Heine-Borel, Bolzano Weierstrass, and Examples of Convergence of Functions

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1 Review of Convergence of Functions

Definition 1 (Pointwise convergence) *Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f_n)_{n=1}^{\infty}$ converges pointwise to f on X if we have*

$$\lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0.$$

for all $x \in X$. Equivalently, for all $x \in X$ and all $\epsilon > 0$, there exists N such that $d_Y(f_n(x), f(x)) < \epsilon$ for all $n > N$.

Definition 2 (Uniform convergence) *Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on X if for every $\varepsilon > 0$ there exists $N > 0$ such that*

$$d_Y(f_n(x), f(x)) < \varepsilon$$

for all $n > N$ and all $x \in X$. We call the function f the uniform limit of the functions f_n.

2 Examples of Convergence of Functions

Which of the following sequences of functions are pointwise convergent? Uniformly convergent? If they are convergent, then what are their limits?

- $\{f_n\}$, where $f_n : [0, 1] \to \mathbb{R}$ is defined by $f_n(x) = x^n$.

  Answer: $f_n(x)$ converges pointwise to $f(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}$. It doesn't converge uniformly which can be proved by contradiction. Assume that $\{f_n\}$ converges uniformly. For any of $\epsilon \in (0, 1/2)$, there exists N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0, 1]$ and all $n > N$. For $x \in [0, 1)$, we have $f_n(x) < \epsilon < 1/2$ since $f(x) = 0$. For $x = 1$, we have $f_n(x) > 1 - \epsilon > 1/2$. It means that f_n is not continuous which contradicts the fact that x^n is continuous.

- $\{f_n\}$, where $f_n : [0, 0.99] \to \mathbb{R}$ is defined by $f_n(x) = x^n$.
  Answer: $f_n(x)$ converges pointwise and uniformly to $f(x) = 0$. To prove that it converges uniformly, we show that for any $\epsilon > 0$, there exists an N such that $|f_n(x) - f(x)| = f_n(x) \leq 0.99^n < \epsilon$ for all $n > N$.

- $\{f_n\}$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = x + n$.
  Answer: Diverge.

- $\{f_n\}$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = \frac{1}{n}$.
  Answer: Converge both pointwise and uniformly.

- Let $\triangle(x) = \begin{cases} 0 & x \leq -1 \\ 1 + x & -1 \leq x \leq 0 \\ 1 - x & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$ Consider $\{f_n\}$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = \triangle(x - n)$.
  Answer: Converge pointwise to $f(x) = 0$. It doesn't converge uniformly since there will always be some x with $f_n(x) = 1 > \epsilon$ for all $\epsilon < 1$.

3 Uniform Convergence in Proof of the Independence of Binary Digits

This section is covered on page 147 of ToP.

On page 147, the proof of the independence of binary digits uses the fact that $\sum_{k=1}^{n} x r_k(t) 2^{-k}$ converges uniformly to $x(1 - 2t)$. To show the uniform convergence, we begin by defining $f_n(t) = \sum_{k=1}^{n} x r_k(t) 2^{-k}$ and $f(t) := \sum_{k=1}^{\infty} x r_k(t) 2^{-k}$. $|f(t) - f_n(t)| = |\sum_{k=n+1}^{\infty} x r_k(t) 2^{-k}| \leq |\sum_{k=n+1}^{\infty} x 2^{-k}| \leq |x| \sum_{k=n+1}^{\infty} 2^{-k} = |x| 2^{-n}$ which goes to zero as n goes to infinity.

The proof of the independence of binary digits relies on the uniform convergence to interchange the integral and limit, i.e.,

$$\lim_{n\to\infty} \int_0^1 \exp\left(i f_n(t)\right) dt = \int_0^1 \lim_{n\to\infty} \exp\left(i f_n(t)\right) dt = \int_0^1 \exp\left(i f(t)\right) dt$$

To prove the interchange is permissible, we want to prove that $|\lim_{n\to\infty} \int_0^1 \exp\left(i f_n(t)\right) dt - \int_0^1 \exp\left(i f(t)\right) dt| = 0$. The proof eventually comes to

$$\left| \lim_{n\to\infty} \int_0^1 \exp\left(i f_n(t)\right) dt - \int_0^1 \exp\left(i f(t)\right) dt \right| < \lim_{n\to\infty} \int_0^1 |f_n(t) - f(t)|\, dt$$

By uniform convergence, we know that for any $\epsilon > 0$, there exists N such that $|f_n(t) - f(t)| < \epsilon$ for all $n > N$ and $t \in [0, 1]$. It implies that $\int_0^1 |f_n(t) - f(t)|\, dt < \epsilon$ for any ϵ.

