Affine Stratifications and Cohomological Dimension of Morphisms

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0 Introduction

In this paper, we discuss affine stratifications of quasi-compact, quasi-separated schemes and introduce a relative stratification number of morphisms.We will apply this definition to generalize two statements on cohomological dimension of morphisms in the étale setting:

Gabber (see [1] 219) showed that for an affine morphism of finite type $f : Y \to X$, where X is a quasi-excellent scheme and $\mathcal{F}$ is a torsion sheaf on the étale site of Y, we can find certain bounds for the higher direct images $R^q f_* \mathcal{F}$, namely

$$\mathbf{d}(R^q f_* \mathcal{F}) \leq f^* \mathbf{d}(\mathcal{F}) - q$$

Here, $\mathbf{d}, f^*\mathbf{d}$ denote dimension functions on the schemes X and Y. We will give a short introduction of dimension functions and discuss when they exist. This is for instance the case for quasi-excellent schemes, who are of classical interest, as they are closely related to the problem of resolution of singularities, see Grothendiecks motivation in [5] 218 and Hironakas work of the characteristic zero case in [6] 109-203.

Moreover, Roth and Vakil (see [7] 14) proved in their paper that for a torsion étale sheaf $\mathcal{F}$ on a variety Y over a separably closed field k,

$$H^q(Y, \mathcal{F}) = 0 \text{ for } q > \mathbf{asn}(Y) + \mathbf{d}(\mathcal{F})$$

We will first give a small recap on affine stratifications of schemes and will later introduce a relative version of this quantity. The goal of this thesis is a theorem that generalizes the two above mentioned statements, using this newly defined relative version of an affine stratification.

1 A Relative Affine Stratification Number

1.1 Affine stratifications of schemes

In this introduction we start by recalling some basic facts on stratification of schemes. The material is taken from [8] 0F2R, where you can find more details.

By a stratification of a scheme X, we mean a stratification of the underlying topological space into locally closed subspaces. In the category of schemes, we can endow these strata with their induced locally closed subscheme structure.

Definition 1.1 (Affine Stratification)**.** An affine stratifictation of a scheme X is a stratification of the underlying topological space into locally closed affine subspaces, i.e. $X = \bigsqcup_{i \in I} X_i$ for affine schemes X_i, such that the inclusion maps $X_i \to X$ are affine morphisms. Here, I is a partially ordered index set.

Throughout this paper, we will often be interested in cases where I is a finite set. The length of a partially ordered index set is defined as the supremum p of chains $i_0 < i_1 < \cdots < i_p$ of elements of I. We propose the following with proof (see [8] 0F2T), as the proof gives us another essential corollary:

Proposition 1.2. *Suppose $X = \bigsqcup_{i \in I} X_i$ is a finite affine stratification of length n. Then there exists an affine stratification of X with index set $\{0, \ldots, n\}$.*

Proof. By definition of a stratification of a topological space X, we have a partial ordering on I such that the closure of each X_i is contained in the union $\bigcup_{j \leq i} U_j$ for all $i \in I$. Now denote by $I' \subset I$ the set of maximal indices. Then for each $i \in I'$ the set X_i is open, as the closed set $\bigcup_{j \neq i} \overline{X_j}$ is its complement in X. Thus we can view $U := \bigcup_{i \in I'} X_i$ as an open subscheme of X with reduced subscheme structure $U_{red} = \bigsqcup_{i \in I'} X_i$. Then U is affine with affine natural inclusion $U \to X$, as for each $i \in I'$, $X_i \to X$ is affine by definition 1.1. Its complement $Z := X \setminus U$ has therefore affine stratification $Z = \bigcup_{i \in I \setminus I'} X_i$ where the partially ordered set $I \setminus I'$ has length exactly one less than I.

