Practical-01 (Limits & Continuity)

(1) $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

(2) $\lim_{x \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

(3) $\lim_{x \to \frac{\pi}{6}} \left[\frac{\cos x - \sqrt{3} \sin x}{\pi - 6x} \right]$

(4) $\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \right]$

(5) Examine the continuity of the following function of given points

i) $f(x) = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 < x \le \frac{\pi}{2}$

$= \cancel{\frac{x^2-9}{x-3}} \frac{\cos x}{\pi - 2x}$, for $\frac{\pi}{2} < x < \pi$ $\Big\}$ at $x = \frac{\pi}{2}$

ii) $f(x) = \frac{x^2-9}{x-3}$ $\quad 0 < x < 3$

$= x+3$ $\quad 3 \le x < 6$ $\Big\}$ at $x = 3$ & $x = 6$

$= \frac{x^2-9}{x+3}$ $\quad 6 \le x < 9$

6) Find value of k, so that the function $f(x)$ is continuous at the indicated point:-

i) $f(x) = \frac{1-\cos 4x}{x^2}$, $x < 0$

$= k$, $x = 0$ $\Big\}$ at $x = 0$

ii) $f(x) = (\sec^2 x)^{\cot^2 x}$, $x \neq 0$
$= k$, $x = 0$ } at $x = 0$

iii) $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$, $x \neq \frac{\pi}{3}$
$= k$, $x = \frac{\pi}{3}$ } at $x = \frac{\pi}{3}$

3) Discuss the continuity of foll. function? Which of the functions have a removable discontinuity? Redefine the function so as to remove the discontinuity.

i) $f(x) = \dfrac{1 - \cos 2x}{x \tan x}$, $x \neq 0$
$= 9$, $x = 0$ } at $x = 0$

ii) $f(x) = (e^{3x} - 1) \sin x^\circ$, $x \neq 0$
$= \dfrac{\pi}{60}$, $x = 0$ } at $x = 0$

iii) If $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$ for $x \neq 0$ is continuous at $x = 0$, find $f(0)$

iv) If $f(x) = \dfrac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$ for $x \neq \frac{\pi}{2}$ is continuous at $x = \frac{\pi}{2}$. Find $f\left(\frac{\pi}{2}\right)$.

Solutions:-

1) $\lim_{x \to a} \left[\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

∴ Rationalising the denominator & numerator

$\Rightarrow \lim_{x \to a} \left[\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \dfrac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \dfrac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$

$\Rightarrow \lim_{x \to a} \dfrac{(a + 2x - 3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a+x - 4x)(\sqrt{a+2x} + \sqrt{3x})}$

$\Rightarrow \frac{1}{3} \lim_{x \to a} \dfrac{\cancel{(a-x)}(\sqrt{3a+x} + 2\sqrt{x})}{\cancel{(a-x)}(\sqrt{a+2x} + \sqrt{3x})}$

$\Rightarrow \frac{1}{3} \lim_{x \to a} \dfrac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}$

$\Rightarrow \frac{1}{3} \lim_{x \to a} \dfrac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$

$\Rightarrow \frac{1}{3} \times \dfrac{2\sqrt{a} + 2\sqrt{a}}{2\sqrt{3a}}$

$\Rightarrow \frac{1}{3} \times \dfrac{\cancel{4}^{2}\sqrt{a}}{\cancel{2}\sqrt{3a}}$

$\Rightarrow \dfrac{2}{3\sqrt{3}}$

② $\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

∴ Rationalising the numerator

$\lim_{y \to 0} \left(\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right)$

$\Rightarrow \lim_{y \to 0} \left(\frac{(a+y) - a}{y\sqrt{a+y}\,(\sqrt{a+y} + \sqrt{a})} \right)$

$\Rightarrow \lim_{y \to 0} \left(\frac{y}{y\sqrt{a+y}\,(\sqrt{a+y} + \sqrt{a})} \right)$

$\Rightarrow \lim_{y \to 0} \left(\frac{1}{a+y + \sqrt{a+y}\sqrt{a}} \right)$

$\Rightarrow \left(\frac{1}{a+0 + \sqrt{a+0} \cdot \sqrt{a}} \right)$

$\Rightarrow \left(\frac{1}{a + \sqrt{a} \cdot \sqrt{a}} \right)$

$\Rightarrow \left(\frac{1}{\cancel{a} + a} \right)$

$\Rightarrow \frac{1}{2a}$

③ $\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3} \sin x}{\pi - 6x}$

∴ By substituting $x - \frac{\pi}{6} = h \Rightarrow x = h + \frac{\pi}{6}$

where $h \to 0$

$\Rightarrow \lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{6}\right) - \sqrt{3} \sin\left(h + \frac{\pi}{6}\right)}{\pi - 6\left(h + \frac{\pi}{6}\right)}$

$\Rightarrow \lim_{h \to 0} \frac{\cos h \cdot \cos\frac{\pi}{6} - \sin h \cdot \sin\frac{\pi}{6} - \sqrt{3} \sin h \cdot \cos\frac{\pi}{6} + \cos h \cdot \sin\frac{\pi}{6}}{\pi - \cancel{6}\left(\frac{6h + \pi}{\cancel{6}}\right)}$

: Using $\Rightarrow \cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \cdot \sin B$

$\Rightarrow \lim_{h \to 0} \frac{\cos h \cdot \frac{\sqrt{3}}{2} - \sin h \cdot \frac{1}{2} - \sqrt{3}\left(\sin h \times \frac{\sqrt{3}}{2} + \cos h \cdot \frac{1}{2}\right)}{\cancel{\pi} - 6h - \cancel{\pi}}$

$\Rightarrow \lim_{h \to 0} \frac{\cancel{\cosh \cdot \frac{\sqrt{3}}{2}} - \sin\frac{h}{2} - \sin\frac{3h}{2} \cancel{-} \cancel{\cosh\frac{\sqrt{3}}{2}}}{-6h}$

$\Rightarrow \lim_{h \to 0} \frac{\cancel{-}\frac{\sin 4h}{2}}{\cancel{-}6h} \Rightarrow \lim_{h \to 0} \frac{\sin 4h}{\cancel{12}h_3} \Rightarrow \frac{1}{3} \lim \frac{\sin h}{h}$

$\Rightarrow \frac{1}{3} \times 1 = \frac{1}{3}$

(4) $\lim_{x\to\infty} \left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$

∴ Rationalising the numerator & denominator

$\Rightarrow \lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}} \times \frac{\sqrt{x^2+5}+\sqrt{x^2-3}}{\sqrt{x^2+5}+\sqrt{x^2-3}} \times \frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}+\sqrt{x^2+1}}\right]$

$\Rightarrow \lim_{x\to\infty}\left[\frac{(\cancel{x^2}+5-\cancel{x^2}+3)(\sqrt{x^2+3}+\sqrt{x^2+1})}{(\cancel{x^2}+3-x^2-1)(\sqrt{x^2+5}+\sqrt{x^2-3})}\right]$

