A PERFECT OBSTRUCTION THEORY FOR SU(2)-HIGGS SHEAVES

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Contents

Abstract. We redefine the perfect obstruction theory for $\mathbf{SU}(r)$-Higgs sheaves constructed by [TT] using a different method in the rank 2 case: The key step here is a modification of $\mathbf{C}^{\times}$-localisation formula from [GP], replacing the torus action by an involution $(E, \phi) \mapsto (E^*, -\phi^*)$.

0. Introduction

0.1. **Background.** Virtual cycles have been one of the key tools in enumerative geometry: the parameter or moduli spaces of the objects one wants to parametrize, such as moduli spaces of stable maps or coherent sheaves, are "wrong" in some sense: they don't have the *expected dimension* that one would hope for. The right thing to do here is to replace the fundamental cycle $[\mathcal{M}]$ by a "better cycle" what is called a *virtual fundamental class.* This has been introduced by [LT] and [BF] in the late 90's.

0.2. **Obstruction sheaves.** The data attached to a space $\mathcal{M}$ that gives a virtual fundamental class is a complex $V^\bullet$ on $\mathcal{M}$ called a *perfect obstruction complex* together with a map $\psi : V^\bullet \to \mathbf{L}_{\mathcal{M}}$ to the truncated cotangent complex $\mathbf{L}_{\mathcal{M}}$ of $\mathcal{M}$, such that ψ is an iso. in degree 0 and a surjection in degree -1. Here, *perfect* means that $V^\bullet$ can be represented by a 2-term complex of vector bundles $[V^{-1} \to V^0]$.
Suppose now $\mathcal{M}$ is a moduli space parametrizing stable rank r sheaves E with fixed invariants defined on a smooth projective variety X. Restricting to a fibre $\mathcal{M}_L$ of $\det : \mathcal{M} \to \mathbf{Pic}(X)$ for $\det(E) = L$, a point $y = [E] \in \mathcal{M}_L$ has deformations given by $\mathrm{Ext}^1(E,E)_0$ and obstructions $\mathrm{Ext}^2(E,E)_0$ where the suffix 0 means trace-free.
Assuming that this dimension is constant over $\mathcal{M}_L$, we call the difference $\mathrm{vd} := \mathrm{ext}^1_0 - \mathrm{ext}^2_0$ the *virtual* or *expected dimension* of the moduli space.
If X is a curve of genus $g \geq 2$, then $\mathrm{ext}^2(E,E)_0 = 0$ and $\mathcal{M}_L$ is smooth. In this case, the virtual dimension agrees with the actual one of $\mathcal{M}_L$ and is equal to

$$\mathrm{ext}^1(E,E)_0 = (r^2-1)(g-1).$$

In general, there are bounds

$$\mathrm{ext}^1(E,E)_0 \geq \dim_{[E]} \mathcal{M}_L \geq \mathrm{ext}^1(E,E)_0 - \mathrm{ext}^2(E,E)_0$$

and we see that the vanishing of ext^2_0 implies smoothness of $\mathcal{M}_L$ at $[E]$ [1].
In case ext^2_0 does not vanish, we want to find a complex $V = [V^{-1} \to V^0]$ on $\mathcal{M}_L$ computing ext^*_0 at every point $m := [E] \in \mathcal{M}$, i.e. such that $h^0(V(m)) = \mathrm{ext}^1(E,E)_0$ and $h^{-1}(V(m)) = \mathrm{ext}^2(E,E)_0$.
Such a V [2] is called a perfect obstruction complex. Lets assume such a V exists and the number $vd = \mathrm{ext}^1(E,E)_0 - \mathrm{ext}^2(E,E)_0$ is constant on $\mathcal{M}_L$. This holds for instance for stable E on a smooth projective surface X: here, we have that

$$\mathrm{vd} = \mathrm{ext}^1_0 - \mathrm{ext}^2_0 = 2rc_2(E) - (r-1)c_1(E)^2 - (r^2-1)\cdot\chi(\mathcal{O}_X)$$

is a topological constant and depends only on X and the Chern classes on $\mathcal{M}_L$. In such a case, V endows $\mathcal{M}_L$ with a virtual cycle

$$[\mathcal{M}_L]^{vir} \in A_{\mathrm{vd}}(\mathcal{M}_L).$$

The existence of perfect obstruction complexes giving virtual cycles for fine moduli of stable E of $\mathrm{rank}(E) > 0$ under the hypothesis

$$\mathrm{Ext}^i(E,E)_0 = 0, \quad i \neq 1, 2$$

is proved in [HT] 4.3.

[1] for this and the statement before see [H] 4.5.4 and 4.5.5

[2] together with conditions on h^0 on h^{-1} making V into what is called a "perfect obstruction theory", which we'll discuss in detail in 5

0.3. **Torsion sheaves.** In this article, we are interested in understanding and defining virtual cycles for a moduli space $\mathcal{N}$ parametrising pairs (E, ϕ) on a surface S consisting of a vector bundle E and $\phi : E \to E \otimes K_S$ an endomorphism, twisted by the canonical K_S of S. Such a pair is equivalent to a compactly supported (i.e. torsion) sheaf $\mathcal{E}$ on $X = \mathrm{Tot}(K_S)$. As X is a Calabi-Yau threefold, we have

$$\mathrm{ext}^1 - \mathrm{ext}^2 = 0$$

and inherit by the rank 0 case in [HT] 4.4 a perfect obstruction complex V and a virtual cycle $[\mathcal{N}]^{vir}$ of $\dim = 0$.
If $\mathcal{N}$ was compact, we could compute its degree

$$\int_{[\mathcal{N}]^{vir}} 1 \in \mathbf{Z}.$$

However, $\mathcal{N}$ is non-compact: it admits a $\mathbf{C}^\times$-action

$$(E, \phi) \mapsto (E, \lambda\phi),$$

or equvialently for torsion sheaves, scaling the fibres $X \to S$. Due to this non-compactness, the resulting virtual cycle is uninteresting.

0.4. **Vafa-Witten invariants.** The interesting virtual cycle arises after passing to the $\mathbf{C}^\times$-fixed locus $\mathcal{N}^{\mathbf{C}^\times}$. This procedure of "$\mathbf{C}^\times$-localisation of virtual cycles", i.e. localising the obstruction complex V on $\mathcal{N}$ to the fixed locus $\mathcal{N}^{\mathbf{C}^\times}$ goes back to [GP] and allows us to compute invariants in the $\mathbf{C}^\times$-equivariant setting via a (virtual) Bott residue formula

$$\int_{[\mathcal{N}^{\mathbf{C}^\times}]^{vir}} \frac{1}{e(N^{vir})}.$$

Here, e is the Euler or top Chern class of N^{vir}, the non-zero weight part of the $\mathbf{C}^\times$-equivariant perfect obstruction complex V after its restriction to $[\mathcal{N}^{\mathbf{C}^\times}]$.
The space $\mathcal{N}^{\mathbf{C}^\times}$ has two components, the "*instaton branch*" $\phi = 0$ and the "*monopol branch*" $\phi \neq 0$.
This would be a first try to define what is a Vafa-Witten invariant [VW] [3] on $\mathcal{N}$

$$\mathsf{VW}_{\mathcal{N}} := \int_{[\mathcal{N}^{\mathbf{C}^\times}]^{vir}} \frac{1}{e(N^{vir})}$$

However, this invariant is zero unless $h^{0,1}(S) = 0 = h^{0,2}(S)$, which in some sense has to do with the fact that the obstruction complex V giving $[\mathcal{N}^{\mathbf{C}^\times}]^{vir}$ is not fixing $\det(E)$. The right way to go around this seems to be passing from $\mathbf{U}(r)$-pairs (E, ϕ) to $\mathbf{SU}(r)$-pairs, which are defined as

$$\mathcal{N}^\perp = \{(E, \phi) \in \mathcal{N} : \det(E) \cong \mathcal{O}_S, \mathrm{tr}(\phi) = 0\}$$

[3] solutions of the "Vafa-Witten" equations correspond to certain stable holomorphic Higgs pairs (E, ϕ).

before doing the $\mathbf{C}^\times$-localisation.
This gives a better virtual cycle $[\mathcal{N}^{\perp,\mathbf{C}^\times}]^{vir}$ and a more sensible Vafa-Witten invariant

$$\mathsf{VW}_{\mathcal{N}^\perp} := \int_{[\mathcal{N}^{\perp,\mathbf{C}^\times}]^{vir}} \frac{1}{e(N^{vir})}$$

Under the assumption that S is simply connected,we refer to [GSY] for its relation to Donaldson-Thomas invariants.

0.5. **Goal.** Defining a perfect obstruction theory for $\mathcal{N}^\perp$ takes over thirty pages in [TT]. The aim of this article is to *redefine* this perfect obstruction theory for $\mathbf{SU}(2)$-pairs by identifying them as fixed points in $\mathcal{N}$: Applying the ideas from [GP]'s $\mathbf{C}^\times$-localisation, we replace the torus action by an involution $\iota \circlearrowright \mathcal{N}$ generically defined as

$$\iota : (E, \phi) \mapsto (E^*, -\phi^*)$$

and one sees quite easily that the fixed locus $\mathcal{N}^\iota$ contains $\mathcal{N}^\perp$ as a connected component.
Taking the perfect obstruction theory $V^\bullet$ on $\mathcal{N}$ and making it ι -equivariant should split, just like in the [GP] case,

$$V^\bullet = V^{\bullet,\iota} \oplus V^{\bullet,mov}$$

into fixed and moving part over $\mathcal{N}^\perp$. As ι has square equal to the identity, we expect that the moving part $V^{\bullet,mov}$(i.e. the non-zero weight part in [GP]) should come up as some -1 eigensheaf of ι inside $V^\bullet$(this is sections 7-9).
[TT] show in their paper that the differentials on tangent complexes of

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\mathrm{tr}"] \arrow[d, "\det \pi_*"] & \Gamma(K_S) \\ \mathbf{Pic}(S) & \end{tikzcd}$$

commute with the following maps of *virtual* tangent complexes via Atiyah classes:

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathbf{R}p_{S_*}K_S \\ \mathbf{R}p_{S_*}\mathcal{O}_S[1]. & \end{tikzcd} \tag{0.5.1}$$

Our approach instead is a lift of ι as

$$\theta_\iota \circlearrowright \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$$

whose restriction to $\mathcal{N}^\perp$ gives

$$N^{vir} := (\mathbf{R}p_{S,*}\mathcal{O}_S[1] \oplus \mathbf{R}p_{X,*}K_S)|_{\mathcal{N}^\perp}$$

as the moving part, i.e. the -1 *eigensheaf.*
The virtual cycle of $\mathcal{N}^\perp$ is then constructed by taking $V^\bullet$ over $\mathcal{N}^\perp$ and

remove the moving part $V^{\bullet,mov}$. We'll then show that the fixed part $V^{\bullet}|^{\iota}_{\mathcal{N}^{\perp}}$ representing $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})^{\perp}[1]|_{\mathcal{N}^{\perp}}$ defines a perfect obstruction theory for $\mathcal{N}^{\perp}$.

The challenges here were

- to generalise $(E,\phi)\mapsto(E^*,-\phi^*)$ to torsion free sheaves and phrase this in terms of their spectral sheaves on X
- to define the correct lift θ_ι, compatible with Atiyah classes. For the deformation of trϕ, we had to use one result of [TT].

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1. Preliminaries

1.1. **Setup.** Let S be a smooth projective surface over $\mathbf{C}$ with polarisation $\mathcal{O}_S(1)$. We denote by $X := K_S \xrightarrow{\pi} S$ the total space of the canonical bundle K_S with structure map π.

1.2. **Higgs bundles and their spectral sheaves.** A Higgs pairs (E,ϕ) on S is a torsion free sheaf E of rank n together with a map $\phi : E \to E\otimes K_S$. Instead of working with a pair (E,ϕ), we consider their *spectral sheaves* $\mathcal{E}_\phi$ on X, which are roughly built as follows: Over a point $s\in S$, we attach the eigenspaces of ϕ_s acting on E_s to their eigenvalues on the fibre $X_s\cong\mathbf{C}$. Globally on S, we make E into a $\pi_*\mathcal{O}_X=\oplus_i K_S^{-i}-$module via

$$E\otimes K_S^{-i}\xrightarrow{\phi^i}E.$$

This give a torsion sheaf $\mathcal{E}_\phi$ on X, preserving stability.
This defines an equivalence of categories

$$\mathbf{Coh}_c(\mathrm{X})\leftrightarrow\mathbf{Higgs}(S)$$

between Higgs pairs on S and coherent sheaves on X of compact support, where the arrow from right to left is the spectral construction.
Conversely, starting with a compactly supported coherent sheaf $\mathcal{E}$ on X, its push-down $E:=\pi_*\mathcal{E}$ is torsion free coherent and we get $\phi:=\pi_*(\tau\cdot\mathrm{id})$ from the action $\tau\cdot\mathrm{id}\circlearrowright\mathrm{X}$ of the tautological section $\tau\in\pi^*K_S$. It can be shown that these two constructions are mutually inverse.
In this paper, we'll restrict to rank$(E)=2$

1.3. **Gieseker Stability.** A pair (E,ϕ) on S is *Gieseker stable* with respect to $\mathcal{O}_S(1)$ if

$$\frac{\chi(F(n))}{\operatorname{rank}(F)} < \frac{\chi(E(n))}{\operatorname{rank}(E)} \text{ for } n \gg 0, \tag{1.3.1}$$

and all ϕ-invariant proper non-zero subsheaves $F \subset E$.
A Gieseker stable Higgs pair (E,ϕ) with respect to $\mathcal{O}_S(1)$ is equivalent to a Gieseker stable $\mathcal{E}_\phi$ with respect to the polarisation defined by $\pi^*\mathcal{O}_S(1)$ on X. This is the condition

$$\frac{\chi(\mathcal{F}(n))}{r(\mathcal{F})} < \frac{\chi(\mathcal{E}(n))}{r(\mathcal{E})} \text{ for } n \gg 0, \tag{1.3.2}$$

and all proper non-zero subsheaves $\mathcal{F} \subset \mathcal{E}$.
Here, $r(\mathcal{E}) = \int_{\mathrm{X}} c_1(\mathcal{E})h^2$ is the leading coefficient of the Hilbert polynomial of $\mathcal{E}$, which agrees with the one of E. Indeed, as $\pi_*(\mathcal{E}(n)) = \pi_*\mathcal{E} \otimes \mathcal{O}(1) = E(n)$, we have $\chi(\mathcal{E}(n)) = \chi(\pi_*\mathcal{E}(n)) = \chi(E(n))$ and we can write $r(\mathcal{E}) = \operatorname{rank}(E)\int_S h^2 = \operatorname{rank}(E)\deg(S)$.

1.4. **Moduli spaces.** Let $\mathcal{N}$ be the moduli space of (Gieseker)-stable Higgs sheaves (E,ϕ) over S with fixed invariants (r, c_1, c_2) - equivalently, $\mathcal{N}$ is the moduli space of spectral sheaves $\mathcal{E}_\phi$ over X with invariants given by a simple Grothendieck-Riemann-Roch computation for $(E, \phi = 0)$ on S, which we identify with a spectral sheaf supported on S via push-forward along the zero section $i : S \hookrightarrow X$.
We'll restrict to the case $r = 2$ in this discussion.
Then $\operatorname{ch}(\mathcal{E}) = i_*(\operatorname{ch}(E)\cdot \operatorname{td}(T_i))$ where $\operatorname{td}(T_i) = \operatorname{td}(K_S)^{-1}$, which gives
$c_1(\mathcal{E}) = 2[S]$ for the cycle class $[S] \in H^2(\mathrm{X}, \mathbf{Z})$
$c_2(\mathcal{E}) = -i_*(-3c_1(S) - c_1)$
$c_3(\mathcal{E}) = i_*(c_1^2 - 2c_2 + 3c_1 \cdot c_1(S) + 4c_1(S)^2)$
where $c_1(S) = -c_1(K_S)$. Furthermore, we choose the Chern classes c_i such that stability = semi-stability.