By induction, we see that Z has affine stratification $Z = Z_0 \sqcup \ldots Z_{n-1}$ with index set $\{0, \ldots n-1\}$. Now $Z_n := U$ gives us the desired stratification. $\square$

Corollary 1.3. *This proof shows that if a scheme X has affine stratification of length n, then we can write*

$$Z \xhookrightarrow{i} Y \xhookleftarrow{j} U$$

where U is affine open and Z is closed with affine stratification of length $n-1$. Furthermore, the natural maps i, j are a closed immersion and an open affine immersion respectively.

We will merely restrict our attention to quasi-compact and quasi-separated schemes, as the following lemma explains:

Lemma 1.4. *For a scheme X, the following are equivalent:*

- *X is quasi compact and quasi separated*
- *X has a finite affine stratification*

This enlightens the following definition:

Definition 1.5 (Affine stratification)**.** For a quasi-compact and quasi-separated scheme X we define the *affine stratification number* $\mathbf{asn}(X)$ as the smallest integer $n \geq 0$, such that there exists an affine stratification $X = X_0 \sqcup X_1 \cdots \sqcup X_n$ of length n, so $\mathbf{asn}(X) = n$.

Now one might wonder how this definition is related to other numerical values that measure the "size" or "affiness" of the scheme X. Indeed, we find bounds from above for the values of $\mathbf{asn}$.

Lemma 1.6. *If X is a separated scheme that is covered by $n+1$ affine subschemes, then we have $\mathbf{asn}(X) \leq n$*

Remark 1.7. To construct a stratification for a given cover of affine opens, use that for every pair of affine opens U, V of a separated scheme X, their intersection $U \cap V$ is an affine open subscheme of X.

Corollary 1.8. *If $Y \to X$ is an affine morphism then $\mathbf{asn}(Y) \leq \mathbf{asn}(X)$.*

Proof. the case where $\mathbf{asn}(X) = \infty$ being trivial, the finite case follows from the observation that pulling back an affine stratification under an affine morphism gives again an affine stratification of the source. □

In addition, the affine stratification index is also related to the Krull dimension of a scheme.

Lemma 1.9. *If X is a Noetherian scheme of positive dimension $d \geq 0$ then we have $\mathbf{asn}(X) \leq d$.*

1.2 A relative version of the affine stratification

We now want to construct a relative version of **asn**:

That means, given a morphism $f : Y \to X$ of quasi-compact and quasi-separated schemes (which we will now abbreviate with qcqs), we want to see how this morphism can be related to given stratifications for X, Y.

First note that if Y is quasi-compact und X is quasi-separated, then f is a quasi-compact morphism, i.e. inverse images of open quasi-compact schemes under f are quasi-compact.

Definition 1.10 (affine stratification number for a morphism of schemes)**.** Given $f : Y \to X$ as above we define the integer value

$$\mathbf{asn}(f) := \inf_{\substack{\{U_i\} \text{ affine open} \\ \text{cover of X}}} \left\{ \sup_{U_i} \{\mathbf{asn}(f^{-1}U_i)\} \right\}$$

We first need to clarify that this value is well-defined: As we take the infimum over all affine open covers, we may may restrict our attention to finite covers $X \subseteq \{U_i\}$, as we assume X to be quasi-compact. As $f^{-1}U_i$ is a qcqs open subscheme of Y, it also admits a finite stratification, so the supremum is taken over a finite set of integers, hence is finite.

First of all, we observe that if $X = \mathbf{Spec}(k)$,where k is a field, then there exists only one trivial open cover, thus

$$\mathbf{asn}(f) = \sup_i \mathbf{asn}(f^{-1}U_i) = \mathbf{asn}(f^{-1}\mathrm{Spec}(k)) = \mathbf{asn}(Y)$$

Hence this relative version indeed generalizes the original notion of **asn**. Furthermore, we have that

Proposition 1.11. $\mathbf{asn}(f) = 0$ *if and only if* f *is an affine morphism.*

Proof. If f is an affine morphism, then for any affine U_i, we have that $f^{-1}U_i$ is affine, so for all i we have $\mathbf{asn}(f^{-1}U_i) = 0$ and hence, taking the infimum over all affine covers $\{U_i\}$ of X, $\mathbf{asn}(f) = 0$.

