\ ,$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\Re)\ \}$$

However, we must also consider the separate cases!

Let's get rid of the upper part of *r* for some space, but keep in mind it's still there.

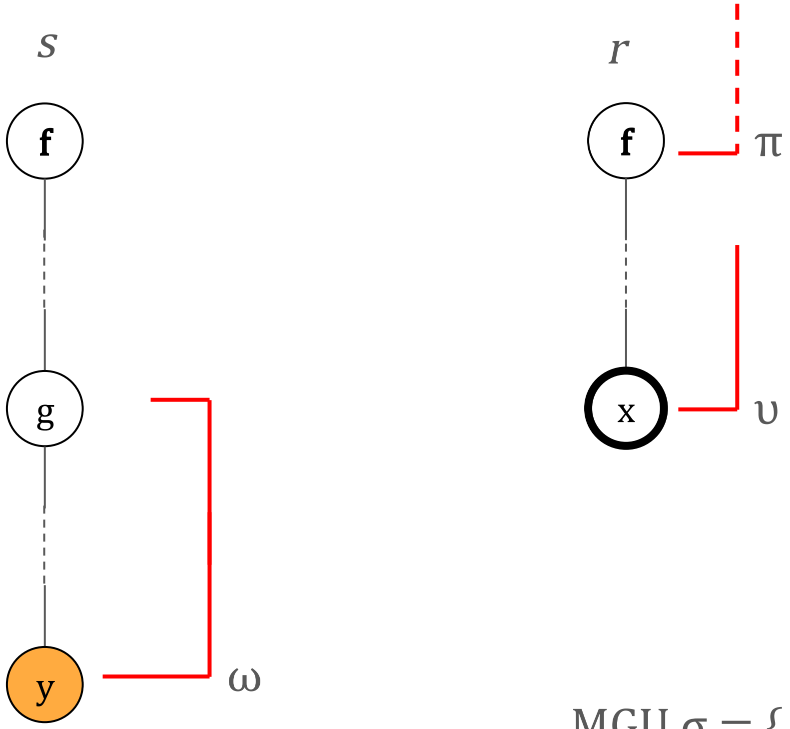


$$\{\ (\alpha, R, \tau) \mid \alpha = l \to r \in \Re,\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \to t \in \Re,\ \exists\, MGU\ for\ \Re|_{(\alpha,R,\pi)}\ and\ \Re|_{(\beta,L,\varepsilon)},$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\,(\beta, L, \omega) \in LEFT_{i-1}(\Re)\ \}$$

I. $\omega \geq \upsilon$

Here only $\omega \geq \upsilon$ would hold.

Since we don't know what symbols occur between υ and ω, we can consider two important cases.

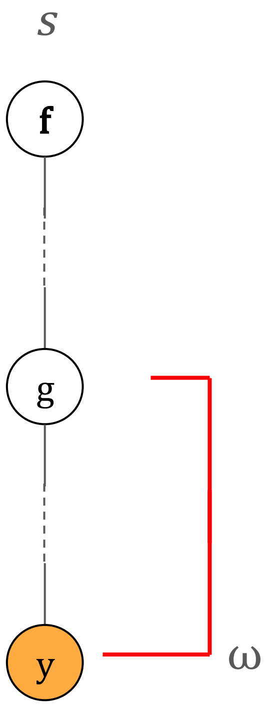


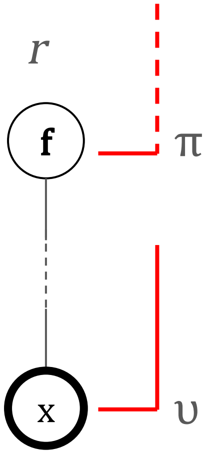
MGU $\sigma = \{ X/g(\ldots(y)), \ldots\}$

$$\{\ (\alpha, R, \tau) \mid \alpha = l \rightarrow r \in \Re,\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \rightarrow t \in \Re,\ \exists\, MGU\ for\ \Re|_{(\alpha,R,\pi)}\ and\ \Re|_{(\beta,L,\varepsilon)},$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\Re)\ \}$$

I. $\omega \geq \upsilon$

1. If no defined symbol occurs in between, then there is a sequence, which takes this rewrite step.



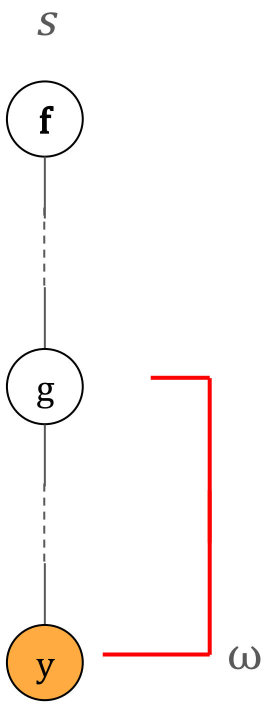


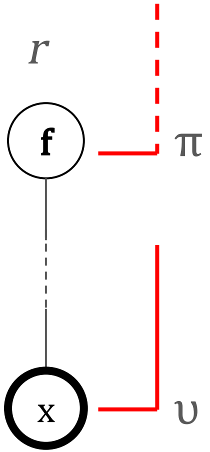
MGU $\sigma = \{ X/g(...(y)), ...\}$

$$\{ (\alpha, R, \tau) \mid \alpha = l \to r \in \mathfrak{R},\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \to t \in \mathfrak{R},\ \exists\, MGU\ for\ \mathfrak{R}|_{(\alpha,R,\pi)}\ and\ \mathfrak{R}|_{(\beta,L,\varepsilon)},$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\,(\beta, L, \omega) \in LEFT_{i-1}(\mathfrak{R}) \}$$

I. $\omega \geq \upsilon$

2. If a defined symbol *h* occurs in between, than it is possible that *x* never receives *h* as input and thus no sequence exist, where this rewrite step is taken.