1.5. **Universal sheaves.** After the following sections 1 and 2, everything will be phrased entirely for spectral sheaves on X and then in terms of their universal family $\mathscr{E}$. As $\mathcal{E}$ is simple, note that $\operatorname{Aut}(\mathcal{E}) \cong \mathbf{C}^\times$.
We remark that $\mathcal{N}$ is a quasi-projective, non compact variety and whose closed points parametrize equivalence classes of spectral sheaves $[\mathcal{E}]$ on X. We choose a twisted universal sheaf

$$\mathscr{E} \in \mathbf{D}^b(X \times \mathcal{N}),$$

for the moduli space $\mathcal{N}$. I.e. $\mathscr{E}$ is locally well-defined and exists globally as a twisted family, see e.g. [HT] 17 for a precise definition.
We remark that the sheaves we are mostly interested in are $\mathcal{E}xt^i_{p_X}(\mathscr{E}, \mathscr{E})$ which always exist globally [HL] 10.2, independent of any choice.
Under the spectral construction, $\mathscr{E}$ is equivalent to a universal Higgs sheaf

$$(\mathsf{E}, \Phi)$$

over $S \times \mathcal{N}$.
Both objects are related via the forget map

$$\Pi : \mathcal{N} \to \mathcal{M}$$

sending $\mathcal{E} \mapsto \pi_* \mathcal{E}$.
Here, $\mathcal{M}$ is the moduli stack of coherent sheaves on S with the above Chern classes. The fibres of this map at $E \in \mathcal{M}$ are given by $\mathrm{Hom}(E, E \otimes K_S)$. In families, let $\mathscr{E}$ denote a universal sheaf on $X \times \mathcal{N}$ and E the corresponding family over $S \times \mathcal{M}$ (or its pullback $\Pi^* \mathsf{E}$ to $S \times \mathcal{N}$).
We denote by p_S, p_X the canonical projections $S \times \mathcal{N} \to \mathcal{N}$ and $\mathrm{X} \times \mathcal{N} \to \mathcal{N}$ respectively. π will be the structure map $X = K_S \xrightarrow{\pi} S$ as well as its base change $\pi \times \mathrm{id} : X \times \mathcal{N} \to S \times \mathcal{N}$. The derived sheaves $\mathbf{R}p_{S,*}\mathcal{O}_S$ and $\mathbf{R}p_{S,*}K_S$ denote the push-downs to $\mathcal{N}$ of the sheaves $\mathcal{O}_{S \times \mathcal{N}}$ and $K_S \otimes \mathcal{O}_{\mathcal{N}}$ on $S \times \mathcal{N}$. $\mathbf{D}^b(X \times \mathcal{N})$ and $\mathbf{D}^b(\mathcal{N})$ are the bounded derived categories of coherent sheaves on $X \times \mathcal{N}$ and $\mathcal{N}$ respectively.

1.6. **Deformation theory.** There is an exact triangle

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E}) \xrightarrow{\pi_*} \mathbf{R}\mathcal{H}om_{p_S}(\mathsf{E}, \mathsf{E}) \xrightarrow{[\_,\Phi]} \mathbf{R}\mathcal{H}om_{p_S}(\mathsf{E}, \mathsf{E} \otimes K_S) \xrightarrow[{[1]}]{\partial}$$

relating (in families over $\mathcal{N}$) deformations of $\mathcal{E}$ on X with the one of $\pi_* \mathcal{E}$ on S. Here, the map to the cokernel is given by the bracket $g \mapsto g \circ \phi - \phi \circ g$ which parametrises deformations of the Higgs field $\phi : \pi_* \mathcal{E} \to \pi_* \mathcal{E} \otimes K_S$. We remark that this diagram equals itw own Serre dual (i.e. replacing all objects by their duals gives the same triangle, just shifted) and refer to [TT] 2.20 and 2.21 for a proof of these statements.

2. The involution

2.0. **Summary.** We'll define an involution $\iota \circlearrowright \mathcal{N}$ extending the classcial map

$$(E, \phi) \mapsto (E^*, -\phi^*)$$

to all torsion free Higgs pairs on S. This will be rephrased as

$$\mathcal{E} \mapsto \sigma^*_{\mathrm{tr}\phi} \mathcal{E} \otimes \pi^* \det(\pi_* \mathcal{E})^{-1}$$

for their corresponding spectral sheaves on X, where $\sigma_{\mathrm{tr}\phi} \circlearrowright X$ translates the points in the fibres of $\pi : X = K_S \to S$ by $\mathrm{tr}\phi$.

2.1. **Skew maps.** Let $\alpha : E \to E^*$ be a map from a locally free rank 2 sheaf E to its dual. Then $\alpha \mapsto \alpha^*$ defines an involution on $\mathcal{H}om(E, E^*)$, i.e. splits

$$\mathcal{H}om(E, E^*) \cong \mathcal{S}ym(E^*) \oplus \wedge^2 E^*$$

into ± 1 eigenspaces: sections of the former are self-dual maps $\alpha = \alpha^*$ and sections of the latter skew maps $\alpha^* = -\alpha$.

As E is rank 2, a section $\alpha \in \wedge^2 E^*$ defines an isomorphism wherever its non-zero, which gives the natural map $E \otimes (\wedge^2 E^*) \xrightarrow{\sim} E^*$ given by evaluation. Now for $\phi \in \mathcal{E}nd(E)$, we claim for α skew:

Lemma 2.1.1. $\alpha\phi - (\alpha\phi)^* = \mathrm{tr}(\phi)\alpha$

which we proof in the appendix 11.
Replacing ϕ by a *twisted* endomorphism $\phi : E \to E \otimes K_S$, we find:

Corollary 2.1.2. The following diagram is commutative

$$\begin{tikzcd} E \otimes \bigwedge^2 E^* \arrow[r, "(\phi - \mathrm{tr}(\phi)\cdot \mathrm{id})\otimes 1"] \arrow[d, equal] & E \otimes K_S \otimes \bigwedge^2 E^* \arrow[d, equal] \\ E^* \arrow[r, "-\phi^*"] & E^* \otimes K_S \end{tikzcd}$$

Proof. Indeed, for $e \otimes \alpha$, going down the LHS side gives $-\phi^*\alpha(e)$. On the RHS, we get $\alpha(\phi - \mathrm{tr}(\phi)\cdot\mathrm{id})(e) = \alpha\phi(e) - \mathrm{tr}(\phi)\alpha(e) = -\phi^*\alpha(e)$ by 2.1.1. $\square$

Remark 2.1.3. This implies that

$$(E^*, -\phi^*) \cong (E \otimes \wedge^2 E^*, (\phi - \mathrm{tr}(\phi)\cdot \mathrm{id}) \otimes 1)$$

are isomorphic Higgs bundles.
Splitting

$$\mathcal{E}nd(E) \otimes K_S = (\mathcal{E}nd_0(E) \otimes K_S) \oplus K_S \cdot \mathrm{id}$$

we can write $\phi = \phi_0 \oplus \frac{1}{2}\mathrm{tr}(\phi)\cdot\mathrm{id}$ and see that $\phi - \mathrm{tr}(\phi)\cdot\mathrm{id} = \phi_0 \oplus -\frac{1}{2}\mathrm{tr}(\phi)\cdot\mathrm{id}$.

Definition 2.1.4. In view of the above, we may redefine the involution

$$\iota : (E, \phi) \mapsto (E^*, -\phi^*)$$

on Higgs bundles (locally frees are reflexive) as

$$\iota : (E, \phi) \mapsto (E \otimes \wedge^2 E^*, (\phi - \mathrm{tr}(\phi)\cdot\mathrm{id}) \otimes 1)$$

under the isomorphism stated in 2.1.2.

This allows an easy extension to torsion free sheaves, which we'll establish in the next step:

2.2. **Torsion free sheaves.** Let E be a torsion free sheaf on S. Then we have [4]

$$\mathrm{dh}(E) \leq \dim(S) - 1.$$

As S is a surface, we have $\mathrm{dh}(E) = \max\{\mathrm{dh}(E_s) : s \in S\} \leq 1$ where $\mathrm{dh}(E_s)$ denotes the minimal length of a projective resolution of the local $\mathcal{O}_s$-module E_s. Further note that as S is smooth, such a resolution can chosen to consist of locally frees.
Thus, either E itself is locally free (which brings us back to to 2.1) or $\mathrm{dh}(E) = 1$, so there is 2-step resolution $E_{-1} \hookrightarrow E_0 \twoheadrightarrow E$ of locally frees E_i.

[4] see see [HL] 4-6 for this section

Definition 2.2.1. We define the determinant bundle of E to be

$$\det(E) := \det(E_0) \otimes \det(E_{-1})^{-1}$$

which is independent of the choice of resolution and agrees with $\wedge^2(E)$ whenever E is locally free, see e.g. [K] 149-152.

2.3. **The general involution.**

Definition 2.3.1. Denoting $\phi^{\mathrm{t}} := \phi - \mathrm{tr}(\phi) \cdot \mathrm{id}$, we can extend the involution

$$\iota : (E, \phi) \mapsto (E^*, -\phi^*)$$

of **Higgs bundles** to all torsion free **Higgs sheaves** on S by the formula

$$\iota : (E, \phi) \mapsto (E \otimes \det(E)^{-1}, \phi^{\mathrm{t}} \otimes 1),$$

which now makes sense for all rank 2 torsion frees and extends the original involution defined for Higgs bundles by the diagram stated in 2.1.2.

Remark 2.3.2. We remark that we actually have $\iota^2 = \mathrm{id}$ as

$$\begin{aligned}&\iota^2(E, \phi) = \iota(E \otimes \det(E)^{-1}, \phi^{\mathrm{t}} \otimes 1)\\ &= (E \otimes \det(E)^{-1} \otimes \det(E \otimes \det(E)^{-1})^{-1}, (\phi^{\mathrm{t}} + \mathrm{tr}(\phi) \cdot \mathrm{id}) \otimes 1) = (E, \phi)\end{aligned}$$

2.4. **For spectral sheaves.** For the rest of this discussion, we need to define ι in terms of spectral sheaves $\mathcal{E}_\phi$ on X. We start with the trace shift:

Definition 2.4.1. Write $x \in X$ as $x = (s, t)$ with $s \in S$, $t \in K_S$. Define the **trace shift**

$$\sigma_{\mathrm{tr}\phi} : X \to X$$

as

$$(s, t) \mapsto (s, t - \mathrm{tr}(\phi_s))$$

We see that $\sigma_{\mathrm{tr}\phi}$ preserves the fibres of $X \xrightarrow{\pi} S$, i.e. $\pi\sigma_{\mathrm{tr}\phi} = \pi$. It is an invertible map on X with inverse $\sigma_{-\mathrm{tr}\phi}$, acting on spectral sheaves $\mathcal{E}_\phi$ as

$$\mathcal{E}_\phi \mapsto \sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi.$$

In particular, we have $\mathcal{E}_\phi = \sigma_{\mathrm{tr}\phi,*}\sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi$. Wee see now how this map keeps track of the trace shift of Higgs fields on S, namely:

Lemma 2.4.2. We have $\sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi = \mathcal{E}_{\phi^{\mathrm{t}}}$

Proof. Let τ be the tautological section of $\pi^* K_S$ on X and choose local coordinates (s, t). Recall $\tau_{(s,t)} = t$ and $\pi_*(\tau \cdot \mathrm{id})_s = \phi_s$. So $(\sigma^*_{\mathrm{tr}\phi}\tau)_{(s,t)} = \tau_{(s,t-\mathrm{tr}(\phi_s))} = t - \mathrm{tr}(\phi_s)$. Thus $\sigma^*_{\mathrm{tr}\phi}(\tau \cdot \mathrm{id})$ acting on $\sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi$ gives

$$\pi_*(\sigma^*_{\mathrm{tr}\phi}(\tau \cdot \mathrm{id})) = \phi - \mathrm{tr}(\phi) \cdot \mathrm{id} = \phi^{\mathrm{t}}.$$

In addition we compute $\pi_*(\sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi) = \pi_*\sigma_{\mathrm{tr}\phi,*}(\sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi) = \pi_*(\sigma_{\mathrm{tr}\phi,*}\sigma^*_{\mathrm{tr}\phi})\mathcal{E}_\phi = \pi_*\mathcal{E}_\phi = E$.
This shows that $\sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi$ is the spectral sheaf for the Higgs pair (E, ϕ^{t}). □

Definition 2.4.3. We define ι for spectral sheaves $\mathcal{E}_\phi$ on X as

$$\iota : \mathcal{E}_\phi \mapsto \sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi \otimes \pi^* \det(\pi_*\mathcal{E}_\phi)^{-1} = \mathcal{E}_{\phi^{\mathrm{t}}} \otimes \pi^* \det(\pi_*\mathcal{E}_\phi)^{-1}$$

2.5. **Action on the moduli.** After having found the right definition of ι, we discuss how this map acts on $\mathcal{N}$. Namely, let $\mathcal{N}(2, c_1, c_2)$ be the moduli space of torsion frees (E, ϕ) on S. We see that ι defines a map

$$\iota : \mathcal{N}(2, c_1, c_2) \to \mathcal{N}(2, -c_1, c_2)$$

as we compute

$$c_1(E \otimes \det(E)^{-1}) = c_1(E) + 2c_1(\det(E)) = c_1(E) - 2c_1(E) = -c_1(E) \text{ and}$$
$$c_2(E \otimes \det(E))^{-1}) = c_1(\det(E))^2 - c_1(\det(E))c_1(E) + c_2(E) = c_2(E)$$

Using 1.4, we get a similar involution on the Chern classes of $\mathcal{E}$.

Remark 2.5.1. We remark that $\iota = \lambda \circ \sigma_{\mathrm{tr}\Phi}$ and further $\lambda \circ \sigma_{\mathrm{tr}\Phi} = \sigma_{\mathrm{tr}\Phi} \circ \lambda$, thus $\iota^2 = \mathrm{id}$.

Theorem 2.5.2. *The action ι on $\mathcal{N}$ respects the spectral correspondence $\mathcal{E}_\phi \leftrightarrow (E, \phi)$. That is, there is a commutative square*

$$\begin{array}{ccc} X \times \mathcal{N} & \xrightarrow{\mathrm{id}\times\iota} & X \times \mathcal{N} \\ \big\downarrow{\scriptstyle \pi\times\mathrm{id}} & & \big\downarrow{\scriptstyle \pi\times\mathrm{id}} \\ S \times \mathcal{N} & \xrightarrow{\mathrm{id}\times\iota} & S \times \mathcal{N} \end{array} \tag{2.5.3}$$

Remark 2.5.4. We omitted from notation that the horizontal arrows do not preserve the chern classes.

Proof. We've already seen that $\pi_* \mathcal{E}_{\phi^{\mathrm{t}}} = \pi_* \mathcal{E}_\phi = E$.
Then $\pi_*(\iota\mathcal{E}_\phi) = \pi_*(\mathcal{E}_{\phi^{\mathrm{t}}} \otimes \pi^* \det(\pi_*\mathcal{E}_{\phi^{\mathrm{t}}})^{-1}) = E \otimes \det(E)^{-1}$. Now the action $\tau{\cdot}\mathrm{id} \circlearrowright \mathcal{E}_\phi$ induces $\tau{\cdot}\mathrm{id}\otimes 1$ acting on $\mathcal{E}_\phi \otimes \pi^* \det(\pi_*\mathcal{E}_\phi)^{-1}$, so $\pi_*\sigma^*_{\mathrm{tr}\phi}(\tau{\cdot}\mathrm{id}\otimes 1) = \phi^{\mathrm{t}} \otimes 1$. This shows that $\mathcal{E}_{\phi^{\mathrm{t}}} \otimes \pi^* \det(\pi_*\mathcal{E}_\phi)^{-1}$ on X corresponds to the Higgs pair $(E \otimes \det(E)^{-1}, \phi^{\mathrm{t}} \otimes 1)$ on S. $\square$

Remark 2.5.5. Although the c_i on $\mathcal{N}$ are not fixed under ι, replacing $\mathcal{N}$ by $\mathcal{N}(c_1) \sqcup \mathcal{N}(-c_1)$ on S (similar for $\mathcal{E}$ on X, using 1.4) defines a genuine involution $\iota \circlearrowright \mathcal{N}$.

