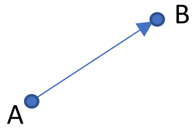
VEKTOROVÁ ALGEBRA

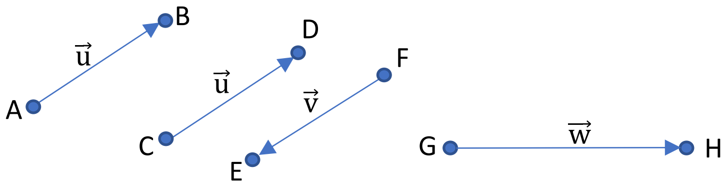
Orientovaná úsečka $\overrightarrow{AB}$



Když A = B -> Nulová orientovaná úsečka $\overrightarrow{AA}$

1 vektor je zástupce nekonečné množiny orientovaných úseček!

Nenulový (volný) vektor



$A[\mathrm{a}_1; \mathrm{a}_2; \ldots; \mathrm{a}_n]$
$B[\mathrm{b}_1; \mathrm{b}_2; \ldots; \mathrm{b}_n]$
$\vec{v} = \overrightarrow{AB} = B - A = (\mathrm{b}_1 - \mathrm{a}_1; \mathrm{b}_2 - \mathrm{a}_2; \ldots; \mathrm{b}_n - \mathrm{a}_n)$

Příklad:

$A[3; 1]$
$B[-1; 2]$

$\vec{v} = \overrightarrow{AB} = B - A = \big((-1) - 3; 2 - 1\big) = (-4; 1)$

Počítání s vektory

$\vec{u} + \vec{v} = (\mathrm{u}_1 + \mathrm{v}_1; \mathrm{u}_2 + \mathrm{v}_2; \ldots; \mathrm{u}_n + \mathrm{v}_n)$
$k \cdot \vec{u} = (k \cdot \mathrm{u}_1; k \cdot \mathrm{u}_2; \ldots; k \cdot \mathrm{u}_n)$
$-\vec{u} = (-\mathrm{u}_1; -\mathrm{u}_2; \ldots; -\mathrm{u}_n)$

Příklad:

$A[0; 0], B[-2; 1], C[4; 4], k = 3, l = -1$

$\overrightarrow{AB} = B - A = (-2 - 0; 1 - 0) = (-2; 1)$
$\overrightarrow{CB} = B - C = (-2 - 4; 1 - 4) = (-6; -3)$
$\overrightarrow{AC} = C - A = (4 - 0; 4 - 0) = (4; 4)$
$\overrightarrow{AB} + \overrightarrow{AC} = (-2 + 4; 1 + 4) = (2; 5)$
$k \cdot \overrightarrow{AB} = (3 \cdot (-2); 3 \cdot 1) = (-6; 3)$
$l \cdot \overrightarrow{CB} = \big(-1 \cdot (-6); -1 \cdot (-3)\big) = (6; 3)$
$k \cdot \overrightarrow{AB} - l \cdot \overrightarrow{CB} = (-6 - 6; 3 - 3) = (-12; 0)$

Lineární kombinace vektorů

$\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$
$c_1, c_2, \dots, c_n \in R$
$\vec{v} = c_1 \cdot \overrightarrow{v_1} + c_2 \cdot \overrightarrow{v_2} + \dots + c_n \cdot \overrightarrow{v_n}$

Příklady:

1)

Zapište $\vec{w}$ jako lineární kombinaci $\vec{u}$ a $\vec{v}$

$\vec{w} = (2; 10), \vec{u} = (1; 3), \vec{v} = (-2; 2)$
$\vec{w} = c_1 \cdot \vec{u} + c_2 \cdot \vec{v}$

$2 = c_1 \cdot 1 + c_2 \cdot (-2)$
$10 = c_1 \cdot 3 + c_2 \cdot 2$

$2 = c_1 - 2c_2 \longrightarrow c_1 = 2 + 2c_2$
$10 = 3c_1 + 2c_2$

$10 = 3(2 + 2c_2) + 2c_2$
$10 = 6 + 6c_2 + 2c_2$
$10 = 6 + 8c_2$
$8c_2 = 4$
$c_2 = \frac{1}{2}$