Conversely, if $\mathbf{asn}(f) = 0$, then there exists an open affine cover $\{U_i\}$ of Y such that $\sup_i\{\mathbf{asn}(f^{-1}U_i)\} = 0$. As **asn** is non-negative, we see for all i that

$\mathbf{asn}(f^{-1}U_i) = 0$, thus every $f^{-1}U_i$ is affine, which is equivalent to the fact that f is affine, as affineness is local on the target. $\square$

In addition, we propose the following

Proposition 1.12 (stratification and base change). *Suppose we have a fibre product*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

where we assume $g : X' \to X$ to be an affine morphism, then $\mathbf{asn}(f') \leq \mathbf{asn}(f)$.

Proof. First note that being affine is stable under arbitrary base change, so g' is also an affine morphism.

Now if for any chosen open affine cover $\{U_i\}$ of X, observe that the values for $\mathbf{asn}(f^{-1}U_i)$ can only drop, when we pull it back to Y' via g': Indeed, any given affine stratification $\mathbf{asn}(f^{-1}U_i) = \bigsqcup_j A_{ij}$ into affine spaces A_{ij} pulls back to an affine cover of $g'^{-1}(f^{-1}U_i)$, which has therefore a smaller or equal stratification number, see corollary 1.8 .

On the other hand, pulling back the cover $\{U_i\}$ under g gives us again a cover on X' of affine opens $\{V_i\}$, where $V_i := g^{-1}U_i$ and by commutativity of the square and what we just saw, $f'^{-1}V_i = g'^{-1}(f^{-1}U_i)$ has smaller or equal stratification number than $f^{-1}U_i$. Hence, taking the infimum over all covers of X', we conclude that $\mathbf{asn}(f') \leq \mathbf{asn}(f)$ $\square$

1.3 Affine stratifications and quasi-coherent cohomology

We just observed that we can view the relative affine stratification of a morphism f as a measure of its affiness.

Recall that for any quasi-coherent sheaf of modules $\mathcal{M}$ on a nonempty scheme X, we have for $q > 0$, $H^q(X, \mathcal{M}) = 0$ if and only if X is affine. We now state a generalized relative version of this fact:

Theorem 1.13. *Let $f : Y \to X$ be a morphism of non-empty qcqs schemes, where $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_Y$-module. Then we have $R^q f_* \mathcal{M} = 0$ for all $q > \mathbf{asn}(f)$.*

Proof. We claim that without loss of generality, we only have to show that

$$H^q(Y, \mathcal{M}) = 0 \text{ for all } q > \mathbf{asn}(Y)$$

Indeed, first note that $R^q f_* \mathcal{M}$ is the sheaf associated to the presheaf rule

$$U \mapsto H^q(f^{-1}U, \mathcal{F}|_{f^{-1}U}) \text{ for } U \subseteq X \text{ open.}$$

Now choose an open affine cover $\{U_i\}$ of X, such that we have

$$\max_i \{\mathbf{asn}(f^{-1}U_i)\} < q$$

Observe that this can be chosen to be a finite cover, by quasi-compactness of X. But this means precisely that we have for all i: $\mathbf{asn}(f^{-1}U_i) < q$. Setting $Y_i := f^{-1}U_i$, we see that it is enough to show that $H^q(Y_i, \mathcal{M}_i) := H^q(Y_i, \mathcal{M}|_{Y_i}) = 0$ for $q > \mathbf{asn}(Y_i)$.