Definition 2.5.6. We may rewrite ι for $\mathcal{E}$ in simpler terms by composing the two involutions $\lambda, \sigma_{\mathrm{tr}\Phi}$ defined as

$$\lambda : \mathcal{E} \mapsto \mathcal{E} \otimes \pi^* \det(\pi_*\mathcal{E})^{-1}$$

and

$$\sigma_{\mathrm{tr}\Phi} : \mathcal{E} \mapsto \sigma^*_{\mathrm{tr}\phi}\mathcal{E}$$

3. Deformations of ι & the fixed locus

3.0. **Summary.** We'll se how $d\iota$ acts on first order deformations $Ext^1(\mathcal{E}, \mathcal{E})$: We'll then identify one of the components of $\mathcal{N}^\iota$ as

$$\mathcal{N}^\perp = \{(E, \phi) : \det(E) \cong \mathcal{O}_S, \mathrm{tr}(\phi) = 0\}.$$

Equivalently, $\mathcal{N}^\perp$ consists of those $\mathcal{E}$ that have "centre of mass zero" on each fibre $X \to S$ and $\det(\pi_*\mathcal{E}) \cong \mathcal{O}_S$.
After that, we study the fixed locus $\mathcal{N}^\iota$ and identify one of its components with $\mathcal{N}^\perp$.

3.1. **Artinian deformations.** We'll observe now how ι acts on Artinian families $f : \mathbf{Spec}(A) \to \mathcal{N}$ for A an Artinian local ring.
Let $X_A \xrightarrow{\pi_A} S_A$ be the corresponding base change of $\pi \times \mathrm{id}_{\mathcal{N}}$ along $\mathrm{id}_S \times f$. Let $\mathcal{E}_A$ be an Artinian family over X_A corresponding to such f, which is in turn the same as a family of Higgs pairs (E_A, ϕ_A) over S_A. Now ϕ_A defines an invertible map $\sigma_{\mathrm{tr}\phi_A} : X_A \to X_A$ shifting the trace on the fibres of $\pi_A : X_A \to S_A$ and we define the line bundle $L_A := \det(E_A)^{-1}$ on S_A.
We identify $\mathcal{E}$ on X with its push-forward $i_{A,*}\mathcal{E}$ on X_A for $i_A : X = X \times 0 \hookrightarrow X_A$ the inclusion of the closed point of $\mathbf{Spec}(A)$, analogously for $\iota\mathcal{E}$. Note that by base change along i_A, we have $\mathrm{T}_{\phi_A} i_A = i_A \mathrm{T}_\phi$ and $\pi_A i_A = i_A \pi$.

Remark 3.1.1. We see that ι acts on Artinian families as

$$\mathcal{E}_A \mapsto \sigma^*_{\mathrm{tr}\phi_A}\mathcal{E}_A \otimes \pi_A^* L_A$$

Furthermore, for $A = \mathbf{C}[t]/(t^2)$ this is the differential ι_* of ι, given by $\iota_* : \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^1(\iota\mathcal{E}, \iota\mathcal{E})$

Proof. Let $A = \mathbf{C}[t]/(t^2)$. Indeed, take an exact sequence $0 \to \mathcal{E} \to \mathcal{E}_A \to \mathcal{E} \to 0$ on X_A which stays exact after applying $\sigma^*_{\mathrm{tr}\phi_A}$ and $\_ \otimes \pi^* L_A$. Furthermore, we compute for $\mathcal{E} = i_{A,*}\mathcal{E}$

$$\begin{aligned}\sigma^*_{\mathrm{tr}\phi_A} i_{A,*}\mathcal{E} \otimes \pi_A^* L_A &= i_{A,*}\sigma^*_{\mathrm{tr}\phi}\mathcal{E} \otimes \pi_A^* L_A \\ &= i_{A,*}(\sigma^*_{\mathrm{tr}\phi}\mathcal{E} \otimes i_A^*\pi_A^* L_A) = i_{A,*}(\sigma^*_{\mathrm{tr}\phi}\mathcal{E} \otimes \pi^* L) = i_{A,*}(\iota\mathcal{E})\end{aligned}$$

This gives the differential map $d\iota : \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^1(\iota\mathcal{E}, \iota\mathcal{E})$ acting as

$$\begin{array}{c} 0 \to \mathcal{E} \to \mathcal{E}_A \to \mathcal{E} \to 0 \\ \Big\downarrow{\scriptstyle d\iota} \\ 0 \to \iota\mathcal{E} \to \sigma^*_{\mathrm{tr}\phi_A}\mathcal{E}_A \otimes \pi_A^* L_A \to \iota\mathcal{E} \to 0 \end{array}$$

□

3.2. **SU**(2)**-Higgs pairs.** The following part will be phrased in terms of Higgs sheaves, where we investigate the scheme-theoretic fixed locus $\mathcal{N}^\iota$ and its relation to **SU**(2)-Higgs pairs, which are defined as

$$\mathcal{N}^\perp := \{(E, \phi) : \det(E) \cong \mathcal{O}_S \text{ and } \mathrm{tr}(\phi) = 0\} \subset \mathcal{N}$$

Now let (E, ϕ) be ι-fixed: Then $E \cong E \otimes \det(E)^{-2}$, hence taking determinants gives $\det(E) \cong \det(E \otimes \det(E)^{-1}) = \det(E) \otimes \det(E)^{-2}$, so we see that $\det(E)$ is a 2-torsion line bundle. We also find $\phi = \phi^{\mathrm{t}}$, so $\mathrm{tr}(\phi) = 0$. If in addition $\det(E) \cong \mathcal{O}_S$ holds, we conclude that $(E, \phi) \in \mathcal{N}^\perp$.We can state the following:

Proposition 3.2.1. The ι-fixed locus $\mathcal{N}^\iota$ consists of trace-free Higgs pairs (E,ϕ) where $\det(E)$ is 2-torsion. Conversely, if $(E,\phi) \in \mathcal{N}^\perp$, then there exists an isomorphism $\iota(E,\phi) \cong (E,\phi)$, so $(E,\phi) \in \mathcal{N}^\iota$.

Proof. We've already described $\mathcal{N}^\iota$ and divide the proof for $\mathcal{N}^\perp \subset \mathcal{N}^\iota$ into two steps.
We start with locally free sheaves, as this case is more illusive: i.e. we show first that a pair (E,ϕ) with $\det(E) \cong \mathcal{O}_S$ and $\mathrm{tr}(\phi) = 0$ is contained in $\mathcal{N}^\iota$. Recall that for locally free E, ι can be written as $(E,\phi) \mapsto (E^*, -\phi^*)$.
In the second step, we'll generalise to torsion frees and show that an Artinian family (E_A, ϕ_A), fixed under ι is contained in $\mathcal{N}^\iota$.
Now let $(E,\phi) \in \mathcal{N}^\perp$ with E a vector bundle. Recalling the splitting $\mathcal{H}om(E,E^*) = \mathcal{S}ym^2(E^*) \oplus \det(E)^{-1}$, $\det(E)$ being trivial implies it admits a nowhere vanishing section α, that is, a skew isomorphism $\alpha : E \xrightarrow{\sim} E^*$ endowing E with a symplectic structure.
As $\mathrm{tr}(\phi) = 0$, the key lemma 2.1.1 reads as $\alpha\phi = \alpha(-\phi^*)$, so

$$\alpha : (E,\phi) \xrightarrow{\sim} (E^*, -\phi^*)$$

defines an isomorphism of Higgs bundles. [5]
Now let (E_A, ϕ_A) be an Artinian family of torsion frees corresponding to a map $\mathbf{Spec}(A) \to \mathcal{N}^\perp$. We need to show this family is ι-fixed: As $\mathrm{tr}(\phi_A) = 0$, we have $\phi_A = \phi_A^{\mathrm{t}}$. Furthermore, there exists a trivialisation $\alpha : \det(E_A)^{-1} \xrightarrow{\sim} \mathcal{O}_{S_A}$ and we claim that

$$\begin{tikzcd} E_A \otimes \det(E_A)^{-1} \arrow[r, "\mathrm{id}\otimes\alpha"] \arrow[d, "\phi_A^{\mathrm{t}}"] & E_A \arrow[d, "\phi_A"] \\ E_A \otimes \det(E_A)^{-1} \otimes K_{S_A} \arrow[r, "\mathrm{id}\otimes f\otimes 1"] & E_A \otimes K_{S_A} \end{tikzcd}$$

commutes. As $\alpha, \alpha^{-1} \in \mathcal{O}_{S_A}$ and ϕ_A is $\mathcal{O}_{S_A}$-linear, we see that $\alpha\phi_A^{\mathrm{t}}\alpha^{-1} = \phi_A^{\mathrm{t}} = \phi_A$, so α defines an isomorphism

$$(E_A \otimes \det(E_A)^{-1}, \phi_A^{\mathrm{t}}) \xrightarrow{\sim} (E_A, \phi_A)$$

thus $(E_A, \phi_A) \in \mathcal{N}^\iota$. □

Corollary 3.2.2. We observe that this identifes $\mathcal{N}^\perp$ with a component of $\mathcal{N}^\iota$, as it is open therein: It is also closed:

Proof. Indeed, restricting the map $\det : \mathcal{N} \to \mathbf{Pic}(S)$ which sends $(E,\phi) \mapsto \det(E)$ to $\mathcal{N}^\iota$ has image in the discrete set of 2-torsion line bundles $\mathbf{Pic}(S)[2]$ and thus decomposes $\mathcal{N}^\iota$ into different components. In particular, $\mathcal{N}^\perp = \det^{-1}([\mathcal{O}_S]) \cap \mathcal{N}^\iota$ is closed. Note that $\mathcal{N}^\iota \subset \mathcal{N}$ itself is closed being the fixed locus of a finite group action. □

[5] We remark that over $\mathcal{N}^\perp$, the map $\alpha : (E,\phi) \xrightarrow{\sim} (E^*, -\phi^*)$ linearises the functor $(E,\phi) \mapsto \iota(E,\phi)$. ι-linearised functors will be discussed in detail later.

Remark 3.2.3. Equivalently, $\mathcal{N}^\perp \subset \mathcal{N}$ are those spectral sheaves $\mathcal{E}$ that have *center of mass* (i.e. the sum over the points in each fibre $X \xrightarrow{\pi} S$, weighted by multiplicity) equal to zero and $\det(\pi_*\mathcal{E}) \cong \mathcal{O}_S$.

4. Equivariant sheaves & the Atiyah class

4.0. **Summary.** We want to generalise $\iota \circlearrowright \mathcal{N}$ to families over $\mathcal{N}$, i.e. we want to lift ι*-equivariantly* to the universal sheaf $\mathscr{E}$ on $X \times \mathcal{N}$ and later to the obstruction complex $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})$ on $\mathcal{N}$.
This section introduces the right notion of ι*-equivariance* and after that, the objects for defining the obstruction theory. As a first consequence, we'll apply this to our problem by making $\mathscr{E}$ equivariant with respect to $\sigma_{\mathrm{tr}\Phi} \circlearrowright \mathcal{N}$.

4.1. **Equivariant sheaves.** Let G be an algebraic group acting on $\mathcal{N}$. Its elements $g \in \mathrm{G}$ act via pull-back $\mathscr{E} \mapsto g^*\mathscr{E}$ on $\mathbf{Coh}(\mathcal{N})$.
Now let $\mathrm{G} \cong \mathbf{Z}/2\mathbf{Z}$, i.e. $\mathrm{G} = \langle \iota \rangle$ for an involution $\iota \circlearrowright \mathcal{N}$.

Definition 4.1.1. We call $\mathscr{E}$ a ι-equivariant sheaf or ι-linearised sheaf if there is an isomorphism $\theta_\iota : \mathscr{E} \to \iota^*\mathscr{E}$ such that

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "\theta_\iota"] \arrow[dr, equal] & \iota^*\mathscr{E} \arrow[d, "\iota^*\theta_\iota"] \\
 & (\iota^*)^2\mathscr{E}
\end{tikzcd}
\tag{4.1.2}
$$

commutes

Definition 4.1.3. We call a morphism $f : \mathscr{E} \to \mathscr{E}'$ a morphism of ι-equivariant sheaves, if for $\mathscr{E}, \mathscr{E}'$ are linearised as above, the triangles induced by $\theta_\iota : \mathscr{E} \to \iota^*\mathscr{E}$ and $\theta'_\iota : \mathscr{E}' \to \iota^*\mathscr{E}'$ map to each other via

$$
\begin{tikzcd}
\mathscr{E} \arrow[r, "\theta_\iota"] \arrow[d, "f"] & \iota^*\mathscr{E} \arrow[d, "\iota^* f"] \\
\mathscr{E}' \arrow[r, "\theta'_\iota"] & \iota^*\mathscr{E}'
\end{tikzcd}
\tag{4.1.4}
$$

and its pullback by ι.

Remark 4.1.5. We note that above definitions generalise to $\mathbf{D}^b(\mathcal{N})$, such that pairs $(\mathscr{E}, \theta_\iota)$ with their compatible maps f form a category denoted by $\mathbf{D}^b(\mathcal{N})^{\langle \iota \rangle}$. For a thorough introduction, we refer the reader to [R] 3-6.

4.2. **Equivariant embeddings.** A ι-equivariant invertible sheaf $\mathcal{L}$ is called ι*-linearised*: If $\mathcal{L}$ is very ample on $\mathcal{N}$, ι-linearised $\theta_\iota : \mathcal{L} \xrightarrow{\sim} \iota^*\mathcal{L}$, then it satisfies 4.1.2: this makes $H^0(\mathcal{L})$ into a $\langle \iota \rangle$-vector space, defined by the action

$$H^0(\mathcal{L}) \xrightarrow{\theta_\iota} H^0(\iota^*\mathcal{L}) \xrightarrow{\sim} H^0(\mathcal{L})$$

where the second arrow is the natural pullback. The induced embedding $\mathcal{N} \hookrightarrow \mathbf{P}(H^0(\mathcal{L})^\vee)$ lifts ι to a projective ambient space $\mathbf{P}$ of $\mathcal{N}$, extending by construction the $\mathbf{Z}/(2\mathbf{Z})$-action ι.
As an application we construct an equivariant embedding $\mathcal{N} \hookrightarrow \mathcal{A}$ into a smooth ambient space. By 4.2, this follows if there exists a ι-equivariant very ample line bundle on $\mathcal{N}$. We recall that $\mathcal{N}$ is quasi-projective and fix a very ample line bundle $\mathcal{L}$ on $\mathcal{N}$.

Lemma 4.2.1. There exists a ι-linearised line bundle on $\mathcal{N}$.

Proof. Indeed, the line bundle $\mathcal{L} \otimes \iota^*\mathcal{L}$ is again very ample and now ι^* linearised by swapping the factors.
As explained in 4.2, this gives a projective embedding $\mathcal{N} \hookrightarrow \mathcal{A}$ into a smooth ambient space $\mathcal{A}$, lifting the action of ι. □

4.3. **Illusie's cotangent complex.** We denote by $\mathbf{L}_\mathcal{N}$ Illusie's truncated cotangent complex. As $\mathcal{N} \subset \mathcal{A}$ admits an embedding into a smooth $\mathcal{A}$, it is represented by the two term

$$\mathbf{L}_\mathcal{N} := [\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega_\mathcal{A}|_\mathcal{N}] \in \mathbf{D}^{[-1,0]}(\mathcal{N})$$

for $\mathcal{I} \subset \mathcal{O}_\mathcal{A}$ the ideal sheaf of this embedding; we note that $h^0(\mathbf{L}_\mathcal{N}) \cong \Omega_\mathcal{N}$.
Similarly, we may define $\mathbf{L}_{X\times\mathcal{N}}$.
It is a well known fact that up to quasi-isomorphism, $\mathbf{L}_\mathcal{N}$ is independent of the choice of $\mathcal{A}$, see eg. [R] 17.
Furthermore, $\mathbf{L}_\mathcal{N}$ is functorial, i.e. for morphisms $f : \mathcal{N} \to \mathcal{N}'$ there are differentials

$$f_* : f^*\mathbf{L}_{\mathcal{N}'} \to \mathbf{L}_\mathcal{N}.$$

For later computations,we denote by $\mathbf{T} := \mathbf{L}^\vee$ the tangent complex, dual to $\mathbf{L}$ and refer to [I] 160-172 for more details. Similar to above, we have natural maps $f_* : \mathbf{T}_\mathcal{N} \to f^*\mathbf{T}_{\mathcal{N}'}$ We cite the following lemma from [R] 19:

Remark 4.3.1. If $\mathcal{N} \subset \mathcal{A}$ is a smooth embedding extending any involution ι as in 4.2.1, then $\mathbf{L}_\mathcal{N}$ is canonically ι-equivariant.