$\Rightarrow \lim_{x\to\infty} \frac{\cancel{8}\,4\,(\sqrt{x^2+3}+\sqrt{x^2+1})}{\cancel{2}\,(\sqrt{x^2+5}+\sqrt{x^2-3})}$

$\Rightarrow 4\lim_{x\to\infty} \frac{\left(\sqrt{x^2\left(1+\frac{3}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}\right)}{\left(\sqrt{x^2\left(1+\frac{5}{x^2}\right)}+\sqrt{x^2\left(1-\frac{3}{x^2}\right)}\right)}$

After applying limit we get $= \underline{\underline{4}}$

Q2) i) $f(x) = \left.\begin{cases} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}, & \text{for } 0 < x \le \pi/2 \\ \dfrac{\cos x}{\pi - 2x}, & \text{for } \frac{\pi}{2} < x < \pi \end{cases}\right\}$ at $x = \pi/2$

$\Rightarrow f(\pi/2) = \frac{\sin 2(\pi/2)}{\sqrt{1-\cos 2(\pi/2)}}$

f at $x = \pi/2$ define

RHL :-

$\Rightarrow \lim_{x\to\pi/2^+} f(x) = \frac{\cos x}{\pi - 2x}$

By substitution method

$x - \pi/2 = h$

$x = h + \pi/2$

where $h \to 0$

$\Rightarrow \lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi - 2\left(h+\frac{\pi}{2}\right)}$

$\Rightarrow \lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi - \cancel{2}\left(\frac{2h+\pi}{\cancel{2}}\right)}$

$\Rightarrow \lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{-2h}$

$\Rightarrow \lim_{h\to 0} \frac{\left(\cos h \cdot \cos\frac{\pi}{2} - \sin h \cdot \sin\frac{\pi}{2}\right)}{-2h}$: Using $\cos(A+B)$

$\Rightarrow \lim_{h\to 0} \frac{\cancel{\cosh \cdot 0} - \sin h}{-2h}$

$\Rightarrow \lim_{h\to 0} \frac{\cancel{-}\sin h}{\cancel{+}2h}$

$\Rightarrow \frac{1}{2}\lim_{h\to 0} \frac{\sin h}{h} \Rightarrow \underline{\underline{\frac{1}{2}}}$

: L.H.D:-

$\Rightarrow \lim_{x \to \pi/2^-} = f(x) = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$: Using $\sin 2x = 2\sin x \cdot \cos x$

$\lim_{x \to \pi/2^-} \frac{2\sin x \cos x}{\sqrt{2\sin^2 x}}$

$\Rightarrow \lim_{x \to \pi/2^-} \frac{2\sin x \cos x}{\sqrt{2}\sin x}$

$\Rightarrow \lim_{x \to \pi/2^-} \frac{2\cos x}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} \lim_{x \to \pi/2^-} \cos x$

$\therefore$ L.H.D $\neq$ R.H.D

$\therefore f$ is not continuous at $x = \pi/2$

ii) $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & 0 \le x < 3 \\ = x+3, & 3 \le x \le 6 \\ = \frac{x^2-9}{x+3}, & 6 \le x < 9 \end{cases}$ at $x = 3$ & $x = 6$

at $x = 3$

$\Rightarrow f(3) = \frac{x^2-9}{x-3} = 0$

f at $x = 3$ define.

R.H.D

$\Rightarrow \lim_{x \to 3^+} = f(3) = x+3 = 6$

$\lim_{x \to 3^-} = f(3) = \frac{x^2-9}{x-3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} = (x+3) = 6$

$\therefore$ L.H.D = R.H.D for $x = 3$

$\therefore f$ is continuous at $x = 3$

$\Rightarrow$ For $x = 6$

$\lim_{x \to 6^+} = f(6) = x+3 = 9$

$\lim_{x \to 6^-} = f(6) = \frac{x^2-9}{x-3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} = x+3 = 9$

$\therefore$ L.H.D = R.H.D for $x = 6$

$\therefore f$ is continuous at $x = 6$.

iii) $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & x < 0 \\ = k & x = 0 \end{cases}$ at $x = 0$

Solⁿ: f is continuous at $x = 0$

$\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = k \Rightarrow \lim_{x \to 0} \frac{2\sin^2 2x}{x^2} = k$

18.

$\Rightarrow 2\lim_{x\to 0} \frac{\sin^2 2x}{x^2} = k$

$\Rightarrow 2\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2 = k$

$\Rightarrow 2(2)^2 = k$

$\Rightarrow \boxed{k = 8}$

ii) $f(x) = \left.\begin{matrix} (\sec^2 x)^{\cot^2 x} & x \neq 0 \\ = k & x = 0 \end{matrix}\right\}$ at $x = 0$

Sol^n $f(x) = (\sec^2 x)^{\cot^2 x}$

Using: $\tan^2 x - \sec^2 x = 1$

$\therefore \sec^2 x = 1 + \tan^2 x$

$\cot^2 x = \frac{1}{\tan^2 x}$

$\therefore \lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$

We know, that

$\lim_{x\to 0} (1+px)^{1/px} = e$

$\therefore = e$

$k = e$

iii) $f(x) = \left.\begin{matrix} \frac{\sqrt{3} - \tan x}{\pi - 3x} & x \neq \pi/3 \\ = k & x = \pi/3 \end{matrix}\right\}$ at $x = \pi/3$

$x - \pi/3 = h$

at $x = h + \pi/3$

when $h \to 0$

$f\left(\frac{\pi}{3} + h\right) = \frac{\sqrt{3} - \tan\left(\frac{\pi}{3} + h\right)}{\pi - 3\left(\frac{\pi}{3} + h\right)}$

$\lim_{h\to 0} = \frac{\sqrt{3} - \tan\left(\frac{\pi}{3} + h\right)}{\pi - \cancel{3}\left(\frac{\pi}{\cancel{3}} + h\right)}$

$\lim_{h\to 0} \frac{\sqrt{3} - \frac{\tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3}\cdot\tan h}}{\cancel{\pi} - \cancel{\pi} + 3h}$: Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\cdot\tan B}$

$\lim_{h\to 0} \frac{\frac{\sqrt{3}\left(1 - \tan\frac{\pi}{3}\cdot\tan h\right) - \left(\tan\frac{\pi}{3} + \tan h\right)}{1 - \tan\frac{\pi}{3}\cdot\tan h}}{-3h}$

$\lim_{h\to 0} \left(\frac{\frac{\sqrt{3} - \sqrt{3}\times\sqrt{3}\cdot\tan h - (\sqrt{3} + \tan h)}{1 - \sqrt{3}\cdot\tan h}}{-3h}\right)$

$\lim_{h\to 0} \frac{\left(\cancel{\sqrt{3}} - 3\tan h - \cancel{\sqrt{3}} - \tan h\right)}{\frac{1 - \sqrt{3}\tan h}{-3h}}$