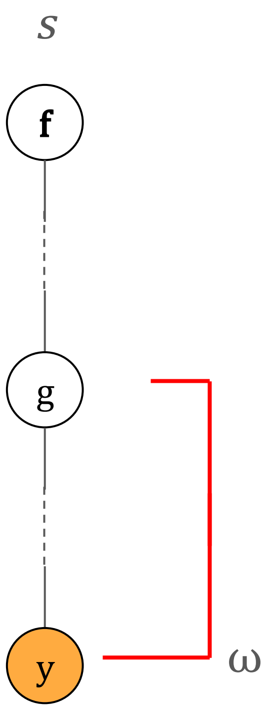
MGU $\sigma = \{ X/g(...(y)), ...\}$

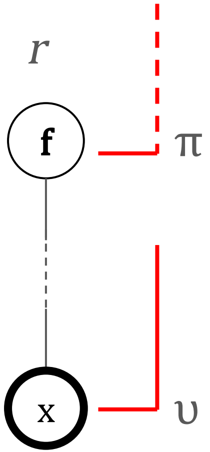
$$\{\ (\alpha, R, \tau)\ |\ \alpha = l \to r \in \Re,\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \to t \in \Re,\ \exists\, MGU\ for\ \Re|_{(\alpha,R,\pi)}\ and\ \Re|_{(\beta,L,\varepsilon)},$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\Re)\ \}$$

I. $\omega \geq \upsilon$

Which would mean that there is no need to mark *x*.

For now we choose to overapproximate these markings.





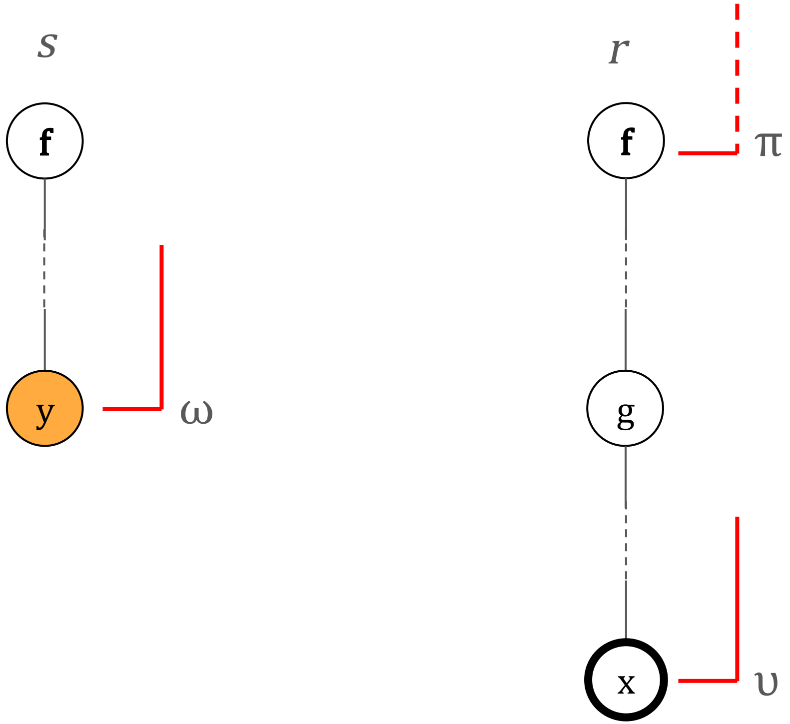
MGU $\sigma = \{\ X/g(\ldots(y)),\ \ldots\}$

$$\{\ (\alpha, R, \tau) \mid \alpha = l \rightarrow r \in \Re,\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \rightarrow t \in \Re,\ \exists\, MGU\ for\ \Re|_{(\alpha,R,\pi)}\ and\ \Re|_{(\beta,L,\varepsilon)},$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\Re)\ \}$$

II. $\upsilon \geq \omega$

Here only $\upsilon \geq \omega$ would hold.

In this case it is irrelevant they don't match exactly, since the term g(...(x)) flows directly into *y*.