4 Heine Borel Theorem

A **cover** of a subset $\mathbb{I}$ on the real line is a collection $\{\mathbb{U}_n, n \geq 1\}$ of sets whose union contains $\mathbb{I}$. It is an **open cover** if each $\mathbb{U}_n$ is an open interval (a_n, b_n). A **subcover** is a subcollection whose union also contains $\mathbb{I}$. A **finite subcover** contains only a finite number of sets. If every open cover of a set $\mathbb{I}$ has a finite subcover, we say that $\mathbb{I}$ is **compact**.

Theorem 1 ***Heine-Borel Theorem:*** *Suppose $\mathbb{I}$ is a closed and bounded interval. Then $\mathbb{I}$ is compact, i.e. every open cover of $\mathbb{I}$ has a finite subcover.*

Proof sketch: Let $\mathbb{I} = [a, b]$, and $\{\mathbb{U}_n\}$ be an open cover for $\mathbb{I}$. Let

$$\mathbb{A} = \{x \in \mathbb{I} | [a, x] \text{ can be covered by a finite subcollection of } \{\mathbb{U}_n | n \geq 1\}\}$$

Note that $\mathbb{A}$ is nonempty, and bounded above and below by a and b respectively. Then the supremum $c = \sup \mathbb{A} \in [a, b]$ exists. To prove that $\mathbb{I}$ has a finite subcover, we want to show 1) $c \in \mathbb{A}$ and 2) $c = b$.

Since c is in $\mathbb{I}$, it's in some open interval $\mathbb{U}_i = (c - \epsilon, c + \delta) \in \{\mathbb{U}_n\}$. Since c is supremum of $\mathbb{A}$, we know there exists $x \in (c - \epsilon, c)$ such that $x \in \mathbb{A}$. Let denote the finite cover of $[a, x]$ as $\mathbb{U}_1, \cdot, \mathbb{U}_N$. Then, it's clear that $\mathbb{U}_1, \cdot, \mathbb{U}_N, \mathbb{U}_i$ is a finite cover of $[a, c]$, i.e., $c \in \mathbb{A}$.

We assume for contradiction that $c \neq b$. Then, there exists an element $y \in (c, \min(c+\delta, b))$. Then, $\mathbb{U}_1, \cdot, \mathbb{U}_N, \mathbb{U}_i$ is also a finite cover of $[a, y]$, i.e. $y \in \mathbb{A}$. Since $y > c$, it contradicts with the condition that c is the supremum of $\mathbb{A}$. $\blacksquare$

Example 1 *Let $I = [0, 1]$ be a closed and bounded interval in $\mathbb{R}$.*

1. Consider the countable collection of open intervals $\{A_n\}_{n=1}^{\infty}$, where $A_n = \left(\frac{1}{n+1} - 0.1, \frac{1}{n} + 0.1\right)$ for each natural number $n \in \mathbb{N}$. Show that $\{A_n\}_{n=1}^{\infty}$ forms an open cover of the interval $[0, 1]$.

2. Find a finite subcover $\{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\}$ that still covers the interval $[0, 1]$.

Example 2 *Let $I = (0, 1)$.*

1. Is $\{A_n\}_{n=1}^{\infty}$, where $A_n = \left(\frac{1}{n}, 1\right)$ for each $n \in \mathbb{N}$ an open cover of $(0, 1)$? ***Answer:*** *Yes.*

2. Is there a finite subcover? ***Answer:*** *No.*

Example 3 *Consider the set $[0, \infty)$. Is this set closed? Is this set bounded? Now consider the open cover $\{\mathbb{U}_n, n \geq 1\}$ where $\mathbb{U}_n = (-0.01, n)$. Justify that this is a cover. Does there exist a finite sub-cover?*

5 Bolzano-Weierstrass Theorem

Theorem 2 ***Bolzano-Weierstrass Theorem*** *Every bounded sequence has a convergent subsequence.*