Proving this can be found similarly in [8] 0F2Y, but we do it here anyway: By abuse of notation, we will now omit the subscript i:

We will proceed via induction over the affine stratification number **asn** of the scheme Y: If $\mathbf{asn}(Y) = 0$ this means precisely that Y is affine and we know that the cohomology groups $H^q(Y, \mathcal{M})$ of a quasi coherent sheaf $\mathcal{M}$ on an affine scheme vanish for $q > 0$. If $\mathbf{asn}(Y) > 0$, then there exists by definition an open affine subscheme U of Y and a closed subscheme $Z := Y \setminus U$ with $\mathbf{asn}(Z) = \mathbf{asn}(Y) - 1$, which we display in the following diagram, where i, j are the natural inclusions and we can choose j to be an affine morphism, see 1.3:

$$Z \xhookrightarrow{i} Y \xhookleftarrow{j} U$$

We also have a natural adjunction map $\mathcal{M} \to j_* j^* \mathcal{M}$, of sheaves that is an isomorphism over U. This map induces an exact sequence of quasi-coherent $\mathcal{O}_Y$-modules:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{M} \longrightarrow j_* \mathcal{M}|_U \longrightarrow \mathcal{Q} \longrightarrow 0$$

Here $\mathcal{K}, \mathcal{Q}$ are the kernel and cokernel of the adjunction map above. As the morphism in the centre is an isomorphism on U, we observe that $\mathcal{K}, \mathcal{Q}$ are supported on the complement Z.

First we compute the cohomology of $j_* \mathcal{M}$: A special case of the Leray spectral sequence (see [4] 202) says that $H^*(U, \mathcal{M}|_U) \cong H^*(X, Rj_*(\mathcal{M}|_U))$ in the derived category of quasi-coherent sheaves; but as j is an affine morphism, $R^q j_*(\mathcal{M}|_U) = 0$ for $q > 0$, hence

$$H^q(X, j_* \mathcal{M}|_U) \cong H^q(U, \mathcal{M}|_U) = 0$$

for $q > 0$, where the last equality follows from the fact that U is affine. Breaking this sequence into short exact sequences and considering the long exact sequence in cohomology, we see that we are only left to proof that $H^q(Z, \mathcal{K}) = H^q(Z, \mathcal{Q}) = 0$ for $q > \mathbf{asn}(Y)$.
But as $\mathbf{asn}(Z) < \mathbf{asn}(Y)$, this follows from induction hypothesis, so we are done. $\square$

2 Relations to Étale Cohomology

2.1 Dimension functions

In the following discussion, we consider schemes X,Y as sites with their ètale topology and by cohomology of a sheaf of abelian groups $\mathcal{F}$ on X, we mean the étale cohomology of the sheaf $\mathcal{F}$ on the ètale site.

First of all, we will give some definitions and a theorem by Gabber.

Definition 2.1 (Quasi-excellent scheme)**.** We call a scheme X quasi-excellent, if every open affine subscheme of X is the spectrum of a quasi-excellent ring. For the definition of a quasi-excellent ring, see e.g. [1] 9. Note that quasi-excellent, quasi-compact schemes are in particular noetherian.

Definition 2.2. (Dimension functions) If X is a scheme, we call a function $\delta : X \to \mathbb{Z}$ a dimension function for the étale topology on X, if for every immediate étale specialization $x \rightsquigarrow y$ of two points $x, y \in X$ we have

$$\delta(y) = \delta(x) - 1$$

Here, an étale specialization between two geometric points $\overline{x}$ and $\overline{y}$ of X is given by a morphism over X of strictly henselian local schemes $X_{\overline{x}} \to X_{\overline{y}}$. It is an immediate étale specialization if the closure of $\overline{x}$ in $X_{\overline{y}}$ is a scheme of dimension 1.

We see that the difference of two dimension functions on X is a function invariant under specialization, hence locally constant.

Although it is not guaranteed that such a δ exists, for a quasi-excellent scheme X, this is (at least locally) always the case (see Gabbers theorem [1] 213).

If X is scheme of finite type over a field k (which is quasi-excellent), or is universally catenary (e.g. local Henselian schemes, see [1] 208) then there exists a rather natural dimension function, namely

$$\begin{aligned} \delta : X &\to \mathbb{Z}_{\geq 0}, \\ x &\mapsto \mathbf{dim}\overline{\{x\}} \end{aligned}$$

Definition 2.3. If $f : Y \to X$ is a finite type morphism over a field k with X admitting a dimension function δ_X, then the assignment

$$f^*\delta_X : y \mapsto \delta_X(f(y)) + \mathbf{degtr}(k(y)|k(f(y)))$$

gives a dimension function on Y. Note that the property of f being finite type is necessary for the RHS to be well defined

For more on dimension functions, see [1] 207 and the following pages.