$c_1 = 2 + 2\left(\frac{1}{2}\right)$
$c_1 = 2 + 1$
$c_1 = 3$

$\vec{w} = 3 \cdot \vec{u} + \frac{1}{2} \cdot \vec{v}$

2)

Zapište $\vec{w}$ jako lineární kombinaci $\vec{u}$, $\vec{v}$ a $\vec{z}$

$\vec{w} = (2; -2; 10), \vec{u} = (2; 1; -1), \vec{v} = (2; 3; 2), \vec{z} = (4; 5; -2)$

$$\begin{array}{lr} 2 & -2 = 1\cdot c_1 + 3\cdot c_2 + 5\cdot c_3 \\ 3 & -10 = -1\cdot c_1 + 2\cdot c_2 - 2\cdot c_3 \\ 1 & 2 = 2\cdot c_1 + 2\cdot c_2 + 4\cdot c_3 \end{array} = \left(\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ -1 & 2 & -2 & -10 \\ 2 & 2 & 4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ -1 & 2 & -2 & -10 \\ 2 & 2 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 5 & 3 & -12 \\ 2 & 2 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 5 & 3 & -12 \\ 0 & -4 & -6 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 5 & 3 & -12 \\ 0 & -2 & -3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 5 & 3 & -12 \\ 0 & 0 & -9 & -9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 5 & 3 & -12 \\ 0 & 0 & -9 & -9 \end{array}\right) = \begin{array}{r} 1\cdot c_1 + 3\cdot c_2 + 5\cdot c_3 = -2 \\ 0\cdot c_1 + 5\cdot c_2 + 3\cdot c_3 = -12 \\ 0\cdot c_1 + 0\cdot c_2 - 9\cdot c_3 = -9 \end{array} \Rightarrow$$

$$\begin{array}{r} c_1 + 3c_2 + 5c_3 = -2 \\ 5c_2 + 3c_3 = -12 \\ -9\cdot c_3 = -9 \end{array} \Rightarrow \mathbf{c_3 = 1} \Rightarrow 5c_2 + 3 = -12 \Rightarrow \mathbf{c_2 = -3} \Rightarrow c_1 - 9 + 5 = -2 \Rightarrow \mathbf{c_1 = 2} \Rightarrow$$

$$\vec{w} = 2\cdot\vec{u} - 3\cdot\vec{v} + \vec{z}$$

3)

Zapište $\vec{u}$ jako lineární kombinaci $\vec{v}$

$\vec{u} = (-5; 10), \vec{v} = (1; -2)$

$\vec{u} = -5\cdot\vec{v}$

4)

Zapište $\vec{0}$ (nulový vektor) jako lineární kombinaci $\vec{u}, \vec{v}$ a $\vec{w}$

$\vec{u} = (3; 6), \vec{v} = (-1; -2), \vec{w} = (1; 4)$

$c_1\cdot\vec{u} + c_2\cdot\vec{v} + c_3\cdot\vec{w} = \vec{0}$

$3c_1 - c_2 + c_3 = 0 \quad \cdot(-2)$

$6c_1 - 2c_2 + 4c_3 = 0 \quad +$

$0c_1 + 0c_2 + 2c_3 = 0$

$2c_3 = 0$

$c_3 = 0$

$$\begin{array}{l} 3c_1 - c_2 + 0 = 0 \\ 3c_1 = c_2 \longrightarrow \end{array} \left\{ \begin{array}{l} c_1 = 1 \\ c_2 = 3 \end{array} \right.$$

$\vec{u} + 3\vec{v} + 0\vec{w} = \vec{0}$

5)