Proof sketch: Suppose that $\{x_n\}$ is a bounded sequence. Then there exist some $M > 0$ such that $|x_n| \leq M/2$ for all n. Then at least one of the subintervals $[-M/2, 0]$ and $[0, M/2]$ contains an infinite number of members of the sequence. These members form a subsequence $\{x_{1n}|n \geq 1\}$. This is now a bounded sequence, so we may continue to bisect the interval and find a subsequence contained in one of the halves, labeling the jth such subsequence $\{x_{jn}\}$. The jth subsequence will be in an interval of length $\frac{M}{2^j}$. Then if we define the subsequence $\{y_n\}$ by $y_n = x_{nn}$, we see that $\{y_n\}$ is a subsequence of $\{x_n\}$ and that $\{y_n\}$ is a Cauchy sequence: given any $\epsilon > 0$, there exists N such that $|y_n - y_m| \leq \epsilon$ for $n, m \geq N$. In particular, if we set $N = \log_2(M/\epsilon)$, then for $n, m \geq N$, y_n and y_m are contained in an inteval of length $M/2^{\log_2(M/\epsilon)} = \epsilon$. The sequence $\{y_n\}$ is thus convergent by the completeness of the real numbers.

Example 4 *Let $\{x_n\}$ be the sequence defined by $x_n = (-1)^n + \frac{1}{n}$. Show that the sequence $\{x_n\}$ has a convergent subsequence and find the limit of that subsequence.*
Answer: *It has a convergence subsequence since it's bounded between -2 and +2. The subsequence $\{x_{2k}\}$ converges to 1. The subsequence $\{x_{2k+1}\}$ converges to -1.*

More complete proofs of both Heine-Borel and Bolzano-Weierstrass may be found on page 776 of the text.

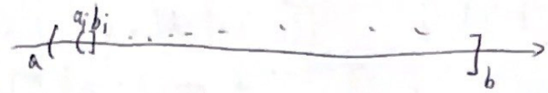
Notation: $\{I_i; i \geq 1\}$ be half closed, disjoint intervals. Siming

$I = \bigcup_{i=1}^{\infty} I_i$, $I_i = (a_i, b_i]$, $I = (a, b]$

WTS $F(I) = \sum_i F(I_i)$ Countable Additivity (ToP 382 Lemma 5. part 1)

Typical proof procedure $\boxed{?}$ ① showing $\sum_i F(I_i) \leq F(I)$. trivial as in ToP 382.

② showing $\sum_i F(I_i) \geq F(I)$.



we have countably infinite $(a_i, b_i]$. and we don't know how to work with it.

WE KNOW "Finite Additivity".

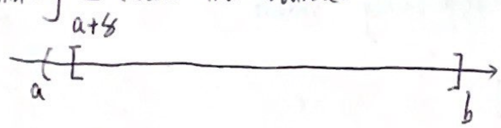
Can we somehow convert the problem so we can exploit finiteness $\boxed{?}$

Heine-Borel Theorem!

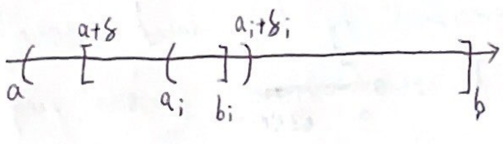
If I^* is compact, every open cover of I_i^* has finite subcover.

The big idea of the proof:

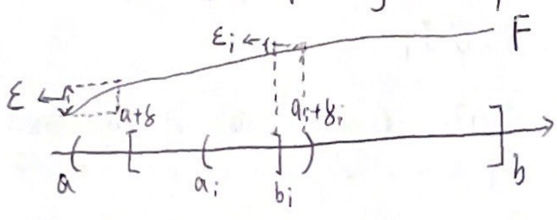
① making I closed and bounded



② making I_i open



③ show the impacts of making I compact and I_i open are arbitrarily small.



ε and ε_i can be arbitrarily small as δ, δ_i are arbitrarily small

due to $\boxed{?}$ right continuity

④ By Heine-Borel Theorem, there is a finite subcover. Then apply finite subadditivity

Formally: ① $(a+\delta, b] \subset [a+\delta, b] \subseteq \bigcup_{i=1}^{n} (a_i, b_i+\delta_i) \subset \bigcup_{i=1}^{n} (a_i, b_i+\delta_i]$.