Definition 2.4. Let δ_X be a dimension function on X and let $\mathcal{F}$ be an étale n-torsion sheaf on X. Define

$$\mathbf{d}(\mathcal{F}) := \sup \{\delta_X(x), x \in X : \mathcal{F}_{\bar{x}} \neq 0\}$$

to be the support function associated to δ_X.

Furthermore, we will often use this helpful observation:

Lemma 2.5. *For an exact sequence of sheaves*

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$$

we have

$$\mathbf{d}(\mathcal{F}_2) \leq \max\{\mathbf{d}(\mathcal{F}_1), \mathbf{d}(\mathcal{F}_3)\}$$

Definition 2.6. A sheaf $\mathcal{F}$ on a scheme Y is called locally constant, when there is an étale covering $U_i \to X$ such that each $\mathcal{F}|U_i$ is constant. Furthermore, if all $\mathcal{F}(U_i)$ are finite, we call this a finite locally constant sheaf.

We call $\mathcal{F}$ constructible, if we can write Y as a union of finitely many locally closed subschemes $Y' \in Y$ such that $\mathcal{F}|Y'$ is finite locally constant. For more equivalent definitions, see e.g. [2] 40.

Definition 2.7. For the rest of this paper, $\mathcal{F}$ is going to be a constructible étale sheaf of $\mathbb{Z}/n\mathbb{Z}$ modules on Y.

Here, Y is a qcqs scheme, where n is invertible in Y. In addition, we will consider $\mathcal{F}$ as an object of $\mathcal{D}_{\text{ét}}(Y)$, the derived category of étale sheaves on X.

To end this section, we will state a theorem by Gabber (see [1] 219).

Theorem 2.8 (Gabber)**.** *If $f : Y \to X$ is an affine morphism of finite type of qcqs schemes with X quasi-excellent, admitting a dimension function δ_X. Then for $q > 0$ we have*

$$\mathbf{d}(R^q f_* \mathcal{F}) \leq f^* \mathbf{d}(\mathcal{F}) - q$$

In the next and final section, we will try to generalize this formula to non-affine morphisms, which involves the stratifications of schemes and morphisms we introduced before.

2.2 The generalized theorem of Gabber

Keeping in mind what we just discussed in the previous section, ending with theorem 2.8, we now state a generalized version of this:

Our following discussions will use this theorem of Gabber as an induction base case: We find a more general way of this in terms of the affine stratification number **asn**. Namely, we will prove the following:

Theorem 2.9 (Main theorem)**.** *Let X, Y be qcqs schemes, where X is quasi-excellent with dimension function δ and let $f : Y \to X$ be a finite-type morphism over a field k. Furthermore, let $\mathcal{F}$ be a constructible n-torsion étale sheaf on Y, where n is invertible on Y. Then for $q > 0$ we have*

$$\mathbf{d}(R^q f_* \mathcal{F}) \leq f^* \mathbf{d}(\mathcal{F}) + \mathbf{asn}(f) - q$$

We will also state this in the following simpler version:

Theorem 2.10 (Main theorem, version II)**.** *for every geometric point $\overline{x} \to X$, we have*

$$(R^q f_* \mathcal{F})_{\overline{x}} = 0$$

for all $q > \mathbf{dim}(Y) + \mathbf{asn}(f) + 1$

We will quickly show how we deduce version II from the main theorem. For this, we need the following lemma from commutative algebra (see [3] 307 for a proof).