MGU $\sigma = \{\ Y/g(...(x)), ...\}$

Going back to our definition of $\mathrm{RIGHT}_i(\mathfrak{R})$

$$\begin{aligned}
RIGHT_i(\mathfrak{R}) \quad := \ & \{ (\alpha, R, \tau) \mid \alpha = l \rightarrow r \in \mathfrak{R},\ \tau \in pos(r),\ \tau = \pi.\upsilon \\
& \qquad \alpha \neq \beta = s \rightarrow t \in \mathfrak{R},\ \exists\, MGU\ for\ \mathfrak{R}|_{(\alpha,R,\pi)}\ and\ \mathfrak{R}|_{(\beta,L,\varepsilon)}, \\
& \qquad \upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\mathfrak{R}) \} \\
\cup \ & \{ (l \rightarrow r, R, \tau) \mid \exists (\beta, P, \pi) \in M_{i-1}(\mathfrak{R}),\ root(l) = root(\mathfrak{R}|_{(\beta,P,\pi)}) \in D,\ \tau \in pos(r) \} \\
\cup \ & RIGHT_{i-1}(\mathfrak{R})
\end{aligned}$$

The second subset is quite a bit more simple. It marks all locations on the right-hand side of a rule, which is a function call of a marked defined symbol elsewhere in the TRS.

Here a new set $\mathrm{M}_i(\mathfrak{R})$ is mentioned, which we define as the union of both $\mathrm{RIGHT}_i(\mathfrak{R})$ and $\mathrm{LEFT}_i(\mathfrak{R})$

$$M_i(\mathfrak{R}) \quad := \quad LEFT_i(\mathfrak{R}) \cup RIGHT_i(\mathfrak{R})$$

Going back to our definition of $\text{RIGHT}_i(\mathfrak{R})$

$$
\begin{aligned}
RIGHT_i(\mathfrak{R}) \quad := \; & \{\, (\alpha, R, \tau) \mid \alpha = l \to r \in \mathfrak{R},\ \tau \in pos(r),\ \tau = \pi.\upsilon \\
& \qquad \alpha \neq \beta = s \to t \in \mathfrak{R},\ \exists\, MGU\ for\ \mathfrak{R}|_{(\alpha,R,\pi)}\ and\ \mathfrak{R}|_{(\beta,L,\varepsilon)}, \\
& \qquad \upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\mathfrak{R}) \,\} \\
\cup \; & \{\, (l \to r, R, \tau) \mid \exists (\beta, P, \pi) \in M_{i-1}(\mathfrak{R}),\ root(l) = root(\mathfrak{R}|_{(\beta,P,\pi)}) \in D,\ \tau \in pos(r) \,\} \\
\cup \; & RIGHT_{i-1}(\mathfrak{R})
\end{aligned}
$$

Since $\text{RIGHT}_i(\mathfrak{R})$ also contains the previous $\text{RIGHT}_{i-1}(\mathfrak{R})$ the same holds true as with $\text{LEFT}_i(\mathfrak{R})$ for some $m \in N$:

$$
(2) \qquad \ldots \subseteq \text{RIGHT}_{m-1}(\mathfrak{R}) \subseteq \text{RIGHT}_m(\mathfrak{R}) = \text{RIGHT}_{m+1}(\mathfrak{R}) = \ldots
$$

Let $n, m \in N$ be the smallest for which (1) and (2) hold, then:

$$M(\Re) \quad := \bigcup_{i \in N} M_i(\Re) = LEFT_n(\Re) \cup RIGHT_m(\Re)$$

Now we can define our final set $F(\Re)$, which the algorithm outputs:

$$F(\Re) \quad := M(\Re) \setminus \left(\left\{ (l \to r, L, \tau) \mid root(l|_\tau) \in V \right\} \cup \left\{ (\alpha, P, \tau) \mid root(\Re|_{(\alpha, P, \tau)}) \in C \right\} \right)$$

The two subsets we want to remove are:

- Marked variables on the left-hand side of rules
- Marked constructors on either side

The output of this algorithm should mostly be equivalent to the one presented last week.

There is however one notable change that ,I think, made the output of this version more precise. Namely in determining the data flows.

```
For each τ ∈ pos(term( α, R, ε)) {

    π = getParent(τ);
    //τ = π . τ' where τ' ∈ N (natural numbers)

    τ' = τ - π ;

    For each rule β ∈ A {
        if( ∃ MGU for term( α, R, π) and term( β, L, ε) ){
            For each ω ∈ pos(term( β, L, τ')) {
                if( ( β, L, ω) ∈ F) {
                    add (α, R, τ) to F ;
                    add all sub-locations of (α, R, τ) to F ;
                    flag = true;
                }...}
```

$$\{\ (\alpha, R, \tau) \mid \alpha = l \rightarrow r \in \mathfrak{R},\ \tau \in pos(r),\ \tau = \pi.\upsilon$$
$$\alpha \neq \beta = s \rightarrow t \in \mathfrak{R},\ \exists\, MGU\ for\ \mathfrak{R}|_{(\alpha,R,\pi)}\ and\ \mathfrak{R}|_{(\beta,L,\varepsilon)},$$
$$\upsilon \nparallel \omega \in pos(s),\ \exists\, (\beta, L, \omega) \in LEFT_{i-1}(\mathfrak{R})\ \}$$

I think that what I wrote originally doesn't take into account parallel positions, which could in turn mark a position for no reason. I believe with the explanations provided earlier, that the new definition on the right is better.