4.4. **Definition.** Let $i_{\Delta_{X\times\mathcal{N}}} : X\times\mathcal{N} \to \Delta_{X\times\mathcal{N}} \subset (X\times\mathcal{N})^2$ be the diagonal map with canonical projections p_1, p_2 to $X \times \mathcal{N}$.
The *universal Atiyah class* is given by a morphism

$$\alpha_{X\times\mathcal{N}} : \mathcal{O}_{\Delta_{X\times\mathcal{N}}} \to i_{\Delta_{X\times\mathcal{N}},*}\mathbf{L}_{X\times\mathcal{N}}[1],$$

see [HT] chapter 5 for details.
The *full Atiyah class* $\mathrm{At}_\mathscr{E}$ of $\mathscr{E}$ is

$$\mathrm{At}_\mathscr{E} := p_{2,*}(p_1^*\mathscr{E} \otimes \alpha_{X\times\mathcal{N}}) : \mathscr{E} \to \mathscr{E} \otimes \mathbf{L}_{X\times\mathcal{N}}[1]$$

which can be seen as a morphism

$$\mathbf{R}\mathcal{H}om(\mathscr{E}, \mathscr{E}) \to \mathbf{L}_{X\times\mathcal{N}}[1].$$

4.5. **Naturality.** A simple but important observation is the fact that the Atiyah class is natural: If $g : \mathscr{F} \to \mathscr{E}$ is a morphism, then

$$\begin{tikzcd} \mathscr{F} \arrow[rr, "g"] \arrow[d, "\mathrm{At}_{\mathscr{F}}"] & & \mathscr{E} \arrow[d, "\mathrm{At}_{\mathscr{E}}"] \\ \mathscr{F} \otimes \mathbf{L}_{X\times\mathcal{N}}[1] \arrow[rr, "g\otimes 1"] & & \mathscr{E} \otimes \mathbf{L}_{X\times\mathcal{N}}[1] \end{tikzcd} \tag{4.5.1}$$

commutes, by functoriality of $p_{2,*}, p_1^*$ and tensor product.

4.6. **Functoriality.** Let $f \circlearrowright X \times \mathcal{N}$ be a map. By functoriality of the Atiyah class we mean that the pullback of $\mathbf{R}\mathcal{H}om(\mathscr{E}, \mathscr{E}) \to \mathbf{L}_{X\times\mathcal{N}}[1]$ by f composed with the natural differential $f^*\mathbf{L}_{X\times\mathcal{N}} \xrightarrow{f_*} \mathbf{L}_{X\times\mathcal{N}}$ equals the Atiyah class of the pullback, i.e.

$$\mathrm{At}_{f^*\mathscr{E}} = f_* \circ f^*\mathrm{At}_{\mathscr{E}}.$$

4.7. **The partial Atiyah class.** Composing the natural $\mathbf{L}_{X\times\mathcal{N}} \to p_X^*\mathbf{L}_{\mathcal{N}}$ with $\mathrm{At}_{\mathscr{E}}$ gives

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E}) \to p_X^*\mathbf{L}_{\mathcal{N}}[1]$$

By Grothendieck-Verdier duality [6] along the projective morphism p_X which is of dimension 3 gives

$$\mathbf{R}p_{X,*}(\mathbf{R}\mathcal{H}om(\mathscr{E}, \mathscr{E}) \otimes \omega_{p_X})[2] \to \mathbf{L}_{\mathcal{N}}.$$

Now ω_{p_X} is trivial, as X is Calabi-Yau and we call the resulting morphism

$$\mathrm{At}_{\mathscr{E},\mathcal{N}} : \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[2] \to \mathbf{L}_{\mathcal{N}}$$

the *partial* Atiyah class on $\mathcal{N}$.

Remark 4.7.1. Whenever possible, we omit the subscript $\mathcal{N}$ from notation.

To close this section, we make the trace shift $\sigma_{\mathrm{tr}\Phi} \circlearrowright \mathcal{N}$ of 2.5.6 into an equivariant action on the universal sheaf $\mathscr{E}$.

Definition 4.7.2. We recall from 2.5.6 that $\sigma_{\mathrm{tr}\phi} \circlearrowright X$ acts on local coordinates as $(s, t) \mapsto (s, t - \mathrm{tr}(\phi_s))$. This defined on $\mathcal{N}$ the involution

$$\sigma_{\mathrm{tr}\Phi} : \mathcal{E}_\phi \mapsto \sigma_{\mathrm{tr}\phi}^*\mathcal{E}_\phi = \mathcal{E}_{\phi^{\mathrm{t}}}$$

We lift this to an action $\sigma \circlearrowright X \times \mathcal{N}$ where we define $\sigma := (\sigma_{-\mathrm{tr}\phi}, \sigma_{\mathrm{tr}\phi}^*)$. Here, $\sigma_{-\mathrm{tr}(\phi)} \circlearrowright X$ is the inverse to $\sigma_{\mathrm{tr}\phi}$.

Lemma 4.7.3. There exists a linearisation $\mathscr{E} \cong \sigma^*\mathscr{E}$ satisfying 4.1.2.

Proof. Write $n = [\mathcal{E}] \in \mathcal{N}$ for the class of a sheaf.
Let $(\sigma_{\mathrm{tr}\phi} \times \mathrm{id}) \circlearrowright X \times \mathcal{N}$ be the map that sends $(x, n) \mapsto (\sigma_{\mathrm{tr}\phi}(x), n)$. Pulling back gives the family $(\sigma_{\mathrm{tr}\phi} \times \mathrm{id})^*\mathscr{E}$ over $X \times \mathcal{N}$ which corresponds by the universal property of $\mathcal{N}$ to a unique morphism

$$f : \mathcal{N} \to \mathcal{N}$$

[6] for a precise statement of Grothendieck Verdier duality we refer to [H] 86-90

such that

$$(\mathrm{id} \times f)^* \mathscr{E} \cong (\sigma_{\mathrm{tr}\phi} \times \mathrm{id})^* \mathscr{E} \otimes p_{\mathcal{N}}^* \mathscr{P}$$

for some $\mathscr{P} \in \mathbf{Pic}(\mathcal{N})$.
By comparing stalks, we see that f is the map $n \mapsto \sigma_{\mathrm{tr}\Phi}^* n$ and $\mathscr{P}$ is in fact trivial. Applying $(\sigma_{-\mathrm{tr}\phi} \times \mathrm{id})^*$ on both sides yields

$$(\sigma_{-\mathrm{tr}\phi} \times \sigma_{\mathrm{tr}\Phi}^*)^* \mathscr{E} \cong \mathscr{E},$$

where the LHS agrees now with $\sigma^* \mathscr{E}$. Thus, we can pick an isomorphism $\Psi : \mathscr{E} \cong \sigma^* \mathscr{E}$. By construction, its inverse is $\sigma^* \Psi$, so there is a commutative square

$$\begin{tikzcd} \mathscr{E} \arrow[r, "\Psi"] \arrow[dr, equal] & \sigma^* \mathscr{E} \arrow[d, "\sigma^* \Psi"] \\ & (\sigma^*)^2 \mathscr{E} \end{tikzcd} \tag{4.7.4}$$

proving the equivariance □

Lemma 4.7.5. There exists a smooth ambient space for $\mathrm{X} \times \mathcal{N} \subset \mathcal{A}$ extending σ

Proof. As X is smooth, it is enough by 4.2.1 to find a σ-linearised very ample line bundle on $\mathcal{N}$. As in 4.2.1, we may consider $\mathcal{L} \otimes \sigma^* \mathcal{L}$, σ-linearised by swapping the factors. □

By 4.3.1, this makes $\mathbf{L}_{\mathcal{N}}$ σ-linearised. We can show:

Corollary 4.7.6. The partial Atiyah class $\mathrm{At}_{\mathscr{E}}$ on $\mathcal{N}$ admits a σ-linearisation:

Proof. Together with a σ-equivariant smooth embedding $X \times \mathcal{N} \subset \mathcal{A}$, the linearisation $\Psi : \mathscr{E} \cong \sigma^* \mathscr{E}$, makes the full Atiyah class $\mathrm{At}_{\mathscr{E}}$ on $X \times \mathcal{N}$ is compatible with σ ([R] Cor. 4.4.).
This makes $\mathrm{At}_{\mathscr{E}}$ into an element of $\mathrm{Ext}^1(\mathscr{E}, \mathscr{E} \otimes \mathbf{L}_{X \times \mathcal{N}})^{\langle \sigma \rangle}$. The rest is similar to 4.7, but now equivariant: Composing with the (naturally σ-equivariant) projection $\mathbf{L}_{X \times \mathcal{N}} \to p_{\mathcal{N}}{}^* \mathbf{L}_{\mathcal{N}}$ gives

$$\mathbf{R}\mathcal{H}om(\mathscr{E}, \mathscr{E}) \to p_{\mathcal{N}}^* \mathbf{L}_{\mathcal{N}}[1].$$

Applying equivariant Grothendieck duality to p_X [7], which is a morphism of dim 3 and noting again that the relative canonical ω_{p_X} is trivial gives a σ-equivariant partial Atiyah class

$$\mathrm{At}_{\mathscr{E}, \mathcal{N}} : \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[2] \to \mathbf{L}_{\mathcal{N}}$$

□

[7] for a statement of this duality, see [R] Thm. 2.27

Remark 4.7.7. After taking duals, the σ-linearisation can now be written as

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\sigma_*"] & \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})[1] \\ \mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E},\mathcal{N}}"] \arrow[r, "\sigma_*"] & \sigma^*\mathbf{T}_{\mathcal{N}} \arrow[u, "\sigma^*\mathrm{At}_{\mathscr{E},\mathcal{N}}"'] \end{tikzcd} \tag{4.7.8}$$

Here, we remark that the upper horizontal arrow is the push-down via p_X of the adjoint action of Ψ,

$$\mathbf{R}\mathcal{H}om(\mathscr{E},\mathscr{E}) \to \mathbf{R}\mathcal{H}om(\sigma^*\mathscr{E},\sigma^*\mathscr{E}),$$

given by $g \mapsto \Psi g \Psi^{-1}$ for $g \in \mathbf{R}\mathcal{H}om(\mathscr{E},\mathscr{E})$.

5. Perfect obstruction theories

5.0. **Summary.** Let $n = [\mathcal{E}] \in \mathcal{N}$ be a closed point. It has deformations given by $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ and obstructions by $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$. We call the integer

$$\mathrm{vd} := \mathrm{ext}^1(\mathcal{E},\mathcal{E}) - \mathrm{ext}^2(\mathcal{E},\mathcal{E}) = 1 - h^3(\mathcal{O}_X) - \int \mathrm{ch}(\mathcal{E}^\vee)\mathrm{ch}(\mathcal{E})\mathrm{Td}_X$$

the *virtual* or *expected* dimension of $\mathcal{N}$ and remark that this is a (topological) constant on $\mathcal{N}$.
Globally, we want to find a 2-term complex of vector bundles on $\mathcal{N}$

$$V = [V^{-1} \to V^0]$$

such that $h^0(V^\vee(n)) = \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ and $h^{-1}(V^\vee(n)) = \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$. Here, $V^\vee(n) = j_n^*V$ for $j : \{n\} \hookrightarrow \mathcal{N}$ the class of a sheaf $[\mathcal{E}] = n$.

5.1. **Virtual cycles.** The right data relating extrinsic obstructions (those coming from V) with intrinsic ones (coming from $\mathbf{L}_{\mathcal{N}}$) is a *perfect obstruction theory* [8]:

Definition 5.1.1. This is a pair (V,ψ) on $\mathcal{N}$ consisting of a

- a 2-term complex $V = [V^{-1} \to V^0]$ of vector bundles in $\mathbf{D}^{[-1,0]}(\mathcal{N})$
- a morphism $\psi : V \to \mathbf{L}_{\mathcal{N}}$ to the truncated cotangent complex, such that $h^0(\psi)$ is an isomorphism and $h^{-1}(\psi)$ onto.

The moduli space $\mathcal{N}$ then inherits a *virtual cycle*

$$[\mathcal{N}]^{vir} := 0^!_{V_1}[C] \in A_{vd}(\mathcal{N}),$$

in the Chow group $A_*(\mathcal{N})$, where $0^!_{V_1}$ is the Gysin map for $V_1 = V^{-1,\vee}$ and $C = C(V^\bullet)$ is the Behrend-Fantechi cone, which is a closed subcone of V_1 [9]. $[\mathcal{N}]^{vir}$ is often called the *virtual fundamental class* $[\mathcal{N}]^{vir}_V$ of $\mathcal{N}$ with respect to V.

[8] we omit the definition of a more general *obstruction theory* which is stated in [BF] Def. 4.4

[9] see [BF] Def. 5.2

Remark 5.1.2. The important observation here is that V (if it exists with the desired properties stated in 5.0) equips $\mathcal{N}$ with a virtual cycle of dimension 0, since on the Calabi-Yau threefold X we have $\mathrm{vd} = 0$ as

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \cong \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})^*.$$

Furthermore, if $\mathcal{N}$ was compact, this would give an actual sheaf count

$$\int_{[\mathcal{N}]^{vir}} 1 \in \mathbf{Z}$$

Remark 5.1.3. The complex V is often called the virtual cotangent bundle $\mathbf{T}^{vir}_{\mathcal{N}}$. A scheme with a perfect obstruction theory V is called *virtually* smooth. Note that this depends on the choice of V.
Furthermore, we remark that if $\mathcal{N}$ is smooth, it admits a natural perfect obstruction theory given by $\Omega_{\mathcal{N}}$ and $[\mathcal{N}]^{vir} = [\mathcal{N}]$ agrees with the usual fundamental class. If $\mathcal{N}$ is l.c.i., $\mathbf{L}_{\mathcal{N}}$ is perfect and can taken to be the obstruction complex with ψ being the identity.

Proposition 5.1.4. The obstruction theory given by the trucated partial Atiyah class

$$\mathrm{At}_{\mathscr{E},\mathcal{N}} : \tau^{[-1,0]}\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[2] \to \mathbf{L}_{\mathcal{N}}$$

admits a 2-term representation of vector bundles.

Proof. As X is non-compact, we embed $j : X \subset \overline{X} = \mathbf{P}(K_S^* \oplus \mathcal{O}_S)$ into its projective completion and identify the spectral sheaves $\mathcal{E}$ with their push-forward $j_*\mathcal{E}$. Then $\overline{\pi} : \overline{X} \to S$ is a $\mathbf{P}^1$-bundle containing X as an open. Let $\mathcal{O}(1)$ be a polarisation[10] on $\overline{X}$ or its pull-back to $\overline{X} \times \mathcal{N}$.
Although $\overline{X}$ is not of Calabi-Yau type, we see that

$$\mathcal{E} \otimes \omega_{\overline{X}} \cong j_*(\mathcal{E} \otimes j^*\omega_{\overline{X}}) = j_*(\mathcal{E} \otimes \omega_X) = j_*\mathcal{E} = \mathcal{E},$$

so its canonical $\omega_{\overline{X}}$ is trivial when restricted to $\mathrm{supp}(\mathcal{E})$.
Fix a universal family $\mathscr{E}$ on $\overline{X} \times \mathcal{N}$ for spectral sheaves $n = [\mathcal{E}] \in \mathcal{N}$, which is by definition supported on $X \times \mathcal{N} \subset \overline{X} \times \mathcal{N}$. Similar to the computation above, we remark that

$$\mathscr{E} \otimes \omega_{p_{\overline{X}}} \cong \mathscr{E}$$

for the *relative* dualising sheaf $\omega_{p_{\overline{X}}} = p_{\overline{X}}^*\omega_{\overline{X}}$ on $\overline{X} \times \mathcal{N}$, again because

$$(j \times \mathrm{id})^* p_{\overline{X}}^*\omega_{\overline{X}} \cong p_X^* j^*\omega_{\overline{X}} \cong p_X^*\omega_X \cong \mathcal{O}$$

$\mathcal{N}$ admits a closed embedding into a smooth $\mathcal{A}$ with ideal sheaf $\mathcal{I}$, so the *truncated* cotangent complex admits an explicit 2-term representation

$$\mathbf{L}_{\mathcal{N}} = [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}}|_{\mathcal{N}}]$$

in degrees $-1, 0$.
We choose a sufficiently negative finite resolution $F^\bullet$ on $\overline{X} \times \mathcal{N}$ of locally frees representing $\mathbf{R}\mathcal{H}om(\mathscr{E}, \mathscr{E})$ such that

[10] this could be $\pi^*\mathcal{O}_S(k) \otimes H^l$ for suitable $k, l > 0$ and H the relative hyperplane bundle for $X \to S$

- the full Atiyah class $\mathrm{At}_{\mathscr{E}} : \mathbf{R}\mathcal{H}om(\mathscr{E},\mathscr{E}) \to \mathbf{L}_{\mathcal{N}}[1]$ is represented by complexes.
- the push-downs $p_{\overline{X},*}F^k$ to $\mathcal{N}$ are again locally free for all k.