Lemma 2.11. *If $A \subseteq B$ are domains with A being noetherian and B a finitely generated A algebra, then we have*

$$\mathbf{dim}(B) + 1 \geq \mathbf{dim}(A) + \mathbf{degtr}(B|A)$$

Corollary 2.12. *If $f : Y \to X$ is of finite type, where δ_X is a dimension function on X, then $\mathbf{dim}(Y) + 1 \geq f^*\delta_X(y)$ for all $y \in Y$.*

Proof of corollary. Locally, we have $\mathbf{Spec}(B) \to \mathbf{Spec}(A)$, where any $y \in Y$ corresponds to a prime $Q \subseteq B$ getting mapped to some $f(y) = x \in X$, the latter corresponding to a prime $P \subseteq A$. We apply now the lemma above to the inclusion $A/P \hookrightarrow B/Q$: we get

$$\begin{aligned}\mathbf{dim}(Y) + 1 \geq \mathbf{dim}(B/Q) + 1 \geq\ & \mathbf{dim}(A/P) + \mathbf{degtr}((B/Q)|(A/P)) \\ =\ & \mathbf{dim}(\overline{\{f(y)\}}) + \mathbf{degtr}(k(y)|k(x)) \\ =\ & f^*\delta_X(y)\end{aligned}$$

□

Proof of version II. Consider the following diagram

$$\begin{tikzcd} Y_{\overline{x}} \arrow[r] \arrow[d, "f_{\overline{x}}"] & Y \arrow[d, "f"] \\ X_{\overline{x}} \arrow[r] & X \end{tikzcd}$$

Note that the horizontal maps above are affine morphisms and $f_{\overline{x}}$ is also of finite type. Here $Y_{\overline{x}} := Y \times_X X_{\overline{x}}$ and $X_{\overline{x}}$ denotes the strictly Henselian local scheme at $\overline{x}$. Then $(R^q f_* \mathcal{F})_{\overline{x}} \cong H^q(Y_{\overline{x}}, \mathcal{F}|_{Y_{\overline{x}}})$.

As we observed, the difference of every two dimension functions is locally constant. That means that the function $\delta_{X_{\overline{x}}} : t \mapsto \mathbf{dim}\overline{\{t\}}$ is the unique dimension function such that $\delta_{X_{\overline{x}}}(x) = 0$ on the local strictly Henselian scheme $X_{\overline{x}}$.

Now keep in mind 1.12 and $\dim(Y_{\overline{x}}) + 1 \geq f_{\overline{x}}^* \delta_{X_{\overline{x}}}$ at all points of $Y_{\overline{x}}$ by corollary 2.12. Applying the main theorem 2.9 to the map $f_{\overline{x}}$, for

$$q > \mathbf{dim}(Y) + \mathbf{asn}(f) + 1 \geq f_{\overline{x}}^* \delta_{X_{\overline{x}}}(\mathcal{F}_{\overline{x}}) + \mathbf{asn}(f_{\overline{x}})$$

we have $\mathbf{d}(R^q f_{\overline{x},*}\mathcal{F}) < 0$, thus indeed $(R^q f_* \mathcal{F})_{\overline{x}} = 0$. □

2.3 First application

Roth and Vakil (see [7] 14) proved already in their paper that for any n-torsion étale sheaf $\mathcal{F}$ on a variety Y over a separably closed field k,

$$H^q(Y, \mathcal{F}) = 0 \text{ for } q > \mathbf{asn}(Y) + \mathbf{d}(\mathcal{F})$$

Here, they used the special choice of dimension function we discussed before, namely $\delta_Y(y) := \mathbf{dim}(\overline{\{y\}})$ and its associated function $\mathbf{d}$ evaluated at $\mathcal{F}$.

Assuming 2.9, we observe that if $X = \mathbf{Spec}(k)$ then we have $R^q f_*(\_) \cong H^q(Y, \_)$ and $\mathbf{asn}(f) = \mathbf{asn}(Y)$, so we get

$$\mathbf{d}(H^q(Y, \mathcal{F})) = \mathbf{d}(R^q f_* \mathcal{F}) \leq \mathbf{d}(\mathcal{F}) + \mathbf{asn}(Y) - q < 0$$

using the assumed bound for q, so we get as a matter of fact $H^q(Y, \mathcal{F}) = 0$. This shows that 2.9 contains the claim in [7] as a special case.