$\Rightarrow \lim_{h\to 0} \frac{-4\tan h}{3h(1-\sqrt{3}\tan h)}$

$\Rightarrow 4 \lim_{3h\to 0} \frac{4\cancel{\tan h}}{3\cancel{h}} \frac{\tan h}{h} \lim_{h\to 0} \frac{1}{(1-\sqrt{3}\tan h)} \left(\frac{\tan h}{h}\right)$

$= \frac{4}{3} \times \frac{1}{(1-\sqrt{3}(0))}$

$= \frac{4}{3} \times \frac{1}{1} = \frac{4}{3}$

(7) $f(x) = \frac{1-\cos 3x}{x\tan x}$; $x \neq 0$
$= 9$; $x = 0$ } at $x = 0$

$f(x) = \frac{1-\cos 3x}{x\tan x}$

$\lim_{x\to 0} = \frac{2\sin^2 3/2x}{x\tan x}$

$\lim_{x\to 0} = \frac{\frac{2\sin^2 \frac{3x}{2}}{x^2} \times x^2}{\frac{x \cdot \tan x}{x^2} \times x^2}$

$= 2 \lim_{x\to 0} \frac{\left(\frac{3}{2}\right)^2}{2} = \cancel{2} \times \frac{9}{4_2} = \frac{9}{2}$

$\Rightarrow \lim_{x\to 0} f(x) = 9/2 \quad 9 = f(0)$

$\therefore f$ is not continuous at $x = 0$

$\therefore$ Redefine function

$f(x) = \begin{cases} \frac{1-\cos 3x}{x\tan x} & x \neq 0 \\ 9/2 & x = 0 \end{cases}$

Now $\lim_{x\to 0} f(x) = f(0)$

f has removable discontinuity at $x = 0$

7) ii) $f(x) = \frac{(e^{3x}-1)\sin x^\circ}{x^2}$; $x \neq 0$
$= \pi/6$; $x = 0$ } $x = 0$

$\Rightarrow \lim_{x\to 0} \frac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$

$\Rightarrow \lim_{x\to 0} \frac{e^{3x}-1}{x} \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$

$\Rightarrow 3 \lim_{x\to 0} \frac{e^{3x}-1}{3x} \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$

$\Rightarrow \cancel{3}\log e \frac{\pi}{\cancel{180}_{60}} = \frac{\pi}{60} = f(0)$

$\therefore f$ is continuous at $x = 0$

8) $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$; $x = 0$

is continuous at $x = 0$

$\therefore$ Given, f is continuous at $x = 0$

$\lim_{x \to 0} f(x) = f(0)$

$= \lim_{x \to 0} \dfrac{e^{x^2} - \cos x}{x^2} = f(0)$

$= \lim_{x \to 0} \dfrac{e^{x^2} - \cos x - 1 + 1}{x^2}$

$= \lim_{x \to 0} \dfrac{(e^{x^2} - 1) + (1 - \cos x)}{x^2}$

$= \lim_{x \to 0} \dfrac{e^{x^2} - 1}{x^2} + \lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$

$\log e + \lim_{x \to 0} \dfrac{2 \sin^2 x/2}{x^2}$

$\log e + 2 \lim_{x \to 0} \left(\dfrac{\sin x/2}{x}\right)^2$

~~Multiply with 2 on Num & deno.~~

$= 1 + 2 \times \dfrac{1}{4/2} = \dfrac{3}{2} = f(0)$

9) $f(x) = \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$ $\quad x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$\lim_{x \to 0} f(x) = f(0)$

$\Rightarrow \lim_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$

$\Rightarrow \lim_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} \times \dfrac{\sqrt{2} + \sqrt{1 + \sin x}}{\sqrt{2} + \sqrt{1 + \sin x}}$

$\Rightarrow \lim_{x \to 0} \dfrac{2 - 1 + \sin x}{\cos^2 x (\sqrt{2} + \sqrt{1 + \sin x})}$

$\Rightarrow \lim_{x \to \pi/2} \dfrac{\cancel{(1 + \sin x)}}{(1 - \sin x)\cancel{(1 + \sin x)}(\sqrt{2} + \sqrt{1 + \sin x})}$

$\Rightarrow \lim_{x \to \pi/2} \dfrac{1}{(1 - \sin x)(\sqrt{2} + \sqrt{1 + \sin x})}$

$= \dfrac{1}{2(\sqrt{2} + \sqrt{2})}$

$= \dfrac{1}{2(2\sqrt{2})} = \dfrac{1}{4\sqrt{2}}$

$= f(\pi/2) = \dfrac{1}{4\sqrt{2}}$

AV 22/01/2020

E.3 # Practical - 2

Topic :- Limits.

Q1) Show that the foll. function defined from $\mathbb{R}$ to $\mathbb{R}$ is differentiable:-

i) $\cot x$ ii) $\operatorname{cosec} x$ iii) $\sec x$

Q2) If $f(x) = 4x+1 \; ; x \leq 2$
$= x^2+5 \; , x > 2$
at $x=2$, then find f is diff or not?

Q3) If $f(x) = 4x+7 \; , x < 3$
$= x^2+3x+1 \; , x \geq 3$
at $x=3$, then find f is diffrentiable or not

Q4) If $f(x) = 8x-5 \; , x \leq 2$
$= 3x^2-4x+7 \; , x > 2$
at $x=2$, then find f is diffrentiable or not

⇒ Solutions:

i) $f(x) = \cot x$

$\because \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$

$\therefore$ As per given condition

$\Rightarrow \lim_{x \to a} \frac{\cot x - \cot a}{x-a}$

$\Rightarrow \lim_{x \to a} \frac{\frac{\cos x}{\sin x} - \frac{\cos a}{\sin a}}{x-a}$

$\Rightarrow \lim_{x \to a} \frac{\frac{\cos x \cdot \sin a - \cos a \cdot \sin x}{\sin x \cdot \sin a}}{x-a}$

$\Rightarrow \lim_{x \to a} \frac{\sin(x-a)}{x-a} \times \frac{1}{\sin x \cdot \sin a}$

$\Rightarrow$ Let $x - a = h$
$x = h+a$
when $x \to a$ $h \to 2a$

$\lim_{h \to 0} \cancel{\frac{\sin h}{h}} \to 1 \times \operatorname{cosec} x \cdot \operatorname{cosec} a$

$\Rightarrow \operatorname{cosec}(a+h) \cdot \operatorname{cosec}(a)$

$\Rightarrow \operatorname{cosec}(a+0) \cdot \operatorname{cosec}(a)$

$\Rightarrow \operatorname{cosec}^2 a$

ii) $f(x) = \operatorname{cosec} x$

$\therefore$ As per given condition

$\therefore \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$

$\therefore \lim_{x \to a} \dfrac{\dfrac{1}{\sin x} - \dfrac{1}{\sin a}}{x-a}$

$\therefore \lim_{x \to a} \dfrac{\dfrac{\sin a - \sin x}{\sin x \cdot \sin a}}{x-a}$

$\therefore \lim_{x \to a} \dfrac{2\cos\left(\frac{a+x}{2}\right) \cdot \sin\left(\frac{a-x}{2}\right)}{(x-a)} \times \dfrac{1}{\sin x \cdot \sin a}$