The latter can be achieved e.g. by making sure $F^{\bullet,\vee}$ has no fibre-wise cohomology $H^i(\overline{X}_n, F^k|_{\overline{X}_n}) = 0$ for $i > 0$ and all k. This forces $p_{\overline{X},*}(F^{\bullet,\vee})$ to consist of locally frees, hence the same holds for $p_{\overline{X},*}F^{\bullet}$.
Applying Verdier duality to $p_{\overline{X}}$, this is isomorphic to

$$\mathbf{R}\mathcal{H}om_{p_{\overline{X}}}(\mathscr{E},\mathscr{E}) \otimes \omega_{p_{\overline{X}}}[2] \to \mathbf{L}_{\mathcal{N}} \tag{5.1.5}$$

By the above computation we have

$$\mathbf{R}\mathcal{H}om_{p_{\overline{X}}}(\mathscr{E},\mathscr{E}) \otimes \omega_{p_{\overline{X}}} \cong \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})$$

so the LHS of 5.1.5 is represented by a finite complex of locally frees $p_{\overline{X},*}F^{\bullet}[2]$. Thus, truncation to degrees $-1, 0$ gives the truncated partial Atiyah class on $\mathcal{N}$

$$\mathrm{At}_{\mathscr{E},\mathcal{N}} : \tau^{[-1,0]}\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[2] \to \mathbf{L}_{\mathcal{N}}$$

where the LHS admits a 2-term representation of vector bundles, as we made sure that $p_{\overline{X},*}F^{\bullet}[2]$ consist of locally frees.

□

Remark 5.1.6. First observe that the $\mathrm{At}_{\mathscr{E},\mathcal{N}}$ is by [HT] Lemma 4.2 an obstruction theory and now perfect.
We denote by $V^{\bullet}$ the resulting 2-term representation of

$$\tau^{[-1,0]}\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[2]$$

and remark that the proof gives a map

$$[V^{-1} \to V^0] \xrightarrow{\psi} [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}}|_{\mathcal{N}}] \text{ in } \mathbf{D}^{[-1,0]}(\mathcal{N})$$

We'll work in the next section with the non-truncated partial Atiyah class only and will often use the dual

$$\mathrm{At}_{\mathscr{E},\mathcal{N}} : \mathbf{T}_{\mathcal{N}} \to \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$$

The 2-term representation of the virtual tangent bundle is going to be important the last section only.
Again, we drop the subscript $\mathcal{N}$ whenever possible.

6. The trace-identity splitting

6.0. **Summary.** The following section sets up the right notation for a ι-*equivariant Atiyah class* for $\mathscr{E}$, in order to make the obstruction theory from 5.1.4 compatible ι.
In order to do so, we'll split $\iota = \sigma \circ \lambda$ into the line bundle twist λ and the trace shift σ defined in 2.5.6 and observe there is a natural lift of these maps

to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})$.
Furthermore, we'll have a closer look at the trace-identity split maps

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathbf{R}p_{S_*}K_S[-1] \\ \mathbf{R}p_{S_*}\mathcal{O}_S. & \end{tikzcd} \tag{6.0.1}$$

and observe that the *natural* lifts of σ, λ act as the identity on these two summands.
We conclude with the observation that the right choice of an equivariant lift of λ and σ to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})$ should deform $\mathcal{O}_S$ and K_S respectively. The "correction" terms to the natural lift will be introduced in the next two sections for λ, σ separately.

6.1. **Setup.** Let $\iota \circlearrowright \mathcal{N}$, now seen as a $\mathbf{Z}/(2\mathbf{Z})$ action $\langle\iota\rangle$.

Definition 6.1.1. As in 4.1.1, we call $\mathrm{At}_{\mathscr{E}}$ ι-equivariant if there is a commutative diagram

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\theta_\iota"] & \mathbf{R}\mathcal{H}om_{p_X}(\iota^*\mathscr{E},\iota^*\mathscr{E})[1] \\ \mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E}}"] \arrow[r, "\iota_*"] & \iota^*\mathbf{T}_{\mathcal{N}} \arrow[u, "\iota^*\mathrm{At}_{\mathscr{E}}"'] \end{tikzcd} \tag{6.1.2}$$

such that the two commuting triangles

$$\begin{tikzcd} \mathbf{T}_{\mathcal{N}} \arrow[r, "\iota_*"] \arrow[dr, equal] & \iota^*\mathbf{T}_{\mathcal{N}} \arrow[d, "\iota^*(\iota_*)"] \\ & (\iota^{*,2})\mathbf{T}_{\mathcal{N}} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \arrow[r, "\theta_\iota"] \arrow[dr, equal] & \mathbf{R}\mathcal{H}om_{p_X}(\iota^*\mathscr{E},\iota^*\mathscr{E}) \arrow[d, "\iota^*\theta_\iota"] \\ & \mathbf{R}\mathcal{H}om_{p_X}(\iota^{2,*}\mathscr{E},\iota^{2,*}\mathscr{E}) \end{tikzcd}$$

map to each other vial $\mathrm{At}_{\mathscr{E}}$ and $\iota^*\mathrm{At}_{\mathscr{E}}$.

Remark 6.1.3. For an involution, the composition of the differential maps $\mathbf{T}_{\mathcal{N}} \to \iota^*\mathbf{T}_{\mathcal{N}} \to \iota^{*,2}\mathbf{T}_{\mathcal{N}}$ gives naturally the identity, so we get the first triangle for free. Thus, we are left to construct a map

$$\theta_\iota : \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \to \mathbf{R}\mathcal{H}om_{p_X}(\iota^*\mathscr{E},\iota^*\mathscr{E})$$

compatible with $\mathrm{At}_{\mathscr{E}}$ in the above sense.

6.2. **Strategy.** We'll decompose $\iota = \sigma \circ \lambda$ into two maps (from the definition 2.4.3) and construct $\theta_\lambda, \theta_\sigma$ separately.
The determinant twist $\lambda \circlearrowright \mathcal{N}$ was the map

$$\mathcal{E} \mapsto \mathcal{E} \otimes \pi^* \det(\pi_* \mathcal{E})^{-1}.$$

The trace shift $\sigma \circlearrowright \mathcal{N}$ is

$$\mathcal{E} \mapsto \sigma^*_{\operatorname{tr}\phi} \mathcal{E}.$$

We remark that both λ and σ have square equal to id.
Furthermore, a similar computation to the one done in 2.5.2 shows that

Lemma 6.2.1. $\lambda\sigma = \sigma\lambda$ holds.

6.3. **Trace and determinant.** We start with the global trace and determinant maps

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\operatorname{tr}} & \Gamma(K_S) \\ \Big\downarrow{\scriptstyle \det \pi_*} & & \\ \mathbf{Pic}(S) & & \end{array}$$

defined by $\mathcal{E} \mapsto \det(\pi_* \mathcal{E})$ and $\mathcal{E} \mapsto \operatorname{tr}(\pi_*(\tau \cdot \mathrm{id}))$ for $\tau \cdot \mathrm{id}$ the tautological endomorphism on X.
At the level of tangent spaces at a fixed sheaf $\mathcal{E} \in \mathcal{N}$, this is given by

$$\begin{array}{ccc} \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) & \longrightarrow & H^0(K_S) \\ \Big\downarrow & & \\ H^1(\mathcal{O}_S) & & \end{array}$$

Globally on $\mathcal{N}$, the cohomology at each point $[\mathcal{E}] \in \mathcal{N}$ of a representation $V^\bullet$ of the virtual tangent bundle $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[1]$ computes $\operatorname{Ext}^*(\mathcal{E}, \mathcal{E})$. Thus the maps

$$\begin{array}{ccc} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E}) & \longleftrightarrow & \mathbf{R}p_{S_*} K_S[-1] \\ \Big\updownarrow & & \\ \mathbf{R}p_{S_*} \mathcal{O}_S. & & \end{array} \tag{6.3.1}$$

can be seen as the *virtual* differential of the trace and determinant maps.[11].
The arrow

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E}) \xrightarrow{\operatorname{tr}\pi_*} \mathbf{R}p_{S,*} \mathcal{O}_S$$

is split with right-inverse $\frac{1}{\operatorname{rank}(\pi_* \mathcal{E})}(\pi^* \cdot \mathrm{id}) = \frac{1}{2}\pi^*(\cdot \mathrm{id})$.
Replacing the arrows by their duals gives after applying Grothendieck-Verdier duality the splitting

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E}) \leftrightarrow \mathbf{R}p_{S,*} K_S[-1]$$

[11]although this might be clear from an intuitive point of view, showing that above split maps are indeed the *virtual* tangent maps of trace and determinant is very difficult. A partial answer (for det) can be found in [STV] and the entire discussion on over 30 pages in [TT]

Note that this shifts the RHS by -1 as p_X is of dimension 3 and p_S of dimension 2. We denote the resulting arrows by a and b respectively, now being Serre dual to $\pi^*(\cdot\mathrm{id})$ and $\mathrm{tr}\pi_*$, i.e. $a \circ b = 2 \cdot \mathrm{id}$.

Lemma 6.3.2. The composition

$$\mathbf{R}p_{S,*}K_S[-1] \xrightarrow{b} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \xrightarrow{\mathrm{tr}(\pi_*)} \mathbf{R}p_{S,*}\mathcal{O}_S$$

is zero.

Proof. We observe that this map factors over

$$\mathbf{R}\mathcal{H}om_{p_S}(\mathsf{E},\mathsf{E}\otimes K_S)[-1] \xrightarrow{\pi_*\circ\partial} \mathbf{R}\mathcal{H}om_{p_S}(\mathsf{E},\mathsf{E}) \xrightarrow{\mathrm{tr}} \mathbf{R}p_{S,*}\mathcal{O}_S$$

which is zero, following from [TT] 2.21 as explained in 1.6. □

Definition 6.3.3. This gives a splitting denoted as

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) = \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})^{\perp} \oplus \mathbf{R}p_{S,*}K_S[-1] \oplus \mathbf{R}p_{S,*}\mathcal{O}_S$$

The following paragraph relates this splitting to σ and λ.

6.4. **The line bundle twist.** On $X \times \mathcal{N}$, define the line bundle

$$\mathscr{L} := \pi^* \det(\pi_*\mathscr{E})^{-1}.$$

and observe that $(\mathrm{id}\times\lambda)^*\mathscr{E} \cong \mathscr{E}\otimes\mathscr{L}$. Therefore there are canonical maps

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \leftrightarrow \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}\otimes\mathscr{L},\mathscr{E}\otimes\mathscr{L})$$

where the arrow from left to right is $\lambda_* : g \mapsto g\otimes 1$ with inverse λ^* given by cancelling $\mathscr{L}^\vee \otimes \mathscr{L} \cong \mathcal{O}$.

Claim 6.4.1. In view of above splitting 6.3.3, λ_* is diagonal and acts as $+1$ on the second two factors.

Proof. This is clear for the $\mathbf{R}p_{S,*}\mathcal{O}_S$ term. Now we claim that the maps λ_*, λ^* fix $\mathbf{R}p_{S,*}K_S$ because the inclusions

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) & & \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}\otimes\mathscr{L},\mathscr{E}\otimes\mathscr{L}) \arrow[ll, "\lambda^*"'] \\
& \mathbf{R}p_{S_*}K_S[-1] \arrow[ul, "b"] \arrow[ur, "b"'] &
\end{tikzcd}$$

are dual to the trace maps

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \arrow[rr, "\lambda_*"] \arrow[dr, "\mathrm{tr}\pi_*"] & & \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}\otimes\mathscr{L},\mathscr{E}\otimes\mathscr{L}) \arrow[dl, "\mathrm{tr}\pi_*"'] \\
& \mathbf{R}p_{S_*}\mathcal{O}_S &
\end{tikzcd}$$

and the latter commutes because taking traces factors over the natural evaluation $\mathscr{L}^\vee \otimes \mathscr{L} \cong \mathcal{O}$.

□

6.5. **The trace shift.** The map $\sigma \circlearrowright \mathrm{X} \times \mathcal{N}$ with linearisation map $\Psi : \mathscr{E} \xrightarrow{\sim} \sigma^*\mathscr{E}$ induces natural isomorphisms

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \leftrightarrow \mathbf{R}\mathcal{H}om_{p_X}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})$$

where σ_*, the arrow from left to right, is induced by (the pushdown of) $g \mapsto \Psi g \Psi^{-1}$. As before, we claim

Claim 6.5.1. $\sigma_* : \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \to \mathbf{R}\mathcal{H}om_{p_X}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})$ is diagonal with respect to the splitting 6.3.1 and acts again trivially on the second two sumands.

Proof. This is similar to the previous one and reduces to the fact that

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_\mathrm{X}}(\mathscr{E},\mathscr{E}) \arrow[dr, "\mathrm{tr}\pi_*"'] & & \mathbf{R}\mathcal{H}om_{p_X}(\sigma^*\mathscr{E},\sigma^*\mathscr{E}) \arrow[ll, "\sigma_*"'] \arrow[dl, "\mathrm{tr}\pi_*"] \\
& \mathbf{R}p_{S_*}\mathcal{O}_S &
\end{tikzcd}$$

commutes because $\mathrm{tr}(\pi_*(\Psi^{-1}g\Psi)) = \mathrm{tr}(\pi_* g)$.
Dually, this is

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_\mathrm{X}}(\mathscr{E},\mathscr{E}) \arrow[rr, "\sigma_*"] & & \mathbf{R}\mathcal{H}om_{p_X}(\sigma^*\mathscr{E},\sigma^*\mathscr{E}) \\
& \mathbf{R}p_{S,*}K_S[-1] \arrow[ul, "b"] \arrow[ur, "b"'] &
\end{tikzcd}$$

Then replace again the arrows by their duals. ∎

Remark 6.5.2. We end the section with the remark that we can write the natural maps λ_* and σ_* as diagonal maps $(\lambda_* \oplus 1 \oplus 1)$ and $(\sigma_* \oplus 1 \oplus 1)$ on

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) = \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})^\perp \oplus \mathbf{R}p_{S,*}K_S[-1] \oplus \mathbf{R}p_{S,*}\mathcal{O}_S$$

6.6. **Goal.** The goal of the next two sections is to see that the virtual differential of ι, i.e. the correct ι-linearisation of

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})$$

acts as -1 on

$$\mathbf{R}p_{S,*}K_S[-1] \oplus \mathbf{R}p_{S,*}\mathcal{O}_S$$

7. The determinant

7.0. **Summary.** We'll define the correct notion of equivariance for $\lambda \circlearrowright \mathcal{N}$, i.e. we construct a lift $\theta_\lambda : \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \to \mathbf{R}\mathcal{H}om_{p_X}(\lambda^*\mathscr{E},\lambda^*\mathscr{E})$ replacing the natural lift λ_* of the previous section as follows:
As $\lambda(\mathcal{E}) = \mathcal{E} \otimes \pi^* \det(\pi_*\mathcal{E})^{-1}$, we observe

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\det \pi_*"] \arrow[d, "\lambda"] & \mathbf{Pic}(S) \arrow[d, "-1"] \\ \mathcal{N} \arrow[r, "\det \pi_*"] & \mathbf{Pic}(S) \end{tikzcd}$$

which is at tangent spaces for a fixed point $[\mathcal{E}] \in \mathcal{N}$ given by

$$\begin{tikzcd} \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \arrow[r, "\mathrm{tr}\pi_*"] \arrow[d, "(d\lambda)_{[\mathcal{E}]}"] & H^1(\mathcal{O}_S) \arrow[d, "-1"] \\ \mathrm{Ext}^1(\lambda\mathcal{E},\lambda\mathcal{E}) \arrow[r, "\mathrm{tr}\pi_*"] & H^1(\mathcal{O}_S) \end{tikzcd}$$

As $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \leftrightarrow H^1(\mathcal{O}_S)$ is split, $d\lambda$ should act as -1 on $H^1(\mathcal{O}_S)$ on *all* points $\mathcal{E}$ of $\mathcal{N}$. However, the natual map

$$\lambda_* : \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^1(\lambda\mathcal{E},\lambda\mathcal{E})$$

is just the identity for all $\mathcal{E}$ where $\det(\pi_*\mathcal{E}) \cong \mathcal{O}_S$ holds.
For families $\mathscr{E}$ of $\mathcal{E}$, we want to replace λ_* of 6.4 by the correct map θ_λ, whose action on $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$ is now -1 on $\mathbf{R}p_{S,*}\mathcal{O}_S[1]$. We'll need the following ingredient:

7.1. **Deformations of the determinant.** By [STV] Proposition 3.2, there is a commutative square

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\mathrm{tr}\pi_*"] & \mathbf{R}p_{S,*}\mathcal{O}_S[1] \\ \mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E}}"] \arrow[r, "\det_*"] & \det^* \mathbf{T}_{\mathbf{Pic}(S)} \arrow[u, "\det^* \mathrm{At}_{\det(\pi_*\mathscr{E})}"'] \end{tikzcd} \tag{7.1.1}$$

relating deformations of $\mathcal{E} \in \mathcal{N}$ with the one of $\det(\pi_*\mathcal{E}) \in \mathbf{Pic}(S)$.