2.4 The key step

We will assume the same properties for all objects as in the section before and will now proof the key step of theorem 2.9. By abuse of notation, we will write $\mathbf{d}$ instead $f^*\mathbf{d}$ for the remaining of this section.

Theorem 2.13 (Key step). *Let $f : Y \to X$ be as in theorem 2.9 and X affine, then we have*

$$\mathbf{d}(R^n f_*\mathcal{F}) \leq \mathbf{d}(\mathcal{F}) + \mathbf{asn}(Y) - n$$

We will show now how 2.9 follows from this key step:
Namely for a general qcqs scheme X, choose an open affine cover $\{U_i\}$ of X. Then we observe that

$$\begin{aligned}\mathbf{d}(R^n f_*\mathcal{F}) = \sup_{U_i}\{\mathbf{d}(R^n f_*\mathcal{F}|_{f^{-1}U_i}\} &\leq \sup_{U_i}\{\mathbf{d}(\mathcal{F}|_{f^{-1}U_i})\} + \sup_{U_i}\{\mathbf{asn}(f^{-1}U_i)\} - n \\ &= \mathbf{d}(\mathcal{F}) + \sup_{U_i}\{\mathbf{asn}(f^{-1}U_i)\} - n\end{aligned}$$

Here we used the key step in the first inequality. Passing to the infimum over all such covers on both sides of this inequality gives finally

$$\mathbf{d}(R^n f_*\mathcal{F}) \leq \mathbf{d}(\mathcal{F}) + \mathbf{asn}(f) - n$$

as the left hand side does not depend on these covers. So we are left to prove the key step:

Proof of key step 2.13. Recall first that as Y is qcqs, it admits a finite affine stratification, thus $\mathbf{asn}(Y) < \infty$.

Let U be an open affine subscheme such that for its complement Z we have

$$\mathbf{asn}(Z) < \mathbf{asn}(Y)$$

Note we can choose the natural map $U \overset{j}{\hookrightarrow} X$ to be an affine morphism. Recall that this is all possible due to 1.3.

We can summarise the maps in the following diagram, where g, h are defined as indicated. Observe further that h is also an affine morphism:

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & Y \arrow[d, "f"] & U \arrow[l, hook', "j"'] \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

We will do this via induction over $\mathbf{asn}(Y)$: If $\mathbf{asn}(Y) = 0$ i.e. Y is affine, then f is an affine morphism and Gabbers theorem 2.8 proves this base case.

If $\mathbf{asn}(Y) > 0$ we have to put in a little more work:

Recall that we have for an injective étale sheaf $\mathcal{I}$ the following exact sequence:

$$0 \longrightarrow i_*i^!\mathcal{I} \longrightarrow \mathcal{I} \longrightarrow j_*j^*\mathcal{I} \longrightarrow 0$$

Recall that these functors preserve injectives.

To calculate the cohomology of the sheaf $\mathcal{F}$, we embed it in an injective resolution $\mathcal{F} \hookrightarrow \mathcal{I}^\bullet$ of étale sheaves over Y and use the exact sequence above to get the following distinguished triangle of derived functors in $\mathcal{D}_{\text{ét}}(Y)$:

$$i_*Ri^!\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_*j^*\mathcal{F} \xrightarrow{[1]} \tag{2.1}$$

Here we are using that i_* is an exact functor, as $i : Z \hookrightarrow Y$ is a closed immersion.