$\therefore -2 \lim_{x \to a} \dfrac{\cos\left(\frac{x+a}{2}\right) \cdot \sin\left(\frac{x+a}{2}\right)}{x-a} \times \dfrac{1}{\sin a \cdot \sin x}$

$\therefore x-a=h \quad x=h+a$

when $x \to a$ then $h \to 0$

$\therefore 2 \lim_{h \to 0} \dfrac{\cos\left(\frac{x+a}{2}\right) \cdot \sin\left(\frac{a-x}{2}\right)}{x-a} \times \dfrac{1}{\sin a \cdot \sin x}$

$\therefore 2 \lim_{h \to 0} \dfrac{\cos\left(\frac{h+2a}{\cancel{2}}\right) \cdot \sin\left(\frac{-h}{2}\right)}{h}$

$\Rightarrow \lim_{h \to 0} \dfrac{\cos\left(\frac{h+2a}{\cancel{2}}\right)}{h/2} \cdot \dfrac{\sin\left(-h/2\right)}{h/2} \times \operatorname{cosec} x \cdot \operatorname{cosec} a$

$\Rightarrow -1 \times \cos\left(\frac{h+2a}{2}\right) \times \operatorname{cosec} a \cdot \operatorname{cosec}(a+h)$

$\Rightarrow -\cos a \times \dfrac{1}{\sin a} \times \operatorname{cosec} a$

$\Rightarrow -\cot a \times \operatorname{cosec} a$

iii) $\sec x$

$f(x) = \sec x$

$\lim_{x \to a} \dfrac{\sec x - \sec a}{x-a}$

$\lim_{x \to a} \dfrac{\dfrac{1}{\cos x} - \dfrac{1}{\cos a}}{x-a}$

$\lim_{x \to a} \dfrac{\dfrac{\cos a - \cos x}{\cos x \cdot \cos a}}{x-a}$

$\lim_{x \to a} \dfrac{\cos a - \cos x}{x-a} \times \dfrac{1}{\cos x \cdot \cos a}$

$\lim_{x \to a} \dfrac{2\sin\left(\frac{x+a}{2}\right) \cdot \sin\left(\frac{x-a}{2}\right)}{x-a} \times \sec x \cdot \sec a$

$2\lim_{x \to a} \dfrac{\sin\left(\frac{x+a}{2}\right) \cdot \sin\left(\frac{x-a}{2}\right)}{(x-a)}$

$\therefore x-a=h, \quad x \to a \quad h \to 0$

$\therefore \lim_{h \to 0} \dfrac{\sin\left(\frac{h+2a}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{h/2} \cdot \sec(a+h) \cdot \sec a$

$\Rightarrow \dfrac{\sin(h/2)}{h/2} \cdot \dfrac{\sin\left(\frac{h+2a}{2}\right)}{h/2} \cdot \sec(a+0)\ldots$

$\Rightarrow 1 \times \dfrac{\sin\left(\frac{2a}{2}\right)}{h/2 \to 0} \times \sec^2 a$

$\Rightarrow \sin a \times \sec^2 a$

Q) $f(x) = 4x+9 \quad , x \le 2$

$\quad = x^2+5 \quad , x > 2$

at $x = 2$

$\therefore$ LHD $= \dfrac{(4x+9)-(4a+9)}{x-a}$

$\lim\limits_{x\to 2^-} = \dfrac{4x+\cancel{9}-4a-\cancel{9}}{x-a}$

$\lim\limits_{x\to 2^-} \dfrac{4.\cancel{(x-a)}}{\cancel{(x-a)}}$

$\Rightarrow 4$

$\therefore$ RHD $= \lim\limits_{x\to 2^+} \dfrac{x^2+\cancel{5}-a^2-\cancel{5}}{x-a}$

$\lim\limits_{x\to 2^+} \dfrac{\cancel{(x-a)}(x+a)}{\cancel{(x-a)}}$

$\lim\limits_{x\to 2^+} x+a$

$2+2$

$= 4$

$\therefore$ LHD = RHD

$\therefore f$ is continuous

Q3) $f(x) = 4x+7 \quad , x < 3$

$\quad = x^2+3x+7 \quad , x \ge 3$

$\therefore$ LHD $= \dfrac{4x+\cancel{7}-4a-\cancel{7}}{x-a} = \lim\limits_{x\to 3^-}$

$\lim\limits_{x\to 3^-} = \dfrac{4\cancel{(x-a)}}{\cancel{(x-a)}}$

$= 4$

RHD

$\lim\limits_{x\to 3^+} = \dfrac{x^2+3x+\cancel{7}-a^2-3a-\cancel{7}}{x-a}$

$= \dfrac{x^2+3x-(3)^2-3(3)}{x-3}$

$= \dfrac{x^2+3x-9-9}{x-3}$

$= \dfrac{x^2+3x-18}{x-3}$

$= \dfrac{x(x+6)-3(x+6)}{(x-3)}$

$= \dfrac{\cancel{(x-3)}(x+6)}{\cancel{(x-3)}}$

$\lim\limits_{x\to 3^+} = x+6$

$= 3+6$

$= 9$

$\therefore$ L.H.D $\neq$ R.H.D

$\therefore f$ is not continuous at $x = 3$

Q4) $f(x) = 8x-5 \quad , x \le 2$
$= 3x^2-4x+7 \quad , x > 2$

$\therefore$ LHD $= \lim_{x \to 2^-} \frac{8x-5-8(2)-5}{x-2}$

$\lim_{x \to 2^-} = \frac{8x-16}{x-2}$

$= \frac{8(x-2)}{x-2}$

$= 8$

RHD $= \frac{3x^2-4x+\cancel{7}-3(2)^2+4(2)-\cancel{7}}{x-2}$

$= \frac{3x^2-4x-4}{x-2}$

$= \frac{3x^2-6x+2x-4}{x-2}$

$= \frac{3x(x-2)+2(x-2)}{(x-2)}$

$= \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

$= 3x+2$

$= 3(2)+2$

$= 6+2$

$= 8$

$\therefore$ L.H.D $=$ R.H.D

$\therefore f$ is continuous & diff at $x = 2$.