7.2. **The differential of λ.**

Definition 7.2.1. We define

$$\theta_\lambda : \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \to \mathbf{R}\mathcal{H}om_{p_X}(\lambda^*\mathscr{E},\lambda^*\mathscr{E})$$

as

$$f \mapsto \lambda_* f \otimes 1 - \pi^*(\mathrm{tr}(\pi_* f) \cdot \mathrm{id}) \otimes 1$$

i.e. it is the natural map λ_* of the previous section minus the differential of det, according to 7.1.1.

Remark 7.2.2. Corresponding to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E}) \leftrightarrow \mathbf{R}p_{S,*}\mathcal{O}_S$, write $f \in \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$ as $f = f_0 \oplus \frac{1}{2}\pi^*(\mathrm{tr}(\pi_* f)) \cdot \mathrm{id}$. We observe

$$\theta_\lambda : f_0 \oplus \frac{1}{2}\pi^*(\mathrm{tr}(\pi_* f)) \cdot \mathrm{id} \mapsto \lambda_* f_0 \oplus -\frac{1}{2}\pi^*(\mathrm{tr}(\pi_* f)) \cdot \mathrm{id},$$

so $\theta_\lambda = \lambda_* \oplus (-1)$ is diagonal.

Lemma 7.2.3. We have $\lambda^*\theta_\lambda \circ \theta_\lambda = \mathrm{id}$, i.e.

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\theta_\lambda"] \arrow[dr, equal] & \mathbf{R}\mathcal{H}om_{p_X}(\lambda^*\mathscr{E},\lambda^*\mathscr{E})[1] \arrow[d, "\lambda^*\theta_\lambda"] \\
& \mathbf{R}\mathcal{H}om_{p_X}(\lambda^{*,2}\mathscr{E},\lambda^{*,2}\mathscr{E})[1]
\end{tikzcd}$$

commutes

Proof. As $\lambda^2 = \mathrm{id}$, the vertical arrrow maps indeed to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$. As $\theta_\lambda = \lambda_* \oplus (-1)$, we get $\theta_\lambda^2 = \mathrm{id}$. □

7.3. **The Atiyah class.** Via $\mathrm{At}_\mathscr{E}$, θ_λ relates to the natural differential map $\lambda_* : \mathbf{T}_\mathcal{N} \to \lambda^*\mathbf{T}_\mathcal{N}$:

Lemma 7.3.1. There is a commutative square

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\theta_\lambda"] & \mathbf{R}\mathcal{H}om_{p_X}(\lambda^*\mathscr{E},\lambda^*\mathscr{E})[1] \\
\mathbf{T}_\mathcal{N} \arrow[u, "\mathrm{At}_\mathscr{E}"] \arrow[r, "\lambda_*"] & \lambda^*\mathbf{T}_\mathcal{N} \arrow[u, "\lambda^*\mathrm{At}_\mathscr{E}"']
\end{tikzcd} \tag{7.3.2}$$

Proof. By functoriality of the Atiyah class, following the RHS up gives $\lambda^*\mathrm{At}_\mathscr{E} \circ \lambda_* = \mathrm{At}_{\lambda^*\mathscr{E}} = \mathrm{At}_{\mathscr{E}\otimes\mathscr{L}}$, so we are left to prove that $\theta_\lambda \circ \mathrm{At}_\mathscr{E} = \mathrm{At}_{\mathscr{E}\otimes\mathscr{L}}$. We compute

$$\theta_\lambda \circ \mathrm{At}_\mathscr{E} = \mathrm{At}_\mathscr{E} \otimes 1 - \pi^*(\mathrm{tr}(\mathrm{At}_{\pi_*\mathscr{E}}) \cdot \mathrm{id}) \otimes 1$$

and

$$\mathrm{At}_{\mathscr{E}\otimes\mathscr{L}} = \mathrm{At}_\mathscr{E} \otimes 1 + \mathrm{At}_\mathscr{L} \cdot \mathrm{id} \otimes 1.$$

Now by 7.1.1, relating det and tr we have

$$-\pi^*(\mathrm{tr}(\mathrm{At}_{\pi_*\mathscr{E}})) = \pi^*\mathrm{At}_{\det(\pi_*\mathscr{E})^{-1}} = \mathrm{At}_{\pi^*\det(\pi_*\mathscr{E})^{-1}} = \mathrm{At}_\mathscr{L}$$

and the claim follows. □

7.4. **The equivariance.** We are now ready to prove that

Corollary 7.4.1. $\mathrm{At}_\mathscr{E}$ is λ-equivariant in the sense of definition 6.1.1.

We see that the commutative triangle

$$\begin{tikzcd} \mathbf{T}_{\mathcal{N}} \arrow[r, "\lambda_*"] \arrow[dr, equal] & \lambda^*\mathbf{T}_{\mathcal{N}} \arrow[d, "\lambda^*(\lambda_*)"] \\ & (\lambda^{*,2})\mathbf{T}_{\mathcal{N}} \end{tikzcd}$$

maps via $\mathrm{At}_{\mathscr{E}}$ to the triangle

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\theta_\lambda"] \arrow[dr, equal] & \mathbf{R}\mathcal{H}om_{p_X}(\lambda^*\mathscr{E},\lambda^*\mathscr{E})[1] \arrow[d, "\lambda^*\theta_\lambda"] \\ & \mathbf{R}\mathcal{H}om_{p_X}(\lambda^{*2}\mathscr{E},\lambda^{*2}\mathscr{E})[1] \end{tikzcd}$$

Indeed, the second triangle commutes by 7.2.3 and the compatibility between both triangles via $\mathrm{At}_{\mathscr{E}}$ follows from 7.3.1. This makes $\mathrm{At}_{\mathscr{E}}$ λ-equivariant in the sense of 6.1.1.

Remark 7.4.2. Restricting to $\mathcal{N}^\perp$, we see that $\lambda(\mathcal{E}) = \mathcal{E}$ as $\det(\pi_*\mathcal{E}) \cong \mathcal{O}_S$. By 7.2.3, $\theta_\lambda = 1 \oplus (-1)$ acts as an endomorphism

$$\theta_\lambda \circlearrowright \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]|_{\mathcal{N}^\perp}$$

giving a splitting into ± 1 eigensheaves

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]^0|_{\mathcal{N}^\perp} \oplus \mathbf{R}p_{S,*}\mathcal{O}_S[1]|_{\mathcal{N}^\perp}.$$

8. The trace

8.0. **Summary.** Having dealt with the determinant, lifting $\sigma_{\mathrm{tr}\Phi} \circlearrowright \mathcal{N}$ to the tangent-obstruction complex is of similar nature:
We recall from 2.4.3 that $\sigma_{\mathrm{tr}\Phi} \circlearrowright \mathcal{N}$ is $\sigma_{\mathrm{tr}\Phi}(\mathcal{E}_\phi) = \sigma^*_{\mathrm{tr}\phi}\mathcal{E}_\phi = \mathcal{E}_{\phi-\mathrm{tr}(\phi)\cdot\mathrm{id}}$. Thus

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\mathrm{tr}"] \arrow[d, "\sigma_{\mathrm{tr}\Phi}"] & \Gamma(K_S) \arrow[d, "-1"] \\ \mathcal{N} \arrow[r, "\mathrm{tr}"] & \Gamma(K_S) \end{tikzcd}$$

commutes as $\mathrm{tr}(\phi - \mathrm{tr}(\phi)\cdot\mathrm{id}) = -\mathrm{tr}(\phi)$. Identifying $\mathbf{T}_{\Gamma(S),\mathrm{tr}\phi} \cong H^0(K_S)$ at each point $\mathrm{tr}\phi \in \Gamma(K_S)$, this is

$$\begin{tikzcd} \mathrm{Ext}^1(\mathcal{E}_\phi,\mathcal{E}_\phi) \arrow[r, "d\mathrm{tr}"] \arrow[d, "(d\sigma_{\mathrm{tr}\Phi})_{[\mathcal{E}]}"] & H^0(K_S) \arrow[d, "-1"] \\ \mathrm{Ext}^1(\sigma^*_{\mathrm{tr}\phi}\mathcal{E},\sigma^*_{\mathrm{tr}\phi}\mathcal{E}) \arrow[r, "d\mathrm{tr}"] & H^0(K_S) \end{tikzcd}$$

on tangent spaces.
As $\mathrm{Ext}^1(\mathcal{E}_\phi,\mathcal{E}_\phi) \leftrightarrow H^0(K_S)$, $d\sigma_{\mathrm{tr}\Phi}$ should act as -1 on $H^0(K_S)$ for all points

$\mathcal{E}_\phi$ of $\mathcal{N}$. But as in the previous section, we remark that the canonical map σ_* of 6.5 at single points

$$\sigma_{\mathrm{tr}\phi,*} : \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^1(\sigma^*_{\mathrm{tr}\phi}\mathcal{E}, \sigma^*_{\mathrm{tr}\phi}\mathcal{E})$$

is simply the pullback $\sigma^*_{\mathrm{tr}\phi}$, which acts trivially whenever $\mathcal{E}_\phi$ has centre of mass zero (i.e. $\mathrm{tr}\phi = 0$).
In terms of families $\mathscr{E}$, we replace σ_* by the correct map θ_σ, whose action on $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[-1]$ is now -1 on $\mathbf{R}p_{S,*}K_S$.

8.1. **First step.** We recall the equivariant lift for $\sigma_{\mathrm{tr}\Phi} \circlearrowleft \mathcal{N}$ to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$ stated in 4.7.8

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\sigma_*"] & \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})[1] \\
\mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E}}"] \arrow[r, "\sigma_*"] & \sigma^*\mathbf{T}_{\mathcal{N}} \arrow[u, "\sigma^*\mathrm{At}_{\mathscr{E}}"']
\end{tikzcd} \tag{8.1.1}$$

Remark 8.1.2. The right fix to above stated problem is using the same map $\sigma_{\mathrm{tr}\Phi} \circlearrowleft \mathcal{N}$, but writing it as a composition

$$\mathcal{N} \xrightarrow{\mathrm{tr}\oplus\mathrm{id}} \Gamma(K_S) \oplus \mathcal{N} \xrightarrow{\mathit{translation}} \mathcal{N}$$

such that its differential is

$$\tilde{\sigma}_*(v) = \sigma_{\mathrm{tr}\Phi,*}(v) + \mathrm{tr}_*(v)\cdot \mathrm{id}.$$

This will be established in the next step. In the third step, we connect this map to the (virtual) differential of the trace $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \to \mathbf{R}p_{S,*}K_S$ and find a lift θ_σ to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$ that acts as -1 on $\mathbf{R}p_{S,*}K_S$

8.2. **Second Step.** We'll use a single result of [TT] at this point.

Claim 8.2.1. $\mathrm{tr} : \mathcal{N} \to \Gamma(K_S)$ induces split maps $\mathbf{T}_{\mathcal{N}} \overset{\frac{1}{2}\mathrm{tr}_*}{\underset{\mathrm{id}_*}{\longleftrightarrow}} (\mathrm{tr}\Phi)^*\mathbf{T}_{\Gamma(K_S)}$ compatible with $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \leftrightarrow \mathbf{R}p_{S,*}K_S$ via At, i.e.

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "a"] & \mathbf{R}p_{S,*}K_S \arrow[r, "b"] & \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \\
\mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E}}"] \arrow[r, "\mathrm{tr}_*"] & (\mathrm{tr}\Phi)^*\mathbf{T}_{\Gamma(K_S)} \arrow[u, "\mathrm{At}_{\Gamma(K_S)}"] \arrow[r, "\mathrm{id}_*"] & \mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E}}"]
\end{tikzcd} \tag{8.2.2}$$

commutes, following from [TT] 5.29 and 5.30.

Corollary 8.2.3. The trace-identity maps $\mathbf{T}_{\mathcal{N}} \leftrightarrow (\mathrm{tr}\Phi)^*\mathbf{T}_{\Gamma(K_S)}$ are compatible with λ_* in the following sense

$$\begin{tikzcd} & \mathbf{T}_{\mathcal{N}} \arrow[dr, "\mathrm{tr}_*"] \arrow[dd, "\lambda_*"] & \\ (\mathrm{tr}\Phi)^*\mathbf{T}_{\Gamma(K_S)} \arrow[ur, "\mathrm{id}_*"] \arrow[dr] & & (\mathrm{tr}\Phi)^*\mathbf{T}_{\Gamma(K_S)} \\ & \lambda^*\mathbf{T}_{\mathcal{N}} \arrow[ur] & \end{tikzcd} \tag{8.2.4}$$

Here, the lower composition agrees with the upper one and is equal to $2 \cdot \mathrm{id}$.

Proof. Assuming the splitting, this follows from the simple fact that $\mathrm{tr} \circ \lambda = \mathrm{tr}$ i.e.

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\lambda"] \arrow[dr, "\mathrm{tr}"] & \mathcal{N} \arrow[d, "\mathrm{tr}"] \\ & \mathcal{N} \end{tikzcd}$$

commutes as $\mathrm{tr}(\phi) = \mathrm{tr}(\phi \otimes 1)$. $\square$

8.3. **Third step.** We review the construction of the trace shift $\sigma_{\mathrm{tr}\Phi}$ on spectral sheaves from 2.3.1: We'll make more explicit that this map factors over $\Gamma(K_S)$. More precisely, deformations of $\mathrm{tr}\phi \in \Gamma(K_S)$ induce deformations of $\mathcal{E}_\phi \in \mathcal{N}$ in the following sense:

We recall that a global section $\alpha \in \Gamma(K_S)$ acts on a Higgs pair (E, ϕ) via translating ϕ, i.e.

$$\alpha : (E, \phi) \mapsto (E, \phi - \alpha \cdot \mathrm{id})$$

induces a global map

$$\sigma : \Gamma(K_S) \times \mathcal{N} \to \mathcal{N}.$$

In terms of their corresponding spectral sheaves $\mathcal{E}_\phi$ on X, this is expressed via the pullback

$$(\alpha, \mathcal{E}_\phi) \mapsto \sigma_\alpha^* \mathcal{E}_\phi = \mathcal{E}_{\phi - \alpha \cdot \mathrm{id}}$$

where $\sigma_\alpha \circlearrowright X$ is translation on the fibres by α,

$$(s, t) \mapsto (s, t - \alpha_s)$$

for local coordinates (s, t) of X.

Again, we'll phrase the rest of this discussion in terms of spectral sheaves entirely.