Now we apply $Rf_*(\_)$ to this triangle, noting that this functor is exact as a functor $\mathcal{D}_{\text{ét}}(Y) \to \mathcal{D}_{\text{ét}}(X)$, so we get another distinguished triangle

$$\mathcal{H}^n(g_*Ri^!\mathcal{F}) \longrightarrow Rf_*\mathcal{F} \longrightarrow Rh_*j^*\mathcal{F} \xrightarrow{[1]}$$

Every distinguished triangle in $\mathcal{D}_{\text{ét}}(X)$ induces a long exact sequence in cohomology: To establish a bound for $\mathbf{d}(R^nf_*\mathcal{F})$ we can consider a piece of this exact sequence

$$R^n g_* Ri^! \mathcal{F} \longrightarrow R^n f_* \mathcal{F} \longrightarrow R^n h_* j^* \mathcal{F}$$

Keeping in mind 2.5, we get a bound for $Rf_*\mathcal{F}$ by establishing bounds for the other to terms: First, we bound $\mathbf{d}(Rh_*j^*\mathcal{F})$: Because h is an affine morphism, we get:

$$\mathbf{d}(R^n h_* j^* \mathcal{F}) \leq \mathbf{d}(j_*\mathcal{F}) - n \leq \mathbf{d}(\mathcal{F}) - n$$

where we used 2.8 and $j^*\mathcal{F} \cong \mathcal{F}|_U$.

Now we need to establish a bound for $\mathbf{d}(Rg_*Ri^!\mathcal{F})$: We will use the induction hypothesis here to assume the theorem for Z when $\mathbf{asn}(Z) > 0$.

Note that for $p, q > 0$ we have a spectral sequence with second page converging to

$$E_2^{p,q} = R^p g_* R^q i^! \mathcal{F} \to E_\infty^{p,q} = \mathcal{H}^{p+q}(Rg_* Ri^! \mathcal{F})$$

This shows that it is enough to prove the bound

$$\mathbf{d}(R^p g_* R^q i^! \mathcal{F}) \leq \mathbf{d}(\mathcal{F}) + \mathbf{asn}(Y) - (p+q)$$

We will start with analyzing the term $R^q i^! \mathcal{F}$: Applying i^* to the the distinguished triangle 2.1 (i^* being left inverse to i_*) results in

$$Ri^! \mathcal{F} \longrightarrow i^* \mathcal{F} \longrightarrow i^* Rj_* j^* \mathcal{F} \xrightarrow{[1]}$$

As a derived functor, $\mathcal{F}$ is concentrated in degree zero, so its higher derivations vanish. That means we get from the above triangle isomorphisms for $q \geq 1$:

$$i^* R^{q-1} j_* \mathcal{F}|_Z \cong i^* R^{q-1} j_* j^* \mathcal{F} \cong R^q i^! \mathcal{F}$$

The case $q = 0$ follows from the fact that $i^! \mathcal{F} \hookrightarrow i^* \mathcal{F}$.

To the higher direct images Rj_* of the sheaf $\mathcal{F}|_Z$ we can again Gabbers theorem, because j is affine: Combining this with the above isomorphisms gives us for $q > 1$

$$\begin{aligned}\mathbf{d}(R^q i^! \mathcal{F}) = \mathbf{d}(i^* R^{q-1} i^! \mathcal{F}) &\leq \mathbf{d}(R^{q-1} j_* \mathcal{F}|_Z) \\ &\leq \mathbf{d}(\mathcal{F}|_Z) - (q-1) \\ &\leq \mathbf{d}(\mathcal{F}) - q + 1\end{aligned}$$

We can combine this now with the induction hypothesis on Z and apply it to the higher direct images $R^p g_*$ of $R^q i^! \mathcal{F}$ (which is a sheaf on Z). This gives us finally (keep in mind that $p + q = n$)

$$\begin{aligned}\mathbf{d}(R^p g_* R^q i^! \mathcal{F}) &\leq \mathbf{d}(R^q i^! \mathcal{F}) + \mathbf{asn}(Z) - p \\ &\leq \mathbf{d}(\mathcal{F}) - (p+q) + (\mathbf{asn}(Z) + 1) \\ &\leq \mathbf{d}(\mathcal{F}) - n + \mathbf{asn}(Y)\end{aligned}$$

These two bounds give us finally the desired bound

$$\mathbf{d}(R^n f_* \mathcal{F}) \leq \mathbf{d}(\mathcal{F}) + \mathbf{asn}(Y) - n$$

□

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