11/12/19

Practical - 3

Topic :- Application of Derivatives

(1) Find the intervals in which function is increasing or decreasing

i) $f(x) = x^3-5x-11$ ii) $f(x) = x^2-4x$

iii) $f(x) = 2x^3+x^2-20x+4$

iv) $f(x) = x^3-27x+5$

v) $f(x) = 69-24x-9x^2+2x^3$

(2) Find intervals in which function is concave upwards or concave downwards:

i) $y = 3x^2-2x^3$

ii) $y = x^4-6x^3+12x^2+5x+7$

iii) $y = x^3-27x+5$

iv) $y = 69-24x-9x^2+2x^3$

v) $y = 2x^3+x^2-20x+4$

→ Solutions:

Q1)

i) $f(x) = x^3-5x-11$

$f'(x) = 3x^2-5$

$\therefore$ for increasing function

$f'(x) > 0$

$3x^2-5 > 0$

$3x^2 > 5 \Rightarrow x^2 > \frac{5}{3} \quad x > \pm\sqrt{\frac{5}{3}}$

$x \in (-\infty, -\sqrt{5/3}) \cup (+\sqrt{5/3}, \infty)$

For decreasing function

$\cancel{x^2}\ f'(x) < 0$

$x^2 < \pm\sqrt{\frac{5}{3}}$

$x < -\sqrt{5/3}$ $\quad x > +\sqrt{5/3}$

$x \in \left(-\sqrt{5/3}, \sqrt{5/3}\right)$

ii) $f(x) = x^2 - 4x$

$\therefore f'(x) = 2x - 4$

$\therefore$ For increasing function

$f'(x) > 0$

$2x - 4 > 0$

$2x > 4_2$

$x > 2$

$\therefore x \in (2, \infty)$

$\therefore$ For decreasing function

$f'(x) < 0$

$2x - 4 < 0$

$2x < 4_2$

$x < 2$

$\therefore x \in (-\infty, 2)$

iii) $f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\therefore$ for increasing function

$\rightarrow f'(x) > 0$

$\therefore 6x^2 + 2x - 20 > 0$

$\therefore 3x^2 + x - 10 > 0$

$\therefore 3x^2 + 6x - 5x - 10 > 0$

$\therefore 3x(x+2) - 5(x+2) > 0$

$\therefore (3x - 5)(x+2) > 0$

$\therefore x > \frac{5}{3}, -2$

Case II:

$3x - 5 < 0$ and $x + 2 < 0$

$x < \frac{5}{3}$ and $x < -2$

$\therefore f(x)$ is increasing in the interval $x \in (2/3, \infty) \cup (-\infty, -2)$

$\therefore$ for decreasing function

$\rightarrow f'(x) < 0$

$\therefore 6x^2 - 2x - 20 < 0$

$\therefore 3x^2 - x - 10 < 0$

$\therefore 3x^2 + 6x - 5x - 10 < 0$

$\therefore 3x(x+2) - 5(x+2) < 0$

$\therefore (3x - 5)(x+2) < 0$

$x < \frac{5}{3}$ & $x < -2$

$\therefore$ Case-II: $3x-5>0$ & $x+2<0$

$3x>5$ $x<-2$

$x>\frac{5}{3}$

$\therefore$ Case-II is not possible

$\therefore f(x)$ is decreasing in interval $(-2, 5/3)$.

iii) $f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$\therefore$ for increasing function

$\rightarrow f'(x) > 0$

$\therefore 3x^2 - 27 > 0$

$\therefore 3x^2 > 27$

$\therefore x^2 > \frac{27}{3}$

$\therefore x > \sqrt{9}$

$x<-3$ & $x>3$

$\therefore f(x)$ is increasing in the interval $x \in (-\infty, -3) \cup (3, \infty)$

$\therefore$ for decreasing funct

$f'(x) < 0$

$3x^2 - 27 < 0$

$3x^2 < 27$

$x^2 < \frac{27}{3}$

$x^2 < 9 \Rightarrow x < \pm 9$ $-3<x<3$

$\therefore f(x)$ is decreasing in the interval $(-3, 3)$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

$f'(x) = 6x^2 - 18x - 24$

$\therefore$ for increasing functn

$f'(x) > 0$

$6x^2 - 18x - 24 > 0$

$x^2 - 3x - 4 > 0$

$x^2 - 4x + x - 4 > 0$

$x(x-4) + 1(x-4) > 0$

$(x+1)(x-4) > 0$

Case I: $x+1>0$ & $x-4>0$

$x>-1$ $x>4$

$x>-1$

Case II, $x-4<0$ $x+1<0$

$\cancel{x<4}$ $x<-1$

$x<-1$

$\therefore f(x)$ is increasing in the interval $x \in (-\infty, -1) \cup (4, \infty)$

$\therefore$ for decreasing

$f'(x) < 0$

$6x^2 - 18x - 24 < 0$

$x^2 - 3x - 4 < 0$

$(x-4)(x+1) < 0$

Case-I: $x-4<0$ and $x+1>0$
$x<4$ $\quad x>-1$
$x \in (-1, 4)$

Case-II: $x-4>0$ and $x+1<0$
$x>4$ $\quad x<-1$
$\therefore$ Case II is invalid

$f(x)$ is decreasing in the interval $x \in (-1, 4)$

Q2) Check the concavity of equation for its respective intervals:-

i) $y = 3x^2 - 2x^3$
$y' = 6x - 6x^2$
$y'' = 6 - 12x$

$\therefore$ For concave upwards
$y'' > 0$
$6-12x > 0$
$6 > 12x$
$x < \frac{1}{2}$

y is concave upwards for $x \in (-\infty, 1/2)$

$\therefore$ for concave downwards
$y'' < 0$
$6-12x < 0$
$6 < 12x$
$x > \frac{1}{2}$

$\therefore$ y is concave downwards for $x \in (1/2, \infty)$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$
$\therefore y' = 4x^3 - 18x^2 + 24x + 5$
$\therefore y'' = 12x^2 - 36x + 24$
$\therefore$ for concave upwards
$y'' > 0$
$12x^2 - 36x + 24 > 0$
$x^2 - 3x + 2 > 0$
$x^2 - 2x - x + 2 > 0$
$x(x-2) - 1(x-2) > 0$
$(x-1)(x-2) > 0$

Cas I: $x-2>0$ & $x-1>0$
$x>2$ $\quad x>1$
$x>2$

Cas II: $x-2<0$ $\quad x-1<0$
$x<2$ $\quad x<1$
$x<1$.

∴ y is concave upwards for $x \in (-\infty, 1) \cup (2, \infty)$

∴ for concave downwards

∴ $y'' < 0$

$12x^2 - 36x + 24 < 0$

$x^2 - 3x + 2 < 0$

$x^2 - 2x - x + 2 < 0$

$x(x-2) - 1(x-2) < 0$

$(x-2)(x-1) < 0$

Case I: $x - 2 < 0$ and $x - 1 > 0$

$x < 2$ $x > 1$

$1 < x < 2$

$x \in (1, 2)$

Case II: $x - 2 > 0$ and $x - 1 < 0$

$x > 2$ $x < 1$

Case 2 is not valid

∴ y is concave downwards for $x \in (1, 2)$

iii) $y = x^3 - 27x + 5$

$y' = 3x^2 - 27$

$y'' = 6x$

∴ for concave upwards

$y'' > 0$

$6x > 0$

$x > 0$

∴ ~~f(x)~~ y is concave upwards for $x \in (0, \infty)$

∴ for concave downwards

$y'' < 0$

$6x < 0$

$x < 0$

y is concave downwards for $x \in (-\infty, 0)$

iii) $y = 69 - 24x - 9x^2 + 2x^3$

$y' = 6x^2 - 18x - 24$

$y'' = 12x - 18$

for concave upwards

$y'' > 0$

$12x - 18 > 0$

$12x > 18$

$x > \frac{18}{12} = \frac{3}{2}$

∴ $f(x)$ is concave upwards for $x \in (3/2, \infty)$

for concave downwards

$y'' < 0$

$12x - \cancel{18} < 0$

$12x < 18$

$x < \frac{\cancel{18}\,3}{\cancel{12}\,2}$

$\therefore y$ is concave downwards for $x \in (-\infty, 3/2)$

(v) $f\; y = 2x^3 + x^2 - 20x + 5$

$y' = 6x^2 + 2x - 20$

$y'' = 12x + 2$

$\therefore$ for concave upwards

$y'' > 0$

$12x + 2 > 0$

$12x > -2$

$x > \frac{-2}{12}$

$x > -1/6$

$\therefore y$ is concave ~~upwards~~ for $x \in (-1/6, \infty)$ ''