We rewrite the trace shift as

$$\sigma_{\mathrm{tr}\Phi} : \mathcal{N} \xrightarrow{\mathrm{tr}\Phi \times \mathrm{id}} \Gamma(K_S) \times \mathcal{N} \xrightarrow{\sigma} \mathcal{N}$$

sending

$$\mathcal{E}_\phi \mapsto (\mathrm{tr}\phi, \mathcal{E}_\phi) \mapsto \sigma_{\mathrm{tr}\phi}^* \mathcal{E}_\phi = \mathcal{E}_{\phi - \mathrm{tr}\phi \cdot \mathrm{id}}$$

This agrees with the previous definition on points, but now factors explicitly over $\Gamma(K_S)$. This induces a map on cotangent complexes

$$\tilde{\sigma} : \mathbf{T}_{\mathcal{N}} \to (\mathrm{tr}\Phi \times \mathrm{id})^*[\mathbf{T}_{\Gamma(K_S)} \oplus \mathbf{T}_{\mathcal{N}}] \to \sigma_{\mathrm{tr}\Phi}{}^* \mathbf{T}_{\mathcal{N}} \tag{8.3.1}$$

and we claim that

Claim 8.3.2. $\tilde{\sigma}_*(v) = \sigma_{\mathrm{tr}\Phi,*}(v) + \mathrm{tr}_*(v)\cdot \mathrm{id}$

Proof. Identifying the vector space $\Gamma(K_S)$ with its own tangent space we may write over $\Gamma(K_S)\times\mathcal{N}$ the differential of σ as

$$\Gamma(K_S)\oplus \mathbf{T}_{\mathcal{N}} \to \sigma^*\mathbf{T}_{\mathcal{N}};\ \ (\alpha, v)\mapsto \sigma_*(v-\alpha\cdot\mathrm{id}).$$

As $(\mathrm{tr}\Phi)^*\mathbf{T}_{\Gamma(K_S)} \cong \Gamma(K_S)\otimes\mathcal{O}$ pulling back by $(\mathrm{tr}\Phi\times\mathrm{id})$ gives the second arrow in 8.3.1, as $(\mathrm{tr}\Phi\times\mathrm{id})^*\sigma^* = \sigma^*_{\mathrm{tr}\Phi}$. So precomposing with

$$\mathbf{T}_{\mathcal{N}} \to (\mathrm{tr}\Phi\times\mathrm{id})^*[\mathbf{T}_{\Gamma(K_S)}\oplus\mathbf{T}_{\mathcal{N}}];\ \ v\mapsto(\mathrm{tr}_*v, v)$$

gives

$$v\mapsto \sigma_{\mathrm{tr}\Phi,*}(v-\mathrm{tr}_*(v)\cdot\mathrm{id}) = \sigma_{\mathrm{tr}\Phi,*}(v)+\mathrm{tr}_*(v)\cdot\mathrm{id}$$

as $\sigma_{\mathrm{tr}\Phi,*}\mathrm{tr}_* = -\mathrm{tr}_*$. □

Remark 8.3.3. We'll leave out the index $\mathrm{tr}\Phi$ in the notation from now on and rewrite the differential simpler as

$$\tilde{\sigma}:\mathbf{T}_{\mathcal{N}}\to\sigma^*\mathbf{T}_{\mathcal{N}}$$

sending

$$v\mapsto\tilde{\sigma}_*(v) = \sigma_*(v)+\mathrm{tr}_*(v)\cdot\mathrm{id}.$$

It is easy so see that $\tilde{\sigma}_*$ again has square equal to the identity on $\mathbf{T}_{\mathcal{N}}$[12]. Now, $\tilde{\sigma}_*$ contains the "correction" term $\mathrm{tr}_*(v)\cdot\mathrm{id}$, the differential of the trace.
We remark that restricting the second arrow in 8.3.1 to the first factor gives a map

$$\sigma_*\circ\mathrm{id}_*:(\mathrm{tr}\Phi)^*\mathbf{T}_{\Gamma(K_S)}\to\sigma^*\mathbf{T}_{\mathcal{N}}.$$

We can relate this to the previous result:

8.4. **Atiyah classes.** Composing 8.2.2 with 4.7.8 gives
(8.4.1)

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "a"] & \mathbf{R}p_{S,*}K_S \arrow[r, "b"] & \mathbf{R}\mathcal{H}om_{p_X}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})[1] \\
\mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E}}"] \arrow[r, "\mathrm{tr}_*"] & (\mathrm{tr}\Phi)^*\mathbf{T}_{\Gamma(K_S)} \arrow[u, "\mathrm{At}_{\Gamma(K_S)}"] \arrow[r, "\sigma_*\circ\mathrm{id}_*"] & \sigma^*\mathbf{T}_{\mathcal{N}} \arrow[u, "\sigma^*\mathrm{At}_{\mathscr{E}}"]
\end{tikzcd}$$

as $\sigma_* b = b$ by 6.5.

Remark 8.4.2. We remark that the lower horizontal composition is

$$v\mapsto\sigma_*(\mathrm{tr}_*(v)\cdot\mathrm{id}) = -\mathrm{tr}_*(v)\cdot\mathrm{id}$$

[12] in fact, we have $\sigma_*^2=\mathrm{id}$. Then note $\sigma_{\mathrm{tr}\Phi,*}\circ\mathrm{tr}_* = \mathrm{tr}_*\circ\sigma_{\mathrm{tr}\Phi,*} = -\mathrm{tr}_*$

8.5. **The differential of σ.**

Definition 8.5.1. We define

$$\begin{aligned}\theta_\sigma : \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\mathscr{E},\mathscr{E})[1] &\to \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})[1]\\ f &\mapsto \sigma_* f - ba(f)\end{aligned}$$

Remark 8.5.2. Corresponding to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \leftrightarrow \mathbf{R}p_{S,*}K_S$, write $f \in \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$ as $f = f_0 \oplus \frac{1}{2}ba(f)$. We observe

$$\theta_\sigma : f_0 \oplus \frac{1}{2}ba(f) \mapsto \sigma_* f_0 \oplus -\frac{1}{2}ba(f),$$

i.e. its -1 on second summand.

Proposition 8.5.3. We have $\sigma^*\theta_\sigma \circ \theta_\sigma = \mathrm{id}$, i.e.

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\theta_\sigma"] \arrow[dr, equal] & \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})[1] \arrow[d, "\sigma^*\theta_\sigma"] \\ & \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\sigma^{*2}\mathscr{E},\sigma^{*2}\mathscr{E})[1] \end{tikzcd}$$

commutes

Proof. This is similar to the case of λ: As $\sigma^2 = \mathrm{id}$, the vertical arrow maps indeed to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$.
Now $\theta_\sigma = \sigma_* \oplus (-1)$ for the above splitting by the remark, thus $\theta_\sigma^2 = \mathrm{id}$. □

8.6. **The Atiyah class.** Again, we relate θ_σ to the differential action on $\mathbf{T}_{\mathcal{N}}$.

Lemma 8.6.1. $\theta_\sigma = \sigma_* - ba$ commutes with $\tilde{\sigma}_* : \mathbf{T}_{\mathcal{N}} \to \sigma^*\mathbf{T}_{\mathcal{N}}$ constructed in 8.3, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\theta_\sigma"] & \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})[1] \\ \mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E}}"] \arrow[r, "\tilde{\sigma}_*"] & \sigma^*\mathbf{T}_{\mathcal{N}} \arrow[u, "\sigma^*\mathrm{At}_{\mathscr{E}}"'] \end{tikzcd} \tag{8.6.2}$$

Proof. We recall the diagram discussed in 4.7.8:

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\sigma_*"] & \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})[1] \\ \mathbf{T}_{\mathcal{N}} \arrow[u, "\mathrm{At}_{\mathscr{E}}"] \arrow[r, "\sigma_*"] & \sigma^*\mathbf{T}_{\mathcal{N}} \arrow[u, "\sigma^*\mathrm{At}_{\mathscr{E}}"'] \end{tikzcd} \tag{8.6.3}$$

The claim follows by subtracting the rows of 8.4.1, as we then get that

$$\tilde{\sigma}_* = \sigma_* + \mathrm{tr}_{*-} \cdot \mathrm{id} = \sigma_* - \sigma_* \circ \mathrm{tr}_{*-} \cdot \mathrm{id} \;\text{ maps to }\; \theta_\sigma = \sigma_* - ba$$

via $\mathrm{At}_{\mathscr{E}}$ and its pullback by σ^*. □

8.7. **The equivariance.** As $\tilde{\sigma}_*$ again has square equal to id,

$$\begin{tikzcd} \mathbf{T}_{\mathcal{N}} \arrow[r, "\tilde{\sigma}_*"] \arrow[dr, equal] & \sigma^*\mathbf{T}_{\mathcal{N}} \arrow[d, "\sigma^*\tilde{\sigma}_*"] \\ & (\sigma^{*,2})\mathbf{T}_{\mathcal{N}} \end{tikzcd}$$

maps via $\mathrm{At}_{\mathscr{E}}$ to

$$\begin{tikzcd} \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \arrow[r, "\theta_\sigma"] \arrow[dr, equal] & \mathbf{R}\mathcal{H}om_{p_X}(\sigma^*\mathscr{E},\sigma^*\mathscr{E})[1] \arrow[d, "\sigma^*\theta_\sigma"] \\ & \mathbf{R}\mathcal{H}om_{p_X}(\sigma^{*2}\mathscr{E},\sigma^{*2}\mathscr{E})[1] \end{tikzcd}$$

with everything commutative thanks to 8.5.3 and 8.6.1, which proves the equivariance.

Remark 8.7.1. Restricting θ_σ to $\mathcal{N}^\perp$, we see that the translation by $\mathrm{tr}(\phi)$ of spectral sheaves $\sigma_{\mathrm{tr}\phi} : \mathcal{E} \mapsto \sigma^*_{\mathrm{tr}\phi}\mathcal{E}$ is trivial as $\mathrm{tr}(\phi) = 0$ on $\mathcal{N}^\perp$. Thus $\theta_\sigma = 1 \oplus (-1)$ acts as an endomorphism

$$\theta_\sigma \circlearrowright \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]|_{\mathcal{N}^\perp}$$

giving a splitting into ± 1 eigensheaves

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]_0|_{\mathcal{N}^\perp} \oplus \mathbf{R}p_{S,*}K_S|_{\mathcal{N}^\perp}.$$

9. The equivariance of ι

9.1. **Summary.** We combine the results of the previous two sections. We've seen that there are linearisation maps $\theta_\lambda, \theta_\sigma$ lifting the actions of λ, σ on $\mathcal{N}$ to the virtual tangent complex $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$. This was done in such a way that $\theta_\lambda, \theta_\sigma$ are compatible with the actions $\lambda_* : \mathbf{T}_{\mathcal{N}} \to \lambda^*\mathbf{T}_{\mathcal{N}}$ and $\tilde{\sigma}_* : \mathbf{T}_{\mathcal{N}} \to \sigma^*\mathbf{T}_{\mathcal{N}}$ via $\mathrm{At}_{\mathscr{E}}$.

9.2. **The equivariance of ι.**

Definition 9.2.1. Recalling that $\iota = \lambda \circ \sigma$, we define

$$\theta_\iota := \theta_\lambda \circ \theta_\sigma$$

The equivariant lift of ι is now an easy corollary:

Corollary 9.2.2. This gives a linearisation

$$\theta_\iota : \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] \to \mathbf{R}\mathcal{H}om_{p_X}(\iota^*\mathscr{E},\iota^*\mathscr{E})[1]$$

for $\iota \circlearrowright \mathcal{N}$ that acts as -1 on $\mathbf{R}p_{S,*}K_S \oplus \mathbf{R}p_{S,*}\mathcal{O}_S[1]$

Proof. Note that $\lambda \circ \sigma = \sigma \circ \lambda$. Again split

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1] = \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]^\perp \oplus \mathbf{R}p_{S,*}K_S \oplus \mathbf{R}p_{S,*}\mathcal{O}_S[1],$$

then we've seen that we can write λ_*, σ_* as $\lambda_* \oplus 1 \oplus 1$ and $\sigma_* \oplus 1 \oplus 1$. By the previous results of 7.2.3 and 8.5.3, we have

$$\theta_\lambda = \lambda_* \oplus 1 \oplus (-1),\ \theta_\sigma = \sigma_* \oplus (-1) \oplus 1$$

Thus,

$$\theta_\iota = \theta_\sigma \circ \theta_\lambda = (\sigma_* \lambda_*) \oplus (-1) \oplus (-1) = \iota_* \oplus (-1) \oplus (-1)$$

which also shows that $\iota^* \theta_\iota \circ \theta_\iota = \mathrm{id}$. □

Definition 9.2.3. Furthermore, we define $\iota_* = (\tilde{\sigma}_* \circ \lambda_*) : \mathbf{T}_\mathcal{N} \to \iota^* \mathbf{T}_\mathcal{N}$

and remark

Corollary 9.2.4. $\tilde{\sigma}_* \circ \lambda_* = \lambda_* \circ \tilde{\sigma}_*$ holds. Thus $\iota_*^2 = \mathrm{id}$.

Proof. Follows from $\sigma_* \circ \lambda_* = \lambda_* \circ \sigma_*$ and corollary 8.2.3.As $\lambda_*^2 = \mathrm{id}$ and $\tilde{\sigma}_*^2 = \mathrm{id}$ by 8.3.3, we conclude $\iota_*^2 = \lambda_*^2 \tilde{\sigma}_*^2 = \mathrm{id}$. □

Proposition 9.2.5. $\mathrm{At}_\mathscr{E} : \mathbf{T}_\mathcal{N} \to \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[1]$ is ι equivariant in the sense of 6.1.1.

Proof. Indeed, composing the squares in 7.3.1 and 8.6.2 gives

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[1] \arrow[r, "\theta_\iota"] & \mathbf{R}\mathcal{H}om_{p_X}(\iota^*\mathscr{E}, \iota^*\mathscr{E})[1] \\
\mathbf{T}_\mathcal{N} \arrow[u, "\mathrm{At}_\mathscr{E}"] \arrow[r, "\iota_*"] & \iota^* \mathbf{T}_\mathcal{N} \arrow[u, "\iota^*\mathrm{At}_\mathscr{E}"']
\end{tikzcd} \tag{9.2.6}$$

such that

$$\begin{tikzcd}
\mathbf{T}_\mathcal{N} \arrow[r, "\iota_*"] \arrow[dr, equal] & \iota^* \mathbf{T}_\mathcal{N} \arrow[d, "\iota^*(\iota_*)"] \\
 & (\iota^{*,2}) \mathbf{T}_\mathcal{N}
\end{tikzcd}$$

maps to the triangle

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[1] \arrow[r, "\theta_\iota"] \arrow[dr, equal] & \mathbf{R}\mathcal{H}om_{p_X}(\iota^*\mathscr{E}, \iota^*\mathscr{E})[1] \arrow[d, "\iota^*\theta_\iota"] \\
 & \mathbf{R}\mathcal{H}om_{p_X}(\iota^{*2}\mathscr{E}, \iota^{*2}\mathscr{E})[1]
\end{tikzcd}$$

via $\mathrm{At}_\mathscr{E}$ with everything commutative. □

Remark 9.2.7. Restricting to $\mathcal{N}^\perp$ gives $\theta_\iota = 1 \oplus (-1) \oplus (-1)$ an endomorphism

$$\theta_\iota \circlearrowright \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[1]|_{\mathcal{N}^\perp}$$

splitting

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[1]|_{\mathcal{N}^\perp} \cong \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[1]^\perp|_{\mathcal{N}^\perp} \oplus N^{vir}$$

where

$$N^{vir} := (\mathbf{R}p_{S,*}\mathcal{O}_S[1] \oplus \mathbf{R}p_{X,*}K_S)|_{\mathcal{N}^\perp}$$

is the virtual normal sheaf, i.e. the -1-eigensheaf for the action of θ_ι.

Remark 9.2.8. For the final chapter we phrase everything again in terms of their duals. Let $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[2]$ denote the virtual ι-equivariant cotangent bundle and $\mathbf{L}_{\mathcal{N}}$ the truncated cotangent complex of $\mathcal{N}$.

10. Application to the localisation formula

In this chapter we find a perfect obstruction theory for $\mathcal{N}^\perp$. The proof is an adaption of [GP] Prop.1 replacing the $\mathbf{C}^\times$-action by ι.
For the $\mathbf{C}^\times$-action, the idea is to split the obstruction bundle V over the fixed locus into weight zero and non-zero part. Then remove the non-zero part, the (virtual) conormal bundle to the fixed locus.
We do the same for ι, where we take away the -1 part of $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[2]|_{\mathcal{N}^\perp}$, which are the deformations of trace and determinant, according to remark 9.2.7 above.
We recall the definition of a perfect obstruction theory again. It consists of

1. A two-term complex of vector bundles $V^\bullet = [V^{-1} \to V^0] \in \mathbf{D}^{[-1,0]}(\mathcal{N})$.
2. A morphism $\psi : V^\bullet \to \mathbf{L}_{\mathcal{N}}$ in $\mathbf{D}^b(\mathcal{N})$ to the truncated cotangent complex $\mathbf{L}_{\mathcal{N}}$ inducing an isomorphism on h^0 and a surjection on h^{-1}.

In order to take ι-invariants and apply [GP], everything needs to be equivariant and represented by complexes.

