Chapter 2: Curving

Distances Determine Geometry

The spacetime interval is the “distance” between events

> "Distance" here means not just spatial but also temporal. Its the **Spacetime** distance between events.

Reference Frames are Secondary

- Spacetime geometry exists regardless of reference frame (and even without a frame)
  - As such, all frames are equally valid

Free Float Frames

- Relative accelerations are called “tidal accelerations”
- Reference frame can be as
  - tiny as a nucleus
  - short as a gamma ray emission
- Schwarzschild coordinates apply to slowly spinning or non-rotating bodies
  - We can talk about the Earth and Sun using Schwarzschild coordinates

The r-coordinate: the Reduced Circumference

- Due to space stretching in the presence of massive objets, we don’t measure radial distances (Euclidean Geometry doesn’t work for spacetime)
  - The is especially apparent with black holes
- walk around the shell of constant radius, measuring the Circumference. Divide by 2π to obtain.
  - Radial distance between shells is $d\sigma$ which will be greater than the difference from two concentric shells
- We can do this for spherically symmetric objects that are rotating “slow enough”
  - angle doesn’t matter
  - only depends on r-coordinate
  - density doesn’t have to be uniform
    - change with r okay; change with ϕ or θ is violates symmetry

Schwarzschild Metric describes spacetime external to any **isolated**, slowly rotating, and spherically symmetric body.

Gravitational Red Shift

- dt_{shell} time between clock ticks > $dt_{far-away}$ time between clock ticks

Mass in Units of Length

- Conversion factor:

$$\frac{G}{c^2}$$

Where:

- G is Gravitational constant: $6.6726 \times 10^{-11} \frac{m^3}{kg\ s^2}$
- c is the speed of light: $2.99792458 \times 10^8 \frac{m}{s}$

Such that:

$$M_{in\ m} = \frac{G}{c^2} M_{in\ kg}$$

Satellite Motion in a Plane

- A satellite is restricted to its plane of motion because of spherical symmetry
  - There is nothing to displace or alter it from its plane of motion
  - So we only need two coordinates to describe the location of the satellite: r, ϕ

> **??**
> How do we know if a given distance is a Euclidean or Non-Euclidean distance? E.g. the distance from Earth to Moon?

Metric for Flat Spacetime

- From Ch1, we have the spacetime interval as:

$$\tau^2 = t^2 - s^2$$
$$\sigma^2 = -t^2 + s^2$$

Expanding into two dimensions of space:

$$s^2 = x^2 + y^2$$
$$\tau^2 = t^2 - (x^2 + y^2) = t^2 - x^2 - y^2$$
$$\sigma^2 = -t^2 + (x^2 + y^2) = -t^2 + x^2 + y^2$$

Changing into spherical coordinates:

$$(d\tau)^2 = (dt)^2 - (dr)^2 - (rd\phi)^2$$
$$(d\sigma)^2 = -(dt)^2 + (dr)^2 + (rd\phi)^2$$

Schwarzschild Metric

We take the [Metric for Flat Spacetime]:

$$(d\tau)^2 = (dt)^2 - (dr)^2 - (rd\phi)^2$$

and account for the curvature between two concentric shells:

$$(d\tau)^2 = \left(1 - \frac{2M}{r}\right)(dt)^2 - \left(1 - \frac{2M}{r}\right)^{-1}(dr)^2 - (rd\phi)^2$$

And then we can describe two events close to one another, where $d\phi$ is small such that $rd\phi$ describes an arc length for a straight line. r is the r-coordinate which is a Non-Euclidean distance.

$$(d\sigma)^2 = -\left(1 - \frac{2M}{r}\right)(dt)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(dr)^2 + (rd\phi)^2$$

Sense Making

1. The **curvature factor** only depends on r, not ϕ
    - $\left(1 - \frac{2M}{r}\right)$
    - We expect only r dependence from a spherically symmetric body.

2. As $r \to \infty$, curvature factor approaches unity and the Schwarzschild metric becomes the [Metric for Flat Spacetime].

3. As $M \to 0$, curvature factor approaches unity and the Schwarzschild metric becomes the [Metric for Flat Spacetime].

4. Two events only separated by an r distance:

    - $d\phi = 0,\ dt = 0$

$$d\sigma = dr_{shell} = \left(1 - \frac{2M}{r}\right)^{-1/2} dr$$

5. Two events only separated by time on the same shell:
    - $d\phi = 0,\ dr = 0$

$$(d\tau)^2 = (dt_{shell})^2 = \left(1 - \frac{2M}{r}\right)(dt)^2 - \left(1 - \frac{2M}{r}\right)^{-1}(dr)^2 - (rd\phi)^2$$

$$d\tau = dt_{shell} = \left(1 - \frac{2M}{r}\right)^{1/2} dt$$

> **Note** that dt is the time between ticks with respect to [Far-away Time].

Far-away Time

- t-coordinate
  - also called “ephemeris time”
- time lapse between two events, dt, is recorded by a far-away observer on their clock (far-away time)
  - this is the clock tick rate for flat spacetime