$\therefore$ for concave downwards

$y'' < 0$

$12x + 2 < 0$

$12x < -2$

$x < -1/6$

18/12/19

$\therefore y$ is concave downwards for $x \in (-1/6, -\infty)$ ~~$(-\infty, -1/6)$~~

Practical 4

→ Find the maximum & minimum:-

i) $f(x) = x^2 + \frac{16}{x^2}$

ii) $f(x) = 3 - 5x^3 + 3x^5$

iii) $f(x) = x^3 - 3x^2 + 1$ in $[-1/2, 4]$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

Q) Find the roots of the following equations using the Newton's Method:

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$

iii) $f(x) = x^3 - 18x^2 - 10x + 17$ in $[1, 2]$

Q1) i) Solutions:

$f(x) = x^2 + \frac{16}{x^2}$

$\therefore f'(x) = 2x - \frac{32}{x^3}$

For maxima / minima

$f'(x) = 0$

$\therefore 2x - \frac{32}{x^3} = 0 \Rightarrow 2x = \frac{32}{x^3} \Rightarrow x^4 = 16$

$\boxed{x = \pm 2}$

$\therefore f''(x) = 2 + \frac{96}{x^4}$

$\therefore f''(2) = f''(-2) = 2 + \frac{96}{(\pm 2)^4} = 2 + \frac{96}{16} = 8 > 0$

$\therefore f(x)$ is maximum at $x = \pm 2$

$f''(2) = 8$ is the minimum value

Q2) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = -15x^2 - 15x^4$

$f''(x) = -60x + 30x^3$

For Maxima / Minima

$f'(x) = 15x^4 - 15x^6 = 0$

$x^4 - x^6 = 0$

$x^4(1 - x^2) = 0$

$x = 0, -1, +1$

$\therefore f''(x) = +60x^3 - 30x^2$

$f''(0) = 0$

$f''(-1) = -60 + 30 = -30 < 0$

$f''(1) = 60 - 30 = 30 > 0$

$\therefore f(x)$ is maximum at -1 & minimum at $+1$

$f(-1) = 3 + 5 - 3 = \underline{5}$

$f(1) = 3 - 5 + 3 = \underline{1}$

iii) $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

$\therefore$ For maxima & minima

$f'(x) = 0$

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x(x - 2) = 0$

$\boxed{x = 0, 2}$

$f''(x) = 6x - 6$

$f''(0) = 6(0) - 6 = -6$

$f''(2) = 6(2) - 6 = 12 - 6 = \underline{6}$

$f''(0) < 0$ & $f''(2) > 0$

$f(x)$ is maximum at 0 & minimum at 2

$f(0) = 1$

$f(2) = -3$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$\therefore$ For maxima & minima

$\therefore f'(x) = 0$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$(x+1)(x-2) = 0$

$\boxed{x = -1, 2}$

$\therefore f''(x) = 12x - 6$

$f''(-1) = 12(-1) - 6$

$= -18 < 0$

$f''(2) = 12(2) - 6$

$= 18 > 0$

$\therefore f(x)$ is maximum at $x = -1$ & minimum at $x = 2$

$\rightarrow f(-1) = 8$

$f(2) = -19$

Q2)

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$

$f'(x) = 3x^2 - 6x - 55$

$f''(x) = 6x - 6$

$\therefore$ At $x = 0$

$f(x) = 9.5$

$f'(x) = -55$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0 - \frac{9.5}{-55}$

$= 0.1727$

$\therefore f(x_1) = -0.0528$

$f'(x_1) = -55.9467$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.1727 - \frac{-0.0528}{-55.9467}$

$x_2 = 0.1712$

$f(x_2) = 0.0011$

$f'(x_2) = -55.9393$

$x_3 = x_2 - \frac{f(x_2)}{f'(x)}$

$= 0.1712 - \frac{0.0011}{-55.9393}$

$\therefore x_3 = 0.1712$

$\therefore x = 0.1712$ is the root of the equation

ii) $f(x) = x^3 - 4x - 9 \rightarrow$ ~~roots~~ $[2, 3]$

$f'(x) = 3x^2 - 4$

$f''(x) = 6x$

$f(2) = -9$

$f(3) = 6$

∴ 6 is closer to 0 in the number line

∴ $x_0 = 3$

∴ $f(x_0) = 6$

$f'(x_0) = 23$

∴ $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = 2.7391$

∴ $f(x_1) = 0.5942$

∴ $f'(x_1) = 18.508$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 2.7391 - \dfrac{0.5942}{18.508}$

$x_2 = 2.707$

$f(x_2) = 0.0095$

$f(x_2) = 17.9835$

∴ $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 2.707 - \dfrac{0.0085}{17.9835}$

$x_3 = 2.7065$

$f(x_3) = -0.0001$

$f'(x_3) = 17.9757$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)}$

$= 2.7065 - \dfrac{-0.0001}{17.9757}$

$x_4 = 2.7065$

∴ 2.7065 is the root of the given equation

iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$

$f'(x) = 3x^2 - 36x - 10$

$f(1) = 1 - 1.8 - 10 + 17 = 6.2$

$f(2) = -2.2$

∴ -2.2 is closer to 0, on the number line

$x_0 = 2$

$f(x_0) = -2.2$

$f'(x_0) = -5.2$

$\therefore x_1 = x_0 - \frac{f(x_1)}{f(x_2)}$

$= 2 - \frac{-2.2}{-5.2}$

$x_1 = 1.5769$

$f(x_1) = (1.5769)^3 - 1.8(1.5769)^2 - 3(1.5769) + 17$

$= 0.6762$

$f'(x_1) = -8.217$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.5769 - \frac{0.6762}{-8.217}$

$x_2 = 1.6592$

$f(x_2) = 0.0204$

$f'(x_2) = -7.7143$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.6592 - \frac{0.0204}{-7.7143}$