10.1. Equivariant representation.

We sum up the results of the previous sections: In 5.1.4 we've found a 2-term representation of vector bundles for

$$\mathrm{At}_{\mathscr{E}} : \tau^{[-1,0]}\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[2] \to \mathbf{L}_{\mathcal{N}}$$

making it into a perfect obstruction theory on $\mathcal{N}$, represented by

$$[V^{-1} \to V^0] \xrightarrow{\psi} [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}}|_{\mathcal{N}}] \in \mathbf{D}^{[-1,0]}(\mathcal{N}). \tag{10.1.1}$$

We've studied the involution $\iota = \sigma \circ \lambda \circlearrowright \mathcal{N}$ and identified one component of $\mathcal{N}^\iota$ with the $\mathbf{SU}(2)$-locus

$$\mathcal{N}^\perp = \{(E,\phi)|\det(E) \cong \mathcal{O}_S \text{ and } \operatorname{tr}(\phi) = 0\} \subset \mathcal{N}$$

We constructed a lift θ_ι of ι to $\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[1]$, compatible with the differential map $\iota_* : \iota^*\mathbf{L}_{\mathcal{N}} \to \mathbf{L}_{\mathcal{N}}$, lifting $\mathrm{At}_{\mathscr{E}}$ to $\mathbf{D}^b(\mathcal{N})^{\langle\iota\rangle}$.
Finally, we've seen in the last chapter that the restriction θ_ι to $\mathcal{N}^\perp$ is $1\oplus(-1)$ for

$$\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[2]|_{\mathcal{N}^\perp} = \mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})[2]^\perp|_{\mathcal{N}^\perp} \oplus N^{vir,\vee}$$

Remark 10.1.2. The above presentation ψ of the obstruction theory can chosen to be ι-equivariant: Indeed, we've seen this for $\mathbf{L}_{\mathcal{N}}$ by finding an ι-equivariant smooth embedding $\mathcal{N} \subset \mathcal{A}$.
Concerning the obstruction bundle $V^\bullet$ in 10.1.1, going back to the proof 5.1.4, we may choose a very negative *ι-equivariant* resolution $F^\bullet \to \mathbf{R}\mathcal{H}om(\mathscr{E}, \mathscr{E})$ which makes

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\iota^*\mathscr{E}, \iota^*\mathscr{E}) \arrow[r, "\iota^*\theta_\iota"] & \mathbf{R}\mathcal{H}om_{p_{\mathrm{X}}}(\mathscr{E}, \mathscr{E}) \\
p_{\overline{X},*}(\iota^* F^\bullet) \arrow[u] \arrow[r, dashed] & p_{\overline{X},*}F^\bullet \arrow[u]
\end{tikzcd}$$

commutative, after replacing the polarisation $\mathcal{O}(1)$ with a ι-linearised one. [13]

Then following the proof of 5.1.4, we end up with a genuine map of complexes

$$\iota^*[V^{-1} \to V^0] \to [V^{-1} \to V^0] \in \mathbf{D}^{[-1,0]}(\mathcal{N})$$

representing the lift of ι to the obstruction complex

$$\iota^*\theta_\iota : \tau^{[-1,0]}\mathbf{R}\mathcal{H}om_{p_X}(\iota^*\mathscr{E}, \iota^*\mathscr{E})[2] \to \tau^{[-1,0]}\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E}, \mathscr{E})[2].$$

By the previous section 9, this maps to

$$\iota^*[\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}}|_{\mathcal{N}}] \to [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}}|_{\mathcal{N}}] \in \mathbf{D}^{[-1,0]}(\mathcal{N})$$

representing

$$\iota^*\mathbf{L}_{\mathcal{N}} \to \mathbf{L}_{\mathcal{N}}$$

equivariantly, via the ι-linearised truncated relative Atiyah class $\mathrm{At}_{\mathscr{E}}$ of 9.2.5.

Corollary 10.1.3. To sum up, we remark that this together with the ι-equivariance shown in 9 lifts

$$[V^{-1} \to V^0] \xrightarrow{\psi} [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}}|_{\mathcal{N}}]$$

to $\mathbf{D}^{[-1,0]}(\mathcal{N})^{\langle\iota\rangle}$, i.e. is a morphism of ι-linearised complexes.

10.2. **G-sheaves.** Let G be a finite group and V a coherent $\mathcal{O}[G]$-module.

Lemma 10.2.1. The functor of taking fixed parts $V \mapsto V^G$ is exact.

Proof. We remark that $V \mapsto V^G$ is naturally isomorphic to $\mathcal{H}om_{\mathcal{O}[G]}(\mathcal{O}, \_)$, where $\mathcal{O}$ is endowed with the trivial G-action. Thus, taking fixed parts is exact if and only if $\mathcal{O}$ is projective as an $\mathcal{O}[G]$-module. The natural projection $\mathcal{O}[G] \twoheadrightarrow \mathcal{O}$ sending $g \mapsto 1$ has a section given by $1 \mapsto \frac{1}{|G|}\sum_{g\in \mathrm{G}} g$, thus $\mathcal{O}$ is projective, being direct summand of the free module $\mathcal{O}[G]$. $\square$

[13] fixing such a line bundle $\mathcal{O}(1)$ as in 4.2.1 on $X \times \mathcal{N}$, we can resolve equivariantly as

$$\cdots \to F^0 = H^0(\mathscr{E}^\vee \otimes \mathscr{E}(l)) \otimes \mathcal{O}(-l) \twoheadrightarrow \mathscr{E}^\vee \otimes \mathcal{E}$$

Corollary 10.2.2. We see this easily extends to a 2-term complex $V^\bullet$ of $\mathcal{O}[G]$-modules $[V^{-1} \xrightarrow{d} V^0]$ for any $\mathcal{O}[G]$-linear map d.

10.2.1. *Application to ι-linearisation.* Let $V^\bullet \in \mathbf{D}^b(\mathcal{N})^{\langle\iota\rangle}$ on $\mathcal{N}$ as described above in 10.1.3. By definition,the linearisation $\theta_\iota \circlearrowright V^\bullet|_{\mathcal{N}^\perp}$ makes $V^\bullet|_{\mathcal{N}^\perp}$ into a 2-term complex of $\mathcal{O}_{\mathcal{N}^\perp}[\langle\iota\rangle]$-modules [14]. Thus by lemma 10.2.1, taking fixed parts

$$V^\bullet|_{\mathcal{N}^\perp} \mapsto V^\bullet|^{\iota}_{\mathcal{N}^\perp}$$

is exact, thus $V^\bullet|^{\iota}_{\mathcal{N}^\perp}$ is a 2-term complex of vector bundles on $\mathcal{N}^\perp$.
To end this section, let us remark that the sheaves $h^{-1}(V^\bullet|^{\iota}_{\mathcal{N}^\perp})$, $h^0(V^\bullet|^{\iota}_{\mathcal{N}^\perp})$ are independent of the choice $V^\bullet \sim W^\bullet \in \mathbf{D}^b(\mathcal{N})^{\langle\iota\rangle}$.

10.3. **The localisation formula.** We'll now adapt the construction of [GP] mentioned at the beginning, which defines obstruction theories of $\mathbf{C}^\times$-fixed loci. It is a general fact that taking fixed part of a $\mathbf{C}^\times$-equivariant map is exact. Based on what we've said above, this also applies to $\langle\iota\rangle$. To sum up:

Corollary 10.3.1. The 2-term complex of locally frees

$$V^\bullet|^{\iota}_{\mathcal{N}^\perp}$$

represents the θ_ι-fixed part

$$\tau^{[-1,0]}\mathbf{R}\mathcal{H}om_{p_X}(\mathscr{E},\mathscr{E})^\perp[2]|_{\mathcal{N}^\perp}$$

computing the cohomology sheaves $\mathcal{E}xt^i_{p_X}(\mathscr{E}|_{\mathcal{N}^\perp},\mathscr{E}|_{\mathcal{N}^\perp})^\perp$ for $i=1,2$.

Theorem 10.3.2. *There is a map $V^\bullet|^{\iota}_{\mathcal{N}^\perp} \to \mathbf{L}_{\mathcal{N}^\perp}$ defining a perfect obstruction theory on $\mathcal{N}^\perp$.*

Proof. The desired map will be constructed along the way, as well as the explicit representation of $\mathbf{L}_{\mathcal{N}^\perp}$. We start with choosing a ι-equivariant embedding $\mathcal{N} \subset \mathcal{A}$ as in 4.2.1,making

$$\mathbf{L}_{\mathcal{N}} = [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}}|_{\mathcal{N}}]$$

equivariant.
As ι lifts to $\mathcal{A}$, let $\mathcal{A}^\iota = \cup_i \mathcal{A}_i$ be the decomposition into irreducible components of the fixed locus and let $\mathcal{N}_i := \mathcal{A}_i \cap \mathcal{N}$, such that $\mathcal{N}_i \subset \mathcal{A}_i$ is defined by the ideal sheaf $\mathcal{I}_{\mathcal{N}_i}$. We choose the i corresponding to the relevant component $\mathcal{N}^\perp := \mathcal{N}_i \subset \mathcal{A}_i$ as in 3.2.2. We get the representation

$$\mathbf{L}_{\mathcal{N}^\perp} = [\mathcal{I}_{\mathcal{N}^\perp}/\mathcal{I}^2_{\mathcal{N}^\perp} \to \Omega_{\mathcal{A}_i}|_{\mathcal{N}^\perp}]$$

As $\Omega_{\mathcal{A}}$ is locally free, there is a natural isomorphism $\Omega_{\mathcal{A}}|^{\iota}_{\mathcal{A}_i} \cong \Omega_{\mathcal{A}_i}$. Indeed, over $\mathcal{N}^\perp$ the differential ι_* acts as an involution $\iota_* \circlearrowright \Omega_{\mathcal{A}}|_{\mathcal{A}_i}$ with fixed part

[14] we remark that over $\mathcal{N}^\perp$, ι acts via $\theta_\iota|_{\mathcal{N}^\perp}$

$\Omega_{\mathcal{A}_i}$.
This gives $\Omega_{\mathcal{A}}|^{\iota}_{\mathcal{N}^\perp} \cong \Omega_{\mathcal{A}_i}|_{\mathcal{N}^\perp}$ over $\mathcal{N}^\perp$. We have the following square

$$\begin{tikzcd}
\Omega_{\mathcal{A}_i}|_{\mathcal{N}^\perp} \arrow[r] & \Omega_{\mathcal{N}^\perp} \\
\Omega_{\mathcal{A}}|^{\iota}_{\mathcal{N}^\perp} \arrow[r] \arrow[u, "\sim"] & \Omega_{\mathcal{N}}|^{\iota}_{\mathcal{N}^\perp} \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are induced by the natural projections. Tus, we observe that the RHS arrow is onto.
Let $V^\bullet|_{\mathcal{N}^\perp}$ be the restriction of the obstruction complex and $\psi_\perp : V^\bullet|_{\mathcal{N}^\perp} \to \mathbf{L}_{\mathcal{N}}|_{\mathcal{N}^\perp}$ the pulled back map. As ψ is ι-equivariant, we can take its fixed part

$$\psi^\iota_\perp : V^\bullet|^\iota_{\mathcal{N}^\perp} \to \mathbf{L}_{\mathcal{N}}|^\iota_{\mathcal{N}^\perp}.$$

Furthermore, denote by $\delta : \mathbf{L}_{\mathcal{N}}|_{\mathcal{N}^\perp} \to \mathbf{L}_{\mathcal{N}^\perp}$ the (naturally ι-equivariant) canonical map. Then we claim that the composition

$$V^\bullet|^\iota_{\mathcal{N}^\perp} \xrightarrow{\psi^\iota_\perp} \mathbf{L}_{\mathcal{N}}|^\iota_{\mathcal{N}^\perp} \xrightarrow{\delta^\iota} \mathbf{L}_{\mathcal{N}^\perp}$$

defines a perfect obstruction theory on $\mathcal{N}^\perp$:
We remark again that the LHS is given by the two-term complex of vector bundles $V^\bullet|^\iota_{\mathcal{N}^\perp}$ by the last lemma 10.3.1, showing (1).
We need to check the upper conditions on cohomology in (2), which we check for the two maps separately: This is obvious for the restricted map $\psi_\perp$ and thus for $\psi^\iota_\perp$, as taking ι-invariants is exact by 10.2.1.

The morphism $\mathbf{L}_{\mathcal{N}}|^\iota_{\mathcal{N}^\perp} \xrightarrow{\delta^\iota_\perp} \mathbf{L}_{\mathcal{N}^\perp}$ can be represented by the following diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \ker(a) \arrow[r] \arrow[d, dashed] & \mathcal{I}_{\mathcal{N}}/\mathcal{I}^2_{\mathcal{N}}|^\iota_{\mathcal{N}^\perp} \arrow[r, "a"] \arrow[d, two heads, "d^{-1}"] & \Omega_{\mathcal{A}}|^\iota_{\mathcal{N}^\perp} \arrow[r] \arrow[d, "\sim"] & \Omega_{\mathcal{N}}|^\iota_{\mathcal{N}^\perp} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \ker(b) \arrow[r] & \mathcal{I}_{\mathcal{N}^\perp}/\mathcal{I}^2_{\mathcal{N}^\perp} \arrow[r, "b"] & \Omega_{\mathcal{A}_i}|_{\mathcal{N}^\perp} \arrow[r] & \Omega_{\mathcal{N}^\perp} \arrow[r] & 0
\end{tikzcd}$$

By what we've already discussed, the right most vertical arrow is onto. As the rows are exact and d^{-1} is onto, we actually get $\Omega_{\mathcal{N}}|^\iota_{\mathcal{N}^\perp} \cong \Omega_{\mathcal{N}^\perp}$. This gives the required property on h^0. Again as d^{-1} is surjective, so is the induced map $\ker(a) \to \ker(b)$, i.e. there is an epimorphism on h^{-1}. This finishes the proof of the theorem. $\square$

Remark 10.3.3. As explained in section 5, this endows $\mathcal{N}^\perp$ with a virtual cycle of dimension 0 by [BF] 5.2.

11. Appendix

Let E be a rank 2 vector bundle and E^* its dual. Writing

$$\mathcal{H}om(E, E^*) \cong \mathrm{Sym}(E^*) \oplus \wedge^2 E^*,$$

we'll explain sections of $\wedge^2 E^*$: an element $\alpha_1 \wedge \alpha_2$ becomes a skew map $E \to E^*$ via

$$[\alpha_1 \wedge \alpha_2](e) := \alpha_1(e)\alpha_2 - \alpha_2(e)\alpha_1 \in E^*$$

for $e \in E$. We'll call this map $\alpha := [\alpha_1 \wedge \alpha_2] : E \to E^*$. We've already seen that $\alpha^* = -\alpha$. We need the following fact:

Lemma 11.0.1. Let E be a vector bundle of dim n and $\phi : E \to E \otimes K_S$. Then $\sum_i a_1 \wedge \cdots \wedge \phi(a_i) \wedge \cdots \wedge a_n = \mathrm{tr}(\phi) a_1 \wedge \cdots \wedge a_n$ for $a_i \in V$.

We refer the reader to [FH] 111-112 for a proof.
Now let $\alpha : E \to E^*$ be skew-map corresponding to $\alpha_1 \wedge \alpha_2 \in \wedge^2(E^*)$ and $\phi : E \to E \otimes K_S$

Lemma 11.0.2. We have

$$\alpha\phi - (\alpha\phi)^* = \mathrm{tr}(\phi)\alpha$$

as elements of $E^* \otimes E^* \otimes K_S$.

Proof. As α is skew, we only need to show $\alpha\phi - (-\phi^*\alpha) = \mathrm{tr}(\phi)\alpha$.
Let $e \in E$, the LHS is given by

$$\begin{aligned}(\alpha\phi + \phi^*\alpha)(e) &= ((\phi^*\alpha_1)(e)\alpha_2 - \alpha_2(e)\phi^*\alpha_1) + (\alpha_1(e)\phi^*\alpha_2 - (\phi^*\alpha_2)(e)\alpha_1)\\ &= [\phi^*\alpha_1 \wedge \alpha_2 + \alpha_1 \wedge \phi^*\alpha_2](e)\\ &= [\mathrm{tr}(\phi^*)\alpha_1 \wedge \alpha_2](e) = \mathrm{tr}(\phi)[\alpha_1 \wedge \alpha_2](e) = \mathrm{tr}(\phi)\alpha(e)\end{aligned}$$

□

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