Zapište $\vec{0}$ jako lineární kombinaci $\vec{u}$, $\vec{v}$ a $\vec{w}$

$\vec{u} = (2; -1; 3), \vec{v} = (3; 0; 6), \vec{w} = (7; -5; 10)$

$$\begin{array}{ll} 2 & 0 = -1 \cdot c_1 + 0 \cdot c_2 - 5 \cdot c_3 \\ 1 & 0 = 2 \cdot c_1 + 3 \cdot c_2 + 7 \cdot c_3 \\ 3 & 0 = 3 \cdot c_1 + 6 \cdot c_2 + 10 \cdot c_3 \end{array} = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 3 & 7 \\ 3 & 6 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 2 & 3 & 7 \\ 3 & 6 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 3 & -3 \\ 3 & 6 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 3 & -3 \\ 0 & 6 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 6 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{array}{l} 1 \cdot c_1 + 0 \cdot c_2 + 5 \cdot c_3 = 0 \\ 0 \cdot c_1 + 1 \cdot c_2 - 1 \cdot c_3 = 0 \\ 0 \cdot c_1 + 0 \cdot c_2 + 1 \cdot c_3 = 0 \end{array} \Rightarrow \left. \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{array} \right\}$$

Lineárně nezávislé vektory ⇒
Nelze zapsat lineární kombinací

<u>Velikost vektoru</u>

$\vec{u} = \overrightarrow{AB}$
$|AB| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \ldots + (b_n - a_n)^2}$

<u>Střed vektoru</u>

$$S = \frac{A + B}{2} = \left(\frac{a_1 + b_1}{2}; \frac{a_2 + b_2}{2}; \ldots; \frac{a_n + b_n}{2} \right)$$

<u>Skalární součin vektorů</u>

$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\varphi$
$\varphi \leftarrow \textit{úhel svíraný vektory}, \varphi \epsilon \langle 0;\ \pi \rangle = \langle 0°;\ 180° \rangle$

6)

Spočítejte úhel svíraný vektory $\vec{u}$ a $\vec{v}$

$\vec{u} = (4; -7), \vec{v} = (7; 4)$
$\vec{u} \cdot \vec{v} = 4 \cdot 7 + (-7) \cdot 4 = 0$
$\cos\varphi = 0$
$\varphi = 90° = \frac{\pi}{2}$

7)

Spočítejte úhel svíraný vektory $\vec{u}$ a $\vec{v}$

$\vec{u} = (4; 7; 4), \vec{v} = (3; -5; \sqrt{2})$

$\vec{u} \cdot \vec{v} = 4 \cdot 3 + 7 \cdot (-5) + 4 \cdot \sqrt{2} = 12 - 35 + 4\sqrt{2} = -23 + 4\sqrt{2}$
$|\vec{u}| = \sqrt{4^2 + 7^2 + 4^2} = \sqrt{81} = 9$
$|\vec{v}| = \sqrt{3^2 + (-5)^2 + (\sqrt{2})^2} = \sqrt{9 + 25 + 2} = \sqrt{36} = 6$
$\cos\varphi = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \dfrac{-23 + 4\sqrt{2}}{9 \cdot 6} = \dfrac{-23 + 4\sqrt{2}}{54} \cong -0.3212$
$\varphi \cong 108° \, 44'$

Vektorový součin vektorů

$|\vec{w}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\varphi$
$|\vec{w}| = |\vec{u}| \times |\vec{v}|$

$\vec{u} = (u_1; u_2; u_3)$
$\vec{v} = (v_1; v_2; v_3)$

$|\vec{u}| \times |\vec{v}| = (u_2v_3 - u_3v_2; u_3v_1 - u_1v_3; u_1v_2 - u_2v_1)$

8)

Proveďte vektorový součin vektorů $\vec{u}$ a $\vec{v}$

$\vec{u} = (2; 0; 0), \vec{v} = (0; 3; 0)$
$|\vec{u}| \times |\vec{v}| = (0 - 0; 0 - 0; 6 - 0) = (0; 0; 6)$

Přímky v rovině mohou být zadány

1. Dvěma body
   - A[0;0], B[3;1]
2. Bodem a směrovým vektorem $\overrightarrow{S_p}$
   - A[0;0], $\overrightarrow{S_p} = (3; 1)$
3. Parametrickým tvarem
   - $X = A + t \cdot \overrightarrow{S_p}$
4. Obecným tvarem
   - $ax + bx + c = 0$
5. Směrnicovým tvarem
   - $y = k \cdot x + q, k = \mathrm{tg}\,\varphi$