$\boxed{?}$ Heine-Borel.

② $$F(a+\delta, b] \le F\left(\bigcup_{i=1}^{n} (a_i, b_i+\delta_i]\right) \quad \boxed{?} \text{ monotonicity}$$

$$\le \sum_{i=1}^{n} F(a_i, b_i+\delta_i] \quad \boxed{?} \text{ subadditivity (finite)}$$

$$\cancel{\le \sum_{i=1}^{\infty} F(a_i, b_i+\delta_i] \quad \boxed{?} \text{ positivity}}$$

why not additivity $\boxed{?}$ we expand the set and may not be disjoint.

$$= \sum_{i=1}^{n} F(a_i, b_i] + F(b_i, b_i+\delta_i] \quad \boxed{?} \text{ additivity}$$

$$\le \sum_{i=1}^{\infty} F(a_i, b_i] + \underbrace{\sum_{i=1}^{n} (b_i, b_i+\delta_i]}_{\sum_{i=1}^{n} \varepsilon_i} \quad \boxed{?} \text{ positivity}$$

③ $$F(a, b] = \underbrace{F(a, a+\delta]}_{\varepsilon} + F(a+\delta, b]$$

$$\le \sum_{i=1}^{\infty} F(a_i, b_i] + \varepsilon + \sum_{i=1}^{n} \varepsilon_i$$

Since $\varepsilon + \sum_{i=1}^{n} \varepsilon_i$ can be arbitrarly small,

$$F(a, b] \le \sum_{i=1}^{\infty} F(a_i, b_i].$$ □

Now we show countable additivity of disjoint half close intervals.

We want to extend it to the ring generated by half-closed intervals. (Part 2 ToP 383)

Denote J halfclosed interval and I ~~sa a element~~ a element of the ring.

$$F(I) = \sum_k F(J_k) \quad \boxed{?} \text{ finite additivity; } I = \bigcup_k J_k \text{ by def. of ring.}$$

$$= \sum_k F\left[\bigcup_i (I_i \cap J_k)\right] \quad \boxed{?} \quad I = \bigcup_i I_i$$

$$= \sum_k \sum_i F(I_i \cap J_k) \quad \boxed{?}$$ $I_i \cap J_k$ is a finite union of half closed intervals. By countable additivity of half close interval

$$= \sum_i \sum_k F(I_i \cap J_k)$$

$$= \sum_i F(I_i) \quad \boxed{?} \text{ by finite additivity.}$$

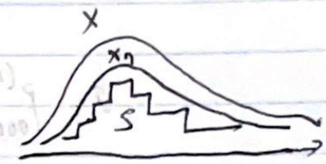
Disjoint . . .

MCT Proof

$0 \le X_n \uparrow X \Rightarrow \lim_{n\to\infty} E(X_n) = E(X)$

① $\lim_{n\to\infty} E(X_n) \le E(X)$

for any X_n dominated by X,
we can find a simple function S dominated by X_n. S is also dominated by X.



$E(X_n) = \sup(E(S) : S \text{ a simple function}, S \le X_n)$
$\le \sup(\quad " \quad " \quad " \quad " \quad X \le X)$
$= E(X)$

② $\lim_{n\to\infty} E(X_n) \ge E(X)$.

Let's pick a simple function S dominated by X.
i.e. $S(w) \le X(w)$.
For any $\varepsilon > 0$, $(1-\varepsilon)S(w) < X(w)$.
There always exists a X_n that dominates $(1-\varepsilon)S(w)$ and is dominated by X: not $\exists\, \delta$ s.t. $\inf\{(1-\varepsilon)S(w) - X(w)\} \ge \delta$.
Since $X_n \uparrow X$, $\exists N$ s.t. $\forall n > N$ $\sup_n\{X_n(w) - X(w)\} \le \delta$.

We now have $\lim_{n\to\infty} E(X_n) \ge E((1-\varepsilon)S) \ge (1-\varepsilon)E(S)$
Since ε can be arbitrarily small, $E(X_n) \ge E(S)$ $\forall S$ simple and $S \le X$.

$\lim_{n\to\infty} E(X_n) \ge \sup\{E(S) : S \text{ is simple}, S \le X\} = E(X)$.

$\lim_{n\to\infty} E(X_n) = E(X)$. □