$x_3 = 1.6618$

$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 3(1.6618) + 17$

$f'(x_3) = 0$

$\therefore$ 1.6618 is the root of the equation

Practical - 5

Q1) Solve the following integrations:-

i) $I = \int \frac{dx}{\sqrt{x^2+2x-3}}$ ii) $I = \int (4e^{3x} + 1)\,dx$

iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

iv) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$ v) $\int t^7 \sin(2t^4)\,dx$

vi) $\int \sqrt{x}(x^2-1)\,dx$ vii) $\int \frac{1}{x^2} \sin\left(\frac{1}{x^2}\right)dx$

viii) $\int \frac{\cos x}{\sqrt{\sin^2 x}}\,dx$ ix) $\int e^{\cos^2 x} \cdot \sin 2x\,dx$

x) $\int \left(\frac{x^2-2x}{x^3-3x+1}\right)dx$

$\Rightarrow$ Solutions:-

2.

i) $I = \int \frac{1}{\sqrt{x^2+2x-3}}\, dx$

$= \int \frac{1}{\sqrt{x^2+2x+1-4}}$

$= \int \frac{1}{\sqrt{(x+1)-2^2}}\, dx$

$I = \ln\left|x+1+\sqrt{x^2+2x-3}\right| + C$

ii) $I = \int (4e^{3x}+1)\, dx$

$= 4\int e^{3x}\, dx + \int 1\, dx$

$= \frac{4e^{3x}}{3} + x + C$

iii) $I = \int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

$= 2\int x^2\, dx - 3\int \sin\, dx + 5\int x^{1/2}\, dx$

$= \frac{2x^3}{3} + 3\cos x + \cancel{\frac{5x\cdot 2}{3}}\ \frac{5\cdot 2x^{3/2}}{3}$

$= \frac{2x^3}{3} + 3\cos x + \frac{10x^{3/2}}{3} + C$

(iv) $I = \int \frac{x^3 + 3x + 4}{\sqrt{x}}\, dx$

Let $\sqrt{x} = t$

$\frac{1}{2\sqrt{x}} = \frac{dt}{dx}$ $= \frac{dx}{\sqrt{x}} = 2dt$

$I = 2\int \frac{(\sqrt{x})^6 + 3(\sqrt{x})^2 + 4}{\sqrt{x}}\, dx$

$= 2\int t^6 + 3t^2 + 4\, dt$

$= 2\left[\frac{t^7}{7} + \frac{3t^3}{3} + 4t\right] + C$

$= 2\left[\frac{x^{7/2}}{7} + 3x^{3/2} + 4x^{1/2}\right] + C$

v) $I = \int t^7 \sin(2t^4)\, dt$

$= \int t^4 \cdot \sin(2t^4) \cdot t^3\, dt$

Put $\cancel{t^4} = x$

$4t^3 = \frac{dx}{dt}$

$t^3 dt = \frac{dx}{4}$

$\therefore I = \frac{1}{4} \int x \sin(2x)\, dx$

$= \frac{1}{4}\left[x \int \sin 2x\, dx - \int \frac{dx}{dt} \int \sin 2x\, dx) dx\right.$

$= \frac{1}{4}\left[-\frac{x \cdot \cos 2x}{2} + \frac{1}{2}\int \cos 2x\, dx\right]$

$= \frac{1}{16}\sin 2x - \frac{x \cdot \cos 2x}{8} + C$

$= \frac{\sin 2t^4}{8} - \frac{t^4 \cdot \cos 2t^4}{8} + C$

vi) $I = \int \sqrt{x}\,(x^2 - 1)\, dx$

$= \int x^2 \sqrt{x}\, dx - \int \sqrt{x}\, dx$

$= \int x^{5/2} dx - \int x^{1/2} dx$

$= \frac{x^{7/2}}{7/2} - \frac{x^{3/2}}{3/2} + C$

$= \frac{2}{7} x^{7/2} - \frac{2}{3} x^{3/2} + C$

:2

vii) $\int \frac{1}{x^2} \sin\left(\frac{1}{x^2}\right) dx$

Let $\frac{1}{x^2} = t$

$\Rightarrow \frac{-2}{x^3} = \frac{dt}{dx}$

$\Rightarrow \frac{dx}{x^3} = \frac{-dt}{2}$

$\therefore I = \frac{-1}{2} \int \sin t \, dt$

$= \frac{-1}{2} - \cos t + C$

$= \frac{\cos t}{2} + C$

$I = \frac{\cos(1/x^2)}{2} + C$

viii) $I = \int \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx$

Put $\sin x = t$

$\therefore \cos x \, dx = dt$

$I = \int \frac{1}{t^{2/3}} dt$

$= \int t^{-2/3} dt$

$\Rightarrow I = \frac{t^{-2/3+1}}{-2/3+1} + C$

$= 3t^{1/3} + C$

$= 3\sqrt[3]{\sin x} + C$

ix) $I = \int e^{\cos^2 x} \cdot \sin 2x \, dx$

Put $\cos^2 x = t$

$-2\cos x \cdot \sin x = \frac{dt}{dx}$

$-dt = \sin 2x \, dx$

$I = -\int e^t \cdot dt$

$= -e^t + C$

$= -e^{\cos^2 x} + C$

x) $I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$

$x^3 - 3x^2 + 1 = t$

$3x^2 - 6x = \frac{dt}{dx}$

$(x^2 - 2x) dx = \frac{dt}{3}$

$\therefore I = \frac{1}{3}\int \frac{1}{t}\,dt$

$= \frac{1}{3}\log|t| + C$

$= \frac{1}{3}\log|x^3 - 3x^2 + 1| + C$

Practical - 6

Q) Find the length of the following curves:-

(1) $x = t\sin t$, $y = 1 - \cos t$; $t \in (0, 2\pi)$

(2) $y = \sqrt{4 - x^2}$; $x \in [-2, 2]$

(3) $y = x^{3/2}$ in $x \in [0, 5]$

(4) $x = 3\sin t$, $y = 3\cos t$; $t \in [0, 2\pi]$

(5) $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1, 2]$

Q) Using Simpson's Rule solve the following:-

(1) $\int_0^2 e^{x^2}\,dx$ with $n = 4$

(2) $\int_0^4 x^2\,dx$ with $n = 4$

(3) $\int_0^{\pi/2} \sqrt{\sin x}\,dx$ with $n = 6$

56

for $x \in [-2, 2]$

Q1) $y = \sqrt{4-x^2}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}} \times (-2x)$

$= \frac{-x}{\sqrt{4-x^2}}$

$\therefore L = \int_{-2}^{2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_{-2}^{2} \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$

$= \int_{-2}^{2} \sqrt{1+\frac{x}{4-x^2}}\, dx$

$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$

$= \int_{-2}^{2} \frac{1}{\sqrt{2^2-x^2}}\, dx$

$= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$

$= 2\left[\sin^{-1}\left(\frac{2}{2}\right) - \sin^{-1}\left(\frac{-2}{2}\right)\right]$

$= \cancel{2}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$

$= 2\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$

$= 2\left[\frac{2\pi}{2}\right]$

$= 2\pi$

Q2) $y = x^{3/2}$ $\quad x \in [0, 4]$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$

$L = \int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx$

$= \frac{1}{2}\int_0^4 \sqrt{4+9x}\, dx$

$= \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{3/2} \times \frac{1}{9}\right]$

$= \frac{1}{27}\left[(4\cancel{tan}+9x)^{3/2}\right]$

$= \frac{1}{27}\left[(4+\overset{36}{\cancel{9}})^{3/2} - (4-0)^{3/2}\right]$

$= \frac{1}{27}\left(40^{3/8} - 8\right)$ unit

Q4) $x = 3\cancel{x}\,\cancel{\sin t}$ $\quad y = 3\cos t$ $\quad t \in [0, 2\pi]$

$\frac{dx}{dt} = 3\cos t$ $\quad \frac{dy}{dx} = -3\sin t$

$L = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$

$$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (3\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} 3\sqrt{1}\, dt$$

$$= 3\int_0^{2\pi} dt$$

$$= 3\left[x\right]_0^{2\pi}$$

$$= 3\left[2\pi - 0\right]$$

$$= \underline{6\pi \text{ units}}$$

Q3) $x = \frac{1}{6}y^3 + \frac{1}{2y}$ $\quad y \in [1, 2]$

$$\therefore \frac{dy}{dx} = \frac{y^3}{2} - \frac{1}{2y^2}$$

$$\frac{dx}{dy} = \cancel{\frac{y^3 - 1}{2y^2}}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{1 + \frac{(y^3-1)^2}{4y^4}}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^6 - 1) + 4xy^4}{4y}}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)}{2y^2}}\, dy$$

$$= \frac{1}{2}\int_1^2 y^2\, dy + \frac{1}{2}\int^2 y^{-2}\, dy$$

$$= \frac{1}{2}\left[\frac{y}{3} - \frac{y^{-2}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{7}{3} + \frac{1}{2}\right]$$

$$= \underline{\frac{17}{12} \text{ units}}$$

Q2) i) $\int_0^2 e^{x^2}\, dx$ with $n = 4$

$$L = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

x	0	~~0.5~~	1	1.5	2
y	1	1.284	2.7183	9.5887	54.5982
	y_0	y_1	y_2	y_3	y_4

$$= \frac{0.5}{2}\left[(1 + 54.5982) + 5(1.284 + 9.487) + 2.7883\right]$$

$$= \underline{17.3535}$$

82.

ii) $\int_0^4 x^2 dx \quad a=4 \quad h=\frac{4-0}{4}=1$

x	0	1	2	3	4
y	0	1	4	9	16
	y_0	y_1	y_2	y_3	y_4

$\Rightarrow L = \frac{h}{3}[(y_0+y_4) + 4(y_1+y_3) + 2y_2)$

$= \frac{1}{3}(16+4(10)+8)$

$= \frac{64}{3}$

$= 21.3333$

iii) $\int_0^{\pi/3} \sqrt{\sin x}\, dx$

$h = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$

0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$
0	0.4167	0.5848	0.7071	0.8702	0.9306
y_0	y_1	y_2	y_3	y_4	y_5

$\Rightarrow N = \frac{\pi/18}{3}[(0.4167 + 0.9306) + 4(0.4167 + 0.7071) + 2(0.5848 + 0.8071)]$

$= \frac{\pi}{54}[1.3473 + 7.996 + 2.773]$

$= \frac{\pi}{54} \times 12.1163 = 0.7049$

Practical 07

Topic :- Differential Equation

Q) Solve the following differential equation :-

1) $x\frac{dy}{dx} + y = e^x$

2) $e^x\frac{dy}{dx} + 2e^x y = 1$

3) $x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

4) $x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

5) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

6) $e^2 x\frac{dy}{dx} + 2e^{2x} y = 2x$

7) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dx = 0$

8) $\frac{dy}{dx} = \sin^2(x-y+1)$

9) $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

Q1) $x\frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$f(x) = 1/x \qquad g(x) = e^x/x$

$I.F. = e^{\int 1/x \, dx}$

$= e^{\ln x}$

$= x$

$y(I.F.) = \int g(x)(I.F.)dx + C$

$= \int \frac{e^x}{x} \times x \, dx + C$

$= \int e^x dx + C$

$\underline{\underline{= e^x dx + C}}$

Q2) $e^x \frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} \cancel{+ 2y} = e^{-x}$

$f(x) = 2 \qquad g(x) = e^{-x}$

$\therefore I.F. = e^{\int f(x)dx}$

$= e^{\int 2dx}$

$= e^{2x}$

$\therefore y(I.F.) = \int g(x) \cdot (I.F.) dx + C$

$= \int e^{-x} \times e^{2x} dx + C$

$= \int e^{-x+2x} dx + C$

$= \int e^x dx + C$

$\underline{\underline{= e^x + C}}$

3) $\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$

$f(x) = \frac{2}{x} \qquad g(x) = \cos x / x^2$

$I.F. = e^{\int f(x)dx}$

$= e^{\int 2/x \, dx}$

$= e^{2 \ln x}$

$= e^{\ln x^2}$

$= x^2$

$y(I.F.) = \int g(x) \cdot (I.F.) dx + C$

$= \int \frac{\cos^2}{x^2} x^2 dx + C$

$= \int \cos x + C \Rightarrow \underline{\underline{-\sin x + C}}$

Q4) $x\dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}$

$\dfrac{dy}{dx} + \dfrac{3}{x}y = \dfrac{\sin x}{x^2}$

$f(x) = \dfrac{3}{x}$ $\quad g(x) = \dfrac{\sin x}{x^2}$

$(I.F.) = e^{\int f(x)\,dx}$

$= e^{\int 3/x\,dx}$

$= e^{3\ln 3}$

$= x^3$

$y(I.F.) = \int g(x)\cdot(I.F.)\,dx + C$

$= \int \dfrac{\sin x}{x^2} \times x^3\,dx + C$

$= \int \sin x\,dx + C$

$= -\cos x + C$

Q5) $e^{2x}\dfrac{dy}{dx} + 2e^{2x}y = 2x$

$\dfrac{dy}{dx} + 2y = \dfrac{2x}{e^{2x}}$

$f(x) = 2$ $\quad g(x) = 2x/e^{2x}$

$(I.F.) = e^{\int f(x)\,dx}$

$= e^{\int 2\,dx}$

$= e^{2x}$

$y\,(I.F.) = \int g(x)\cdot(I.F.)\,dx + C$

$= \int \dfrac{2x}{e^{2x}} \times e^{2x}\,dx + C$

$= \int 2x\,dx + C$

$= x^2 + C$

Q6) $\sec^2 x\cdot\tan y\,dx + \sec^2 y\tan x\,dy = 0$

$\Rightarrow \sec^2 x\cdot\tan y\,dx = -\sec^2 y\cdot\tan x\,dy$

$\dfrac{\sec^2 x\,dx}{\tan x} = \dfrac{-\sec^2 y\,dy}{\tan y}$

$\therefore \int \dfrac{\sec^2 x\,dx}{\tan x} = -\dfrac{\sec^2 y\,dy}{\tan y}$

$\therefore \log|\tan x| = -\log|\tan y| + 1$

$\therefore \log|\tan x \cdot \tan y| = c$

$\tan x\cdot\tan y = e$